WITTEN'S PERTURBATION ON STRATA

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Abstract. The main result is a version of Morse inequalities for the minimum and maximum ideal boundary conditions of the de Rham complex on strata of compact Thom-Mather stratifications, endowed with adapted metrics. An adaptation of the analytic method of Witten is used in the proof, as well as certain perturbation of the harmonic oscillator related with the Dunkl harmonic oscillator.

Contents

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1. Introduction and main results

A Hilbert complex [6] is a differential complex given by a densely defined closed operator $\mathbf{d}$ in a graded separable Hilbert space $\mathfrak{H}$. The corresponding Laplacian $\mathbf{\Delta} = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ is a self-adjoint operator in $\mathfrak{H}$. It is said that $\mathbf{d}$ is discrete when $\mathbf{\Delta}$ has a discrete spectrum[1]; in particular, its homology is of finite dimension by a version of the Hodge decomposition.

Let $(\Omega_0(M), d)$ be the compactly supported de Rham complex of a Riemannian manifold M. Its Hilbert complex extensions in $L^2\Omega(M)$ (the graded Hilbert space of square integrable differential forms) are called its ideal boundary conditions (i.b.c.). There is a minimum i.b.c., $d_{\text{min}} = \overline{d}$, and a maximum i.b.c., $d_{\text{max}} = \delta^*$, where δ is de Rham coderivative acting on $\Omega_0(M)$. The Laplacian defined by $d_{\text{min/max}}$ is denoted by $\Delta_{\text{min/max}}$. It is well known that $d_{\text{min}} = d_{\text{max}}$ if M is complete, but suppose that M may not be complete. The i.b.c. $d_{\text{min/max}}$ defines the min/max-cohomology $H^\bullet_{\text{min/max}}(M)$, min/max-Betti numbers $\beta^r_{\text{min/max}} = \beta^r_{\text{min/max}}(M)$, and min/max-Euler characteristic $\chi_{\text{min/max}} = \chi_{\text{min/max}}(M)$ (if the min/max-Betti numbers are finite); these are quasi-isometric invariants of M. These concepts can indeed be defined for arbitrary elliptic complexes [6]. these are quasi-isometric invariants of M. These concepts can indeed be defined for arbitrary elliptic complexes [6].

From now on, assume that M is a stratum of a compact Thom-Mather stratification A [36, 26, 27, 37]. Roughly speaking, around each $x \in \overline{M}$, there is a chart of A with values in a product $\mathbb{R}^m \times c(L)$, where:

- L is a compact Thom-Mather stratification of lower depth, and $c(L) = L \times [0, \infty)/L \times \{0\}$ (the cone with link L);
- x corresponds to $(0, *)$, where $*$ is the vertex of $c(L)$; and,
- near x, M corresponds to $\mathbb{R}^m \times M'$ for some stratum M' of $c(L)$.

We have, either $M' = N \times \mathbb{R}_+$ for some stratum N of L, or $M' = \{*\}$. Note that $x \in M$ just when $M' = \{*\}$. Let $\rho : c(L) \to [0, \infty)$ be the canonical function induced by the second factor projection $L \times [0, \infty) \to [0, \infty)$. The sum of ρ and the norm of $\mathbb{R}^m$ will be also called the canonical function of $\mathbb{R}^m \times c(L)$.

[1]Recall that a self-adjoint operator has a discrete spectrum when there is no essential spectrum; i.e., the spectrum consists of eigenvalues with finite multiplicity without accumulation points.

Endow M with a Riemannian metric g, which is adapted in the following sense defined by induction on the depth of M [9, 10]: there is a chart around each $x \in \overline{M} \setminus M$ as above such that g is quasi-isometric to a model metric of the form $g_0 + \rho^2 \tilde{g} + (d\rho)^2$ on $\mathbb{R}^m \times N \times \mathbb{R}_+$, where g_0 is the Euclidean metric on $\mathbb{R}^m$ and $\tilde{g}$ an adapted metric on N; this is well defined since $\operatorname{depth} N < \operatorname{depth} M$. Note that g may not be complete. More general adapted metrics are considered in [29, 30, 4]. The first main result of the paper is the following.

Theorem 1.1. *With the above notation, the following properties hold:*

- (i) $d_{\mathrm{min/max}}$ *is discrete.*
- (ii) *Let* $0 \leq \lambda_{\mathrm{min/max},0} \leq \lambda_{\mathrm{min/max},1} \leq \cdots$ *be the eigenvalues of* $\Delta_{\mathrm{min/max}}$*, repeated according to their multiplicities. Then there is some* $\theta > 0$ *such that* $\liminf_k \lambda_{\mathrm{min/max},k}\, k^{-\theta} > 0$.

The discreteness of d_{min} is essentially due to J. Cheeger [9, 10]. Theorem 1.1-(ii) is a weak version of the Weyl's assymptotic formula (see e.g. [33, Theorem 8.16]). Elliptic theory for the case of conformally conic manifolds was studied in [7, 21], and a non-commutative index theorem for the case of conical pseudo-manifolds is given in [13].

A smooth function f on M is called relatively admissible (or rel-admissible) when the functions $|df|$ and $|\operatorname{Hess} f|$ are bounded. In this case, f may not have any continuous extension to $\overline{M}$, but it has a continuous extension to the (componentwise) metric completion $\widehat{M}$ of M. Then it makes sense to say that $x \in \widehat{M}$ is a rel-critical point of f when there is a sequence (y_k) in M such that $\lim_k y_k = x$ in $\widehat{M}$ and $\lim_k |df(y_k)| = 0$. To say that f is a rel-Morse function on M, it should be also required that $\operatorname{Hess} f$ is "rel-non-degenerate" at each rel-critical point x, but a "rel-Morse lemma" is missing. Thus, instead, we require the existence of a local model of $\widehat{M}$ centered at x of the form $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times c(L_+) \times c(L_-)$ so that:

- M corresponds to the stratum $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, where $M_\pm$ is a stratum of $c(L_\pm)$; and
- f corresponds to a constant plus the model function $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, where $\rho_\pm$ is the canonical function on $\mathbb{R}^{m_\pm} \times c(L_\pm)$.

Either $M_\pm$ is the vertex stratum $\{*_\pm\}$ of $c(L_\pm)$, or $M_\pm = N_\pm \times \mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$; in the second case, let $n_\pm = \dim N_\pm$. This local model makes sense because the product of two Thom-Mather stratifications can be endowed with a Thom-Mather structure; in particular, the product of two cones becomes a cone. There is no canonical choice of a product Thom-Mather structure, but all of them have the same adapted metrics.

For each rel-critical point x of f as above and every $r \in \mathbb{Z}$, define $\nu^r_{x,\mathrm{min/max}} = \nu^r_{x,\mathrm{min/max}}(f)$ in the following way. If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = N_- \times \mathbb{R}_+$, let

$$\nu^r_{x,\mathrm{min/max}} = \sum_{r_+, r_-} \beta^{r_+}_{\mathrm{min/max}}(N_+)\, \beta^{r_-}_{\mathrm{min/max}}(N_-)\ ,$$

where (r_+, r_-) runs in the subset of $\mathbb{Z}^2$ determined by the conditions:

$$r = m_- + r_+ + r_- + 1 \,, \tag{1}$$

$$r_+ \leq \begin{cases} \frac{n_+}{2} - 1 & \text{if } n_+ \text{ is even} \\ \frac{n_+-3}{2} & \text{if } n_+ \text{ is odd, in the minimum i.b.c. case} \\ \frac{n_+-1}{2} & \text{if } n_+ \text{ is odd, in the maximum i.b.c. case} \,, \end{cases} \tag{2}$$

$$r_- \geq \begin{cases} \frac{n_-}{2} & \text{if } n_- \text{ is even} \\ \frac{n_+-1}{2} & \text{if } n_- \text{ is odd, in the minimum i.b.c. case} \\ \frac{n_-+1}{2} & \text{if } n_- \text{ is odd, in the maximum i.b.c. case} \,, \end{cases} \tag{3}$$

If $M_+ = \{*_+\}$ and $M_- = N_- \times \mathbb{R}_+$, let $\nu^r_{x,\min/\max} = \sum_{r_+} \beta^{r_+}_{\min/\max}(N_+)$, where r_+ runs in the the set of integers satisfying $r = m_- + r_+$ and (2). If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = \{*_-\}$, let $\nu^r_{x,\min/\max} = \sum_{r_-} \beta^{r_-}_{\min/\max}(N_-)$, where r_- runs in the the set of integers satisfying $r = m_- + r_- + 1$ and (3). If $M_+ = \{*_+\}$ and $M_- = \{*_-\}$, let[2] $\nu^r_{x,\min/\max} = \delta_{r,m_-}$. Finally, let $\nu^r_{\min/\max} = \sum_x \nu^r_{x,\min/\max}$, where x runs in the rel-critical point set of f. Our second main result is the following.

Theorem 1.2. *With the above notation, we have the inequalities*

$$\begin{gathered} \beta^0_{\min/\max} \leq \nu^0_{\min/\max} \,, \\ \beta^1_{\min/\max} - \beta^0_{\min/\max} \leq \nu^1_{\min/\max} - \nu^0_{\min/\max} \,, \\ \beta^2_{\min/\max} - \beta^1_{\min/\max} + \beta^0_{\min/\max} \leq \nu^2_{\min/\max} - \nu^1_{\min/\max} + \nu^0_{\min/\max} \,, \end{gathered}$$

etc., and the equality

$$\chi_{\min/\max} = \sum_r (-1)^r \, \nu^r_{\min/\max} \,.$$

We also show that the existence of rel-Morse functions. For instance, for any smooth action of a compact Lie group G on a closed manifold M, any invariant Morse-Bott function on M whose critical manifolds are orbits induces a rel-Morse function on $G\backslash M$; this provides a rich family of examples where Theorem 1.2 can be applied.

To prove Theorem 1.1, it is first shown that the stated properties are "rel-local" (Section 5), and it is well known that they are invariant by quasi-isometries. Then the spectrum is studied for the local models $\mathbb{R}^m \times N \times \mathbb{R}_+$ with the model metrics $g_0 + \rho^2\tilde{g} + (d\rho)^2$, assuming that the result holds for N with $\tilde{g}$ by induction. In fact, by the min-max principle, it is enough to make this argument for the minimun/maximum i.b.c. $d_{s,\min/\max}$ of the Witten's perturbation d_s ($s > 0$) of d defined by any rel-Morse function [41]; the Laplacian defined by $d_{s,\min/\max}$ is denoted by $\Delta_{s,\min/\max}$. In this way, the proof of Theorem 1.1 becomes a step in the proof of Theorem 1.2 by using the analytic method of E. Witten; specially, as it is described in [33, Chapters 9 and 14].

A part of that method is a local analysis around the rel-critical points; more explicitly, the spectral analysis of the perturbed Laplacian $\Delta_{s,\min/\max}$ defined with the model functions $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$. By the version of the Künneth formula for Hilbert complexes [6], this study can be reduced to the

[2]Kronecker's delta is used.

case of the functions $\pm\frac{1}{2}\rho^2$ on $N \times \mathbb{R}_+$, where ρ is the canonical function of $c(L)$. Then the discrete spectral decomposition for N with $\tilde{g}$ is used to split the Witten's perturbation of the de Rham complex of $N \times \mathbb{R}_+$ into direct sum of simple elliptic complexes of two types (Sections 7, 10 and 11), whose Laplacians are given by the perturbation of the harmonic oscillator on $\mathbb{R}_+$ studied in [1], which is related to the Dunkl harmonic oscillator. We end up with the spectral properties of $\Delta_{s,\min/\max}$ needed to describe the "cohomological contribution" from the rel-critical points (Section 16.3).

Another part of the adaptation of Witten's method is the proof of the "null cohomological contribution" away from the rel-critical points. In this part, some arguments of [33, Chapter 14] cannot be used because there is no version of the Sobolev embedding theorem with the Sobolev spaces $W^m(\Delta_{\min/\max})$ defined with $\Delta_{\min/\max}$; such a result may be true, but the usual way to prove it does not work since $W^m(\Delta_{\min/\max})$ may depend on the choice of the adapted metric (Section 17). Therefore a new method is applied in that part of the proof (Section 16.2), which uses strongly Theorem 1.1-(ii).

By extending f to $\widehat{M}$, Theorem 1.2 can be considered as Morse inequalities on the Thom-Mather stratification $\widehat{M}$. In this sense, it would be interesting to compare it with the Morse inequalities of Goresky-MacPherson [15, Chapter 6, Section 6.12], where they consider intersection homology with lower middle perversity of complex analytic varieties with Whitney stratifications. Another analytic proof of Morse inequalities was made by U. Luwig in [23, 24, 25] on the special case of conformally conic manifolds, but her admissible and Morse functions are different from ours: the norm of their differential is bounded away from zero around the frontier of the stratum, and the norm of their Hessian may be unbounded.

In the future, we hope to extend this work to the case of other types of adapted metrics (those considered in [29, 30, 4], or even more general ones); in the case of $d_{\min}$ with the adapted metrics of [29, 30, 4], it would give Morse inequalities for the intersection homology with arbitrary perversity. This will require the study of a perturbation of the harmonic oscillator on $\mathbb{R}_+$ more general than in [1].

It is also natural to try to extend this work to the case of "rel-Morse-Bott functions", where the rel-critical point set consists of "rel-non-degenerate rel-critical Thom-Mather substratifications".

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2. Preliminaries on Thom-Mather stratifications

2.1. **Thom-Mather stratifications.** Here, we recall the needed concepts introduced by R. Thom [36] and J. Mather [26]. We mainly follow [37], and some new remarks are also made, specially concerning products.

2.1.1. *Thom-Mather stratifications and their morphisms.* Let A be Hausdorff, locally compact and second countable topological space. Let $X \subset A$ be a locally closed subset. Two subsets $Y, Z \subset A$ are said to be *equal near* X (or $Y = Z$ *near*

X) if $Y \cap U = Z \cap U$ for some neighborhood U of X in A. It is also said that two maps, $f : Y \to B$ and $g : Z \to B$, are *equal near* X (or $f = g$ *near* X) when there is some neighborhood U of X in A such that $Y \cap U = Z \cap U$, and the restrictions of f and g to $Y \cap U$ are equal.

Consider triples (T, π, ρ), where T is an open neighborhood of X in A, $\pi : T \to X$ is a continuous retraction, and $\rho : X \to [0, \infty)$ is a continuous function such that $\rho^{-1}(0) = X$. Two such triples, (T, π, ρ) and (T', π', ρ'), are said to be *equal near* X when $T = T'$, $\pi = \pi'$ and $\rho = \rho'$ near X. This defines an equivalence relation whose equivalence classes are called *tubes* of X in A. The notation $[T, \pi, \rho]$ is used for the tube represented by (T, π, ρ). If X is open in A, then $[X, \mathrm{id}_X, 0]$ is its unique tube (the *trivial tube*).

Definition 2.1. A *Thom-Mather stratification*[3] (or *Thom-Mather stratified space*) is a triple $(A, \mathcal{S}, \tau)$, where:

- (i) A is a Hausdorff, locally compact and second countable space,
- (ii) $\mathcal{S}$ is a partition of A into locally closed subspaces with the additional structure of smooth (C^∞) manifolds, called *strata*, and
- (iii) τ is the assignment of a tube τ_X of each $X \in \mathcal{S}$ in A,

such that the following conditions are satisfied with some choice of $(T_X, \pi_X, \rho_X) \in \tau_X$ for each $X \in \mathcal{S}$:

- (iv) For all $X, Y \in \mathcal{S}$, if $X \cap \overline{Y} \neq \emptyset$, then $X \subset \overline{Y}$. The notation $X \leq Y$ is used in this case, and this defines a partial order relation on $\mathcal{S}$. As usual, $X < Y$ means that $X \leq Y$ but $X \neq Y$.
- (v) If $Y \neq X$ in $\mathcal{S}$ and $T_X \cap Y \neq \emptyset$, then $X < Y$ and $(\pi_X, \rho_X) : T_X \cap Y \to X \times \mathbb{R}_+$ is a smooth submersion; in particular, $\dim X < \dim Y$.
- (vi) If $X < Y$ in $\mathcal{S}$, then $\pi_Y(T_X \cap T_Y) \subset T_X$, and $\pi_X\, \pi_Y = \pi_X$ and $\rho_X\, \pi_Y = \rho_X$ on $T_X \cap T_Y$.

It may be also said that $(\mathcal{S}, \tau)$ is a *Thom-Mather stratification* of A.

Remark 1.
- (i) A is paracompact and normal.
- (ii) By the normality of A, we can also assume that, if $X, Y \in \mathcal{S}$ and $T_X \cap T_Y \neq \emptyset$, then $X \leq Y$ or $Y \leq X$.
- (iii) The frontier of a stratum X equals the union of the strata $Y < X$.
- (iv) The connected components of each stratum may have different dimensions.
- (v) The connected components of the strata, with the corresponding restrictions of the tubes, define an induced Thom-Mather stratification $A_{\mathrm{con}} \equiv (A, \mathcal{S}_{\mathrm{con}}, \tau_{\mathrm{con}})$; in this way, we can assume that the strata are connected if desirable.

Remark 2. The following are some variants of the concept "stratification" and related notions:

- (i) A *weak Thom-Mather stratification* is defined by removing the condition $\rho_X\, \pi_Y = \rho_X$ from Definition 2.1-(vi).
- (ii) A *stratification* is a pair $(A, \mathcal{S})$ satisfying Definition 2.1-(i),(ii),(iv); it is also said that $\mathcal{S}$ is a *stratification* of A. Definition 2.1-(iv) is called the *frontier condition*. If moreover τ satisfies the other conditions of Definition 2.1, then it is called *Thom-Mather structure* on $(A, \mathcal{S})$.

[3] This is called *abstract prestratification* in [26] and *abstract stratification* in [37].

(iii) If A is a subspace of a smooth manifold M, then a stratification $\mathcal{S}$ of A is usually required to consist of regular submanifolds of M; the term *stratified subspace* of M is used in this case. In [14], a weaker version of this notion is defined by requiring local finiteness of $\mathcal{S}$ instead of the frontier condition.

(iv) For a stratified subspace $(A,\mathcal{S})$ of a smooth manifold M, the *condition* (B), introduced by H. Whitney [39, 40], is defined as follows[4]. In the case $M=\mathbb{R}^m$, it requires that, for all $X\neq Y$ in $\mathcal{S}$, if (x_i) and (y_i) are sequences in X and Y, respectively, both of them converging in A to some $x\in X$, if the sequence of tangent spaces $T_{y_i}Y$ converges[5] to a linear subspace $T\subset\mathbb{R}^n$, and if the sequence of lines $\mathbb{R}(x_i-y_i)$ converges to a line $L\subset\mathbb{R}^m$, then $L\subset T$. This property is preserved by local diffeomorphisms of $\mathbb{R}^m$, and therefore generalizes to arbitrary smooth manifolds. This condition gives rise to the concept of *Whitney stratification* of a subspace (or *Whitney stratified subspace*) of M.

Example 2.2. (i) Any smooth manifold is a Thom-Mather stratification with one stratum and the trivial tube.

(ii) Any smooth manifold with boundary is a stratification with two strata, the interior and the boundary. It can be endowed with a Thom-Mather structure by using a collar of the boundary.

(iii) Any subanalytic subset of $\mathbb{R}^m$ has a primary and secondary stratifications; the secondary one satisfies condition (B) [22, 27, 17, 16, 18].

(iv) J. Mather [26] has proved that any Whitney stratified subspace of a smooth manifold admits a Thom-Mather structure (see also [14, Proposition 2.6 and Corollary 2.7]).

For a stratification $A\equiv(A,\mathcal{S})$, the *depth* of any $X\in\mathcal{S}$, denoted by $\operatorname{depth}X$, is the supremum of the naturals n such that there exist strata $X_0,\dots,X_n$ with $X_0<X_1<\dots<X_n=X$. Notice that $\operatorname{depth}X\le\dim X$. Moreover $\operatorname{depth}X=0$ (X is minimal in $\mathcal{S}$) if and only if X is closed in A. The *depth* and *dimension* of A are the supremum of the depths and dimensions of its strata, respectively. The dimension of A equals its topological dimension, which may be infinite. The depth of A is zero if and only if all strata are open and closed.

Let $A\equiv(A,\mathcal{S},\tau)$ be a Thom-Mather stratification. Let $B\subset A$ be a locally closed subset. Suppose that, for all $X\in\mathcal{S}$, $X\cap B$ is a smooth submanifold of X, and $B\cap\pi_X^{-1}(X\cap B)$, endowed with the restrictions of π_X and ρ_X, defines a tube $\tau_{X\cap B}$ of $X\cap B$ in B. Then let $\mathcal{S}|_B=\{\,X\cap B\mid X\in\mathcal{S}\,\}$, and let $\tau|_B$ be defined by the assignment of $\tau_{X\cap B}$ to each $X\cap B\in\mathcal{S}|_B$. If $(B,\mathcal{S}|_B,\tau|_B)$ satisfies the conditions of a stratification, it is said that the stratification A (or $(\mathcal{S},\tau)$) can be *restricted* to B, and $B\equiv(B,\mathcal{S}|_B,\tau|_B)$ is called a *restriction* of A (or $(\mathcal{S}|_B,\tau|_B)$ is called the *restriction* of $(\mathcal{S},\tau)$); it may be also said that B is a *Thom-Mather substratification* of A. For instance, A can be restricted to any open subset and to any locally closed union of strata. A restriction of a restriction of A is a restriction of A.

For a stratum X of A, we can consider the restriction of A to $\overline{X}$. In this way, to study X, we can assume that X is dense in A and $\dim X=\dim A$ if desirable.

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[4] Certain condition (A) was also introduced by H. Whitney in [39, 40], but J. Mather [26] has observed that it follows from condition (B).

[5] The convergence of linear subspaces of $\mathbb{R}^m$ is considered in the appropriate Grassmannians.

A locally closed subset $B \subset A$ is said to be *saturated* if the stratification A can be restricted to B and, for every $X \in \mathcal{S}$, there is a representative (T_X, π_X, ρ_X) of τ_X such that $\pi_X^{-1}(X \cap B) = T_X \cap B$.

Let $A' \equiv (A', \mathcal{S}', \tau')$ be another Thom-Mather stratification. A continuous map $f : A \to A'$ is called a *morphism* if, for any $X \in \mathcal{S}$, there is some $X' \in \mathcal{S}'$ such that $f(X) \subset X'$, the restriction $f : X \to X'$ is smooth, and there are $(T_X, \pi_X, \rho_X) \in \tau_X$ and $(T'_{X'}, \pi'_{X'}, \rho'_{X'}) \in \tau'_{X'}$ such that $f(T_X) \subset T'_{X'}$, $f\pi_X = \pi'_{X'}f$ and $f\rho_X = \rho'_{X'}$. Notice that the continuity of a morphism follows from the other conditions. Morphisms between stratifications form a category with the operation of composition; in particular, we have the corresponding concepts of *isomorphism* and *automorphism*. The set of morphisms $A \to A'$ is denoted by $\mathrm{Mor}(A, A')$, and the group of automorphisms of A is denoted by $\mathrm{Aut}(A)$. The other variants of the concept "stratification" given in Remark 2 also have obvious corresponding versions of morphisms, isomorphisms and automorphisms; in particular, we get the concept of *weak morphism* between weak Thom-Mather stratifications. A (weak) morphism is called *submersive* when it restricts to smooth submersions between the strata.

Example 2.3. Let G be a compact Lie group G acting smoothly on a closed manifold M. Consider the orbit type stratifications of M and $G\backslash M$ [5]. It is well known that $G\backslash M$ admits a Thom-Mather structure [37, Introduction], which can be seen as follows. $G\backslash M$ is locally isomorphic to a semi-algebraic subset of an Euclidean space whose primary and secondary stratifications are equal [2]. By using an invariant smooth partition of unity of M, like in the Whitney's embedding theorem, it follows that $G\backslash M$ is isomorphic to a Whitney stratified subspace of some Euclidean space, and therefore it admits a Thom-Mather structure. This can also be seen by observing that the stratification of M satisfies condition (B), and the proof of [14, Proposition 2.6] can be adapted to produce an invariant[6] Thom-Mather structure on M, which induces a Thom-Mather structure on $G\backslash M$.

The following two lemmas are easy to prove.

Lemma 2.4. *Let A be a Hausdorff, locally compact and second countable space, $\{U_i\}$ an open covering of A, and $(\mathcal{S}_i, \tau_i)$ a Thom-Mather stratification of each U_i.*

- (i) *If $(\mathcal{S}_i, \tau_i)$ and $(\mathcal{S}_j, \tau_j)$ have the same restrictions to $U_{ij} := U_i \cap U_j$ for all i and j, then there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on A whose restriction to each U_i is $(\mathcal{S}_i, \tau_i)$.*
- (ii) *If $((\mathcal{S}_i|_{U_{ij}})_{\mathrm{con}}, (\tau_i|_{U_{ij}})_{\mathrm{con}}) = ((\mathcal{S}_j|_{U_{ij}})_{\mathrm{con}}, (\tau_j|_{U_{ij}})_{\mathrm{con}})$ for all i and j, then there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on A with connected strata such that $((\mathcal{S}|_{U_i})_{\mathrm{con}}, (\tau|_{U_i})_{\mathrm{con}}) = (\mathcal{S}_{i,\mathrm{con}}, \tau_{i,\mathrm{con}})$.*

Lemma 2.5. *Let $(A', \mathcal{S}', \tau')$ be another Thom-Mather stratification.*

- (i) *With the notation of Lemma 2.4-(i), let $f_i : (U_i, \mathcal{S}_i, \tau_i) \to (A', \mathcal{S}', \tau')$ be a morphism for each i. If $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all i and j, then the combination of the maps f_i is a morphism $f : (A, \mathcal{S}, \tau) \to (A', \mathcal{S}', \tau')$.*
- (ii) *With the notation of Lemma 2.4-(ii), let $f_i : (U_i, \mathcal{S}_{i,\mathrm{con}}, \tau_{i,\mathrm{con}}) \to (A', \mathcal{S}', \tau')$ be a morphism for each i. If $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all i and j, then the combination of the maps f_i is a morphism $f : (A, \mathcal{S}, \tau) \to (A', \mathcal{S}', \tau')$.*

[6] G acts by automorphisms.

Remark 3. As a particular case of Lemma 2.4, given a countable family of Thom-Mather stratifications, $\{A_i \equiv (A_i, \mathcal{S}_i, \tau_i)\}$, there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on the topological sum $\bigsqcup_i A_i$ whose restriction to each A_i is $(\mathcal{S}_i, \tau_i)$; this $(\mathcal{S}, \tau)$ will be called the *sum* of the Thom-Mather stratifications $(\mathcal{S}_i, \tau_i)$.

2.1.2. *Products.* The product of two weak Thom-Mather stratifications, A and A', has a weak Thom-Mather stratification $A \times A' \equiv (A \times A', \mathcal{S}'', \tau'')$ with $\mathcal{S}'' = \{\, X \times X' \mid X \in \mathcal{S},\ X' \in \mathcal{S}' \,\}$ and $\tau''_{X\times X'} = [T''_{X\times X'}, \pi''_{X\times X'}, \rho''_{X\times X'}]$, where $T''_{X\times X'} = T_X \times T'_{X'}$, $\pi''_{X\times X'} = \pi_X \times \pi'_{X'}$ and $\rho''_{X\times X'}(x,x') = \rho_X(x) + \rho'_{X'}(x')$.

If A and A' are Thom-Mather stratifications and the depth of at least one of them is zero, then $A \times A'$ is a Thom-Mather stratification, but this is not true when the depths of A and A' are positive [37, Section 1.2.9, pp. 5–6]. Another choice of $\rho_{X\times X'}$ is needed to get the second equality of Definition 2.1-(vi). For instance, $\rho''_{X\times X'} = \max\{\rho_X, \rho'_{X'}\}$ satisfies that condition, but it is not smooth on the intersection of the strata with $T''_{X\times X'}$. To solve this problem, pick up a function $h : [0,\infty)^2 \to [0,\infty)$ that is continuous, homogeneous of degree one, smooth on $\mathbb{R}^2_+$, with $h^{-1}(0) = \{(0,0)\}$, and such that, for some $C > 1$, we have $h(r,s) = \max\{r,s\}$ if $C \min\{r,s\} < \max\{r,s\}$. Then $A \times A'$ becomes a Thom-Mather stratification by setting $\rho''_{X\times X'}(x,x') = h(\rho_X(x), \rho'_{X'}(x'))$; it will be called *a product* of A and A'.

2.1.3. *Cones.* Recall that the *cone* with *link* a non-empty topological space L is the quotient space $c(L) = L \times [0,\infty)/L \times \{0\}$. The class $* = L \times \{0\}$ is called the *vertex* or *summit* of $c(L)$. The element of $c(L)$ represented by each $(x,\rho) \in L \times [0,\infty)$ will be denoted by $[x,\rho]$. The function on $c(L)$ induced by the second factor projection $L \times [0,\infty) \to [0,\infty)$ will be called its *canonical function*, and will be usually denoted by ρ. Notice that $c(L)$ is locally compact if and only if L is compact. It is also declared that $c(\emptyset)$ is the singleton space $\{*\}$, and the above terminology can be obviously adapted to this case.

Now, suppose that L is a compact Thom-Mather stratification. Then $c(L)$ has a canonical Thom-Mather stratification so that $\{*\}$ is a stratum, its restriction to $c(L) \setminus \{*\} = L \times \mathbb{R}_+$ is the product Thom-Mather stratification, and the tube of $\{*\}$ is $[c(L), \pi, \rho]$, where ρ is the canonical function and π is the unique map $c(L) \to \{*\}$. If $L \neq \emptyset$, then $\operatorname{depth} c(L) = \operatorname{depth} L + 1$ and $\dim c(L) = \dim L + 1$. For any $\epsilon > 0$, let $c_\epsilon(L) = \rho^{-1}([0,\epsilon))$.

Let L' be another compact Thom-Mather stratification, and let $*'$ denote the vertex of $c(L')$. If $L \neq \emptyset$, the *cone* of any morphism $f : L \to L'$ is the morphism $c(f) : c(L) \to c(L')$ induced by $f \times \mathrm{id} : L \times [0,\infty) \to L' \times [0,\infty)$. If $L = \emptyset$, $c(f)$ is defined by mapping $*$ to $*'$. Reciprocally, it is easy to check that, for any morphism $h : c(L) \to c(L')$, there is some morphism $f : L \to L'$ such that $h = c(f)$ near $*$; in particular, $h(*) = *'$. Let $c(\mathrm{Aut}(L)) = \{\, c(f) \mid f \in \mathrm{Aut}(L) \,\} \subset \mathrm{Aut}(c(L))$.

Example 2.6. For each integer $m \geq 1$, there is a canonical homeomorphism $\mathrm{can} : c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$ defined by $\mathrm{can}([x,\rho]) = \rho x$. Of course, this is not an isomorphism of Thom-Mather stratifications, but it restricts to a diffeomorphism of the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$ to $\mathbb{R}^m \setminus \{0\}$. Via $\mathrm{can} : c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$, the canonical function of $c(\mathbb{S}^{m-1})$ corresponds to the function $\rho_0(x) = |x|$ on $\mathbb{R}^m$, which will be also called the *canonical function* on $\mathbb{R}^m$ for the scope of this paper. If ρ_1 is the canonical function on $c(L)$ for some compact Thom-Mather stratification L, then the function $\rho = \sqrt{\rho_0^2 + \rho_1^2}$ will be called the *canonical function* on $\mathbb{R}^m \times c(L)$.

The following argument shows that a product of two cones is isomorphic to a cone. With the above notation, let $\rho : c(L) \to [0,\infty)$ and $\rho' : c(L') \to [0,\infty)$ be the canonical functions, and let $\rho'' = h(\rho\times\rho') : c(L)\times c(L') \to [0,\infty)$ for a function h like in Section 2.1.2. Since the restrictions $\rho : L\times\mathbb{R}_+ \to \mathbb{R}_+$ and $\rho' : L'\times\mathbb{R}_+ \to \mathbb{R}_+$ are submersive weak morphisms, and $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ is non-singular, it follows that $\rho'' : c(L)\times c(L')\setminus\{(*,*')\} \to \mathbb{R}_+$ is a submersive weak morphism. Hence $L'' = \rho''^{-1}(1)$ is saturated in $c(L)\times c(L')$ [37, Lemma 2.9, p. 17]. Let $*''$ denote the vertex of $c(L'')$. Since h is homogeneous of degree one, the mapping

$$[([x,r],[x',r']),s] \mapsto ([x,rs],[x',r's])$$

defines an isomorphism $c(L'') \to c(L)\times c(L')$, whose inverse is given by $(*,*') \mapsto *''$, and

$$([x,r],[x',r']) \mapsto \left[\left(\left[x,\frac{r}{h(r,r')}\right],\left[x',\frac{r'}{h(r,r')}\right]\right),h(r,r')\right]$$

if $(r,r') \neq (0,0)$.

2.1.4. *Conic bundles.* Let X be a smooth manifold, L a compact Thom-Mather stratification, and $\pi : T \to X$ a fiber bundle whose typical fiber is $c(L)$ and whose structural group can be reduced to $c(\mathrm{Aut}(L))$. Thus there is a family of local trivializations of π, $\{(U_i,\phi_i)\}$, such that the corresponding transition functions define a cocycle with values in $c(\mathrm{Aut}(L))$; i.e., for all i and j, there is a map $h_{ij} : U_{ij} := U_i\cap U_j \to c(\mathrm{Aut}(L))$ such that $\phi_j\phi_i^{-1}(x,y) = (x,h_{ij}(x)(y))$ for every $x\in U_{ij}$ and $y\in c(L)$. Thus we get another cocycle consisting of maps $g_{ij} : U_{ij} \to \mathrm{Aut}(L)$ so that $h_{ij}(x) = c(g_{ij}(x))$ for all $x\in U_{ij}$. Consider the Thom-Mather stratification on each open subset $\pi^{-1}(U_i)\subset T$ that corresponds by ϕ_i to the product Thom-Mather stratification on $U_i\times c(L)$. For each connected open $V\subset U_{ij}$ and every stratum N_0 of L, there is an stratum N_1 of L such that $g_{ij}(x)(N_0) = N_1$ for all $x\in V$, and suppose also that, in this case, the map $V\times N_0 \to N_1$, $(x,y)\mapsto g_{ij}(x)(y)$, is smooth. Then each mapping $(x,y)\mapsto(x,g_{ij}(x)(y))$ defines an automorphism of $U_{ij}\times L$. This means that the induced Thom-Mather stratifications on $\pi^{-1}(U_i)$ and $\pi^{-1}(U_j)$ have the same restriction to $\pi^{-1}(U_{ij})$. By Lemma 2.4-(i), it follows that there is a unique Thom-Mather stratification on T whose restriction to each $\pi^{-1}(U_i)$ is the above Thom-Mather stratification. Furthermore there is a canonical section of π, called the *vertex* (or *summit*) *section*, which is well defined by $x\mapsto *_x = \phi_i^{-1}(x,*)$ if $x\in U_i$, where $*$ denotes the vertex of $c(L)$; each $*_x$ can be called the *vertex* of the fiber over x. The image of the vertex section is a stratum of T, called the *vertex* (or *summit*) *stratum*, which is diffeomorphic to X.

If $\pi : T\to X$ is endowed with a maximal family Φ of trivializations satisfying the above conditions, it will be called a *conic bundle*, and the corresponding Thom-Mather stratification on T is called its *conic bundle Thom-Mather stratification*. It will be also said that Φ is the *conic bundle structure* of π.

Let $\rho : c(L)\to[0,\infty)$ be the canonical function. Its lift to each $U_i\times c(L)$ is also denoted by ρ. The functions $\phi^*\rho$ on the sets $\pi_X^{-1}(U_i)$ can be combined to define a function $\rho : T\to[0,\infty)$. The tubular neighborhood of X in T is $[T,\pi,\rho]$, and (T,π,ρ) is called its *canonical representative*.

Let $\pi' : T'\to X'$ be another conic bundle, whose structure is given by a family Φ' of trivializations as above. Let $F : T\to T'$ be a fiber bundle morphism over a map $f : X\to X'$. Then we can choose $\{(U_i,\phi_i)\}$ as above and a family $\{(U'_i,\phi'_i)\}\subset\Phi'$ such that $f(U_i)\subset U'_i$ for all i, and therefore $F(\pi^{-1}(U_i))\subset\pi'^{-1}(U'_i)$. Let $h'_{ij} =$

$c(g'_{ij}) : U'_{ij} := U'_i \cap U'_j \to c(\mathrm{Aut}(L'))$ be the maps defined by the transition maps $\phi'_j \, \phi_i'^{-1}$ as above. Suppose that there are maps $\kappa_i : U_i \to \mathrm{Mor}(L, L')$ such that $\kappa_j(x) \, g_{ij}(x) = g'_{ij}(f(x)) \, \kappa_j(x)$ for all $x \in U_{ij}$. For each connected open $V \subset U_i$ and every stratum N of L, there is an stratum N' of L' such that $\kappa_i(x)(N) \subset N'$ for all $x \in V$, and assume also that, in this case, the map $V \times N \to N'$, $(x, y) \mapsto \kappa_i(x)(y)$, is smooth. Then F is called a *morphism of conic bundles*. In this case, each mapping $(x, y) \mapsto (f(x), \kappa_i(x)(y))$ defines a morphism $U_i \times c(L) \to U'_i \times c(L')$. So each restriction $F : \pi^{-1}(U_i) \to \pi'^{-1}(U'_i)$ is a morphism of Thom-Mather stratifications, and therefore $F : T \to T'$ is a morphism of Thom-Mather stratifications by Lemma 2.5-(i). According to Section 2.1.3, any morphism of Thom-Mather stratifications between conic bundles, preserving the vertex stratum, equals a conic bundle morphism near the vertex stratum.

The case of conic bundles is specially important because, as pointed out in [3, Chapitre A, Remarque 3], the proof of [37, Theorem 2.6, pp. 16–17] can be easily adapted to get the following.

Proposition 2.7. *Let $A \equiv (A, \mathcal{S}, \tau)$ be a Thom-Mather stratification with connected strata. Then, for any $X \in \mathcal{S}$, there is some $(T, \pi, \rho) \in \tau_X$ such that $\pi : T \to X$ admits a structure Φ of conic bundle such that the corresponding conic bundle Thom-Mather stratification is $(\mathcal{S}|_T, \tau|_T)$.*

Remark 4. (i) The notation T_X, π_X, ρ_X, L_X and Φ_X will be used when a reference to the stratum X is desired.
(ii) The connectedness of the strata is assumed for the sake of simplicity. In the general case, the description of Proposition 2.7 holds around the connected components of the strata.
(iii) We can choose ρ so that (T, π, ρ) is the canonical representative of the tube around X in T with its conic bundle Thom-Mather stratification.

Definition 2.8. A *chart* or *distinguished neighborhood* of A is a pair (O, ξ), where O is open in A and, for some $X \in \mathcal{S}$ and $\epsilon > 0$, with the notation and conditions of Proposition 2.7, ξ is an isomorphism $O \to B \times c_\epsilon(L)$ defined by some $(U, \phi) \in \Phi$ and some chart (U, ζ) of X with $\zeta(U) = B$, where B is an open subset of $\mathbb{R}^m$ for $m = \dim X$. It is said that (O, ξ) is said to be *centered* at $x \in X$ if B is an open ball centered at 0 and $\xi(x) = (0, *)$, where $*$ is the vertex of $c(L)$. A collection of charts that cover A is called an *atlas* of A.

Remark 5. Definition 2.8 also includes the case where any factor of the product $\mathbb{R}^m \times c(L)$ is missing by taking $m = 0$ or $L = \emptyset$.

Remark 6. The following two assertions follow by using charts and induction on the depth of the strata:

(i) In any Thom-Mather stratification, there is at most one dense stratum, which is open.
(ii) Any stratum with compact closure has a finite number of connected components.

2.1.5. *Uniqueness of Thom-Mather stratifications.*

Lemma 2.9. *Let A be a Hausdorff, locally compact and second countable space, let $(A', \mathcal{S}', \tau')$ be a Thom-Mather stratification with connected strata, and let $f : A \to A'$ be a continuous map. Then there is at most one Thom-Mather stratification*

$(\mathcal{S},\tau)$ on A with connected strata so that $f:(A,\mathcal{S},\tau)\to(A',\mathcal{S}',\tau')$ is a morphism that restricts to local diffeomorphism between corresponding strata.

Proof. Let $(\mathcal{S},\tau)$ be a Thom-Mather stratification on A satisfying the conditions of the statement. Then the elements of $\mathcal{S}$ are the connected components X of the sets $f^{-1}(X')$ for $X'\in\mathcal{S}$, endowed with the differential structure so that $f:X\to X'$ is a local diffeomorphism. Thus the elements of $\mathcal{S}$ are determined by f and the elements of $\mathcal{S}'$.

Let $X\in\mathcal{S}$ and $X'\in\mathcal{S}'$ with $f(X)\subset X'$, and let $(T,\pi,\rho)\in\tau_X$ and $(T',\pi',\rho')\in\tau'_{X'}$ with $f(T)\subset T'$, $\pi' f=f\pi$ and $\rho' f=\rho$; in particular, ρ is determined by f and ρ'. Let $x\in T$ and $x'=f(x)\in T'$, and let $Y\in\mathcal{S}$ such that $x\in Y$. Then $f\pi(x)=\pi'(x')$, obtaining that $\pi(x)$ is the unique point of $X\cap f^{-1}(\pi'(x'))$ that is contained in the connected component of x in $f^{-1}\pi'^{-1}(\pi'(x'))$. It follows that π is also determined by f and π', and therefore τ_X is determined by f and $\tau'_{X'}$. □

2.1.6. *Relatively local properties on strata.* The following kind of terminology will be used for a subspace X of an arbitrary topological space A. Let $\mathcal{P}$ be a property that may hold on open subsets $U\subset X$; for the sake of simplicity, let us say that "U is $\mathcal{P}$" when $\mathcal{P}$ holds on U. It is is said that X is *relatively locally* (or simply, *rel-locally*) $\mathcal{P}$ *at* some $x\in\overline{X}$ if there is a base $\mathcal{U}$ of open neighborhoods of x in A such that $U\cap X$ is $\mathcal{P}$ for all $U\in\mathcal{U}$; if X is rel-locally $\mathcal{P}$ at all points of $\overline{X}$, then X is said to be *relatively locally* (or simply, *rel-locally*) $\mathcal{P}$. Similarly, $\mathcal{P}$ is said to be a *relatively local* (or simply, *rel-local*) property when X is $\mathcal{P}$ if and only if it is rel-locally $\mathcal{P}$.

We will apply this terminology to the case where A is a Thom-Mather stratification and X is a stratum of A. For instance, on X, we will consider functions that are rel-locally bounded or rel-locally bounded away from zero, rel-locally finite open coverings, and rel-local connectedness at points of $\overline{X}$. Any locally finite covering of $\overline{X}$ by open subsets of A restricts to a rel-locally finite open covering of X; thus there exist rel-locally finite open coverings of X by the paracompactness of A. Observe that $\overline{X}$ is compact if and only if any rel-locally finite open covering of X is finite.

2.2. **Adapted metrics on strata.** The definition of adapted metrics was given for the regular stratum of any Thom-Mather stratification that is a pseudomanifold [9, 10, 29, 30]. But its definition has an obvious version for any stratum of a Thom-Mather stratification. In this paper, we will consider only the simplest type of adapted metrics, whose definition is recalled. The corresponding (componentwise) metric completion of strata will be specially studied.

2.2.1. *Adapted metrics on strata and local quasi-isometries between Thom-Mather stratifications.* Let A be a Thom-Mather stratification. The adapted metrics on its strata are combinations of the adapted metrics on their connected components with respect to the Thom-Mather stratification defined by those connected components. Thus we can assume that the strata of A are connected to define adapted metrics. This definition is given by induction on the depth of the strata.

Definition 2.10. Let M be a stratum of A. If $\operatorname{depth} M=0$, then M is a closed manifold, and any Riemannian metric on M is called *adapted*. If $\operatorname{depth} M>0$ and adapted metrics are defined for strata of lower depth, then an *adapted metric* on M is a Riemannian metric g such that, for any point $x\in\overline{M}\setminus M$, there is some chart

(O,ξ) of A centered at x, with $\xi(O) = B \times c_\epsilon(L)$ and $\xi(O \cap M) = B \times N \times (0,\epsilon)$ for some stratum N of L, so that g is quasi-isometric to $\xi^*(g_0 + \rho^2\tilde{g} + (d\rho)^2)$ on O, where g_0 is the standard Riemannian metric on $\mathbb{R}^m$, ρ is the standard coordinate of $\mathbb{R}_+$, and $\tilde{g}$ is some adapted metric on N, which is defined because the depth of N in L is smaller than the depth of M in A.

Remark 7. Since all Riemannian metrics on a smooth manifold are locally quasi-isometric, any metric on $\mathbb{R}^m$ could be used in Definition 2.10 instead of g_0.

Remark 8. The following properties follow by taking charts and using induction on the depth of the strata:

(i) Any pair of adapted metrics on M, g and g', are rel-locally quasi-isometric; in particular, if $\overline{M}$ is compact, then any pair of adapted metrics on M are quasi-isometric.
(ii) Any point in $\overline{M}$ has a countable base $\{ O_m \mid m \in \mathbb{N} \}$ of open neighborhoods such that, with respect to any adapted metric, $\operatorname{vol}(M \cap O_m) \to 0$ and $\max\{ \operatorname{diam} P \mid P \in \pi_0(M \cap O_m) \} \to 0$ as $m \to \infty$; in particular, if $\overline{M}$ is compact, then, with respect to any adapted metric, we have $\operatorname{vol} M < \infty$ and $\operatorname{diam} P < \infty$ for all $P \in \pi_0(M)$.
(iii) Any morphism of Thom-Mather stratifications restricts to rel-locally uniformly continuous maps between corresponding strata with respect to arbitrary adapted metrics.
(iv) If g and g' are adapted metrics on strata M and M' of Thom-Mather stratifications A and A', respectively, then $g \oplus g'$ is an adapted metric on the stratum $M \times M'$ of any product Thom-Mather stratification on $A \times A'$ (Section 2.1.2).

In [4, Appendix], it was proved that there exist adapted metrics on the regular stratum of any Thom-Mather stratification that is a pseudomanifold. It can be easily checked that the same argument proves the existence of adapted metrics on any stratum M of every Thom-Mather stratification A.

Example 2.11. The proof in [4, Appendix] also shows the following:

(i) With the notation of Definition 2.10, the metric $g = g_0 + \rho^2\tilde{g} + (d\rho)^2$ is adapted on the stratum $M = \mathbb{R}^m \times N \times \mathbb{R}_+$ of $c(L)$; it will be called a *model adapted metric*.
(ii) Given a rel-locally finite atlas $\{(O_a,\xi_a)\}$ of $\overline{M}$, a smooth partition of unity $\{\lambda_a\}$ subordinated to the open covering $\{M \cap O_a\}$ of M, and an adapted metric g_a on each $M \cap O_a$, then the metric $\sum_a \lambda_a g_a$ on M is adapted.

Example 2.12. For an integer $m \geq 1$, let $\tilde{g}_0$ be the restriction to $\mathbb{S}^{m-1}$ of the standard metric g_0 of $\mathbb{R}^m$. Then, via $\operatorname{can} : c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$ (Example 2.6), the model adapted metric $g_1 = \rho^2\tilde{g}_0 + (d\rho)^2$ on the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$ corresponds to g_0 on $\mathbb{R}^m \setminus \{0\}$.

Example 2.13. With the notation of Example 2.3, for any invariant Riemannian metric g on M, consider the Riemannian metric $\bar{g}$ on the strata of $G\backslash M$ so that the canonical projection of the strata of M to the strata of $G\backslash M$ is a Riemannian submersion. The proof of [14, Proposition 2.6] can be easily adapted to produce an invariant Thom-Mather structure on M so that the restriction of g to any stratum is adapted. Hence $\bar{g}$ is adapted for the induced Thom-Mather structure of $G\backslash M$.

A weak isomorphism between Thom-Mather stratifications is called a *local quasi-isometry* if it restricts to rel-local quasi-isometries between their strata with respect to adapted metrics; this is independent of the choice of adapted metrics by Remark 8-(i). In particular, a local quasi-isometry between compact Thom-Mather stratifications restricts to quasi-isometries between their strata; thus a local quasi-isometry between compact Thom-Mather stratifications will be called a *quasi-isometry*. The condition of being locally quasi-isometric defines an equivalence relation on the family of Thom-Mather stratifications on any Hausdorff, locally compact and second countable space; each equivalence class will be called a *quasi-isometry type* of Thom-Mather stratifications. By Remark 8-(iv), the product of Thom-Mather stratifications is unique up to local quasi-isometries.

Definition 2.14. Consider an adapted metric on a connected stratum M of a Thom-Mather stratification A, and let d denote the corresponding distance function on M. For each $x \in \overline{M}$ and $\rho > 0$, the *relative ball* (or *rel-ball*) of *radius* ρ and *center* x is the set consisting of the points $y \in M$ such that there is a sequence (z_k) in M with $\lim_k z_k = x$ in $\overline{M}$ and $\limsup_k d(y, z_k) < \rho$. The term *$\rho$-relative neighborhood* (or *ρ-rel-neighborhood*) of x will be also used for this concept.

Example 2.15.
(i) The rel-balls centered at points of M are the usual balls.
(ii) In the case of a model adapted metric on the stratum $M = N \times \mathbb{R}_+$ of $c(L)$, the ρ-rel-neighborhood of the vertex $*$ is $N \times (0, \rho)$.

2.2.2. *Relatively local completion.* Let M be a stratum of a Thom-Mather stratification A, and fix an adapted metric g on M.

Definition 2.16. Assume first that M is connected, and consider the distance function d on M induced by g. The *relatively local completion* (or simply, *rel-local completion*) is the subspace $\widehat{M}$ of the metric completion of M whose points can be represented by Cauchy sequences in M that converge in A; the limits in $\overline{M}$ of those sequences define a canonical continuous map $\lim : \widehat{M} \to \overline{M}$. The canonical dense injection of M into its metric completion restricts to a canonical dense injection $\iota : M \to \widehat{M}$ satisfying $\lim \iota = \mathrm{id}_M$. The more specific notation $\lim_M$ and ι_M may be also used.

If M is not connected, then $\widehat{M}$ is defined as the disjoint union of the rel-local completions of its connected components.

Remark 9.
(i) If $\overline{M}$ is compact, then $\widehat{M}$ is independent of the choice of the adapted metric by Remark 8-(i).
(ii) For any open $O \subset A$, $\widehat{M \cap O}$ can be canonically identified to the open subspace $\lim^{-1}(\overline{M} \cap O) \subset \widehat{M}$.

Example 2.17 (Relatively local completion of the strata of cones)**.** Let L be a compact Thom-Mather stratification and M a stratum of $c(L)$. With the notation of Section 2.1.3, if $M = \{*\}$, then $\widehat{M} = M$, obviously. Now, suppose that $M = N \times \mathbb{R}_+$ for some stratum N of L. Consider the model adapted metric $g = \rho^2 \tilde{g} + (d\rho)^2$ for some adapted metric $\tilde{g}$ on N, and the corresponding rel-local completion $\widehat{M}$. $\pi_0(N)$ is finite by Remark 6-(ii). For each $P \in \pi_0(N)$, let $\widehat{P}$ denote the rel-local completion of P with respect to L_{con}, which is independent of the choice of $\tilde{g}$. Then it is easy to check that

$$M \equiv \bigsqcup_P P \times \mathbb{R}_+ \xrightarrow{\bigsqcup_P \iota_P \times \mathrm{id}} \bigsqcup_P \widehat{P} \times \mathbb{R}_+ \hookrightarrow \bigsqcup_P c(\widehat{P})$$

extends to a homeomorphism $\widehat{M} \to \bigsqcup_{P\in\pi_0(N)} c(\widehat{P})$.

Remark 10. The following properties follow easily by using charts, induction on the depth of the strata, Example 2.17 and Remark 8-(ii):

(i) $\lim : \widehat{M} \to \overline{M}$ is surjective with finite fibers.
(ii) M is rel-locally connected with respect to $\widehat{M}$.
(iii) If $\overline{M}$ is compact, then $\widehat{M}$ is compact, and therefore its connected components are the metric completions of the connected components of M.

Proposition 2.18. (i) *$\widehat{M}$ has a unique Thom-Mather stratification with connected strata such that $\lim : \widehat{M} \to \overline{M}$ is a morphism that restricts to local diffeomorphisms between corresponding strata. In particular, the connected components of M can be considered as strata of $\widehat{M}$ via ι_M.*
(ii) *The restriction of g to the connected components of M are adapted metrics with respect to $\widehat{M}$.*
(iii) *Let M' be a connected stratum of another Thom-Mather stratification A' endowed with an adapted metric. Then, for any morphism $f : A \to A'$ with $f(M) \subset M'$, the restriction $f : M \to M'$ extends to a morphism $\hat{f} : \widehat{M} \to \widehat{M'}$. Moreover $\hat{f}$ is an isomorphism if f is an isomorphism.*

Proof. This is proved by induction on $\operatorname{depth} M$. If $\operatorname{depth} M = 0$, then $\widehat{M} \equiv \overline{M} = M$, and there is nothing to prove.

Suppose that $\operatorname{depth} M > 0$ and the statement holds for strata of lower depth. We can assume that the strata of $\overline{M}$ is connected. For each stratum X of $\overline{M}$, let (T_X, π_X, ρ_X) be a representative of the tube around X in $\overline{M}$ satisfying the conditions of Section 2.1.4 with a compact Thom-Mather stratification L_X and a family $\{(U_i, \phi_i)\}$ of local trivializations of π_X. The corresponding cocycle with values in $c(\operatorname{Aut}(L_X))$ consists of the maps $h_{ij} : U_i \cap U_j \to c(\operatorname{Aut}(L_X))$ defined by $h_{ij}(x) = (\phi_j\,\phi_i^{-1})(x, \cdot)$. We have $h_{ij}(x) = c(g_{ij}(x))$ for a cocycle consisting of maps $g_{ij} : U_i \cap U_j \to \operatorname{Aut}(L_x)$.

By the density of M in $\overline{M}$ and Remark 6-(i), there is a dense stratum N of L_X so that $\phi_i(M \cap \pi_X^{-1}(U_i)) = U_i \times N \times \mathbb{R}_+$ for all i. Consider triples (x, i, P) such that $x \in U_i$ and $P \in \pi_0(N)$. Two triples of this type, (x, i, P) and (y, j, Q), are declared to be equivalent if $x = y$ and $g_{ij}(x)(P) = Q$. The equivalence class of each triple (x, i, P) is denoted by $[x, i, P]$, and let X' denote the corresponding quotient set. There is a canonical map $f_X : X' \to X$, defined by $f_X([x, i, P]) = x$. Consider the topology on X' determined by requiring that the sets $U'_{i,P} = \{\, [x, i, P] \mid x \in U_i \,\}$ are open, and the restrictions $f_X : U'_{i,P} \to U_i$ are homeomorphisms. Notice that f_X is a finite fold covering map; in particular, in the case $X = M$, f_M is a homeomorphism. Consider the differential structure on each X' so that f_X is a local diffeomorphism.

By the induction hypothesis, for each $P \in \pi_0(N)$, $\widehat{P}$ satisfies the statement of the proposition with some Thom-Mather stratification. Consider quadruples (x, i, P, u) such that $x \in U_i$, $P \in \pi_0(N)$ and $u \in c(\widehat{P})$. Two such quadruples, (x, i, P, u) and (y, j, Q, v), are said to be equivalent if $x = y$, $g_{ij}(x)(P) = Q$ and $c(\widehat{g_{ij}(x)})(u) = v$. The equivalence class of each quadruple (x, i, P, u) is denoted by $[x, i, P, u]$, and let T'_X denote the corresponding quotient set. There are canonical maps, $\pi'_X : T'_X \to X'$, $\lim'_X : T'_X \to T_X$, $\rho'_X : T'_X \to [0, \infty)$ and $\iota'_X : M \cap T_X \to T'_X$ defined by $\pi'_X([x, i, P, u]) = [x, i, P]$, $\lim'_X([x, i, P, u]) = \phi_i^{-1}(x, c(\lim_P)(u))$,

$\rho'_X([x,i,P,u]) = \rho(u)$, and $\iota'_X(z) = [x,i,P,(\iota_P(v),r)]$ if $z \in M \cap \pi_X^{-1}(U_i)$ and $\phi_i(z) = (x,v,r) \in U_i \times P \times \mathbb{R}_+$. Notice that $f_X\,\pi'_X = \pi_X\,\lim'_X$ and $\rho_X\,\pi'_X = \rho'_X$.

Let $G \subset \operatorname{Aut}(L_X)$ be the subgroup generated by the above elements $g_{ij}(x)$. Since the canonical action of G on L_X preserves N, we get an induced action of G on $\pi_0(N)$. Since X is connected, there is a bijection between $G\backslash\pi_0(N)$ and the set $\pi_0(X')$ of connected components of X', where any orbit $\mathcal{O} \in G\backslash\pi_0(N)$ corresponds to the connected component $X'_{\mathcal{O}} \in \pi_0(X')$ consisting of the points $[x,i,P] \in X'$ with $P \in \mathcal{O}$. Also, let $T'_{X,\mathcal{O}} = (\pi'_X)^{-1}(X'_{\mathcal{O}}) \subset T'_X$.

Given any $\mathcal{O} \in G\backslash\pi_0(N)$, fix some $P_0 \in \mathcal{O}$. For any other $P \in \mathcal{O}$, there is some $g_P \in G$ such that $g_P(P) = P_0$. Thus the restriction $g_P : P \to P_0$ induces a map $\widehat{g_P} : \widehat{P} \to \widehat{P_0}$, and let $\phi'_{i,P} : (\pi'_X)^{-1}(U'_{i,P}) \to U'_{i,P} \times c(\widehat{P_0})$ be the bijection defined by $\phi'_{i,P}([x,i,P,u]) = ([x,i,P], c(\widehat{g_P})(u))$. Consider the topology on $T'_{X,\mathcal{O}}$ determined by requiring that the sets $(\pi'_X)^{-1}(U'_{i,P})$ are open, and the maps $\phi'_{i,P}$ are homeomorphisms. Then the maps $\phi'_{i,P}$ are local trivializations of the restriction $\pi'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to X'_{\mathcal{O}}$ of π'_X, obtaining that $\pi'_{X,\mathcal{O}}$ is a fiber bundle with typical fiber $c(\widehat{P_0})$. The associated cocycle has values in $c(\operatorname{Aut}(\widehat{P_0}))$; in fact, it consists of the functions $h'_{i,P;j,Q} : U'_{i,P} \cap U'_{j,Q} \to c(\operatorname{Aut}(\widehat{P_0}))$ defined by

$$h'_{i,P;j,Q}([x,i,P])(u) = c(g'_{i,P;j,Q}([x,i,P]))(u)\;,$$

where $g'_{i,P;j,Q} : U'_{i,P} \cap U'_{j,Q} \to \operatorname{Aut}(\widehat{P_0})$ is the cocycle given by

$$g'_{i,P;j,Q}([x,i,P]) = \widehat{g_Q}\,\widehat{g_{ij}(x)}\,\widehat{g_P}^{-1}\;.$$

The conditions of Section 2.1.4 are satisfied, obtaining that $\pi'_{X,\mathcal{O}}$ is a conic bundle, and therefore $T'_{X,\mathcal{O}}$ can be endowed with the corresponding conic bundle Thom-Mather stratification.

Since $N_{X,\mathcal{O}} := \bigcup_{P\in\mathcal{O}} P$ is G-invariant, the set $N_{X,\mathcal{O}} \times \mathbb{R}_+$ is invariant by all transformations $h_{ij}(x)$ for $x \in U_{ij}$, and therefore it defines an open subspace $M_{X,\mathcal{O}} \subset M \cap T_X$. Let $\lim'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to T_X$, $\rho'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to [0,\infty)$ and $\iota'_{X,\mathcal{O}} : M_{X,\mathcal{O}} \to T'_{X,\mathcal{O}}$ be defined by restricting $\lim'_X$, ρ'_X and ι'_X. Then $(T'_{X,\mathcal{O}}, \pi'_{X,\mathcal{O}}, \rho'_{X,\mathcal{O}})$ is the canonical representative of the tube of X' in $T'_{X,\mathcal{O}}$, $\iota'_{X,\mathcal{O}}$ is a dense open embedding, $\lim'_{X,\mathcal{O}}\,\iota'_{X,\mathcal{O}} = \mathrm{id}$, and $\lim'_{X,\mathcal{O}}$ is the conic bundle morphism over $f_X : X'_{\mathcal{O}} \to X$ induced by the maps $\kappa_{i,P} : U'_{i,P} \to \operatorname{Mor}(\widehat{P_0}, L_X)$ given by $\kappa_{i,P}([x,i,P]) = \lim_P\,\widehat{g_P}^{-1}$ (Section 2.1.4). By the induction hypothesis, $\kappa_{i,P}([x,i,P])$ restricts to local diffeomorphisms between corresponding strata, and therefore $\lim'_{X,\mathcal{O}}$ restricts to local diffeomorphisms between corresponding strata.

On $T'_X \equiv \bigsqcup_{\mathcal{O}\in G\backslash\pi_0(N)} T'_{X,\mathcal{O}}$, consider the sum of the topologies and Thom-Mather stratifications of the spaces $T'_{X,\mathcal{O}}$ (Remark 3). By Lemma 2.5-(i), $\lim'_X : T'_X \to T_X$ is a morphism that restricts to local diffeomorphisms between corresponding strata. Observe that the strata of T'_X are connected.

By using the local trivializations of π_X and each $\pi'_{X,\mathcal{O}}$, and Example 2.17, it follows that $\iota'_{X,\mathcal{O}} : M_{X,\mathcal{O}} \to T'_{X,\mathcal{O}}$ extends to an isomorphism $\widehat{M_{X,\mathcal{O}}} \to T'_{X,\mathcal{O}}$ such that $\lim'_{X,\mathcal{O}}$ corresponds to $\lim_{M_{X,\mathcal{O}}}$. Hence $\iota'_X : M \cap T_X \to T'_X$ extends to an isomorphism $\widehat{M \cap T_X} \to T'_X$ such that $\lim'_X$ corresponds to $\lim_{M\cap T_X}$. Then, according to Remark 9-(ii), we can consider the spaces T'_X as open subspaces of $\widehat{M}$, obtaining an open covering of $\widehat{M}$ as X runs in the family of strata of $\overline{M}$. Moreover each restriction $\lim_M : T'_X \to \overline{M} \cap T_X$ restricts to local diffeomorphisms between the corresponding strata. Hence, by Lemma 2.9, for strata X and Y of $\overline{M}$, the restrictions of the Thom-Mather stratifications of T'_X and T'_Y to $T'_X \cap T'_Y$ induce the same Thom-Mather stratification with connected strata. By Lemma 2.4-(ii), it follows that there is a unique Thom-Mather stratification with connected strata on $\widehat{M}$ whose restriction to each T'_X induces the above conic bundle Thom-Mather stratification. By Lemma 2.5-(ii), $\lim_M$ is a morphism because its restriction to each T'_X is a morphism. This completes the proof of (i).

In the above construction, consider every $U'_{i,P} \times P_0$ as a stratum of each $U'_{i,P} \times c(\widehat{P_0})$ via $\mathrm{id} \times \iota_{P_0}$. Let $g'_{i,P}$ be any Riemannian metric on $U'_{i,P}$, and let $\tilde{g}_0$ be an adapted metric on P_0 with respect to $\overline{P_0} \subset L_X$. Thus $g'_{i,P} + \tilde{g}_0$ is an adapted metric on $U'_{i,P} \times P_0$, and therefore, by the induction hypothesis, it is also adapted with respect to $U'_{i,P} \times c(\widehat{P_0})$. Hence, considering each $M_{X,\mathcal{O}}$ as a stratum of $T'_{X,\mathcal{O}}$ via $\iota'_{X,\mathcal{O}}$, the restriction of g to each $M_{X,\mathcal{O}}$ is adapted with respect to $T'_{X,\mathcal{O}}$, and (ii) follows.

Part (iii) follows from (i), (ii) and Remark 8-(iii). □

3. Relatively Morse functions

Our version of Morse functions on strata is introduced and studied in this section.

Let M be a stratum of a Thom-Mather stratification A, and fix an adapted metric g on M. Identify M and its image by the canonical dense open embedding $\iota : M \to \widehat{M}$. Let $f \in C^\infty(M)$.

Definition 3.1. (i) It is said that f is *relatively admissible* (or simply, *rel-admissible*) with respect to g if f, $|df|$ and $|\nabla df|$ are rel-locally bounded.

(ii) A point $x \in \widehat{M}$ is called *relatively critical* (or simply, *rel-critical*) if

$$\liminf_{y \in M,\ y \to x} |df(y)| = 0$$

for some adapted metric. The set of rel-critical points of f is denoted by $\mathrm{Crit}_{\mathrm{rel}}(f)$.

(iii) A point $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$ is said to be *relatively non-degenerate* (or simply, *rel-non-degenerate*) if there is some neighborhood O of x in $\widehat{M}$ and some $c > 0$ such that $|\nabla_v df| \geq c\,|v|$ for all $v \in T(M \cap O)$.

Remark 11. (i) Let O be any open subset of A. If $f \in C^\infty(M)$ is rel-admissible with respect to g, then $f|_{M\cap O}$ is rel-admissible with respect to $g|_{M\cap O}$.

(ii) The rel-local boundedness of $|df|$ is invariant by rel-local quasi-isometries, and therefore it is independent of g, but the rel-local boundedness of $|\nabla df|$ depends on the choice of g. However it follows from Lemma 3.4 and Proposition 3.5 below that the existence of g so that f is rel-admissible with respect to g is a rel-local property.

(iii) If $\operatorname{depth} M = 0$, then any smooth function is admissible, and its (rel-non-degenerate) rel-critical points are its (non-degenerate) critical points.

(iv) A rel-admissible function on M may not have any continuous extension to $\overline{M}$, but it has a continuous extension to $\widehat{M}$ by the rel-local boundedness of $|df|$. Thus it becomes natural to define its rel-critical points in $\widehat{M}$.

(v) The admissible functions on M form a unital subalgebra of $C^\infty(M)$ because d is a derivation and, for $f,h\in C^\infty(M)$,

$$\nabla d(fh) = df\otimes dh + f\nabla dh + dh\otimes df + h\nabla df\ .$$

Example 3.2. With the notation of Example 2.11-(i), for any $h\in C_0^\infty(\mathbb{R}_+)$, the function $h(\rho)$ is rel-admissible on the stratum $\mathbb{R}^m\times N\times\mathbb{R}_+$ of $\mathbb{R}^m\times c(L)$ with respect to any model adapted metric.

Example 3.3. With the notation of Examples 2.3 and 2.13, for any G-invariant smooth function f on M, let $\bar f$ denote the induced function on $G\backslash M$, whose restriction to each stratum is smooth, and df is the pull-back of $d\bar f$ on corresponding strata of M and $G\backslash M$. Fix any invariant metric on M and consider the induced adapted metric on the strata of $G\backslash M$. The restriction of $\operatorname{Hess} f$ to horizontal tangent vectors on the strata of M corresponds via the canonical projection to $\operatorname{Hess}\bar f$ on the strata of $G\backslash M$ by [31, Lemma 1]. It easily follows that $\bar f$ is rel-admissible on the strata of $G\backslash M$.

Lemma 3.4. *For any rel-locally finite covering $\{\,O_a\mid a\in\mathcal{A}\,\}$ of $\overline{M}$ by open subsets of A, there is a smooth partition of unity $\{\lambda_a\}$ on M subordinated to the open covering $\{M\cap O_a\}$ such that, for any adapted metric on M, each function $|d\lambda_a|$ is rel-locally bounded.*

Proof. If $\operatorname{depth} M=0$, then the rel-locally bounded smooth functions on M are the locally bounded ones, and therefore the statement holds in this case because any continuous function is locally bounded. Thus suppose that $\operatorname{depth} M>0$. For $0\le k\le\operatorname{depth} M$, let $\mathfrak{F}_k$ denote the union of all strata $X<M$ with $\operatorname{depth} X\le k$. The lemma is given by the case $k=\operatorname{depth} M$ in the following assertion.

Claim 1. For $0\le k\le\operatorname{depth} M$, there is a family of smooth functions $\{\lambda_{a,k}\}$ on M such that:

(i) $0\le\sum_a\lambda_{a,k}\le 1$ for all k;
(ii) $\lambda_{a,k}$ is supported in $M\cap O_a$ for all $a\in\mathcal{A}$;
(iii) there is some open neighborhood U_k of $\mathfrak{F}_k$ in A so that $\sum_a\lambda_{a,k}=1$ on $U_k\cap M$; and,
(iv) for any adapted metric on M, each function $|d\lambda_{a,k}|$ is rel-locally bounded.

This claim is proved by induction on k. To simplify its proof, observe that it is also satisfied for $k=-1$ with $\mathfrak{F}_{-1}=U_{-1}=\emptyset$, and $\lambda_{a,-1}=0$ for all $a\in\mathcal{A}$.

Now, assume that Claim 1 holds for some $k\in\{-1,0,\dots,\operatorname{depth} M-1\}$. Let V_k be another open neighborhood of $\mathfrak{F}_k$ in A such that $\overline{V_k}\subset U_k$. We can assume that the strata of A are connected by Remark 1-(v).

$\mathfrak{F}_{k+1}\setminus\mathfrak{F}_k$ is the union of the strata X that satisfy $\overline{X}\setminus X\subset\mathfrak{F}_k$, and therefore the sets $X\setminus\overline{V_k}$ are closed in $A\setminus\overline{V_k}$ and disjoint from each other. For the strata $X\subset\mathfrak{F}_{k+1}\setminus\mathfrak{F}_k$, choose representatives $(T_X,\pi_X,\rho_X)\in\tau_X$ satisfying the properties of Definition 2.1-(iv)–(vi), Proposition 2.7 and Remark 4-(iii). Let Φ_X denote the conic bundle structure of π_X. Moreover, like in Remark 1-(ii), we can assume that the sets $T_X\setminus\overline{V_k}$ are disjoint one another.

By refining $\{O_a\}$ if necessary, we can suppose that, for each stratum $X\subset\mathfrak{F}_{k+1}\setminus\mathfrak{F}_k$, any point in $X\setminus\overline{V_k}$ is in some set O_a such that there is a chart of A of the form (O_a,ξ_a), obtained from a local trivialization in Φ_X according to Definition 2.8; in this case, let $\xi_a(O_a)=B_a\times c_{\epsilon_a}(L_X)$ for some open subset $B_a\subset\mathbb{R}^{m_X}$ and some

$\epsilon_a > 0$, where $m_X = \dim X$; let $\mathcal{A}_X$ be the family the indices $a \in \mathcal{A}$ that satisfy this condition. For each $a \in \mathcal{A}_X$, take a smooth function $h_a : [0, \infty) \to [0, 1]$ supported in $[0, \epsilon_a)$ and such that $h_a = 1$ around 0. Let $\{ \mu_a \mid a \in \mathcal{A}_X \}$ be a smooth partition of unity on $\mathfrak{F}_{k+1} \setminus \overline{V_k}$ subordinated to the open covering $\{ O_a \setminus \overline{V_k} \mid a \in \mathcal{A}_X \}$. Set $\lambda_k = \sum_a \lambda_{a,k}$. Then define

$$\lambda_{a,k+1} = \lambda_{a,k} + (1 - \lambda_k) \cdot \rho_X^* h_a \cdot \pi_X^* \mu_a$$

if $a \in \mathcal{A}_X$ for some stratum $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, and $\lambda_{a,k+1} = \lambda_{a,k}$ otherwise. These functions are smooth on M because λ_k is smooth and equals 1 on U_k. It is easy to check that they also satisfy Claim 1-(i)–(iv). □

Proposition 3.5. *Let $\{ O_a \mid a \in \mathcal{A} \}$ be a rel-locally finite covering of $\overline{M}$ by open subsets of A, let $\{\lambda_a\}$ be a partition of unity on M subordinated to the open covering $\{M \cap O_a\}$ satisfying the conditions of Lemma 3.4, and let $f \in C^\infty(M)$ such that each $f|_{M \cap O_a}$ is rel-admissible with respect to some metric g_a on $M \cap O_a$. Then f is rel-admissible with respect to the adapted metric $g = \sum_a \lambda_a g_a$ on M.*

To prove Proposition 3.5, we will use the following lemma.

Lemma 3.6. *Let X be a Riemannian manifold of dimension n, and let $f \in C^\infty(X)$ and $p \in X$. If $(df)(p) \neq 0$, then there is a system of coordinates $(x^1, \dots, x^n)$ of X around p such that $(\partial_1(p), \dots, \partial_n(p))$ is an orthonormal reference and $\partial_i \partial_j f = 0$ for all $i, j \in \{1, \dots, n\}$, where $\partial_i = \partial / \partial x^i$.*

Proof. Because $(df)(p) \neq 0$, the 1-form df defines a codimension one foliation around p (its tangent bundle is $\ker df$). By using a foliation chart around p, it follows that there is a system of coordinates $(x^1, \dots, x^n)$ around p such that the vectors $\partial_1(p), \dots, \partial_{n-1}(p)$ are orthonormal, and $x^n = f / |(df)(p)|$. It is easy to check that these coordinates satisfy the stated properties. □

Proof of Proposition 3.5. Let $|\ |_a$ and ∇^a denote the norm and Levi-Civita connection of each g_a, and let $|\ |$ and ∇ denote the norm and Levi-Civita connection of g. On every $M \cap O_a$, the functions $|df|_a$ and $|\nabla^a df|_a$ are rel-locally bounded. Since g and g_a are rel-locally quasi-isometric on $M \cap O_a$, we get that $|df|$ and $|\nabla^a df|$ are rel-locally bounded on $M \cap O_a$. By shrinking $\{O_a\}$ if necessary, we can assume that there are constants $K_a \geq 0$ and $C_a \geq 1$ such that

$$|df|, |\nabla^a df|, |d\lambda_a| \leq K_a \quad \text{on} \quad M \cap O_a , \tag{4}$$

$$\frac{1}{C_a} |X|_a \leq |X| \leq C_a |X|_a \quad \forall X \in T(M \cap O_a) . \tag{5}$$

For any fixed $a_0 \in \mathcal{A}$, it is enough to prove that $|\nabla df|$ is bounded on $M \cap O_{a_0}$. For each $p \in M \cap O_{a_0}$, take any system of coordinates $(x^1, \dots, x^n)$ on some open neighborhood U of p in M such that $(\partial_1(p), \dots, \partial_n(p))$ is an orthonormal reference with respect to g. Let $g_{a,ij}$ and g_{ij} be the corresponding metric coefficients of g_a and g on $O_a \cap U$ and U, respectively; thus $g_{ij}(p) = \delta_{ij}$, and we can write $g_{ij} = \sum_a \lambda_a g_{a,ij}$ on U. As usual, the inverses of the matrices $(g_{a,ij})$ and (g_{ij}) are denoted by (g_a^{ij}) and (g^{ij}). By (5) and since $g_{ij}(p) = \delta_{ij}$, we have

$$\frac{1}{C_a^2} g_{a,ii}(p) \leq 1 \leq C_a^2 g_{a,ii}(p)$$

for all $i \in \{1,\dots,n\}$ if $p \in O_a$, giving

$$\begin{aligned}|g_{a,ij}(p)| &= \frac{1}{2}\,|\,|\partial_i(p)+\partial_j(p)|_a^2 - g_{a,ii}(p) - g_{a,jj}(p)\,| \\ &\le \frac{1}{2}\,(|\partial_i(p)+\partial_j(p)|_a^2 + g_{a,ii}(p) + g_{a,jj}(p)) \\ &\le \frac{C_a^2}{2}\,(|\partial_i(p)+\partial_j(p)|^2 + 2) = 2C_a^2\end{aligned}$$

for all $i,j \in \{1,\dots,n\}$. Since O_{a_0} meets a finite number of sets O_a, it follows that $|g_{a,ij}(p)|$ and $|g_a^{ij}(p)|$ are bounded by some $C \geq 1$, independent of the point $p \in O_{a_0}$. Similarly, by (4), we get that $|(df)(p)|$, $|(\nabla^a df)(p)|$ and $|(d\lambda_a)(p)|$ are bounded by some $K \geq 0$ independent of the point $p \in O_{a_0}$.

Let $\Gamma^k_{a,ij}$ and Γ^k_{ij} be the Christoffel symbols of g_a and g on $O_a \cap U$ and U, respectively, corresponding to $(x^1,\dots,x^n)$. Since $g_{ij}(p) = \delta_{ij}(p)$, we have[7]

$$\begin{aligned}\Gamma^k_{ij}(p) &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(p) \\ &= \frac{1}{2}\sum_a (g_{a,jk}\,\partial_i\lambda_a + \lambda_a\,\partial_i g_{a,jk} + g_{a,ik}\,\partial_j\lambda_a + \lambda_a\,\partial_j g_{a,ik} \\ &\qquad - g_{a,ij}\,\partial_k\lambda_a - \lambda_a\,\partial_k g_{a,ij})(p) \\ &= \left.\begin{aligned}&\frac{1}{2}\sum_a (g_{a,jk}\,\partial_i\lambda_a + g_{a,ik}\,\partial_j\lambda_a - g_{a,ij}\,\partial_k\lambda_a)(p) \\ &+ \sum_a \lambda_a(p)\,\Gamma^\ell_{a,ij}(p)\,g_{a,\ell k}(p)\ .\end{aligned}\right\}\end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned}\nabla df &= dx^i \otimes \nabla_i(\partial_k f\,dx^k) \\ &= \partial_i\partial_k f\,dx^i \otimes dx^k - \partial_k f\,\Gamma^k_{ij}\,dx^i \otimes dx^j \\ &= (\partial_i\partial_j f - \partial_k f\,\Gamma^k_{ij})\,dx^i \otimes dx^j\ .\end{aligned} \tag{7}$$

Similarly,

$$\nabla^a df = (\partial_i\partial_j f - \partial_k f\,\Gamma^k_{a,ij})\,dx^i \otimes dx^j\ . \tag{8}$$

If $(df)(p) = 0$, then

$$(\nabla df)(p) = (\partial_i\partial_j f\,dx^i \otimes dx^j)(p) = (\nabla^a df)(p)$$

by (7) and (8), and therefore $|(\nabla df)(p)| \leq K$.

If $(df)(p) \neq 0$, by Lemma 3.6, we can assume that the coordinates $(x^1,\dots,x^n)$ also satisfy $(\partial_i\partial_j f)(p) = 0$ for all $i,j \in \{1,\dots,n\}$. So, by (7) and (8),

$$\begin{aligned}(\nabla df)(p) &= -(\partial_k f\,\Gamma^k_{ij}\,dx^i \otimes dx^j)(p)\ , \\ (\nabla^a df)(p) &= -(\partial_k f\,\Gamma^k_{a,ij}\,dx^i \otimes dx^j)(p)\ .\end{aligned}$$

[7]Einstein convention is used for the sums involving local coefficients.

Since $g^{ij}(p)=\delta_{ij}$, it follows that $|(\partial_k f\,\Gamma^k_{a,ij})(p)|\le K$ for all $i,j\in\{1,\dots,n\}$, and it is enough to find a similar bound for each $|(\partial_k f\,\Gamma^k_{ij})(p)|$. But, by (6),

$$\begin{aligned}|(\partial_k f\,\Gamma^k_{ij})(p)| &\le \frac{1}{2}|(df)(p)|\sum_a |(d\lambda_a)(p)|\,(|g_{a,jk}(p)|+|g_{a,ik}(p)|+|g_{a,ij}(p)|)\\ &\quad+\sum_a \lambda_a(p)\,|(\partial_k f\,\Gamma^\ell_{a,ij})(p)|\,|g_{a,\ell k}(p)|\\ &\le \left(\frac{3}{2}K^2C+KC\right)\cdot\#\{\,a\in\mathcal{A}\mid O_a\cap O_{a_0}\ne\emptyset\,\}\,. \quad\square\end{aligned}$$

We would like to define relatively Morse functions on M as rel-admissible functions whose rel-critical points are rel-non-degenerate. However an appropriate version of the Morse lemma [28, Lemma 2.2] is missing (see Problem 3.9 below), and therefore they are defined by giving their "rel-local models" around their rel-critical points.

Definition 3.7. It is said that $f\in C^\infty(M)$ is a *relatively Morse function* (or *rel-Morse function*) if it is rel-admissible with respect to some adapted metric and, for every $x\in\operatorname{Crit}_{\mathrm{rel}}(f)$, there exists a chart (O,ξ) of $\widehat{M}$ centered at x, with $\xi(O)=B\times c_\epsilon(L)$, such that, for some $m_\pm\in\mathbb{N}$ and compact Thom-Mather stratifications $L_\pm$, there exists a pointed diffeomorphism $\theta_0:(\mathbb{R}^m,0)\to(\mathbb{R}^{m_+}\times\mathbb{R}^{m_-},(0,0))$, and a local quasi-isometry $\theta_1:c(L)\to c(L_+)\times c(L_-)$ so that $f_{M\cap O}$ corresponds to a constant plus $\frac{1}{2}(\rho_+^2-\rho_-^2)$ via $(\theta_0\times\theta_1)\,\xi$, where $\rho_\pm$ is the canonical function on $\mathbb{R}^{m_\pm}\times c(L_\pm)$ (Example 2.6).

Example 3.8. With the notation of Examples 2.2-(v), 2.13 and (3.3), the invariant Morse-Bott functions on M whose critical submanifolds are orbits form a dense subset of the space of invariant smooth functions [38, Lemma 4.8]. They induce rel-Morse functions on every orbit type stratum of $G\backslash M$.

Let f be a rel-Morse function on M. For each $x\in\operatorname{Crit}_{\mathrm{rel}}(f)$, with the notation of Definition 3.7, let $M_\pm$ be the strata of $c(L_\pm)$ so that $(\theta_0\times\theta_1)\,\xi$ defines an open embedding of $M\cap O$ into $\mathbb{R}^{m_+}\times\mathbb{R}^{m_-}\times M_+\times M_-$, where either $M_\pm$ is the vertex stratum of $c(L_\pm)$, or $M_\pm=N_\pm\times\mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$ with $n_\pm=\dim M_\pm$. Using this local data, for each $r\in\mathbb{Z}$, the number $\nu^r_{x,\min/\max}=\nu^r_{x,\min/\max}(f)$ was defined in Section 1, before Theorem 1.2. Recall also that $\nu^r_{\min/\max}=\nu^r_{\min/\max}(f)$ was defined as the sum of the numbers $\nu^r_{x,\min/\max}$ for $x\in\operatorname{Crit}_{\mathrm{rel}}(f)$.

Remark 12.
- (i) Every rel-Morse function on M is a Morse function, and the rel-critical points in M are the usual critical points. For such a critical point $x\in M$ with index m_-, we have $\nu^r_{x,\min/\max}=\delta_{r,m_-}$; thus $\sum_{x\in\operatorname{Crit}(f)}\nu^r_{x,\min/\max}$ is the number of critical points with index r. If $\operatorname{depth}M=0$, then any Morse function on M is a rel-Morse function by the Morse lemma.
- (ii) The rel-critical points of rel-Morse functions are isolated.
- (iii) The function $\frac{1}{2}(\rho_+^2-\rho_-^2)$ on $\mathbb{R}^{m_+}\times\mathbb{R}^{m_-}\times M_+\times M_-$ is rel-Morse, and will be called a *model rel-Morse function*.

Problem 3.9 ("Rel-Morse lemma")**.** Let x be a rel-non-degenerate rel-critical point of a rel-admissible function f on M. Does there exist a chart (O,ξ) of $\widehat{M}$ centered at x and maps θ_0 and θ_1 satisfying the conditions of Definition 3.7? An affirmative answer may require a stronger condition in Definition 3.1-(i); for instance, the rel-local boundedness of $|\nabla^k f|$ for all $k \in \mathbb{N}$.

The existence, and indeed certain abundance, of rel-Morse functions is guaranteed by the following result.

Proposition 3.10. *Let $\mathcal{F} \subset C^\infty(M)$ denote the subset of functions with continuous extensions to $\overline{M}$ that restrict to rel-Morse functions on all strata $\leq M$. Then $\mathcal{F}$ is dense in $C^\infty(M)$ with the weak C^∞ topology.*

Proof. If $\operatorname{depth} M = 0$, then the statement holds by the density of the Morse functions in $C^\infty(M)$ with the strong C^∞ topology [19, Theorem 6.1.2]. Thus suppose that $\operatorname{depth} M > 0$. Let the sets $\mathfrak{F}_k$ be defined like in the proof of Lemma 3.4.

Claim 2. For $0 \leq k \leq \operatorname{depth} M$, there is an open neighborhood U_k of $\mathfrak{F}_k$ in A and some $f_k \in C(U_k \cap \overline{M})$ such that, for each stratum $X \leq M$,

(i) f_k restricts to a rel-Morse function on $U_k \cap X$; and,
(ii) if $\operatorname{depth} X > k$, then:
   (a) the restriction of f_k to $U_k \cap X$ has no critical points, and
   (b) there is some $(T_X, \pi_X, \rho_X) \in \tau_X$ such that f_k is constant on the fibers of $\pi_X : U_k \cap \overline{M} \cap T_X \to X$.

This assertion is proved by induction on k. To simplify its proof, observe that it is also satisfied for $k = -1$ with $\mathfrak{F}_{-1} = U_{-1} = \emptyset$ and $f_{-1} = \emptyset$.

Now, assume that Claim 2 holds for some $k \in \{-1, 0, \dots, \operatorname{depth} M - 1\}$. Let V_k be another open neighborhood of $\mathfrak{F}_k$ in A such that $\overline{V_k} \subset U_k$. We can assume that the strata of A are connected by Remark 1-(v). For the strata $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, choose representatives $(T_X, \pi_X, \rho_X) \in \tau_X$ satisfying the properties stated in the proof of Claim 1. We can also suppose that these (T_X, π_X, ρ_X) satisfy Claim 2-(ii)-(b) with f_k. A fixed adapted metric g on M will be used.

Let X be a stratum contained in $\mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$. By the density of the Morse functions in $C^\infty(X)$ with the strong C^∞ topology and since the restriction of f_k to $U_k \cap X$ has no critical points by Claim 2-(iii), it is easy to construct a Morse function h_X on X such that $h_X = f_k$ on $V_k \cap X$. Since (T_X, π_X, ρ_X) satisfies Claim 2-(ii)-(b) with f_k, we get $\pi_X^* h_X = f_k$ on $U_k \cap \overline{M} \cap T_X$.

Let U_{k+1} be the open neighborhood of $\mathfrak{F}_{k+1}$ given as the union of V_k and the sets T_X for strata $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$. The function f_k on $V_k \cap \overline{M}$ and the functions $\pi_X^* h_X + \rho_X^2$ on the sets $T_X \cap \overline{M}$ can be combined to define a function $f_{k+1} \in C(U_{k+1} \cap \overline{M})$. The function f_{k+1} satisfies Claim 2-(i) and Claim 2-(ii)-(a). Moreover it satisfies Claim 2-(ii)-(b) by Definition 2.1-(vi).

Finally, let us complete the proof of Proposition 3.10. A basic neighborhood $\mathcal{N}$ of any $h \in C^\infty(M)$ with respect to the weak C^∞ topology can be determined by a finite family of charts (U_i, ϕ_i) of M, compact subsets $K_i \subset U_i$, some $k \in \mathbb{N}$ and some $\epsilon > 0$. Precisely, $\mathcal{N}$ consists of the functions $h' \in C^\infty(M)$ such that $|D^\ell((h' - h)\,\phi_i^{-1})| < \epsilon$ on $\phi_i(K_i)$ for all i and $0 \leq \ell \leq k$. By Claim 2, there is some open neighborhood U of $\overline{M} \setminus M$ in A and some $f \in C(U \cap \overline{M})$ that restricts to rel-Morse functions on $U \cap X$ for all strata $X \leq M$, and whose restriction to $U \cap M$ has no critical points. By shrinking U if necessary, we can assume that $\overline{U} \cap K_i = \emptyset$ for all i. Let V be another open neighborhood of $\overline{M} \setminus M$ in A so that $\overline{V} \subset U$. By the density of the Morse functions in $C^\infty(M)$ with the strong C^∞ topology, it is easy to check that there is a Morse function $h' \in \mathcal{N}$ such that $h' = f$ on $V \cap M$. Therefore $h' \in \mathcal{F} \cap \mathcal{N}$. □

For rel-Morse functions, a much better density result should be true as suggested by the following.

Problem 3.11. By using the ideas of this section, define and study a "rel-strong C^∞ topology" on the set of rel-admissible functions on M, and show that the rel-Morse functions form a dense subset.

An approach to Problems 3.9 and 3.11 would take us too far from the main goals of the paper.

4. Preliminaries on Hilbert complexes

Here, we recall from [6] some basic definitions and needed results about Hilbert and elliptic complexes. Some elementary observations are also made.

4.1. **Hilbert complexes.** For each $r \in \mathbb{N}$, let $\mathfrak{H}_r$ be a separable (real or complex) Hilbert space such that, for some $N \in \mathbb{N}$, we have $\mathfrak{H}_r = 0$ for all $r > N$. They give rise to the graded Hilbert space $\mathfrak{H} = \bigoplus_r \mathfrak{H}_r$, where the terms $\mathfrak{H}_r$ are mutually orthogonal. For each degree r, let $\mathbf{d}_r$ be a densely defined closed operator of $\mathfrak{H}_r$ to $\mathfrak{H}_{r+1}$. Let $\mathcal{D}_r = \mathcal{D}(\mathbf{d}_r)$ (its domain) and $\mathcal{R}_r = \mathbf{d}_r(\mathcal{D}_r)$ for each r, and let $\mathcal{D} = \bigoplus_r \mathcal{D}_r$ and $\mathbf{d} = \bigoplus_r \mathbf{d}_r$. Assume that $\mathcal{R}_r \subset \mathcal{D}_{r+1}$ and $\mathbf{d}_{r+1}\mathbf{d}_r = 0$ for all r. Then the complex

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{\mathbf{d}_0} \mathcal{D}_1 \xrightarrow{\mathbf{d}_1} \cdots \xrightarrow{\mathbf{d}_{N-1}} \mathcal{D}_N \longrightarrow 0$$

is called a *Hilbert complex*; its notation is abbreviated as $(\mathcal{D}, \mathbf{d})$, or simply as $\mathbf{d}$. Assuming that $\mathcal{D}_0 \neq 0$, the maximum $N \in \mathbb{N}$ such that $\mathcal{D}_N \neq 0$ will be called the *length* of $(\mathcal{D}, \mathbf{d})$. We may also consider Hilbert complexes with spaces of negative degree or with homogeneous operators of degree -1 without any essential change.

For the adjoint operator $\mathbf{d}_r^*$ of each $\mathbf{d}_r$, let $\mathcal{D}_r^* = \mathcal{D}(\mathbf{d}_r^*) \subset \mathfrak{H}_{r+1}$ and $\mathcal{R}_r^* = \mathbf{d}_r^*(\mathcal{D}_r^*) \subset \mathfrak{H}_r$, and set $\mathcal{D}^* = \bigoplus_r \mathcal{D}_r^*$ and $\mathbf{d}^* = \bigoplus_r \mathbf{d}_r^*$. Then we get a Hilbert complex

$$0 \longleftarrow \mathcal{D}_{-1}^* \xleftarrow{\mathbf{d}_0^*} \mathcal{D}_0^* \xleftarrow{\mathbf{d}_1^*} \cdots \xleftarrow{\mathbf{d}_{N-1}^*} \mathcal{D}_{N-1}^* \longleftarrow 0\,,$$

denoted by $(\mathcal{D}^*, \mathbf{d}^*)$ (or simply $\mathbf{d}^*$), which is called *dual* or *adjoint* of $(\mathcal{D}, \mathbf{d})$.

If $(\mathcal{D}', \mathbf{d}')$ is another Hilbert complex in the graded Hilbert space $\mathfrak{H}' = \bigoplus_r \mathfrak{H}'_r$, a homomorphism of complexes, $\zeta = \bigoplus_r \zeta_r : (\mathcal{D}, \mathbf{d}) \to (\mathcal{D}', \mathbf{d}')$, is called a *map of Hilbert complexes* if it is the restriction of a bounded map $\zeta : \mathfrak{H} \to \mathfrak{H}'$. If moreover ζ is an isomorphism of complexes and ζ^{-1} is a Hilbert complex map, then ζ is called an *isomorphism of Hilbert complexes*. If $\zeta : (\mathcal{D}, \mathbf{d}) \to (\widetilde{\mathcal{D}}', \mathbf{d}')$ is an isomorphism, where $\widetilde{\mathcal{D}}'_r = \mathcal{D}'_{r+r_0}$ for all r and some fixed $r_0 \neq 0$, then it will be said that $\zeta : (\mathcal{D}, \mathbf{d}) \to (\mathcal{D}', \mathbf{d}')$ is an *isomorphismm up to a shift of degree*.

Let

$$\mathfrak{H}_{\mathrm{ev}} = \bigoplus_r \mathfrak{H}_{2r}\,, \quad \mathfrak{H}_{\mathrm{odd}} = \bigoplus_r \mathfrak{H}_{2r+1}\,,$$

$$\mathcal{D}_{\mathrm{ev}} = \bigoplus_r \mathcal{D}_{2r}\,, \quad \mathcal{D}_{\mathrm{odd}}^* = \bigoplus_r \mathcal{D}_{2r-1}^*\,,$$

$$\mathbf{d}_{\mathrm{ev}} = \bigoplus_r \mathbf{d}_{2r}\,, \quad \mathbf{d}_{\mathrm{odd}}^* = \bigoplus_r \mathbf{d}_{2r-1}^*\,.$$

Note that $\mathcal{D}^*_{\text{odd}} \subset \mathfrak{H}_{\text{ev}}$. The operator $\mathbf{D}_{\text{ev}} = \mathbf{d}_{\text{ev}} + \mathbf{d}^*_{\text{odd}}$, with domain $\mathcal{D}_{\text{ev}} \cap \mathcal{D}^*_{\text{odd}}$, is a densely defined closed operator of $\mathfrak{H}_{\text{ev}}$ to $\mathfrak{H}_{\text{odd}}$, whose adjoint is $\mathbf{D}_{\text{odd}} = \mathbf{d}_{\text{odd}} + \mathbf{d}^*_{\text{ev}}$. Thus

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{D}_{\text{ev}} \\ \mathbf{D}_{\text{odd}} & 0 \end{pmatrix} = \mathbf{d} + \mathbf{d}^*$$

is a self-adjoint operator in $\mathfrak{H} = \mathfrak{H}_{\text{ev}} \oplus \mathfrak{H}_{\text{odd}}$ with $\mathcal{D}(\mathbf{D}) = \mathcal{D} \cap \mathcal{D}^*$, and

$$\mathbf{\Delta} = \mathbf{D}^2 = \mathbf{D}_{\text{odd}}\mathbf{D}_{\text{ev}} \oplus \mathbf{D}_{\text{ev}}\mathbf{D}_{\text{odd}} = \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$$

is a self-adjoint non-negative operator, which can be called the *Laplacian* of $(\mathcal{D}, \mathbf{d})$. Observe that $(\mathcal{D}, \mathbf{d})$ and $(\mathcal{D}^*, \mathbf{d}^*)$ define the same Laplacian. The Hilbert complex $(\mathcal{D}, \mathbf{d})$ can be reconstructed from $\mathbf{D}_{\text{ev}}$ [6, Lemma 2.3]. The restriction of $\mathbf{\Delta}$ to each space $\mathcal{D}_r$ will be denoted by $\mathbf{\Delta}_r$. Notice that $\ker \mathbf{\Delta}_r = \ker \mathbf{d}_r \cap \ker \mathbf{d}^*_{r-1}$ for all r. Moreover we have a weak Hodge decomposition [6, Lemma 2.1]

$$\mathfrak{H}_r = \ker \mathbf{\Delta}_r \oplus \overline{\mathcal{R}_{r-1}} \oplus \overline{\mathcal{R}^*_r}\ .$$

If T is a self-adjoint operator in a Hilbert space, then $\mathcal{D}^\infty(T) = \bigcap_{k\geq 1} \mathcal{D}(T^k)$ is a core[8] for T, which is called its *smooth core*. In the case of the Laplacian $\mathbf{\Delta}$ of a Hilbert complex $(\mathcal{D}, \mathbf{d})$ in a graded Hilbert space $\mathfrak{H}$, the smooth core $\mathcal{D}^\infty(\mathbf{\Delta})$, also denoted by $\mathcal{D}^\infty(\mathbf{d})$ or $\mathcal{D}^\infty$, is a subcomplex of $(\mathcal{D}, \mathbf{d})$, and $(\mathcal{D}^\infty, \mathbf{d}) \hookrightarrow (\mathcal{D}, \mathbf{d})$ induces an isomorphism in homology [6, Theorem 2.12]. It will be also said that $\mathcal{D}^\infty$ (respectively, $\mathcal{D}^\infty_r$) is the *smooth core* of $\mathbf{d}$ (respectively, $\mathbf{d}_r$); notice that it is a core of $\mathbf{d}$ (respectively, $\mathbf{d}_r$). Let $\mathcal{R}^\infty_r = \mathbf{d}_r(\mathcal{D}^\infty_r)$ and $\mathcal{R}^{*\infty}_r = \mathbf{d}^*_r(\mathcal{D}^\infty_r)$, which are dense subspaces of $\mathcal{R}_r$ and $\mathcal{R}^*_r$.

The following properties are equivalent [6, Theorem 2.4]:

- The homology of $(\mathcal{D}, \mathbf{d})$ is of finite dimension and $\mathcal{R}$ is closed in $\mathfrak{H}$.
- The homology of $(\mathcal{D}, \mathbf{d})$ is of finite dimension.
- $\mathbf{D}_{\text{ev}}$ is a Fredholm operator.
- $0 \notin \operatorname{spec}_{\text{ess}}(\mathbf{\Delta})$ (the essential spectrum of $\mathbf{\Delta}$).

In this case, $(\mathcal{D}, \mathbf{d})$ is called a *Fredholm complex* and satisfies the following properties:

- $\mathcal{R}$ and $\mathcal{R}^*$ are closed in $\mathfrak{H}$ [6, Corollary 2.5], obtaining the stronger Hodge decompositions
$$\mathfrak{H}_r = \ker \mathbf{\Delta}_r \oplus \mathcal{R}_{r-1} \oplus \mathcal{R}^*_r\ , \quad \mathcal{D}^\infty = \ker \mathbf{\Delta}_r \oplus \mathcal{R}^\infty_{r-1} \oplus \mathcal{R}^{*\infty}_r\ .$$
- $\mathbf{d}_r : \mathcal{R}^{*\infty}_r \to \mathcal{R}^\infty_r$ and $\mathbf{d}^*_r : \mathcal{R}^\infty_r \to \mathcal{R}^{*\infty}_r$ are isomorphisms.
- $\ker \mathbf{\Delta}_r$ is isomorphic to the homology of degree r of $(\mathcal{D}, \mathbf{d})$.

It is said that $(\mathcal{D}, \mathbf{d})$ is *discrete* when $\mathbf{\Delta}$ has a discrete spectrum ($\operatorname{spec}_{\text{ess}}(\mathbf{\Delta}) = \emptyset$). The following properties hold when $(\mathcal{D}, \mathbf{d})$ is discrete:

- For each $\lambda \in \operatorname{spec}(\mathbf{\Delta}|_{\mathcal{R}^\infty_r})$, we get isomorphisms
$$\mathbf{d}_r : E_\lambda(\mathbf{\Delta}|_{\mathcal{R}^{*\infty}_r}) \to E_\lambda(\mathbf{\Delta}|_{\mathcal{R}^\infty_r})\ , \quad \mathbf{d}^*_r : E_\lambda(\mathbf{\Delta}|_{\mathcal{R}^\infty_r}) \to E_\lambda(\mathbf{\Delta}|_{\mathcal{R}^{*\infty}_r})$$
between the corresponding eigenspaces. Thus $\operatorname{spec}(\mathbf{\Delta}|_{\mathcal{R}^\infty_r}) = \operatorname{spec}(\mathbf{\Delta}|_{\mathcal{R}^{*\infty}_r})$.
- We have
$$\operatorname{spec}(\mathbf{d}_r|_{\mathcal{R}^{*\infty}_r} \oplus \mathbf{d}^*_r|_{\mathcal{R}^\infty_r}) = \{\, \pm\sqrt{\lambda} \mid \lambda \in \operatorname{spec}(\mathbf{\Delta}|_{\mathcal{R}^\infty_r})\,\}\ ,$$

[8]Recall that a *core* of a closed densely defined operator T between Hilbert spaces is any subspace of its domain $\mathcal{D}(T)$ which is dense with the graph norm.

and, for each $\lambda \in \operatorname{spec}(\mathbf{\Delta}|_{\mathcal{R}_r^\infty})$, $E_{\pm\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}_r^\infty} \oplus \mathbf{d}_r^*|_{\mathcal{R}_r^{*\infty}})$ consists of the elements of the form $u \pm v$ with $u \in E_\lambda(\mathbf{\Delta}|_{\mathcal{R}_r^\infty})$ and $v \in E_\lambda(\mathbf{\Delta}|_{\mathcal{R}_r^{*\infty}})$ satisfying $\mathbf{d}^*u = \sqrt{\lambda}\,v$ and $\mathbf{d}v = \sqrt{\lambda}\,u$. Moreover the mapping $u+v \mapsto u-v$, for u and v as above, defines an isomorphism

$$E_{\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}_r^{*\infty}} \oplus \mathbf{d}_r^*|_{\mathcal{R}_r^\infty}) \to E_{-\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}_r^{*\infty}} \oplus \mathbf{d}_r^*|_{\mathcal{R}_r^\infty})\ .$$

- Any Hilbert complex $(\mathcal{D}', \mathbf{d}')$ isomorphic to $(\mathcal{D}, \mathbf{d})$ is also discrete, and, if $\operatorname{spec}(\mathbf{\Delta}_r)$ and $\operatorname{spec}(\mathbf{\Delta}'_r)$ consist of the eigenvalues $0 \le \lambda_0 \le \lambda_1 \le \cdots$ and $0 \le \lambda'_0 \le \lambda'_1 \le \cdots$, respectively, then there is some $C \ge 1$ such that $C^{-1}\lambda_k \le \lambda'_k \le C\lambda_k$ for all $k \in \mathbb{N}$ [6, Lemma 2.17].

Consider Hilbert complexes, $(\mathcal{D}', \mathbf{d}')$ and $(\mathcal{D}'', \mathbf{d}'')$, in respective graded Hilbert spaces, $\mathfrak{H}'$ and $\mathfrak{H}''$. The Hilbert space tensor product[9], $\mathfrak{H} = \mathfrak{H}'\widehat{\otimes}\mathfrak{H}''$, has a canonical grading ($\mathfrak{H}_r = \bigoplus_{p+q=r} \mathfrak{H}'_p\widehat{\otimes}\mathfrak{H}''_q$), and

$$\widetilde{\mathcal{D}} = (\mathcal{D}' \otimes \mathfrak{H}'') \cap (\mathfrak{H}' \otimes \mathcal{D}'') \subset \mathfrak{H}$$

is a dense graded subspace. Let $\tilde{\mathbf{d}} = \mathbf{d}' \otimes 1 + \mathbf{w} \otimes \mathbf{d}''$ with domain $\widetilde{\mathcal{D}}$, where $\mathbf{w}$ denotes the degree involution on $\mathfrak{H}'$, and let $\mathbf{d} = \overline{\tilde{\mathbf{d}}}$, whose domain is denoted by $\mathcal{D}$. Then $(\mathcal{D}, \mathbf{d})$ is a Hilbert complex in $\mathfrak{H}$ called the *tensor product* of $(\mathcal{D}', \mathbf{d}')$ and $(\mathcal{D}'', \mathbf{d}'')$. If $\mathbf{\Delta}'$, $\mathbf{\Delta}''$ and $\mathbf{\Delta}$ denote the Laplacians of $(\mathcal{D}', \mathbf{d}')$, $(\mathcal{D}'', \mathbf{d}'')$ and $(\mathcal{D}, \mathbf{d})$, respectively, then $\mathbf{\Delta} = \mathbf{\Delta}' \otimes 1 + 1 \otimes \mathbf{\Delta}''$ on $\widetilde{\mathcal{D}}$. The following result is elementary.

Lemma 4.1. *If $(\mathcal{D}', \mathbf{d}')$ and $(\mathcal{D}'', \mathbf{d}'')$ are discrete, then $(\mathcal{D}, \mathbf{d})$ is discrete. More precisely, given complete orthonormal systems of $\mathfrak{H}'$ and $\mathfrak{H}''$ consisting of eigenvectors e'_k and e''_k ($k \in \mathbb{N}$) of $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$, with corresponding eigenvalues λ'_k and λ''_k, respectively, we get a complete orthonormal system of $\mathfrak{H}$ consisting of the eigenvectors $e'_k \otimes e''_\ell \in \widetilde{\mathcal{D}}$ of $\mathbf{\Delta}$ with corresponding eigenvalues $\lambda'_k + \lambda''_\ell$.*

Let $(\mathcal{E}, d)$ be a densely defined complex in a graded separable Hilbert space $\mathfrak{H}$ ($\mathcal{E}$ is a dense graded linear subspace of $\mathfrak{H}$). Consider the family of Hilbert complexes $(\mathcal{D}, \mathbf{d})$ in $\mathfrak{H}$ extending $(\mathcal{E}, d)$ ($(\mathcal{E}, d)$ is a subcomplex of $(\mathcal{D}, \mathbf{d})$) endowed with the order relation defined by "being a subcomplex". We will be interested in its minimum/maximum elements. Notice that, if $(\mathcal{E}, d)$ has some Hilbert complex extension, then $\overline{d}$ is a Hilbert complex; thus, in this case, $\overline{d}$ is the minimum Hilbert complex extension of $(\mathcal{E}, d)$. Another complex of the form $(\mathcal{E}, \delta)$, with $\delta_r : \mathcal{E}_{r+1} \to \mathcal{E}_r$ for each degree r, will be called a *formal adjoint* of $(\mathcal{E}, d)$ if $\langle du, v\rangle = \langle u, \delta v\rangle$ for all $u, v \in \mathcal{E}$; there is at most one formal adjoint by the density of $\mathcal{E}$ in $\mathfrak{H}$. In this case, if $(\mathcal{E}, \delta)$ has some Hilbert complex extension, then the adjoint of the minimum Hilbert complex extension of $(\mathcal{E}, \delta)$ is the maximum Hilbert complex extension of $(\mathcal{E}, d)$.

Now, consider a countable family of densely defined complexes $(\mathcal{E}^a, d^a)$ in separable graded Hilbert spaces $\mathfrak{H}^a$ ($a \in \mathbb{N}$), and let $(\mathcal{D}^a, \mathbf{d}^a)$ be a Hilbert complex extension of each $(\mathcal{E}^a, d^a)$ in $\mathfrak{H}^a$. Suppose that the Hilbert complexes $(\mathcal{D}^a, \mathbf{d}^a)$ are of uniformly finite length (there is some $N \in \mathbb{N}$ such that $\mathcal{D}^a_r = 0$ for all $r \ge N$ and all a). Let $(\mathcal{E}, d)$ be the complex defined by $\mathcal{E} = \bigoplus_a \mathcal{E}^a$ and $d = \bigoplus_a d^a$. The

[9]Recall that this is the Hilbert space completion of the algebraic tensor product $\mathfrak{H}' \otimes \mathfrak{H}''$ with respect to the scalar product defined by $\langle u' \otimes u'', v' \otimes v''\rangle = \langle u', v'\rangle' \langle u'', v''\rangle''$, where $\langle\ ,\ \rangle'$ and $\langle\ ,\ \rangle''$ are the scalar products of $\mathfrak{H}'$ and $\mathfrak{H}''$, respectively.

Hilbert space direct sum[10], $\mathfrak{H} = \widehat{\bigoplus}_a \mathfrak{H}^a$, has an induced grading ($\mathfrak{H}_r = \widehat{\bigoplus}_a \mathfrak{H}^a_r$). Let $\mathbf{d} = \widehat{\bigoplus}_a \mathbf{d}^a$ (the graph of $\mathbf{d}$ is the Hilbert space direct sum of the graphs of the maps $\mathbf{d}^a$). The domain $\mathcal{D}$ of $\mathbf{d}$ consists of the points $(u^a) \in \mathfrak{H}$ such that $u^a \in \mathcal{D}^a$ for all a and $(\mathbf{d}^a u^a) \in \mathfrak{H}$. Moreover $\mathbf{d}$ is defined by $(u^a) \mapsto (\mathbf{d}^a u^a)$. Clearly, $(\mathcal{D}, \mathbf{d})$ is a Hilbert complex extension of $(\mathcal{E}, d)$ in $\mathfrak{H}$ with

$$\mathcal{D}^\infty(\mathbf{d}) = \widehat{\bigoplus_a} \mathcal{D}^\infty(\mathbf{d}^a) \,, \tag{9}$$

$$\mathbf{d}^* = \widehat{\bigoplus_a} \mathbf{d}^{a*} \,. \tag{10}$$

Lemma 4.2. (i) *If each $(\mathcal{D}^a, \mathbf{d}^a)$ is a minimum Hilbert complex extension of $(\mathcal{E}^a, \mathbf{d}^a)$ in $\mathfrak{H}^a$, then $(\mathcal{D}, \mathbf{d})$ is a minimum Hilbert complex extension of $(\mathcal{E}, \mathbf{d})$ in $\mathfrak{H}$.*

(ii) *If each $(\mathcal{E}^a, d^a)$ has a formal adjoint $(\mathcal{E}^a, \delta^a)$ with some Hilbert complex extension, and each $(\mathcal{D}^a, \mathbf{d}^a)$ is a maximum Hilbert complex extension of $(\mathcal{E}^a, \mathbf{d}^a)$ in $\mathfrak{H}^a$, then $(\mathcal{D}, \mathbf{d})$ is a maximum Hilbert complex extension of $(\mathcal{E}, \mathbf{d})$ in $\mathfrak{H}$.*

Proof. Property (i) follows because d is dense in $\mathbf{d}$ if each d^a is dense in $\mathbf{d}^a$.

Now, assume the conditions of (ii) and let $\delta = \bigoplus_a \delta^a$. Then each $\mathbf{d}^{a*}$ is a minimum Hilbert complex extension of $(\mathcal{E}^a, \delta^a)$. So, by (10) and (i), $(\mathcal{D}^*, \mathbf{d}^*)$ is a minimum Hilbert complex extension of $(\mathcal{E}, \delta)$, and therefore $(\mathcal{D}, \mathbf{d})$ is a maximum Hilbert complex extension of $(\mathcal{E}, d)$. □

4.2. **Elliptic complexes.** Let M be a possibly non-complete Riemannian manifold, and let $E = \bigoplus_r E_r$ be a graded Riemannian (or Hermitean) vector bundle over M, with $E_r = 0$ if $r < 0$ or $r > N$ for some $N \in \mathbb{N}$. The space of smooth sections of each E_r will be denoted by $C^\infty(E_r)$, its subspace of compactly supported smooth sections will be denoted by $C^\infty_0(E_r)$, and the Hilbert space of square integrable sections of E_r will be denoted by $L^2(E_r)$; then $C^\infty(E) = \bigoplus_r C^\infty(E_r)$, $C^\infty_0(E) = \bigoplus_r C^\infty_0(E_r)$ and $L^2(E) = \bigoplus_r L^2(E_r)$. For each r, let $d_r : C^\infty(E_r) \to C^\infty(E_{r+1})$ be a first order differential operator, and set $d = \bigoplus_r d_r$. Suppose that $(C^\infty(E), d)$ is an elliptic complex[11]; however, ellipticity is not needed for several elementary properties stated in this section. The simpler notation (E, d) (or even d) will be preferred. Elliptic complexes with non-zero terms of negative degrees or homogeneous differential operators of degree -1 may be also considered without any essential change.

Consider the formal adjoint $\delta_r = {}^t d_r : C^\infty(E_{r+1}) \to C^\infty(E_r)$ for each r, and set $\delta = \bigoplus_r \delta_r$. Then (E, δ) is another elliptic complex that will be called the *formal adjoint* of (E, d), and its subcomplex $(C^\infty_0(E), \delta)$ is formal adjoint of $(C^\infty_0(E), d)$ in $L^2(E)$ in the sense of Section 4.1. Let $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ on $C^\infty(E)$; Δ can be called the *Laplacian* defined by (E, d). The components of Δ are $\Delta_r = d_{r-1}\delta_{r-1} + \delta_r d_r$.

[10] Recall that this is the Hilbert space completion of the algebraic direct sum, $\bigoplus_a \mathfrak{H}^a$, with respect to the scalar product $\langle (u^a), (v^a) \rangle = \sum_a \langle u^a, v^a \rangle_a$, where each $\langle\ ,\ \rangle_a$ is the scalar product of $\mathfrak{H}^a$. We have $\mathfrak{H} = \bigoplus_a \mathfrak{H}^a$ if the number of terms $\mathfrak{H}^a$ is finite.

[11] Recall that this means that it is a complex and the sequence of principal symbols of the operators d_r is exact in the fiber over each non-zero cotangent vector

Any Hilbert complex extension of $(C_0^\infty(E), d)$ in $L^2(E)$ is called an *ideal boundary condition* (shortly, *i.b.c.*) of (E, d). There always exist a minimum and maximum i.b.c., $d_{\min} = \overline{d}$ and $d_{\max} = \delta^*_{\min}$ [6, Lemma 3.1]. The complex $d_{\min/\max}$ defines the operator $D_{\min/\max} = d_{\min/\max} + \delta_{\max/\min}$ and the Laplacian $\Delta_{\min/\max} = D^2_{\min/\max}$, which extend D and Δ on $C_0^\infty(E)$. The homogeneous components of $\Delta_{\min/\max}$ are

$$\Delta_{\min/\max,r} = \delta_{\max/\min,r}\, d_{\min/\max,r} + d_{\min/\max,r-1}\, \delta_{\max/\min,r-1}\ . \tag{11}$$

The notation $d_{r,\min/\max}$ and $\delta_{r,\max/\min}$ also makes sense for $d_{\min/\max,r}$ and $\delta_{\max/\min,r}$ by considering d_r and δ_r as differential complexes of length one (ellipticity is not needed here); similarly, any first order differential operator can be considered as a differential complex of length one and denote its minimum/maximum i.b.c. with the the min/max subindex, regardless of ellipticity.

For any i.b.c. $(\mathcal{D}, \mathbf{d})$ of (E, d), the map of complexes, $(\mathcal{D} \cap C^\infty(E), d) \hookrightarrow (\mathcal{D}, \mathbf{d})$, induces an isomorphism in homology [6, Theorem 3.5]. We have $\mathcal{D}^\infty \subset \mathcal{D} \cap C^\infty(E)$ by elliptic regularity.

Let (E', d') be another elliptic complex over another Riemannian manifold M'. Consider a vector bundle isomorphism $\zeta : E \to E'$ over a quasi-isometric diffeomorphism $\xi : M \to M'$ such that the restrictions of ζ to the fibers are quasi-isometries. It induces a map $\zeta : C^\infty(E) \to C^\infty(E')$ defined by $(\zeta u)(x') = \zeta(u(\xi^{-1}(x'))$ for $u \in C^\infty(E)$ and $x' \in M'$. If moreover $\zeta : (C^\infty(E'), d') \to (C^\infty(E), d)$ is a homomorphism of complexes, then it will be called a *quasi-isometric isomorphism* of elliptic complexes, and the simpler notation $\zeta : (E', d') \to (E, d)$ will be preferred. In this case, ζ induces a quasi-isometric isomorphism $\zeta : L^2(E') \to L^2(E)$, which restricts to an isomorphism of complexes, $\zeta : (C_0^\infty(E'), d') \to (C_0^\infty(E), d)$. Moreover, for any i.b.c. $(\mathcal{D}', \mathbf{d}')$ of (E', d'), there is a unique i.b.c. $(\mathcal{D}, \mathbf{d})$ of (E, d) so that $\zeta : L^2(E') \to L^2(E)$ restricts to a Hilbert complex isomorphism $\zeta : (\mathcal{D}', \mathbf{d}') \to (\mathcal{D}, \mathbf{d})$. In particular, ζ induces Hilbert complex isomorphisms between the corresponding minimum/maximum i.b.c. If ξ is isometric and the restrictions to the fibers of ζ are isometries, then $\zeta : (E', d') \to (E, d)$ is called an *isometric isomorphism* of elliptic complexes. For instance, for any quasi-isometric (respectively, isometric) diffeomorphism $\xi : M \to M'$, the induced isomorphism ξ^* between the corresponding de Rham complexes is quasi-isometric (respectively, isometric).

Now, let (E', d') and (E'', d'') be elliptic complexes on Riemannian manifolds M' and M'', respectively, and consider the exterior tensor product $E = E' \boxtimes E''$ on $M = M' \times M''$ with its canonical grading ($E_r = \bigoplus_{p+q=r} E'_p \boxtimes E''_q$). With the weak C^∞ topology, $C^\infty(E') \otimes C^\infty(E'')$ can be canonically realized as a dense subspace of $C^\infty(E)$. Then $d = d' \otimes 1 + \mathbf{w} \otimes d''$ on $C^\infty(E') \otimes C^\infty(E'')$ has a unique continuous extension to $C^\infty(E)$, also denoted by d. It turns out that (E, d) is an elliptic complex. Moreover the minimum/maximum i.b.c. of (E, d) is the tensor product, in the sense of Section 4.1, of the minimum/maximum i.b.c. of (E', d') and (E'', d'') [6, Lemma 3.6].

Example 4.3. A particular case of elliptic complex on M is its de Rham complex $(\Omega(M), d)$. In this case, δ is the de Rham coderivative, the subcomplex of compactly supported differential forms is denoted by $\Omega_0(M)$, and the Hilbert space of L^2 differential forms is denoted by $L^2\Omega(M)$. Let $H_{\min/\max}(M)$ denote the cohomology of the minimum/maximum i.b.c., $d_{\min/\max}$, of $(\Omega_0(M), d)$, which is a quasi-isometric invariant of M. $H_{\min}(M)$ is canonically isomorphic to the L^2-cohomology $H_{(2)}(M)$ [9]; (a generalization to arbitrary elliptic complexes is given in [6, Theorem 3.5]).

The dimensions $\beta^r_{\mathrm{min/max}}(M) = \dim H^r_{\mathrm{min/max}}(M)$ can be called *min/max-Betti numbers*; if they are finite, then $\chi_{\mathrm{min/max}}(M) = \sum_r (-1)^r \beta^r_{\mathrm{min/max}}(M)$ is defined and can be called *min/max-Euler characteristic*; the simpler notation $\beta^r_{\mathrm{min/max}}$ and $\chi_{\mathrm{min/max}}$ may be used. Is is known that $d_{\mathrm{min/max}}$ satisfies the following properties for special classes of Riemannian manifolds:

- If M is complete, then $d_{\mathrm{min}} = d_{\mathrm{max}}$ (a particular case of [6, Lemma 3.8]).
- If M is the interior of a compact manifold with boundary, then $d_{\mathrm{min/max}}$ is given by the relative/absolute boundary conditions [6, Theorem 4.1].
- Suppose that $M = \widetilde{M} \setminus \Sigma$, where $\widetilde{M}$ is a closed Riemannian manifold of dimension > 2 and Σ is a closed finite union of submanifolds with codimension ≥ 2. Then $d_{\mathrm{min}} = d_{\mathrm{max}}$ [6, Theorem 4.4].
- Let A be a compact Tham-Mather stratification that is a pseudomanifold. If M is the regular stratum of A endowed with an adapted metric, then $H_{(2)}(M)$ is isomorphic to the intersection homology of A with lower middle perversity [11]. There is a more general isomorphism of this type involving more general types of adapted metrics and intersection homologies with other perversities [29, 30, 4].

5. Sobolev spaces defined by an i.b.c.

Let T be a self-adjoint operator in a Hilbert space $\mathfrak{H}$. For each $m \in \mathbb{N}$, the *Sobolev space* of *order* k associated to T is the Hilbert space completion $W^m = W^m(T)$ of $\mathcal{D}^\infty = \mathcal{D}^\infty(T)$ with respect to the scalar product $\langle\ ,\ \rangle_m$ on $\mathcal{D}^\infty$ defined by $\langle u, v\rangle_m = \langle u, (1+T)^m v\rangle$. The notation $\|\ \|_m$ and Cl_m (or $\|\ \|_{W^m}$ and Cl_{W^m}) will be used for the norm and closure in W^m. There are continuous inclusions $W^{m+1} \hookrightarrow W^m$, and we have $\mathcal{D}^\infty = \bigcap_m W^m$. Moreover T defines a bounded operator $W^{m+2} \to W^m$.

Now, let $(\mathcal{D}, \mathbf{d})$ be an i.b.c. of an elliptic complex (E, d) on a Riemannian manifold M. Its adjoint $(\mathcal{D}^*, \mathbf{d}^*)$ is an i.b.c. of the elliptic complex (E, δ), where $\delta = {}^t d$. We get the operators $D = d + \delta$ and $\mathbf{D} = \mathbf{d} + \mathbf{d}^*$, and the Laplacians $\Delta = D^2$ and $\boldsymbol{\Delta} = \mathbf{D}^2$. Then $W^m = W^m(\boldsymbol{\Delta})$ can be called the *Sobolev space* of *order* m associated to $(\mathcal{D}, \mathbf{d})$, and may be also denoted by $W^m(\mathbf{d})$; the notation $W^m(\mathbf{d}_r)$ will be also used when we consider its subspace of homogeneous elements of degree r. Since $(\mathcal{D}, \mathbf{d})$ and $(\mathcal{D}^*, \mathbf{d}^*)$ define the same Laplacian, we have $W^m(\mathbf{d}) = W^m(\mathbf{d}^*)$ for all m. For $u \in \mathcal{D}^\infty_r$, we have

$$\|u\|_1^2 = \|u\|^2 + \|Du\|^2 = \|u\|^2 + \|d_r u\|^2 + \|\delta_{r-1} u\|^2 .$$

So

$$W^1 = \mathcal{D}(\mathbf{D}) = \mathcal{D} \cap \mathcal{D}^* , \tag{12}$$

$$\|u\|_1^2 = \|u\|^2 + \|\mathbf{D}u\|^2 = \|u\|^2 + \|\mathbf{d}_r u\|^2 + \|\mathbf{d}^*_{r-1} u\|^2 \tag{13}$$

for $u \in W^1(\mathbf{d}_r)$.

Lemma 5.1. *The following properties are equivalent:*

(i) $(\mathcal{D}, \mathbf{d})$ *is discrete.*
(ii) $W^1 \hookrightarrow W^0 = L^2(E)$ *is compact.*
(iii) $W^{m+1} \hookrightarrow W^m$ *is compact for all* m.

Proof. The part "(i) $\Rightarrow$ (iii)" follows with the arguments of the proof of the Rellich's theorem on a torus (see e.g. [33, Theorem 5.8]). The part "(ii) $\Rightarrow$ (i)" follows with the arguments to prove that any Dirac operator on a closed manifold has a discrete spectrum (see e.g. [33, pp. 81–82]). $\square$

The following refinement of Lemma 5.1 is obtained with a deeper analysis.

Lemma 5.2. *Suppose that* $(\mathcal{D},\mathbf{d})$ *is discrete, and let* $0\le\lambda_1\le\lambda_2\le\cdots$ *be the eigenvalues of* $\mathbf{\Delta}$*, repeated according to their multiplicities. Let* B^1 *be the standard unit ball of* W^1*, and* B_r *the standard ball of radius* $r>0$ *in* $L^2(E)$*. Then the following properties are equivalent for* $\theta>0$*:*

(i) $\liminf_k \lambda_k k^{-\theta}>0$.
(ii) *There are some* $C_0,C_1>0$ *such that, for all* $n\in\mathbb{Z}_+$*, there is a linear subspace* $Z_n\subset L^2(E)$ *so that:*
  (a) Z_n *is closed and of codimension* $\le C_0\,n^{1/\theta}$ *in* $L^2(E)$*;*
  (b) $\mathbf{D}(W^1\cap Z_n)\subset Z_n$*; and*
  (c) $B^1\cap Z_n\subset B_{C_1/n}$.
(iii) *There are some* $C_0,\dots,C_4>0$ *and* $A\in\mathbb{Z}_+$ *such that, for all* $n\in\mathbb{Z}_+$*, there is a linear map*[12] $R_n=(R_n^1,\dots,R_n^A):L^2(E)\to\bigoplus_A L^2(E)$ *so that:*
  (a) $\dim\ker R_n\le C_0\,n^{1/\theta}$*;*
  (b) $\|R_nu\|\le C_1\,\|u\|$ *for all* $u\in L^2(E)$*;*
  (c) $\|R_nu\|\ge C_2\,\|u\|$ *for all* $u\in(\ker R_n)^\perp$*;*
  (d) $R_n^a(W^1)\subset W^1$ *and* $\|[\mathbf{D},R_n^a]u\|\le C_3\,\|u\|$ *for all* $u\in W^1$*; and*
  (e) $B^1\cap R_n^a(L^2(E))\subset B_{C_4/n}$.

Proof. Let (e_i) $(i\in\mathbb{Z})$ be a complete orthonormal system of $L^2(E)$ such that $e_{\pm k}$ is a $\pm\sqrt{\lambda_k}$-eigenvector of $\mathbf{D}$ for each $k\in\mathbb{N}$. The mapping $u=\sum_i u_ie_i\mapsto(u_i)$ defines a unitary isomorphism $L^2(E)\cong\ell^2(\mathbb{Z})$. Moreover W^1 consists of the elements $u\in L^2(E)$ with $\sum_k(1+\lambda_k)u_{\pm k}^2<\infty$, and $\|u\|_1^2=\sum_k(1+\lambda_k)(u_k^2+u_{-k}^2)$ for $u\in W^1$.

Suppose that (i) holds. Then there is some $C>0$ so that $1+\lambda_k\ge Ck^\theta$ for all k. For each $n\in\mathbb{Z}_+$, the linear subspace

$$Z_n=\left\{\,u\in L^2(E)\mid u_{\pm k}=0\text{ if }k\le(n/C)^{1/\theta}\,\right\}$$

of $L^2(E)$ satisfies (ii)-(a),(b) with $C_0=2/C^{1/\theta}$. Furthermore, for every $u\in B^1\cap Z_n$,

$$\begin{aligned}\|u\|^2=\sum_{k>(n/C)^{1/\theta}}(u_k^2+u_{-k}^2)&<\frac{C}{n}\sum_{k>(n/C)^{1/\theta}}k^\theta(u_k^2+u_{-k}^2)\\&\le\frac{1}{n}\sum_{k>(n/C)^{1/\theta}}(1+\lambda_k)(u_k^2+u_{-k}^2)=\frac{\|u\|_1^2}{n}<\frac{1}{n}\,,\end{aligned}$$

completing the proof of (ii)-(c) with $C_1=1$.

Now, assume that (ii) is satisfied. By (ii)-(a),

$$L^2(E)=Z_n^\perp\oplus Z_n \tag{14}$$

[12] For $A\in\mathbb{Z}_+$ and any topological vector space L, the notation $\bigoplus_A L$ is used for the direct sum of A copies of L. Similarlarly, for any linear map between topological vector spaces, $T:L\to L'$, the notation $\bigoplus_A T:\bigoplus_A L\to\bigoplus_A L'$ is used for the direct sum of A copies of T.

as topological vector space [34, Chapter I, 3.5]. Furthermore, by (ii)-(a) and the canonical linear isomorphism

$$\frac{W^1}{W^1 \cap Z_n} \cong \frac{W^1 + Z_n}{Z_n} \,,$$

we also get that $W^1 \cap Z_n$ is a closed linear subspace of finite codimension in W^1. Hence

$$W^1 = Y_n \oplus (W^1 \cap Z_n) \tag{15}$$

as topological vector spaces for any linear complement Y_n of $W^1 \cap Z_n$ in W^1 [34, Chapter I, 3.5].

On the other hand, for each $u \in Z_n^\perp$, the linear mapping $v \mapsto \langle u, \mathbf{D}v \rangle$ is bounded on Y_n because Y_n is of finite dimension, and $\langle u, \mathbf{D}w \rangle = 0$ for all $w \in W^1 \cap Z_n$ by (ii)-(b). So $v \mapsto \langle u, \mathbf{D}v \rangle$ is bounded on W^1 by (15), obtaining that $u \in W^1$ by (12) since $\mathbf{D}$ is self-adjoint. Hence $Z_n^\perp \subset W^1$, and therefore we can take $Y_n = Z_n^\perp$ in (15), obtaining

$$W^1 = Z_n^\perp \oplus (W^1 \cap Z_n) \tag{16}$$

as topological vector spaces. Note that $W^1 \cap Z_n$ is dense in Z_n by (14) and (16). So, since $\mathbf{D}$ is self-adjoint, it follows from (ii)-(b) and (16) that $\mathbf{D}$ preserves $Z_n^\perp$.

To get (iii), take $A = 1$ and R_n equal to the orthogonal projection of $L^2(E)$ to Z_n. Then (iii)-(a) follows from (ii)-(a), and properties (iii)-(b),(c) hold with $C_1 = C_2 = 1$ because R_n is an orthogonal projection. By (ii)-(b) and since $\mathbf{D}$ preserves $Z_n^\perp$, we get $R_n(W^1) \subset W^1$ and $DR_n = R_n D$ on W^1, showing (iii)-(d). Property (iii)-(e) is a consequence of (ii)-(c).

Finally, assume that (iii) is true. The following general assertion will be used.

Claim 3. Let $\mathfrak{H}$ be a (real or complex) Hilbert space, Π an orthogonal projection of $\mathfrak{H}$ with finite rank p, and $0 < C < 1$. Then the cardinality of any orthonormal set contained in

$$U_C = \{\, u \in \mathfrak{H} \mid \|\Pi u\| > C\, \|u\| \,\}$$

is $\le p/C^2$.

Suppose $v_1, \dots, v_p$ is an orthonormal basis of $\Pi(\mathfrak{H})$. Let $u_1, \dots, u_k$ be orthonormal vectors in U_C, and Π' the orthogonal projection of $\mathfrak{H}$ to the linear subspace generated by them. We have

$$kC^2 \le \sum_{j=1}^{k} \|\Pi u_j\|^2 = \sum_{j=1}^{k} \sum_{i=1}^{p} |\langle v_i, u_j \rangle|^2 = \sum_{i=1}^{p} \|\Pi' v_i\|^2 \le p \,,$$

showing Claim 3.

Let $p_n = \lfloor C_0\, n^{1/\theta} \rfloor$ and $0 < C < 1$.

Claim 4. There is some $I \subset \mathbb{Z}$ with $\#I \le p_n/C^2$ and $\|R_n e_i\| \ge C_2 C$ for all $i \in \mathbb{Z} \setminus I$.

Let Π_n and $\widetilde{\Pi}_n$ be the orthogonal projections of $L^2(E)$ to $\ker R_n$ and $(\ker R_n)^\perp$, respectively. By Claim 3, the cardinality of the set

$$I = \{\, i \in \mathbb{Z} \mid \|\widetilde{\Pi}_n e_i\| > C \,\}$$

is $\le p_n/C^2$. For $i \in \mathbb{Z} \setminus I$, we have

$$\|R_n e_i\| = \|R_n \Pi_n e_i\| \ge C_2\, \|\Pi_n e_i\| \ge C_2 C$$

by (iii)-(b), showing Claim 4.

From Claim 4, it follows that there is some $i_n \in \mathbb{Z}$ such that

$$|i_n| \leq \frac{p_n}{C^2} + 1 \,, \tag{17}$$

$$\|R_n e_{i_n}\| \geq C_2 C \,. \tag{18}$$

We have

$$\begin{aligned}
\|R_n^a e_{i_n}\|_1^2 &= \|R_n^a e_{i_n}\|^2 + \|\mathbf{D} R_n^a e_{i_n}\|^2 \\
&\leq \|R_n^a e_{i_n}\|^2 + (\|R_n^a \mathbf{D} e_{i_n}\| + \|[\mathbf{D}, R_n^a] e_{i_n}\|)^2 \\
&\leq C_1^2 + \left(C_1 \sqrt{\lambda_{|i_n|}} + C_3\right)^2 \,.
\end{aligned}$$

Hence

$$u_{n,r}^a = \frac{r}{\sqrt{C_1^2 + \left(C_1 \sqrt{\lambda_{|i_n|}} + C_3\right)^2}} \, R_n^a e_{i_n} \in B^1 \cap Z_n$$

for all $r \in [0,1)$, giving

$$\begin{aligned}
\frac{r C_2 C}{\sqrt{C_1^2 + \left(C_1 \sqrt{\lambda_{|i_n|}} + C_3\right)^2}} &\leq \frac{r \, \|R_n e_{i_n}\|}{\sqrt{C_1^2 + \left(C_1 \sqrt{\lambda_{|i_n|}} + C_3\right)^2}} \\
&\leq \frac{r \sum_a \|R_n^a e_{i_n}\|}{\sqrt{C_1^2 + \left(C_1 \sqrt{\lambda_{|i_n|}} + C_3\right)^2}} = \sum_a \|u_{n,r}^a\| < \frac{A C_4}{n}
\end{aligned}$$

for all $r \in [0,1)$ by (18) and (iii)-(e). So there is some $C' > 0$, independent of n, such that

$$\lambda_{|i_n|} \geq \frac{1}{C_2^2} \left(\sqrt{\frac{C_2^2 C^2}{A C_4^2} \, n^2 - C_1^2} - C_3^2 \right)^2 \geq C' n^2 \tag{19}$$

for n large enough. If $|i_{n-1}| \leq k < |i_n|$ for n large enough and $k \in \mathbb{N}$, then

$$\lambda_k \geq \lambda_{|i_{n-1}|} \geq C'(n-1)^2 \geq C' n \geq C' \left(\frac{C^2(|i_n| - 1)}{C_0} \right)^\theta \geq C' \left(\frac{C^2 k}{C_0} \right)^\theta$$

by (19) and (17). This shows (i) because, since $|i_n| \to \infty$ as $n \to \infty$ by (19), there is an increasing sequence (n_ℓ) in $\mathbb{Z}_+$ such that $[|i_{n_0 - 1}|, \infty) = \bigcup_\ell [|i_{n_\ell - 1}|, |i_{n_\ell}|)$. $\square$

For any fixed $f \in C^\infty(M)$, let f also denote the operator of multiplication by f on $C^\infty(E)$ (or on $L^2(E)$ if f is bounded). Observe that $[d, f]$ is of order zero because d is of first order; moreover $[d, f]^* = -[\delta, f]$.

Lemma 5.3. *If f and $|[d, f]|$ are bounded, then:*

(i) *$f \, \mathcal{D}(d_{\min/\max}) \subset \mathcal{D}(d_{\min/\max})$ and $[d_{\min/\max}, f] = [d, f]$; and*
(ii) *$f \, W^1(d_{\min/\max}) \subset W^1(d_{\min/\max})$.*

Proof. For each $u \in \mathcal{D}(d_{\min})$, there is a sequence (u_n) in $C_0^\infty(E)$ such that $u_n \to u$ and (du_n) is convergent in $L^2(E)$; in fact, $d_{\min} u = \lim_n du_n$. Then $f u_n \to f u$ and

$$d(f u_n) = f \, du_n + [d, f] u_n \to f \, d_{\min} u + [d, f] u$$

in $L^2(E)$ because f and $|[d, f]|$ are bounded. So $fu \in \mathcal{D}(d_{\min})$ and $d_{\min}(fu) = f \, d_{\min} u + [d, f] u$.

Now, suppose that $u \in \mathcal{D}(d_{\max})$. Thus there is some $v \in L^2(E)$ such that $\langle u, \delta w\rangle = \langle v, w\rangle$ for all $w \in C_0^\infty(E)$; indeed, $v = d_{\max}u$. Then

$$\begin{aligned}\langle fu, \delta w\rangle = \langle u, f\delta w\rangle &= \langle u, \delta(fw) - [\delta, f]w\rangle \\ &= \langle v, fw\rangle - \langle u, [\delta, f]w\rangle = \langle fv + [d, f]u, w\rangle\end{aligned}$$

for all $w \in C_0^\infty(E)$. So $fu \in \mathcal{D}(d_{\max})$ and $d_{\max}(fu) = f\, d_{\max}u + [d, f]u$. This completes the proof of (i).

Property (ii) follows from (12) by applying (i) to d and δ. □

Let (E', d') be another elliptic complex on a Riemannian manifold M'. The scalar product of $L^2(E')$ will be denoted by $\langle\ ,\ \rangle'$, and let $\delta' = {}^t d'$. Let U and U' be open subsets of M and M', respectively, so that $U \supset \operatorname{supp} f$, and let $\zeta : (E|_U, d) \to (E'|_{U'}, d')$ be a quasi-isometric isomorphism of elliptic complexes whose underlying quasi-isometric diffeomorphism is $\xi : U \to U'$. For each $u \in L^2(E)$, identify fu to $fu|_U$, and identify $\zeta(fu) \in L^2(E'|_{U'})$ with its extension by zero to the whole of M'; in this way, we get a subspace $\zeta(f\,\mathcal{D}(d_{\min/\max})) \subset L^2(E')$.

Lemma 5.4. *If f and $|[d, f]|$ are bounded, then the following properties hold:*

(i) *We have $\zeta(f\,\mathcal{D}(d_{\min/\max})) \subset \mathcal{D}(d'_{\min/\max})$ and $d'_{\min/\max}\zeta = \zeta d_{\min/\max}$ on $f\,\mathcal{D}(d_{\min/\max})$*

(ii) *If moreover ζ is isometric, then $\zeta(f\,W^1(d_{\min/\max})) \subset W^1(d'_{\min/\max})$.*

Proof. Let $u \in f\,\mathcal{D}(d_{\min})$. Then $u \in \mathcal{D}(d_{\min})$ by Lemma 5.3-(i); in fact, according to its proof, there is a sequence (u_n) in $C_0^\infty(E)$ such that $u_n \to u$ and $du_n \to d_{\min}u$ in $L^2(E)$, and with $\operatorname{supp} u_n \subset \operatorname{supp} f$ for all n. Then $\zeta u_n \in C_0^\infty(E')$, $\zeta u_n \to \zeta u$ and $d'\zeta u_n = \zeta du_n \to \zeta d_{\min}u$ in $L^2(E')$. Hence $\zeta u \in \mathcal{D}(d'_{\min})$ and $d'_{\min}\zeta u = \zeta d_{\min}u$.

To prove the case of $d_{\max}$, since $\mathcal{D}(d'_{\max})$ is invariant by quasi-isometric changes of the metrics of M' and E', after shrinking U and U' if necessary, we can assume that $\zeta : (E|_U, d) \to (E'|_{U'}, d')$ is an isometric isomorphism of elliptic complexes. Such a change of metrics can be achieved by taking an open subset $V' \subset M'$ so that $\xi(\operatorname{supp} f) \subset V'$ and $\overline{V'} \subset U'$, and using a smooth partition of unity of M' subordinated to $\{V', M' \setminus \xi(\operatorname{supp} f)\}$ to combine metrics. Let $u \in f\,\mathcal{D}(d_{\max})$. Then $u \in \mathcal{D}(d_{\max})$ by Lemma 5.3-(i); indeed, according to its proof, the support of $v := d_{\max}u$ is contained in $\operatorname{supp} f$. Thus

$$\langle \zeta u, \delta'\zeta w\rangle' = \langle \zeta u, \zeta\delta w\rangle' = \langle u, \delta w\rangle = \langle v, w\rangle = \langle \zeta v, \zeta w\rangle'$$

for each $u \in f\,\mathcal{D}(d_{\max})$ and all $w \in C_0^\infty(E|_U)$. So $\langle \zeta u, \delta' w'\rangle' = \langle \zeta v, w'\rangle'$ for all $w' \in C_0^\infty(E')$, giving $\zeta u \in \mathcal{D}(d'_{\max})$ and $d_{\max}(\zeta u) = \zeta d_{\max}u$. This completes the proof of (i).

If ζ is isometric, then it is also an isometric isomorphism $(E|_U, \delta) \to (E'|_{U'}, \delta')$. So (ii) follows from (12) by applying (i) to d and δ. □

Proposition 5.5. *Let (E, d) be an elliptic complex on a Riemannian manifold M. Let $\{U_a\}$ be a finite open covering of M, and let $\{f_a\}$ be a smooth partition of unity on M subordinated to $\{U_a\}$ such that each $|[d, f_a]|$ is bounded. Assume also that there is another family $\{\tilde f_a\} \subset C^\infty(M)$ such that $\tilde f_a$ and $|[d, \tilde f_a]|$ are bounded, $\tilde f_a = 1$ on $\operatorname{supp} f_a$, and $\operatorname{supp} \tilde f_a \subset U_a$. For each a, let (E^a, d^a) be an elliptic complex on a Riemannian manifold M_a, let $V_a \subset M_a$ be an open subset, and let $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ be a quasi-isometric isomorphism of elliptic complexes over $\xi_a : U_a \to V_a$. Then the following properties hold:*

(i) $\mathcal{D}(d_{\mathrm{min/max}}) = \{ u \in L^2(E) \mid \zeta_a(f_a u) \in \mathcal{D}(d^a_{\mathrm{min/max}}) \ \forall a \}$.
(ii) *If* $d^a_{\mathrm{min/max}}$ *is discrete for all* a*, then* $d_{\mathrm{min/max}}$ *is discrete.*

Proof. The inclusion "$\subset$" of (i) follows from properties (i) of Lemmas 5.3 and 5.4.

Now, take any $u \in L^2(E)$ such that $\zeta_a(f_a u) \in \mathcal{D}(d^a_{\mathrm{min/max}})$ for all a. Let g_a and $\tilde g_a$ be the smooth functions on each M_a, supported in V_a, that correspond to f_a and $\tilde f_a$ via ξ_a. By Lemma 5.3-(i),

$$f_a u = \zeta_a^{-1}\zeta_a(f_a u) = \zeta_a^{-1}(\tilde g_a\, \zeta_a(f_a u)) \in \mathcal{D}(d_{\mathrm{min/max}}) \ .$$

So $u = \sum_a f_a u \in \mathcal{D}(d_{\mathrm{min/max}})$, completing the proof of (i).

To prove (ii), we can make the following reduction. Since discreteness is invariant by quasi-isometric isomorphisms of elliptic complexes, like in the proof of Lemma 5.4-(i), after shrinking $\{U_a\}$ if necessary, we can assume that each $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric. If every $d^a_{\mathrm{min/max}}$ is discrete, then each $W^1(d^a_{\mathrm{min/max}}) \hookrightarrow L^2(F^a)$ is compact by Lemma 5.1. So

$$\mathrm{Cl}_1(g_a\, W^1(d^a_{\mathrm{min/max}})) \hookrightarrow \mathrm{Cl}_0(g_a\, L^2(E^a))$$

is compact for all a by Lemma 5.3-(ii). Therefore

$$\mathrm{Cl}_1(f_a\, W^1(d_{\mathrm{min/max}})) \hookrightarrow \mathrm{Cl}_0(f_a\, L^2(E))$$

is compact by Lemma 5.4-(ii). Since $W^1(d_{\mathrm{min/max}}) = \sum_a f_a W^1(d_{\mathrm{min/max}})$ by Lemma 5.3-(ii), it follows that $W^1(d_{\mathrm{min/max}}) \hookrightarrow L^2(E)$ is compact. Hence $d_{\mathrm{min/max}}$ is discrete by Lemma 5.1. □

Proposition 5.6. *With the notation of Proposition 5.5, suppose that every* $d^a_{\mathrm{min/max}}$ *is discrete, and therefore* $d_{\mathrm{min/max}}$ *is also discrete. Let*

$$0 \le \lambda^a_{\mathrm{min/max},0} \le \lambda^a_{\mathrm{min/max},1} \le \cdots\ , \quad 0 \le \lambda_{\mathrm{min/max},0} \le \lambda_{\mathrm{min/max},1} \le \cdots$$

denote the eigenvalues, repeated according to their multiplicities, of the Laplacians $\Delta^a_{\mathrm{min/max}}$ *and* $\Delta_{\mathrm{min/max}}$ *defined by* $d^a_{\mathrm{min/max}}$ *and* $d_{\mathrm{min/max}}$*, respectively. If there is some*[13] $\theta_a > 0$ *for all* a *such that* $\liminf_k \lambda^a_{\mathrm{min/max},k} k^{-\theta_a} > 0$*, then* $\liminf_k \lambda_{\mathrm{min/max},k}\, k^{-\theta} > 0$ *with* $\theta = \min_a \theta_a$*.*

Proof. According to Sections 4.1 and 4.2, the condition $\liminf_k \lambda^a_{\mathrm{min/max},k} k^{-\theta_a} > 0$ is invariant by quasi-isometric isomorphisms of elliptic complexes. Thus, like in the proof of Proposition 5.5-(ii), we can assume that $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric. Set $D^a_{\mathrm{min/max}} = d^a_{\mathrm{min/max}} + \delta^a_{\mathrm{max/min}}$ and $W^{1,a} = W^1(d^a_{\mathrm{min/max}})$. Let $B^{1,a}$ denote the standard unit ball in $W^{1,a}$, and B^a_r the standard ball of radius $r > 0$ in $L^2(E^a)$. By Lemma 5.2, we get the following.

Claim 5. There are some $C_{a,0}, C_{a,1} > 0$ for every a such that, for all $n \in \mathbb{Z}_+$, there is a linear subspace $Z^a_n \subset L^2(E^a)$ so that:

(a) Z^a_n is closed and of codimension $\le C_{a,0}\, n^{1/\theta_a}$ in $L^2(E^a)$;
(b) $D^a_{\mathrm{min/max}}(W^{1,a} \cap Z^a_n) \subset Z^a_n$; and
(c) $B^{1,a} \cap Z^a_n \subset B^a_{C_{a,1}/n}$.

[13]The notation $\theta_{a,\mathrm{min/max}}$ would be more correct, but, for the sake of simplicity, reference to the maximum/minimum i.b.c. is omitted here and in most of the notation of the proof.

For each a, fix an open subset $O_a \subset M$ such that $\operatorname{supp} f_a \subset O_a$, $\overline{O_a} \subset U_a$ and the frontier of O_a has zero Riemannian measure. Let $P_a = \xi_a(O_a)$,

$$\mathcal{P}^a = \left\{ v \in L^2(E^a) \mid v \text{ is essentially supported in } \overline{P_a} \right\} ,$$

and $Z_n^{a\,\prime} = Z_n^a \cap \mathcal{P}^a$. Each $\mathcal{P}^a$ is a closed linear subspace of $L^2(E^a)$ satisfying

$$D^a_{\mathrm{min/max}}(W^{1,a} \cap \mathcal{P}^a) \subset \mathcal{P}^a . \tag{20}$$

Claim 6. (a) $Z_n^{a\,\prime}$ is closed and of codimension $\leq C_{a,0}\, n^{1/\theta_a}$ in $\mathcal{P}^a$;
(b) $D^a_{\mathrm{min/max}}(W^{1,a} \cap Z_n^{a\,\prime}) \subset Z_n^{a\,\prime}$; and
(c) $B^{1,a} \cap Z_n^{a\,\prime} \subset B^a_{C_{a,1}/n} \cap \mathcal{P}^a$.

Claim 6-(a) follows from Claim 5-(a) and the canonical linear isomorphism

$$\frac{\mathcal{P}^a}{Z_n^{a\,\prime}} \cong \frac{\mathcal{P}^a + Z_n^a}{Z_n^a} .$$

Claim 6-(b) is a consequence of Claim 5-(b) and (20), and Claim 6-(c) follows from Claim 5-(c).

Now, consider the linear spaces

$$\mathcal{O}^a = \left\{ u \in L^2(E) \mid u \text{ is essentially supported in } \overline{O_a} \right\} ,$$
$$Z_n^{a\,\prime\prime} = \left\{ u \in \mathcal{O}^a \mid \exists v \in Z_n^{a\,\prime} \text{ so that } \zeta_a(u|_{U_a}) = v|_{V_a} \right\} .$$

Each $\mathcal{O}^a$ is a closed linear subspace of $L^2(E)$, and we have $L^2(E) = \sum_a \mathcal{O}^a$. Set $D_{\mathrm{min/max}} = d_{\mathrm{min/max}} + \delta_{\mathrm{max/min}}$ and $W^1 = W^1(d_{\mathrm{min/max}})$. Let B^1 denote the standard unit ball in W^1, and B_r the standard ball of radius $r > 0$ in $L^2(E)$. Since $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric for all a, Claim 6 gives the following.

Claim 7. (a) $Z_n^{a\,\prime\prime}$ is closed and of codimension $\leq C_{a,0}\, n^{1/\theta_a}$ in $\mathcal{O}^a$;
(b) $D_{\mathrm{min/max}}(W^1 \cap Z_n^{a\,\prime\prime}) \subset Z_n^{a\,\prime\prime}$; and
(c) $B^1 \cap Z_n^{a\,\prime\prime} \subset B_{C_{a,1}/n} \cap \mathcal{O}^a$.

Let Y_n^a be a linear complement of each $Z_n^{a\,\prime\prime}$ in $\mathcal{O}^a$. By Claim 7-(a), we have

$$\mathcal{O}^a = Y_n^a \oplus Z_n^{a\,\prime\prime} \tag{21}$$

as topological vector spaces [34, Chapter I, 3.5]. On the other hand, for any $m \in \mathbb{Z}_+$, $W^m \cap \mathcal{O}^a$ is dense in $\mathcal{O}^a$ because it contains all sections $u \in C_0^\infty(E)$ with $\operatorname{supp} u \subset O_a$. So we can choose $Y_n^a \subset W^m$ by Claim 7-(a), obtaining

$$W^m \cap \mathcal{O}^a = Y_n^a \oplus (W^m \cap Z_n^{a\,\prime\prime}) \tag{22}$$

as topological vector spaces with respect to the topology induced by $\|\ \|$. The following assertion follows from (21), (22) and the density of $W^m \cap \mathcal{O}^a$ in $\mathcal{O}^a$.

Claim 8. $W^m \cap Z_n^{a\,\prime\prime}$ is $\|\ \|$-dense in $Z_n^{a\,\prime\prime}$.

For the case $m = 1$, observe that (22) is satisfied with

$$Y_n^a = \mathcal{O}^a \cap (W^1 \cap Z_n^{a\,\prime\prime})^{\perp_1} , \tag{23}$$

where $\perp_1$ denotes $\langle\ ,\ \rangle_1$-orthogonality, and therefore (22) also holds with respect to the topology induced by $\|\ \|_1$. From now on, consider the choice (23) for Y_n^a.

Claim 9. $D_{\mathrm{min/max}}(Y_n^a) \subset W^1$.

Since the Riemannian measure of the frontier of O_a is zero, $\mathcal{O}^{a\,\perp}$ consists of the sections $u \in L^2(E)$ whose essential support is contained in $M \setminus O_a$. Hence the set

$$(W^1 \cap \mathcal{O}^{a\,\perp}) + Y_n^a + (W^1 \cap Z_n^a{}'')$$

is dense in $L^2(E)$ by (22) for $m = 1$. It follows that, given any $u \in Y_n^a$, to check that $D_{\min/\max} u \in W^1$, its enough to check that the mapping

$$v \mapsto \langle D_{\min/\max} u, D_{\min/\max} v \rangle$$

is bounded on $W^1 \cap \mathcal{O}^{a\,\perp}$, Y_n^a and $W^1 \cap Z_n^a{}''$. This mapping vanishes on $W^1 \cap \mathcal{O}^{a\,\perp}$ because

$$D_{\min/\max}(W^1 \cap \mathcal{O}^a) \subset \mathcal{O}^a , \quad D_{\min/\max}(W^1 \cap \mathcal{O}^{a\,\perp}) \subset \mathcal{O}^{a\,\perp} .$$

Moreover it is bounded on Y_n^a because this space is of finite dimension. Finally, for $v \in W^1 \cap Z_n^a{}''$, we have

$$\langle D_{\min/\max} u, D_{\min/\max} v \rangle = -\langle u, v \rangle$$

because $u \perp_1 v$. Thus the above mapping is bounded on $W^1 \cap Z_n^a{}''$, which completes the proof of Claim 9.

Claim 10. $D_{\min/\max}(Y_n^a) \subset Y_n^a$.

For $u \in Y_n^a$ and $v \in W^2 \cap Z_n^a{}''$, we have

$$\begin{aligned}\langle D_{\min/\max} u, v \rangle_1 &= \langle D_{\min/\max} u, v \rangle + \langle \Delta_{\min/\max} u, D_{\min/\max} v \rangle \\ &= \langle u, D_{\min/\max} v \rangle + \langle D_{\min/\max} u, \Delta_{\min/\max} v \rangle = \langle u, D_{\min/\max} v \rangle_1 = 0\end{aligned}$$

by Claim 9 and because $D_{\min/\max}$ is self-adjoint. Then Claim 10 follows by Claim 8.

Claim 11. $Y_n^a = \mathcal{O}^a \cap (Z_n^a{}'')^\perp$.

Let $u \in Y_n^a$ and $v \in W^1 \cap Z_n^a{}''$. By Claim 10, $\Delta_{\min/\max}$ is a self-adjoint operator on Y_n^a. Then $u = (1 + \Delta_{\min/\max}) u_0$ for $u_0 = (1 + \Delta_{\min/\max})^{-1} u \in Y_n^a$, obtaining

$$\langle u, v \rangle = \langle (1 + \Delta_{\min/\max}) u_0, v \rangle = \langle u_0, v \rangle_1 = 0 .$$

This shows Claim 11 by Claim 8 and (21).

Let $\Pi_n^a : \mathcal{O}^a \to Z_n^a{}''$ denote the orthogonal projection. The following claim follows from (22) for $m = 1$, and Claims 7-(b), 10 and 11.

Claim 12. $\Pi_n^a(W^1 \cap \mathcal{O}^a) \subset W^1 \cap \mathcal{O}^a$, and $[D_{\min/\max}, \Pi_n^a] = 0$ on $W^1 \cap \mathcal{O}^a$.

Consider each function f_a as the corresponding bounded multiplication operator on $L^2(E)$. Assuming that a runs in $\{1, \dots, A\}$ for some $A \in \mathbb{Z}_+$, we get the bounded operator $T = (f_1, \dots, f_A) : L^2(E) \to \bigoplus_A L^2(E)$. Also, let $\Sigma : \bigoplus_A L^2(E) \to L^2(E)$ be the bounded operator defined by $\Sigma(u_1, \dots, u_A) = \sum_a u_a$. We have $\Sigma T = 1$ because $\{f_a\}$ is a partition of unity.

Claim 13. The image of T is closed.

Let (u^i) be a sequence in $L^2(E)$ such that (Tu^i) converges to some v in $\bigoplus_A L^2(E)$. Then $u^i = \Sigma T u^i \to \Sigma v$ as $i \to \infty$, obtaining $Tu^i \to T\Sigma v$ as $i \to \infty$. Hence $v = T\Sigma v \in T(L^2(E))$, showing Claim 13.

By Claim 13 and the open mapping theorem (see e.g. [12, Chapter III, 12.1] or [34, Chapter III, 2.1]), we get that T is a topological homomorphism[14]. So T :

[14] Recall that a bounded operator between topological vector spaces, $T : \mathfrak{H} \to \mathfrak{G}$, is called a topological homomorphism if the map $T : \mathfrak{H} \to T(\mathfrak{H})$ is open, where $T(\mathfrak{H})$ is endowed with the restriction of the topology of $\mathfrak{G}$.

$L^2(E) \to T(L^2(E))$ is a quasi-isometric isomorphism; its inverse is $\Sigma : T(L^2(E)) \to L^2(E)$. Since $\Pi_n := \bigoplus_a \Pi_n^a$ is an orthogonal projection of $\bigoplus_A L^2(E)$, it follows that $R_n := \Pi_n T$ satisfies Lemma 5.2-(iii)-(b),(c). Moreover, by Claim 7-(a),

$$\dim\ker R_n \le \dim\ker \Pi_n = \sum_a \dim\ker \Pi_n^a \le \sum_a C_{0,a}\, n^{1/\theta_a} \le C_0\, n^{1/\theta}$$

with $C_0 = \sum_a C_{0,a}$ and $\theta = \min_a \theta_a$, which shows that R_n satisfies Lemma 5.2-(iii)-(a).

We have $R_n = (R_n^1, \dots, R_n^A)$ with $R_n^a = \Pi_n^a f_a$. Since each function $|[d, f_a]|$ is uniformly bounded, it follows that $f_a W^1 \subset W^1$ and $[D_{\min/\max}, f_a] : W^1 \to L^2(E)$ extends to a bounded operator on $L^2(E)$. Therefore each R_n^a satisfies Lemma 5.2-(iii)-(d) by Claim 12.

Finally, R_n^a satisfies Lemma 5.2-(iii)-(e) by Claim 7-(c). Now, the result follows from Lemma 5.2. □

6. Preliminaries on a type of perturbation of the harmonic oscillator

To study the Witten's perturbed operators defined by the functions considered in this paper, the main analytic tool is the following perturbation of the harmonic oscillator introduced and studied in [1].

Let x and ρ denote the canonical coordinates of $\mathbb{R}$ and $\mathbb{R}_+$. For each $a \in \mathbb{R}$, the operator of multiplication by the function ρ^a on $C^\infty(\mathbb{R}_+)$ will be also denoted by ρ^a. We have

$$\left[\frac{d}{d\rho}, \rho^a\right] = a\rho^{a-1}\ , \quad \left[\frac{d^2}{d\rho^2}, \rho^a\right] = 2a\rho^{a-1}\,\frac{d}{d\rho} + a(a-1)\rho^{a-2}\ . \tag{24}$$

Recall that the harmonic oscillator, acting on $C^\infty(\mathbb{R}_+)$, is the operator

$$H = -\frac{d^2}{d\rho^2} + s^2\rho^2\ ,$$

depending on a parameter $s > 0$. For $c_1, c_2 \in \mathbb{R}$, consider the perturbation of H given by

$$P = H - 2c_1\rho^{-1}\,\frac{d}{d\rho} + c_2\rho^{-2}\ . \tag{25}$$

By (24), we get an operator of the same type if ρ^{-1} and $\frac{d}{d\rho}$ is interchanged.

Let $\mathcal{S}_{\mathrm{ev/odd}}$ denote the space of even/odd functions in the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$. The restrictions of those functions to $\mathbb{R}_+$ form th space denoted by $\mathcal{S}_{\mathrm{ev/odd},+}$. The scalar product of $L^2(\mathbb{R}_+, \rho^{2c_1}\, d\rho)$ will be denoted by $\langle\ ,\ \rangle_{c_1}$. For each $\sigma > -1/2$, let p_k denote the sequence of orthogonal polynomials associated with the measure $e^{-sx^2}|x|^{\sigma/2}\, dx$ on $\mathbb{R}$ [35], called generalized Hermite polynomials. The corresponding generalized Hermite functions are $\phi_k = p_k e^{-sx^2/2}$.

Proposition 6.1 ([1, Corollary 1.8])**.** *If there is some $a \in \mathbb{R}$ such that*

$$a^2 + (2c_1 - 1)a - c_2 = 0\ , \tag{26}$$

$$\sigma := a + c_1 > -1/2\ , \tag{27}$$

then:

(i) *P, with domain $x^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{2c_1}\, d\rho)$;*

(ii) *the spectrum of its self-adjoint extension, denoted by* $\mathcal{P}$*, consists of the eigenvalues* $(4k+1+2\sigma)s$ $(k\in\mathbb{N})$ *with multiplicity one and normalized eigenfunctions* $\chi_{s,a,\sigma,k} := \sqrt{2}\,\rho^a\phi_{2k,+}$ *(or simply* χ_k*); and*
(iii) $\mathcal{D}^\infty(\mathcal{P}) = \rho^a\mathcal{S}_{\mathrm{ev},+}$.

Remark 13. By Proposition 6.1-(iii), we have $h\,\mathcal{D}^\infty(\mathcal{P}) \subset \mathcal{D}^\infty(\mathcal{P})$ for all $h\in C^\infty(\mathbb{R}_+)$ such that $h'\in C_0^\infty(\mathbb{R}_+)$.

The existence of $a\in\mathbb{R}$ satisfying (26) is characterized by the condition

$$(2c_1-1)^2+4c_2\geq 0\,. \tag{28}$$

Observe that (28) is satisfied if $c_2\geq\min\{0,2c_1\}$. In particular, we have the following special cases:

- If $c_2=0$, then (26) means that $a\in\{0,1-2c_1\}$, and (27) gives
$$\sigma=\begin{cases}c_1 & \text{if } a=0\\ 1-c_1 & \text{if } a=1-2c_1\,.\end{cases}$$
- If $c_2=2c_1$, then (26) means that $a\in\{1,-2c_1\}$, and (27) gives
$$\sigma=\begin{cases}1+c_1 & \text{if } a=1\\ -c_1 & \text{if } a=-2c_1\,.\end{cases}$$

The following property of $\chi_{s,a,\sigma,0}$ will be also used.

Lemma 6.2. *If* h *is a bounded measurable function on* $\mathbb{R}_+$ *with* $h(\rho)\to 1$ *as* $\rho\to 0$*, then* $\langle h\chi_{s,a,\sigma,0},\chi_{s,a,\sigma,0}\rangle_{c_1}\to 1$ *as* $s\to\infty$*.*

Proof. Given any $\epsilon>0$, take some $\rho_0>0$ such that $|h(\rho)-1|\leq\epsilon/2$ for $\rho\leq\rho_0$. For s large enough, we have

$$\int_{\rho_0}^\infty e^{-s\rho^2}\,\rho^{2\sigma}\,d\rho\leq\frac{\epsilon}{4p_0^2\,\max|h-1|}$$

Hence, for s large enough,

$$\begin{aligned}|\langle(1-h)\chi_{s,a,\sigma,0},\chi_{s,a,\sigma,0}\rangle_{c_1}| &\leq 2p_0^2\int_0^\infty|1-h(\rho)|\,e^{-s\rho^2}\,\rho^{2\sigma}\,d\rho\\ &= p_0^2\epsilon\int_0^{\rho_0}e^{-s\rho^2}\,\rho^{2\sigma}\,d\rho+2p_0^2\,(\max|1-h|)\int_{\rho_0}^\infty e^{-s\rho^2}\,\rho^{2\sigma}\,d\rho\\ &< p_0^2\epsilon\int_0^\infty e^{-s\rho^2}\,\rho^{2\sigma}\,d\rho+\frac{\epsilon}{2}=\frac{\epsilon}{2}\,\|\chi_{s,a,\sigma,0}\|_{c_1}^2+\frac{\epsilon}{2}=\epsilon\,.\quad\square\end{aligned}$$

7. Two simple types of elliptic complexes

Here, we study the two types of simple elliptic complexes. They will show up in the direct sum splitting of the local model of Witten's perturbation (Section 11). We could describe better the spectra of the Laplacians associated to the minimum/maximum i.b.c. of these simple elliptic complexes, but this will be done with the local model of the Witten's perturbation (Section 10).

7.1. **Some more results on general elliptic complexes.** Consider the notation of the beginning of Section 4.2.

Lemma 7.1. *Let $\mathcal{G} \subset C^\infty(E) \cap L^2(E)$ be a graded linear subspace containing $C_0^\infty(E)$, preserved by d and δ, and such that $\langle du, v\rangle = \langle u, \delta v\rangle$ for all $u, v \in \mathcal{G}$. Let $d_\mathcal{G}$, $\delta_\mathcal{G}$ and $\Delta_\mathcal{G}$ denote the restrictions of d, δ and Δ to $\mathcal{G}$. Assume that $\Delta_\mathcal{G}$ is essentially self-adjoint in $L^2(E)$, and $\mathcal{G}$ is the smooth core of $\overline{\Delta_\mathcal{G}}$. Then the following properties hold:*

(i) *If $\mathcal{G}_r \subset \mathcal{D}(d_{\min,r})$ and $\mathcal{G}_{r-1} \subset \mathcal{D}(d_{\min,r-1})$ for some degree r, then $\mathcal{G}_r$ is the smooth core of $d_{\min,r}$.*

(ii) *If $\mathcal{G}_r \subset \mathcal{D}(\delta_{\min,r-1})$ and $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\min,r})$ for some degree r, then $\mathcal{G}_r$ is the smooth core of $d_{\max,r}$.*

Proof. For each degree r, the restrictions $d_r : \mathcal{G}_r \to \mathcal{G}_{r+1}$, $\delta_r : \mathcal{G}_{r+1} \to \mathcal{G}_r$ and $\Delta_r : \mathcal{G}_r \to \mathcal{G}_r$ will be denoted by $d_{\mathcal{G},r}$, $\delta_\mathcal{G}, r$ and $\Delta_{\mathcal{G},r}$, respectively. Suppose that $\mathcal{G}_r \subset \mathcal{D}(d_{\min,r})$ and $\mathcal{G}_{r-1} \subset \mathcal{D}(d_{\min,r-1})$, and therefore $d_{\mathcal{G},r} \subset d_{\min,r}$ and $d_{\mathcal{G},r-1} \subset d_{\min,r-1}$. Since $C_0^\infty(E) \subset \mathcal{G}$ and $\langle du, v\rangle = \langle u, \delta v\rangle$ for all $u, v \in \mathcal{G}$, it follows that $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\max,r})$ and $\mathcal{G}_r \subset \mathcal{D}(\delta_{\max,r-1})$, and therefore $\delta_{\mathcal{G},r} \subset \delta_{\max,r}$ and $\delta_{\mathcal{G},r-1} \subset \delta_{\max,r-1}$. By (11), we get $\Delta_{\mathcal{G},r} \subset \Delta_{\min,r}$. So $\overline{\Delta_{\mathcal{G},r}} \subset \Delta_{\min,r}$, and therefore $\overline{\Delta_{\mathcal{G},r}} = \Delta_{\min,r}$ because these operators are self-adjoint in $L^2(E_r)$. Then $\mathcal{G}_r$ is the smooth core of $d_{\min,r}$, completing the proof of (i).

Now, assume that $\mathcal{G}_r \subset \mathcal{D}(\delta_{\min,r-1})$ and $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\min,r})$, and therefore $\delta_{\mathcal{G},r-1} \subset \delta_{\min,r-1}$ and $\delta_{\mathcal{G},r} \subset \delta_{\min,r}$. As above, it follows that $d_{\mathcal{G},r-1} \subset d_{\max,r-1}$ and $d_{\mathcal{G},r} \subset d_{\max,r}$. By (11), we get $\Delta_{\mathcal{G},r} \subset \Delta_{\max,r}$. So $\overline{\Delta_{\mathcal{G},r}} \subset \Delta_{\max,r}$, obtaining $\overline{\Delta_{\mathcal{G},r}} = \Delta_{\max,r}$ as before. Thus $\mathcal{G}_r$ is the smooth core of $d_{\max,r}$, completing the proof of (ii). □

Now, suppose that there is an orthogonal decomposition $E_{r+1} = E_{r+1,1} \oplus E_{r+1,2}$ for some degree $r+1$. Thus

$$\begin{aligned} C^\infty(E_{r+1}) &\equiv C^\infty(E_{r+1,1}) \oplus C^\infty(E_{r+1,2}) \;, \\ C_0^\infty(E_{r+1}) &\equiv C_0^\infty(E_{r+1,1}) \oplus C_0^\infty(E_{r+1,2}) \;, \\ L^2(E_{r+1}) &\equiv L^2(E_{r+1,1}) \oplus L^2(E_{r+1,2}) \;, \end{aligned}$$

giving

$$d_r = \begin{pmatrix} d_{r,1} \\ d_{r,2} \end{pmatrix} \;, \quad \delta_r = \begin{pmatrix} \delta_{r,1} & \delta_{r,2} \end{pmatrix} \;.$$

Lemma 7.2. *We have:*

$$\mathcal{D}(d_{\max,r}) = \mathcal{D}(d_{r,1,\max}) \cap \mathcal{D}(d_{r,2,\max}) \;, \quad d_{\max,r} = \begin{pmatrix} d_{r,1,\max}|_{\mathcal{D}(d_{\max,r})} \\ d_{r,2,\max}|_{\mathcal{D}(d_{\max,r})} \end{pmatrix} \;.$$

Proof. Let $u \in L^2(E_r)$. We have $u \in \mathcal{D}(d_{\max,r})$ if and only if there is some $w \in L^2(E_{r+1})$ such that $\langle u, \delta v\rangle = \langle w, v\rangle$ for all $v \in C_0^\infty(E_{r+1})$, and moreover $d_{\max,r}u = w$ in this case. Writing $w = w_1 \oplus w_2$ and $v = v_1 \oplus v_2$, this condition on u means that $\langle u, \delta_{0,i} v_i\rangle = \langle w_i, v_i\rangle$ for all $v_i \in C_0^\infty(E_{r+1}^i)$ and $i \in \{1, 2\}$. In turn, this is equivalent to $u \in \mathcal{D}(d_{r,1,\max}) \cap \mathcal{D}(d_{r,2,\max})$ with $d_{r,i,\max} u = w_i$. □

For $i \in \{1, 2\}$, let $\Delta_{r,i} = \delta_{r,i} d_{r,i} + d_{r-1}\delta_{r-1}$ on $C^\infty(E_r)$.

Corollary 7.3. *If $a\,\Delta_r = b\,\Delta_{r,i} + c$ for some $a, b, c \in \mathbb{R}$ with $a, b \neq 0$, $d_{\min,r}$ and $d_{r,i,\min}$ have the same smooth core, and $d_{r,i,\min} = d_{r,i,\max}$ for some $i \in \{0, 1\}$, then $d_{\min,r} = d_{\max,r}$.*

Proof. By Lemma 7.2 and since $d_{r,i,\min} = d_{r,i,\max}$, we get $\mathcal{D}(d_{\max,r}) \subset \mathcal{D}(d_{r,i,\min})$. Because $a\,\Delta_r = b\,\Delta_{r,i} + c$ for some $a, b, c \in \mathbb{R}$ with $a, b \neq 0$, it follows that

$$\{\, u \in \mathcal{D}(d_{\max,r}) \cap C^\infty(E_r) \mid \Delta_r^k u \in L^2(E_r)\ \forall k \in \mathbb{N} \,\}$$
$$\subset \{\, u \in \mathcal{D}(d_{r,i,\min}) \cap C^\infty(E_r) \mid \Delta_{r,i}^k u \in L^2(E_r)\ \forall k \in \mathbb{N} \,\} \;.$$

This means that the smooth core of $d_{\max,r}$ is contained in the smooth core of $d_{r,i,\min}$, which equals the smooth core of $d_{\min,r}$. Then $d_{\max,r} = d_{\min,r}$. □

7.2. **An elliptic complex of length two.** Consider the standard metric on $\mathbb{R}_+$. Let E be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose non-zero terms are E_0 and E_1, which are real/complex trivial line bundles endowed with the standard Riemannian/Hemitian metrics. Thus

$$C^\infty(E_0) \equiv C^\infty(\mathbb{R}_+) \equiv C^\infty(E_1)\;, \quad L^2(E_0) \equiv L^2(\mathbb{R}_+, d\rho) \equiv L^2(E_1)\;,$$

where real/complex valued functions are considered in $C^\infty(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+, d\rho)$. For any fixed $s > 0$ and $\kappa \in \mathbb{R}$, let

$$C^\infty(E_0) \underset{\delta}{\overset{d}{\rightleftarrows}} C^\infty(E_1)$$

be the differential operators defined by

$$d = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\;, \quad \delta = -\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\;.$$

It is easy to check that (E, d) is an elliptic complex, whose formal adjoint is (E, δ). By (24), the homogeneous components of the corresponding Laplacian Δ are:

$$\begin{aligned} \Delta_0 = \delta d &\equiv H + \kappa\left[\frac{d}{d\rho}, \rho^{-1}\right] \mp s\left[\frac{d}{d\rho}, \rho\right] + \kappa^2\rho^{-2} \mp 2s\kappa \\ &= H + \kappa(\kappa - 1)\rho^{-2} \mp s(1 + 2\kappa)\;, \qquad (29) \\ \Delta_1 = d\delta &= H - \kappa\left[\frac{d}{d\rho}, \rho^{-1}\right] \pm s\left[\frac{d}{d\rho}, \rho\right] + \kappa^2\rho^{-2} \mp 2s\kappa \\ &= H + \kappa(\kappa + 1)\rho^{-2} \pm s(1 - 2\kappa)\;, \qquad (30) \end{aligned}$$

where H is the harmonic oscillator on $C^\infty(\mathbb{R}_+)$ defined with the constant s. Then Δ_0 and Δ_1 are of the form of P in (25) (with $c_1 = 0$) plus a constant; in particular, for $\kappa = 0$, they are equal to H plus a constant.

For Δ_0, the condition (26) means that $a \in \{\kappa, 1 - \kappa\}$, and (27) gives $\sigma = \kappa$ if $a = \kappa$, and $\sigma = 1 - \kappa$ if $a = 1 - \kappa$. By Proposition 6.1, the following holds:

- If $\kappa > -1/2$, then Δ_0, with domain $\rho^\kappa\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^\kappa\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 3/2$, then Δ_0, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

For Δ_1, the condition (26) means that $a \in \{1 + \kappa, -\kappa\}$, and (27) becomes $\sigma = 1 + \kappa$ if $a = 1 + \kappa$, and $\sigma = -\kappa$ if $a = -\kappa$. Now Proposition 6.1 states the following:

- If $\kappa > -3/2$, then Δ_1, with domain $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 1/2$, then Δ_1, with domain $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

When $\kappa > -1/2$, let $\mathcal{E}_1 \subset C^\infty(E) \cap L^2(E)$ be the dense graded linear subspace with

$$\mathcal{E}_1^0 \equiv \rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;, \quad \mathcal{E}_1^1 \equiv \rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;.$$

When $\kappa < 1/2$, let $\mathcal{E}_2 \subset C^\infty(E) \cap L^2(E)$ be the dense graded linear subspace with

$$\mathcal{E}_2^0 \equiv \rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;, \quad \mathcal{E}_2^1 \equiv \rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;.$$

Observe that, by restricting d and δ, we get complexes $(\mathcal{E}_1, d)$ and $(\mathcal{E}_1, \delta)$ when $\kappa > -1/2$, and complexes $(\mathcal{E}_2, d)$ and $(\mathcal{E}_2, \delta)$ when $\kappa < 1/2$. Thus Δ preserves $\mathcal{E}_1$ when $\kappa > -1/2$, and preserves $\mathcal{E}_2$ when $\kappa < 1/2$.

Proposition 7.4. (i) *If $|\kappa| < 1/2$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are the smooth cores of $d_{\max}$ and $d_{\min}$, respectively.*

(ii) *If $|\kappa| \geq 1/2$, then (E, d) has a unique i.b.c., whose smooth core is $\mathcal{E}_1$ when $\kappa \geq 1/2$, and $\mathcal{E}_2$ when $\kappa \leq -1/2$.*

The following lemma will be used in the proof of Proposition 7.4.

Lemma 7.5. *Suppose that $\theta \geq 1/2$. Then, for each $\xi \in \rho^\theta\,\mathcal{S}_{\mathrm{ev},+}$, considered as subspace of $C^\infty(E_0)$ (respectively, $C^\infty(E_1)$), there is a sequence (ξ_n) in $C_0^\infty(E_0)$ (respectively, $C_0^\infty(E_1)$), independent of κ, such that $\lim_n \xi_n = \xi$ and $\lim_n d\xi_n = d\xi$ in $L^2(E_0)$ (respectively, $\lim_n \delta\xi_n = \delta\xi$ in $L^2(E_1)$). In particular, $\rho^\theta\,\mathcal{S}_{\mathrm{ev},+}$ is contained in $\mathcal{D}(d_{\min})$ (respectively, $\mathcal{D}(\delta_{\min})$).*

Remark 14. In Lemma 7.5, the independence of κ means that (ξ_n) depends only on θ and ξ, whilst the convergences $\lim_n d\xi_n = d\xi$ and $\lim_n \delta\xi_n = \delta\xi$ hold with d and δ defined by any κ.

Proof of Lemma 7.5. The proof is made for $\mathcal{D}(d_{\min})$; the case of $\mathcal{D}(\delta_{\min})$ is analogous.

Let $0 < a < b$ and $f \in C_0^\infty(\mathbb{R}_+)$ such that $0 \leq f \leq 1$, $f(\rho) = 1$ for $\rho \leq a$, and $f(\rho) = 0$ for $\rho \geq b$. For each $n \in \mathbb{N}$, let $g_n, h_n \in C^\infty(\mathbb{R}_+)$ be defined by $g_n(\rho) = f(n\rho)$ and $h_n(\rho) = f(\rho/n)$. It is clear that

$$\chi_{[\frac{b}{n}, na]} \leq (1 - g_n)h_n \leq \chi_{[\frac{a}{n}, nb]}\;, \tag{31}$$

where χ_S denotes the characteristic function of each subset $S \subset \mathbb{R}_+$.

Let $\phi \in \mathcal{S}_{\mathrm{ev},+}$. From (31), we get $(1-g_n)h_n\rho^\theta\phi \in C_0^\infty(E_0)$ and $(1-g_n)h_n\rho^\theta\phi \to \rho^\theta\phi$ in $L^2(E_0)$ as $n \to \infty$. Observe that

$$d((1-g_n)h_n\rho^\theta\phi) = -g_n' h_n\rho^\theta\phi + (1-g_n)h_n'\rho^\theta\phi + (1-g_n)h_n\,d(\rho^\theta\phi)\;.$$

In right hand side of this equality, the last term converges to $d(\rho^\theta\phi)$ in $L^2(E_1)$ as $n\to\infty$ by (31). Moreover

$$\begin{aligned}\|(1-g_n)h_n'\rho^\theta\phi\|^2 &= \int_0^\infty (1-g_n)^2 {h_n'}^2(\rho)\rho^{2\theta}\phi^2(\rho)\,d\rho\\ &\le (\max\rho^{2\theta}\phi^2)\,n^{-2}\int_0^\infty {f'}^2(\rho/n)\,d\rho = (\max\rho^{2\theta}\phi^2)\,n^{-1}\int_0^\infty {f'}^2(x)\,dx\\ &= (\max\rho^{2\theta}\phi^2)\,n^{-1}\,\|f'\|^2\ ,\end{aligned}$$

which converges to zero as $n\to\infty$, and

$$\begin{aligned}\|g_n'h_n\rho^\theta\phi\|^2 &= \int_0^\infty {g_n'}^2(\rho)h_n^2(\rho)\rho^{2\theta}\phi^2(\rho)\,d\rho \le (\max\phi^2)\,n^2\int_0^\infty {f'}^2(n\rho)\rho^{2\theta}\,d\rho\\ &= (\max\phi^2)\,n^{1-2\theta}\int_0^\infty {f'}^2(x)x^{2\theta}\,dx = (\max\phi^2)\,n^{1-2\theta}\,\|f'\rho^\theta\|^2\ ,\end{aligned}$$

which converges to zero as $n\to\infty$ if $\theta>1/2$.

In the case $\theta=1/2$, it is enough to prove that f can be chosen so that $\|f'\rho^{1/2}\|$ is as small as desired. For $m>1$ and $0<\epsilon<1$, observe that there is some f as above such that:

- the support of f' is contained in $[e^{-\epsilon},e^m]$,
- $-\frac{1}{m\rho}\le f'\le 0$, and
- $f'(\rho)=-\frac{1}{m\rho}$ if $1\le\rho\le e^{m-\epsilon}$.

Then

$$\|f'\rho^{1/2}\|^2=\int_{e^{-\epsilon}}^{e^m}{f'}^2(\rho)\rho\,d\rho\le\frac{1}{m^2}\int_{e^{-\epsilon}}^{e^m}\frac{d\rho}{\rho}=\frac{m+\epsilon}{m^2}\ ,$$

which converges to zero as $m\to\infty$. □

Proof of Proposition 7.4. Suppose that $|\kappa|<1/2$. Since $1\pm\kappa>1/2$, by Lemma 7.5, $\mathcal{E}_2^0\subset\mathcal{D}(d_{\min})$ and $\mathcal{E}_1^1\subset\mathcal{D}(\delta_{\min})$. The other conditions of Lemma 7.1 are satisfied by d with $\mathcal{G}=\mathcal{E}_2$, and by δ with $\mathcal{G}=\mathcal{E}_1$ by the discussion previous to Proposition 7.4. So $\mathcal{E}_2$ is the smooth core of $d_{\min}$ and $\mathcal{E}_1$ is the smooth core of $d_{\max}$ by Lemma 7.1.

Now, assume that $\kappa\ge1/2$ (respectively, $\kappa\le-1/2$), giving also $1+\kappa>1/2$ (respectively, $1-\kappa>1/2$). Then, by Lemma 7.5, $\mathcal{E}_1^0\subset\mathcal{D}(d_{\min})$ and $\mathcal{E}_1^1\subset\mathcal{D}(\delta_{\min})$ (respectively, $\mathcal{E}_2^0\subset\mathcal{D}(d_{\min})$ and $\mathcal{E}_2^1\subset\mathcal{D}(\delta_{\min})$). By the discussion previous to Proposition 7.4, the other conditions of Lemma 7.1 are satisfied by d and δ with $\mathcal{G}=\mathcal{E}_1$ (respectively, $\mathcal{G}=\mathcal{E}_2$). So, by Lemma 7.1, $\mathcal{E}_1$ (respectively, $\mathcal{E}_2$) is the smooth core of $d_{\min}$ and $d_{\max}$. □

Remark 15. In the proof of Lemma 7.5 and Proposition 7.4, we have borrowed ideas from the proof of [6, Theorem 4.1]; in fact, in the case with $\kappa=0$, Proposition 7.4 could be proved exactly like [6, Theorem 4.1].

7.3. **An elliptic complex of length three.** Consider again the standard metric on $\mathbb{R}_+$. Let F be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose non-zero terms are F_0, F_1 and F_2, which are trivial real/complex vector bundles of ranks 1, 2 and 1, respectively, endowed with the standard Riemannian/Hermitian metrics. Thus

$$\begin{gathered}C^\infty(F_0)\equiv C^\infty(\mathbb{R}_+)\equiv C^\infty(F_2)\ ,\quad C^\infty(F_1)\equiv C^\infty(\mathbb{R}_+)\oplus C^\infty(\mathbb{R}_+)\ ,\\ L^2(F_0)\equiv L^2(\mathbb{R}_+,d\rho)\equiv L^2(F_2)\ ,\quad L^2(F_1)\equiv L^2(\mathbb{R}_+,d\rho)\oplus L^2(\mathbb{R}_+,d\rho)\ ,\end{gathered}$$

where real/complex valued functions are considered in $C^\infty(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+, d\rho)$. Fix $s, c > 0$ and $\kappa \in \mathbb{R}$, and let

$$C^\infty(F_0) \underset{\delta_0}{\overset{d_0}{\rightleftarrows}} C^\infty(F_1) \underset{\delta_1}{\overset{d_1}{\rightleftarrows}} C^\infty(F_2)$$

be the differential operators defined by

$$d_0 = \begin{pmatrix} d_{0,1} \\ d_{0,2} \end{pmatrix} , \quad \delta_0 = \begin{pmatrix} \delta_{0,1} & \delta_{0,2} \end{pmatrix} , \quad d_1 = \begin{pmatrix} d_{1,1} & d_{1,2} \end{pmatrix} , \quad \delta_1 = \begin{pmatrix} \delta_{1,1} \\ \delta_{1,2} \end{pmatrix} ,$$

where

$$\begin{aligned}
d_{0,1} &= \frac{c}{\sqrt{1+c^2}} \left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho \right) , \\
d_{0,2} &= \frac{1}{\sqrt{1+c^2}} \left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho \right) , \\
\delta_{0,1} &= \frac{c}{\sqrt{1+c^2}} \left(-\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho \right) , \\
\delta_{0,2} &= \frac{1}{\sqrt{1+c^2}} \left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho \right) , \\
d_{1,1} &= \frac{1}{\sqrt{1+c^2}} \left(\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho \right) , \\
d_{1,2} &= \frac{c}{\sqrt{1+c^2}} \left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho \right) , \\
\delta_{1,1} &= \frac{1}{\sqrt{1+c^2}} \left(-\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho \right) , \\
\delta_{1,2} &= \frac{c}{\sqrt{1+c^2}} \left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho \right) .
\end{aligned}$$

A direct computation shows that d_0 and d_1 define an elliptic complex (F, d) of length three. Its formal adjoint is the complex (F, δ) given by δ_0 and δ_1. The homogeneous components Δ_0 and Δ_2 of the corresponding Laplacian Δ can be computed as follows, where the notation of Section 7.1 is used. By (29) and (30),

$$\begin{aligned}
\Delta_{0,1} = \delta_{0,1}d_{0,1} &= \frac{c^2}{1+c^2}\left(-\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\left(\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\\
&= \frac{c^2}{1+c^2}\left(H+\kappa(\kappa+1)\rho^{-2}\mp s(1-2\kappa)\right)\ ,\\
\Delta_{0,2} = \delta_{0,2}d_{0,2} &= \frac{1}{1+c^2}\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\\
&= \frac{1}{1+c^2}\left(H+(\kappa+1)\kappa\rho^{-2}\mp s(1+2(\kappa+1))\right)\ ,\\
\Delta_{2,1} = d_{1,1}\delta_{1,1} &= \frac{1}{1+c^2}\left(\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\\
&= \frac{1}{1+c^2}\left(H+\kappa(\kappa+1)\rho^{-2}\pm s(1-2\kappa)\right)\ ,\\
\Delta_{2,2} = d_{1,2}\delta_{1,2} &= \frac{c^2}{1+c^2}\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\\
&= \frac{c^2}{1+c^2}\left(H+(\kappa+1)\kappa\rho^{-2}\pm s(1+2(\kappa+1))\right)\ ,\\
\Delta_0 = \delta_0 d_0 = \Delta_{0,1}+\Delta_{0,2} &= H+\kappa(\kappa+1)\rho^{-2}\mp s\left(2+\frac{1-c^2}{1+c^2}(1+2\kappa)\right)\ ,\\
\Delta_2 = d_1\delta_1 = \Delta_{2,1}+\Delta_{2,2} &= H+\kappa(\kappa+1)\rho^{-2}\pm s\left(2+\frac{1-c^2}{1+c^2}(1+2\kappa)\right)\ .
\end{aligned}$$

Thus Δ_0 can be identified to Δ_2, and they are of the form of P in (25) (with $c_1 = 0$) plus a constant.

For Δ_0 and Δ_2, the condition (26) means that $a \in \{1+\kappa, -\kappa\}$, and (27) gives $\sigma = 1+\kappa$ if $a = 1+\kappa$, and $\sigma = -\kappa$ if $a = -\kappa$. By Proposition 6.1, the following holds:

- If $\kappa > -3/2$, then Δ_0 and Δ_2, with domain $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectra of their closures are discrete, and the smooth core of their closures is $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 3/2$, then Δ_0 and Δ_2, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectra of their closures are discrete, and the smooth core of their closures is $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

Write

$$\begin{aligned}
\Delta_1 &= d_0\delta_0+\delta_1 d_1\\
&= \begin{pmatrix} d_{0,1}\delta_{0,1}+\delta_{1,1}d_{1,1} & d_{0,1}\delta_{0,2}+\delta_{1,1}d_{1,2}\\ d_{0,2}\delta_{0,1}+\delta_{1,2}d_{1,1} & d_{0,2}\delta_{0,2}+\delta_{1,2}d_{1,2}\end{pmatrix} = \begin{pmatrix}\Delta_{1,1} & A\\ B & \Delta_{1,2}\end{pmatrix}\ .
\end{aligned}$$

By (29) and (30),

$$
\begin{aligned}
\Delta_{1,1} &= \frac{1}{1+c^2}\left(c^2\left(\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\right.\\
&\qquad\left.+\left(-\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\left(\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\right)\\
&= \frac{1}{1+c^2}\big(c^2\left(H+\kappa(\kappa-1)\rho^{-2}\mp s(1+2\kappa)\right)\\
&\qquad + H+\kappa(\kappa-1)\rho^{-2}\mp s(1+2\kappa)\big)\\
&= H+\kappa(\kappa-1)\rho^{-2}\mp s\frac{1-c^2}{1+c^2}(1+2\kappa)\;,\\
\Delta_{1,2} &= \frac{1}{1+c^2}\left(\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\right.\\
&\qquad\left.+c^2\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\right)\\
&= \frac{1}{1+c^2}\big(H+(\kappa+1)(\kappa+2)\rho^{-2}\pm s(1-2(\kappa+1))\\
&\qquad + c^2\left(H+(\kappa+1)(\kappa+2)\rho^{-2}\mp s(1-2(\kappa+1))\right)\big)\\
&= H+(\kappa+1)(\kappa+2)\rho^{-2}\mp s\frac{1-c^2}{1+c^2}(1+2\kappa)\;.
\end{aligned}
$$

So $\Delta_{1,1}$ and $\Delta_{1,2}$ also are of the form of P in (25) (with $c_1=0$) plus a constant.

For $\Delta_{1,1}$, the condition (26) means that $a\in\{\kappa,1-\kappa\}$, and (27) gives $\sigma=\kappa$ if $a=\kappa$, and $\sigma=1-\kappa$ if $a=1-\kappa$. By Proposition 6.1, the following holds:

- If $\kappa>-1/2$, then $\Delta_{1,1}$, with domain $\rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa<3/2$, then $\Delta_{1,1}$, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

For $\Delta_{1,2}$, the condition (26) means that $a\in\{2+\kappa,-1-\kappa\}$, and (27) becomes $\sigma=2+\kappa$ if $a=2+\kappa$, and $\sigma=-1-\kappa$ if $a=-1-\kappa$. Then Proposition 6.1 states the following:

- If $\kappa>-5/2$, then $\Delta_{1,2}$, with domain $\rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa<-1/2$, then $\Delta_{1,2}$, with domain $\rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

Finally, by (24),

$$\begin{aligned} A &= \frac{c}{1+c^2}\left(\left(\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\right.\\ &\quad\left.+\left(-\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\right)\\ &= \pm\frac{2cs}{1+c^2}\left(\left[\frac{d}{d\rho},\rho\right]-1\right)=0\;,\\ B &= \frac{c}{1+c^2}\left(\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\right)\left(-\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\right.\\ &\quad\left.+\left(\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\left(\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho\right)\right)\\ &= \pm\frac{2cs}{1+c^2}\left(\left[\frac{d}{d\rho},\rho\right]-1\right)=0\;. \end{aligned}$$

When $\kappa > -1/2$, let $\mathcal{F}_1 \subset C^\infty(F)\cap L^2(F)$ be the dense graded linear subspace with

$$\mathcal{F}_1^0=\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \mathcal{F}_1^1=\rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \mathcal{F}_1^2=\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;.$$

When $\kappa < -1/2$, let $\mathcal{F}_2 \subset C^\infty(F)\cap L^2(F)$ be the dense graded linear subspace with

$$\mathcal{F}_2^0=\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \mathcal{F}_2^1=\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \mathcal{F}_2^2=\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\;.$$

By restricting d and δ, we get complexes $(\mathcal{F}_1,d)$ and $(\mathcal{F}_1,\delta)$ when $\kappa>-1/2$, and complexes $(\mathcal{F}_2,d)$ and $(\mathcal{F}_2,\delta)$ when $\kappa<1/2$. Thus Δ preserves $\mathcal{F}_1$ when $\kappa>-1/2$, and preserves $\mathcal{F}_2$ when $\kappa<-1/2$.

Proposition 7.6. *Suppose that $\kappa\neq-1/2$. Then (F,d) has a unique i.b.c., whose smooth core is $\mathcal{F}_1$ if $\kappa>-1/2$, and $\mathcal{F}_2$ if $\kappa<-1/2$.*

Proof. We prove only the case with $\kappa>-1/2$; the other case is analogous.

By Lemma 7.5 (using the independence of (ξ_n) on κ in its statement), we get $\mathcal{F}_1^0\subset\mathcal{D}(d_{0,\min})$ and $\mathcal{F}_1^2\subset\mathcal{D}(\delta_{1,\min})$. Then, by the discussion previous to this proposition, the other conditions of Lemma 7.1 are satisfied by the complexes defined by d and δ with $\mathcal{G}=\mathcal{F}_1$, obtaining that $\mathcal{F}_1^0$ and $\mathcal{F}_1^2$ are the smooth cores of $d_{0,\min}$ and $\delta_{1,\min}$, respectively. By Proposition 7.4 and since $1+\kappa,2+\kappa>1/2$, we get $d_{0,2,\min}=d_{0,2,\max}$ with smooth core $\mathcal{F}_1^0$, and $\delta_{2,2,\min}=\delta_{2,2,\max}$ with smooth core $\mathcal{F}_1^2$. So, according to the discussion previous to this proposition, the conditions of Corollary 7.3 are satisfied with d and δ, obtaining $d_{0,\min}=d_{0,\max}$ and $\delta_{1,\min}=\delta_{1,\max}$, which also gives $d_{1,\min}=d_{1,\max}$. □

7.4. **Finite propagation speed of the wave equation.** For the Hermitian bundle versions of E and F, consider the wave equation

$$\frac{du_t}{dt}-iDu_t=0 \tag{32}$$

on any open subset of $\mathbb{R}_+$, where $D=d+\delta$ and u_t is in $C^\infty(E)$ or $C^\infty(F)$, depending smoothly on $t\in\mathbb{R}$.

Proposition 7.7. *For $0<a<b$, suppose that $u_t\in\mathcal{D}^\infty(d_{\min/\max})$, depending smoothly on $t\in\mathbb{R}$, satisfies* (32) *on $(0,b)$. The following properties hold:*

(i) *If* $\operatorname{supp} u_0 \subset [a,\infty)$*, then* $\operatorname{supp} u_t \subset [a-|t|,\infty)$ *for* $0<|t|\le a$*.*
(ii) *If* $\operatorname{supp} u_0 \subset (0,a]$*, then* $\operatorname{supp} u_t \subset (0,a+|t|]$ *for* $0<|t|\le b-a$*.*

Proof. We prove Proposition 7.7 only for E; the proof is clearly analogous for F, but with more cases because F is of length three. Let $u_{t,0}\in C^\infty(E^0)\equiv C^\infty(\mathbb{R}_+)$ and $u_{t,1}\in C^\infty(E^1)\equiv C^\infty(\mathbb{R}_+)$ be the homogeneous components of u_t. From the description of the smooth core of $d_{\text{min/max}}$ in Proposition 7.4, it follows that

$$\lim_{\rho\downarrow 0}(u_{t,0}\,u_{t,1})(\rho)=0\;. \tag{33}$$

We have

$$\begin{aligned}\frac{d}{dt}\int_0^{a-t}|u_t(\rho)|^2\,d\rho&=\int_0^{a-t}((iDu_t,u_t)+(u_t,iDu_t))(\rho)\,d\rho-|u_t(a-t)|^2\\&=i\int_0^{a-t}((Du_t,u_t)-(u_t,Du_t))(\rho)\,d\rho-|u_t(a-t)|^2\;.\end{aligned}$$

But, since d and δ are respectively equal to $d/d\rho$ and $-d/d\rho$ up to the sum of multiplication operators by the same real valued functions,

$$\begin{aligned}(Du_t,u_t)-(u_t,Du_t)&=\frac{du_{t,0}}{dt}\cdot\overline{u_{t,1}}-\frac{du_{t,1}}{dt}\cdot\overline{u_{t,0}}-u_{t,1}\cdot\frac{d\overline{u_{t,0}}}{dt}+u_{t,0}\cdot\frac{d\overline{u_{t,1}}}{dt}\\&=2\,\Im\left(\frac{du_{t,0}}{d\rho}\cdot\overline{u_{t,1}}+u_{t,0}\cdot\frac{d\overline{u_{t,1}}}{d\rho}\right)=2\,\Im\frac{d}{d\rho}(u_{t,0}\,\overline{u_{t,1}})\;,\end{aligned}$$

giving

$$\begin{aligned}\left|\int_0^{a-t}((Du_t,u_t)-(u_t,Du_t))(\rho)\,d\rho\right|&\le 2\left|(u_{t,0}\,\overline{u_{t,1}})(a-t)-\lim_{\rho\downarrow 0}(u_{t,0}\,\overline{u_{t,1}})(\rho)\right|\\&=2\,|(u_{t,0}\,\overline{u_{t,1}})(a-t)|\le|u_{t,0}(a-t)|^2+|u_{t,1}(a-t)|^2=|u_t(a-t)|^2\end{aligned}$$

by (33). So

$$\frac{d}{dt}\int_0^{a-t}|u_t(\rho)|^2\,d\rho\le 0\;,$$

giving

$$\int_0^{a-t}|u_t(\rho)|^2\,d\rho\le\int_0^a|u_0(\rho)|^2\,d\rho=0\;,$$

and (i) follows.

Property (ii) can be proved with the same kind of arguments, but using that

$$\lim_{\rho\to\infty}u(\rho)=0 \tag{34}$$

for all $u\in\mathcal{D}^\infty(d_{\text{min/max}})$ instead of (33). □

Remark 16. The proof of Proposition 7.7 is an adaptation of [33, Proposition 7.20], where (33) and (34) are used to settle the lack of compact support.

8. Preliminaries on Witten's perturbation of the de Rham complex

Let $M \equiv (M, g)$ be a Riemannian manifold of dimension n. For any $x \in M$ and any $\alpha \in T_xM^*$, let

$$\alpha\lrcorner = (-1)^{nr+n+1} \star \alpha\wedge \star \quad \text{on} \quad \bigwedge^r T_xM^* \;,$$

involving the Hodge star operator $\star$ on $\bigwedge T_xM^*$ defined by any choice of orientation of T_xM. Writing $\alpha = g(X, \cdot)$ for $X \in T_xM$, we have $\alpha\lrcorner = -\iota_X$, where ι_X denotes the inner product by X. Moreover let

$$R_\alpha = \alpha\wedge - \alpha\lrcorner \;, \quad L_\alpha = \alpha\wedge + \alpha\lrcorner$$

on $\bigwedge T_xM^*$. Recall that there is an isomorphism between the underlying linear spaces of the exterior and Clifford algebras of T_xM^*,

$$\bigwedge T_xM^* \to \mathrm{Cl}(T_xM^*) \;, \quad e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto e_{i_1} \bullet \cdots \bullet e_{i_r} \;,$$

where $(e_1, \dots, e_n)$ is an orthonormal frame of T_xM^* and "$\bullet$" denotes Clifford multiplication. By this linear isomorphism, L_α and $R_\alpha \mathbf{w}$ correspond to left and right Clifford multiplication by α. So L_α and R_β anticommute for any $\alpha, \beta \in T_xM^*$. Any symmetric bilinear form $H \in T_xM^* \otimes T_xM^*$ induces an endomorphism $\mathbf{H}$ of $\bigwedge T_xM^*$ defined by

$$\mathbf{H} = \sum_{i,j=1}^{n} H(e_i, e_j)\, L_{e_i}\, R_{e_j} \;, \tag{35}$$

by using an orthonormal frame $(e_1, \dots, e_n)$ of T_xM^*. Observe that $|\mathbf{H}| = |H|$.

On the graded algebra of differential forms, $\Omega(M)$, let d and δ be the derivative and coderivative, let $D = d + \delta$ (the de Rham operator), and let $\Delta = D^2 = d\delta + \delta d$ (the Laplacian on differential forms). For any $f \in C^\infty(M)$, E. Witten [41] has introduced the following perturbations of the above operators, depending on a parameter $s \geq 0$:

$$d_s = e^{-sf}\, d\, e^{sf} = d + s\, df\wedge \;, \tag{36}$$

$$\delta_s = e^{sf}\, \delta\, e^{-sf} = \delta - s\, df\lrcorner \;, \tag{37}$$

$$D_s = d_s + \delta_s = D + sR \;,$$

$$\Delta_s = D_s^2 = d_s\delta_s + \delta_s d_s = \Delta + s(RD + DR) + s^2R^2 \;, \tag{38}$$

where $R = R_{df}$. Notice that δ_s is the formal adjoint of d_s, and therefore D_s and Δ_s are formally self-adjoint.

The Hessian of f, with respect to g, is the smooth section of $TM^* \otimes TM^*$ defined by $\operatorname{Hess} f = \nabla df$, which is symmetric and induces an endomorphism $\mathbf{Hess} f$ of $\bigwedge TM^*$ according to (35). Then [33, Lemma 9.17]

$$RD + DR = \mathbf{Hess} f \;, \quad R^2 = |df|^2 \;,$$

obtaining that (38) becomes

$$\Delta_s = \Delta + s\, \mathbf{Hess} f + s^2\, |df|^2 \;. \tag{39}$$

The Witten's perturbed operators also make sense with complex valued differential forms, and the above equalities hold as well.

Example 8.1. Let $d^\pm_{0,s}$, $\delta^\pm_{0,s}$, $D^\pm_{0,s}$, $\Delta^\pm_{0,s}$ denote the Witten's perturbed operators on $\Omega(\mathbb{R}^m)$ defined by the model Morse function $\pm\frac{1}{2}\,\rho_0^2$ and the standard metric g_0. According to [33, Proposition 9.18 and the proof of Lemma 14.11], $\Delta^\pm_{0,s}$, with domain $\Omega_0(\mathbb{R}^m)$, is essentially self-adjoint in $L^2\Omega(\mathbb{R}^m, g_0)$, and its self-adjoint extension has a discrete spectrum of the following form:

- 0 is an eigenvalue of multiplicity one, and the corresponding eigenforms are of degree zero in the case of $\Delta^+_{0,s}$, and of degree m in the case of $\Delta^-_{0,s}$.
- Let $e^\pm_s$ be a 0-eigenform of $\Delta^\pm_{0,s}$ with norm one, and let h be a bounded measurable function on $\mathbb{R}^m$ such that $h(x) \to 1$ as $x \to 0$. Then $\langle he^\pm_s, e^\pm_s\rangle \to 1$ as $s \to \infty$.
- All non-zero eigenvalues, as functions of s, are in $O(s)$ as $s \to \infty$.

Therefore $(\bigwedge T\mathbb{R}^{m*}, d^\pm_{0,s})$ has a unique i.b.c., which is discrete.

9. Witten's perturbation on a cone

For our version of Morse functions, the local analysis of the Witten's perturbed Laplacian will be reduced to the case of the functions $\pm\frac{1}{2}\rho^2$ on a stratum of a cone with a model adapted metric, where ρ denotes the canonical function. That kind of local analysis begins in this section.

9.1. **Laplacian on a cone.** Let L be a non-empty compact Thom-Mather stratification, let ρ be the canonical function on $c(L)$, let N be a stratum of L of dimension $\tilde n$, let $M = N \times \mathbb{R}_+$ be the corresponding stratum of $c(L)$ with dimension $n = \tilde n + 1$, and let $\pi : M \to N$ denote the second factor projection. From $\bigwedge TM^* = \bigwedge TN^* \boxtimes \bigwedge T\mathbb{R}_+^*$, we get a canonical identity

$$\bigwedge^r TM^* \equiv \pi^* \bigwedge^r TN^* \oplus d\rho \wedge \pi^* \bigwedge^{r-1} TN^* \equiv \pi^* \bigwedge^r TN^* \oplus \pi^* \bigwedge^{r-1} TN^* \tag{40}$$

for each degree r, obtaining

$$\Omega^r(M) \equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus d\rho \wedge C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) \tag{41}$$

$$\equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus C^\infty(\mathbb{R}_+, \Omega^{r-1}(N))\,. \tag{42}$$

Here, smooth functions $\mathbb{R}_+ \to \Omega(N)$ are defined by considering $\Omega(N)$ as Fréchet space with the weak C^∞ topology. Let d and $\tilde d$ denote the exterior derivatives on $\Omega(M)$ and $\Omega(N)$, respectively. The following lemma is elementary.

Lemma 9.1. *According to* (42)*,*

$$d \equiv \begin{pmatrix} \tilde d & 0 \\ \frac{d}{d\rho} & -\tilde d \end{pmatrix}\,.$$

Fix an adapted metric $\tilde g$ on N, and let $g = \rho^2\tilde g + (d\rho)^2$ be the corresponding adapted metric on M. The induced metrics on $\bigwedge TM^*$ and $\bigwedge TN^*$ are also denoted by g and $\tilde g$, respectively. According to (40),

$$g = \rho^{-2r}\,\tilde g \oplus \rho^{-2(r-1)}\,\tilde g \tag{43}$$

on $\bigwedge^r TM^*$.

Given an orientation on an open subset $W \subset N$, and denoting by $\tilde\omega$ the corresponding $\tilde g$-volume form on W, consider the orientation on $W \times \mathbb{R}_+ \subset M$ so that the corresponding g-volume form is

$$\omega = \rho^{n-1}\, d\rho \wedge \tilde\omega\ . \tag{44}$$

The corresponding star operators on $\bigwedge T(W \times \mathbb{R}_+)^*$ and $\bigwedge TW^*$ will be denoted by $\star$ and $\tilde\star$, respectively.

Lemma 9.2. *According to* (40),

$$\star \equiv \begin{pmatrix} 0 & \rho^{n-2r+1}\tilde\star \\ (-1)^r \rho^{n-2r-1}\tilde\star & 0 \end{pmatrix}$$

on $\bigwedge^r T(W \times \mathbb{R}_+)^*$.

Proof. Let $\alpha, \alpha' \in \pi^* \bigwedge TN^*$, at the same point $(\rho, x) \in \mathbb{R}_+ \times W$. If α and α' are of degree r, then

$$\begin{aligned} \alpha' \wedge \rho^{n-2r-1}\, d\rho \wedge \tilde\star\alpha &= (-1)^r \rho^{n-2r-1}\, d\rho \wedge \alpha' \wedge \tilde\star\alpha \\ &= (-1)^r \rho^{n-2r-1} \tilde g(\alpha', \alpha)\, d\rho \wedge \tilde\omega = (-1)^r g(\alpha', \alpha)\, \omega \end{aligned}$$

by (43) and (44), giving $\star\alpha = (-1)^r \rho^{n-2r-1} d\rho \wedge \tilde\star\alpha$. Similarly, if α and α' are of degree $r-1$, then

$$d\rho \wedge \alpha' \wedge \rho^{n-2r+1}\tilde\star\alpha = \rho^{n-2r+1}\tilde g(\alpha', \alpha)\, d\rho \wedge \tilde\omega = g(d\rho \wedge \alpha', d\rho \wedge \alpha)\, \omega\ ,$$

obtaining $\star(d\rho \wedge \alpha) = \rho^{n-2r+1}\tilde\star\alpha$. □

Let $L^2\Omega^r(M, g)$ and $L^2\Omega^r(N, \tilde g)$ be simply denoted by $L^2\Omega^r(M)$ and $L^2\Omega^r(N)$. From (43) and (44), it follows that (42) induces a unitary isomorphism

$$\begin{aligned} L^2\Omega^r(M) \cong (L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho) \mathbin{\widehat\otimes} L^2\Omega^r(N)) \\ \oplus (L^2(\mathbb{R}_+, \rho^{n-2r+1}\, d\rho) \mathbin{\widehat\otimes} L^2\Omega^{r-1}(N))\ , \end{aligned} \tag{45}$$

which will be considered as an identity.

Let δ and $\tilde\delta$ denote the exterior coderivatives on $\Omega(M)$ and $\Omega(N)$, respectively.

Lemma 9.3. *According to* (42),

$$\delta \equiv \begin{pmatrix} \rho^{-2}\,\tilde\delta & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \\ 0 & -\rho^{-2}\,\tilde\delta \end{pmatrix}$$

on $\Omega^r(M)$.

Proof. For an oriented open subset $W \subset N$, consider the orientation on $W \times \mathbb{R}_+$ defined as above, and let $\star$ and $\tilde\star$ denote the corresponding star operators on

$\bigwedge T(W\times\mathbb{R}_+)^*$ and $\bigwedge TW^*$. By Lemmas 9.1 and 9.2, on $\Omega^r(W\times\mathbb{R}_+)$,

$$\begin{aligned}
\delta &= (-1)^{nr+n+1}\star d\star\\
&\equiv (-1)^{nr+n+1}\begin{pmatrix} 0 & \rho^{-n+2r-1}\tilde\star\\ (-1)^{n-r+1}\rho^{-n+2r-3}\tilde\star & 0\end{pmatrix}\begin{pmatrix}\tilde d & 0\\ \frac{d}{d\rho} & -\tilde d\end{pmatrix}\\
&\quad\times\begin{pmatrix} 0 & \rho^{n-2r+1}\tilde\star\\ (-1)^r\rho^{n-2r-1}\tilde\star & 0\end{pmatrix}\\
&= (-1)^{nr+n+1}\begin{pmatrix} -(-1)^r\rho^{-2}\tilde\star\tilde d\tilde\star & \rho^{-n+2r-1}\frac{d}{d\rho}\rho^{n-2r+1}\tilde\star^2\\ 0 & (-1)^{n-r+1}\rho^{-2}\tilde\star\tilde d\tilde\star\end{pmatrix}\\
&= \begin{pmatrix}\rho^{-2}\tilde\delta & -\rho^{-n+2r-1}\frac{d}{d\rho}\rho^{n-2r+1}\\ 0 & -\rho^{-2}\tilde\delta\end{pmatrix}\;,
\end{aligned}$$

which equals the matrix of the statement by (24). □

Let Δ and $\widetilde\Delta$ denote the Laplacians on $\Omega(M)$ and $\Omega(N)$, respectively.

Corollary 9.4. *According to* (42),

$$\Delta\equiv\begin{pmatrix} P & -2\rho^{-1}\tilde d\\ -2\rho^{-3}\tilde\delta & Q\end{pmatrix}$$

on $\Omega^r(M)$*, where*

$$\begin{aligned}
P &= \rho^{-2}\widetilde\Delta-\frac{d^2}{d\rho^2}-(n-2r-1)\rho^{-1}\frac{d}{d\rho}\;,\\
Q &= \rho^{-2}\widetilde\Delta-\frac{d^2}{d\rho^2}-(n-2r+1)\frac{d}{d\rho}\rho^{-1}\;.
\end{aligned}$$

Proof. By Lemmas 9.1 and 9.3,

$$\begin{aligned}
\delta d &\equiv \begin{pmatrix}\rho^{-2}\tilde\delta & -\frac{d}{d\rho}-(n-2r-1)\rho^{-1}\\ 0 & -\rho^{-2}\tilde\delta\end{pmatrix}\begin{pmatrix}\tilde d & 0\\ \frac{d}{d\rho} & -\tilde d\end{pmatrix}\\
&= \begin{pmatrix}\rho^{-2}\tilde\delta\tilde d-\frac{d^2}{d\rho^2}-(n-2r-1)\rho^{-1}\frac{d}{d\rho} & (\frac{d}{d\rho}+(n-2r-1)\rho^{-1})\tilde d\\ -\rho^{-2}\tilde\delta\frac{d}{d\rho} & \rho^{-2}\tilde\delta\tilde d\end{pmatrix}\;,\\
d\delta &\equiv \begin{pmatrix}\tilde d & 0\\ \frac{d}{d\rho} & -\tilde d\end{pmatrix}\begin{pmatrix}\rho^{-2}\tilde\delta & -\frac{d}{d\rho}-(n-2r+1)\rho^{-1}\\ 0 & -\rho^{-2}\tilde\delta\end{pmatrix}\\
&= \begin{pmatrix}\rho^{-2}\tilde d\tilde\delta & -\tilde d(\frac{d}{d\rho}+(n-2r+1)\rho^{-1})\\ \frac{d}{d\rho}\rho^{-2}\tilde\delta & -\frac{d^2}{d\rho^2}-(n-2r+1)\frac{d}{d\rho}\rho^{-1}+\rho^{-2}\tilde d\tilde\delta\end{pmatrix}\\
&= \begin{pmatrix}\rho^{-2}\tilde d\tilde\delta & -\tilde d(\frac{d}{d\rho}+(n-2r+1)\rho^{-1})\\ \rho^{-2}\frac{d}{d\rho}\tilde\delta-2\rho^{-3}\tilde\delta & -\frac{d^2}{d\rho^2}-(n-2r+1)\frac{d}{d\rho}\rho^{-1}+\rho^{-2}\tilde d\tilde\delta\end{pmatrix}\;.
\end{aligned}$$

The sum of these matrices is the matrix of the statement. □

9.2. **Witten's perturbation on a cone.** Let $d_s^\pm$, $\delta_s^\pm$, $D_s^\pm$ and $\Delta_s^\pm$ ($s\ge 0$) denote the Witten's perturbations of d, δ, D and Δ induced by the function $f=\pm\frac12\rho^2$ on M. In this case, $df=\pm\rho\,d\rho$. According to (42),

$$\rho\,d\rho\wedge\equiv\begin{pmatrix}0&0\\ \rho&0\end{pmatrix}\;,\quad -\rho\,d\rho\lrcorner\equiv\begin{pmatrix}0&\rho\\ 0&0\end{pmatrix}\;.$$

So the following is a consequence of Lemmas 9.1 and 9.3, (36) and (37).

Corollary 9.5. *According to* (42),

$$d_s^\pm \equiv \begin{pmatrix} \tilde d & 0 \\ \frac{d}{d\rho} \pm s\rho & -\tilde d \end{pmatrix} ,$$

$$\delta_s^\pm \equiv \begin{pmatrix} \rho^{-2}\,\tilde\delta & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \pm s\rho \\ 0 & -\rho^{-2}\,\tilde\delta \end{pmatrix}$$

on $\Omega^r(M)$.

With the notation of Section 8,

$$R = \pm\rho(d\rho\wedge - d\rho\lrcorner) \equiv \pm \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} ,$$

and therefore

$$R^2 \equiv \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \equiv \rho^2 . \tag{46}$$

Lemma 9.6. $RD + DR = \pm(2r-n)$ *on* $\Omega^r(M)$.

Proof. By Lemmas 9.1 and 9.3, and according to (42),

$$\begin{aligned} RD &\equiv \pm \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} \begin{pmatrix} \tilde d + \rho^{-2}\tilde\delta & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \\ \frac{d}{d\rho} & -\tilde d - \rho^{-2}\tilde\delta \end{pmatrix} \\ &= \pm \begin{pmatrix} \rho\,\frac{d}{d\rho} & -\rho\tilde d - \rho^{-1}\tilde\delta \\ \rho\,\tilde d + \rho^{-1}\tilde\delta & -\rho\,\frac{d}{d\rho} - n + 2r - 1 \end{pmatrix} , \\ DR &\equiv \pm \begin{pmatrix} \tilde d + \rho^{-2}\,\tilde\delta & -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \\ \frac{d}{d\rho} & -\tilde d - \rho^{-2}\,\tilde\delta \end{pmatrix} \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} \\ &= \pm \begin{pmatrix} -\frac{d}{d\rho}\,\rho - n + 2r + 1 & \rho\,\tilde d + \rho^{-1}\,\tilde\delta \\ -\rho\,\tilde d - \rho^{-1}\tilde\delta & \frac{d}{d\rho}\,\rho \end{pmatrix} . \end{aligned}$$

So

$$RD + DR \equiv \pm \begin{pmatrix} 2r-n & 0 \\ 0 & 2r-n \end{pmatrix} \equiv \pm(2r-n)$$

by (24). □

Remark 17. The expression of $RD+DR$ can be also obtained by computing $\operatorname{Hess} f$ (Section 8).

The following is a consequence of (39), Corollary 9.4 and Lemma 9.6.

Corollary 9.7. *According to* (42),

$$\Delta_s^\pm \equiv \begin{pmatrix} P_s^\pm & -2u\rho^{-1}\tilde d \\ -2\rho^{-3}\tilde\delta & Q_s^\pm \end{pmatrix}$$

on $\Omega^r(M)$, *where*

$$P_s^\pm = \rho^{-2}\widetilde\Delta + H - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} \mp s(n-2r) ,$$

$$Q_s^\pm = \rho^{-2}\widetilde\Delta + H - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + (n-2r+1)\rho^{-2} \mp s(n-2r) .$$

10. Domains of the Witten's Laplacian on a cone

Theorem 1.1 is proved by induction on the dimension. Thus, with the notation of Section 9, suppose that $\tilde{d}_{\mathrm{min/max}}$ satisfies the statement of Theorem 1.1. Let

$$\widetilde{\mathcal{H}}_{\mathrm{min/max}} = \ker \widetilde{D}_{\mathrm{min/max}} = \ker \widetilde{\Delta}_{\mathrm{min/max}}\;,$$

which is a graded subspace of $\Omega(N)$. For each degree r, let

$$\widetilde{\mathcal{R}}_{\mathrm{min/max},r-1}, \widetilde{\mathcal{R}}^*_{\mathrm{min/max},r} \subset L^2\Omega^r(N)$$

be the images of $\tilde{d}_{\mathrm{min/max},r-1}$ and $\tilde{\delta}_{\mathrm{min/max},r}$, respectively, whose intersections with $\mathcal{D}^\infty(\widetilde{\Delta})$ are denoted by $\widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}$ and $\widetilde{\mathcal{R}}^{*\infty}_{\mathrm{min/max},r}$. According to Section 4.1, $\widetilde{\Delta}$ preserves $\widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}$ and $\widetilde{\mathcal{R}}^{*\infty}_{\mathrm{min/max},r}$, and its restrictions to these spaces have the same eigenvalues. For any eigenvalue $\tilde{\lambda}$ of the restriction of $\widetilde{\Delta}$ to $\widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}$, let

$$\widetilde{\mathcal{R}}_{\mathrm{min/max},r-1,\tilde{\lambda}} = E_{\tilde{\lambda}}(\widetilde{\Delta}_{\mathrm{min/max}}) \cap \widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}\;,$$
$$\widetilde{\mathcal{R}}^*_{\mathrm{min/max},r,\tilde{\lambda}} = E_{\tilde{\lambda}}(\widetilde{\Delta}_{\mathrm{min/max}}) \cap \widetilde{\mathcal{R}}^{*\infty}_{\mathrm{min/max},r}\;.$$

Moreover

$$L^2\Omega^r(N) = \widetilde{\mathcal{H}}^r_{\mathrm{min/max}} \oplus \widehat{\bigoplus_{\tilde{\lambda}}} \left(\widetilde{\mathcal{R}}_{\mathrm{min/max},r-1,\tilde{\lambda}} \oplus \widetilde{\mathcal{R}}^*_{\mathrm{min/max},r,\tilde{\lambda}}\right)\;, \tag{47}$$

where $\tilde{\lambda}$ runs in the spectrum of $\widetilde{\Delta}_{\mathrm{min/max}}$ on $\widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}$; i.e., the positive spectrum of $\widetilde{\Delta}_{\mathrm{min/max},r}$.

Now, consider the Witten's perturbed Laplacian $\Delta^\pm_s$. In the following, suppose that $s > 0$.

10.1. Domains of first type. For some degree r, let $0 \neq \gamma \in \widetilde{\mathcal{H}}^r_{\mathrm{min/max}}$. By Corollary 9.7,

$$\Delta^\pm_s \equiv H - (n-2r-1)\rho^{-1}\frac{d}{d\rho} \mp s(n-2r)$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\gamma \subset \Omega^r(M)$. This operator is of the type of P in (25) with $c_2 = 0$. Thus (28) is satisfied, and (26) means that $a \in \{0, -n+2r+2\}$.

For $a = 0$, we have $2\sigma = n-2r-1$. When $\sigma > -1/2$, which means $r \le \frac{n-1}{2}$, Proposition 6.1 asserts that $\Delta^\pm_s$, with domain $\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + (1 \mp 1)(n-2r))s \tag{48}$$

of multiplicity one, with corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\mathcal{S}_{\mathrm{ev},+}$. For Δ^+_s, (48) becomes $4ks$, which is ≥ 0 for all k and $= 0$ just for $k = 0$. For Δ^-_s, (48) becomes $(4k + 2(n-2r))s$, which is > 0 for all k.

For $a = -n+2r+2$, we have $2\sigma = -n+2r+3$. When $\sigma > -1/2$, which means $r \ge \frac{n-3}{2}$, Proposition 6.1 asserts that $\Delta^\pm_s$, with domain $\rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + 4 - (1 \pm 1)(n-2r))s \tag{49}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}$. For Δ^+_s, (49) becomes $(4k+4-2(n-2r))s$, which is:

- > 0 for all k if $r \geq \frac{n-1}{2}$,
- ≥ 0 for all k and $= 0$ just for $k = 0$ if $r = \frac{n}{2} - 1$, and
- < 0 for $k = 0$ if $r = \frac{n-3}{2}$.

For Δ_s^-, (49) becomes $(4k+4)s$, which are > 0 for all k.

When $\frac{n-3}{2} \leq r \leq \frac{n-1}{2}$, we have got two essentially self-adjoint operators, with $a = 0$ and $a = -n + 2r + 2$. These two operators are equal just when $r = \frac{n}{2} - 1$.

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *first type*.

10.2. **Domains of second type.** With the notation of Section 10.1,

$$\Delta_s^\pm \equiv H - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + (n-2r-1)\rho^{-2} \mp s(n-2r-2)$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,d\rho \wedge \gamma \subset \Omega^{r+1}(M)$ by Corollary 9.7. This is an operator of the type of P in (25) with $c_2 = c_1$. Thus (28) is also satisfied, and (26) becomes $a \in \{1, -n+2r+1\}$.

For $a = 1$, we have $2\sigma = n - 2r + 1$ according to (27). When $\sigma > -1/2$, which means $r \leq \frac{n+1}{2}$, Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho\,\mathcal{S}_{\mathrm{ev},+} = \mathcal{S}_{\mathrm{odd},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + 4 + (1 \mp 1)(n-2r-2))s \tag{50}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (50) is > 0 for all k. For Δ_s^-, (50) is:

- > 0 for all k if $r \leq \frac{n-1}{2}$,
- ≥ 0 for all k and $= 0$ just for $k = 0$ if $r = \frac{n}{2}$, and
- < 0 for $k = 0$ if $r = \frac{n+1}{2}$.

For $a = -n + 2r + 1$, we have $2\sigma = -n + 2r + 1$ according to (27). When $\sigma > -1/2$, which means $r \geq \frac{n-1}{2}$, Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k - (1 \pm 1)(n-2r-2))s \tag{51}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (51) is > 0 for all k. For Δ_s^-, (51) is ≥ 0 for all k and $= 0$ just for $k = 0$.

For $\frac{n-1}{2} \leq r \leq \frac{n+1}{2}$, we have obtained two essentially self-adjoint operators, with $a = 1$ and $a = -n + 2r + 1$. These operators are equal just when $r = \frac{n}{2}$.

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *second type*.

10.3. **Domains of third type.** Let $\mu = \sqrt{\tilde\lambda}$ for an eigenvalue $\tilde\lambda$ of the restriction of $\widetilde\Delta_{\mathrm{min/max}}$ to $\widetilde{\mathcal{R}}^\infty_{\mathrm{min/max},r-1}$. According to Section 4.1, there are non-zero differential forms,

$$\alpha \in \widetilde{\mathcal{R}}_{\mathrm{min/max},r-1,\lambda} \subset \Omega^r(N)\;, \quad \beta \in \widetilde{\mathcal{R}}^*_{\mathrm{min/max},r-1,\lambda} \subset \Omega^{r-1}(N)\;,$$

such that $\tilde d\beta = \mu\alpha$ and $\tilde\delta\alpha = \mu\beta$. By Corollary 9.7,

$$\Delta_s^\pm \equiv -\frac{d^2}{d\rho^2} - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + \mu^2\rho^{-2} \mp (n-2r+2)s$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\beta \subset \Omega^{r-1}(M)$. This operator is of the type of P in (25) with $c_2 = \mu^2 > 0$. Thus (28) is satisfied, and (26) becomes

$$a = \frac{-n + 2r \pm \sqrt{(n-2r)^2 + 4\mu^2}}{2} \;. \tag{52}$$

These two possibilities for a have different sign because $\mu > 0$.

For the choice of positive square root in (52), we get

$$\sigma = \frac{1 + \sqrt{(n-2r)^2 + 4\mu^2}}{2} > \frac{1}{2} \tag{53}$$

according to (27). Then Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k + 2 + \sqrt{(n-2r)^2 + 4\mu^2} \mp (n - 2r + 2)\right) s \;, \tag{54}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. Notice that (54) is > 0 for all k.

For the choice of negative square root in (52), we get

$$\sigma = \frac{1 - \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{55}$$

according to (27). Then $\sigma > -1/2$ if and only if

$$\mu < 1 \quad \text{and} \quad |n - 2r| < 2\sqrt{1 - \mu^2} \;, \tag{56}$$

which is equivalent to $\frac{\sqrt{3}}{2} \le \mu < 1$ and $r = \frac{n}{2}$, or $\mu < \frac{\sqrt{3}}{2}$ and $\frac{n-1}{2} \le r \le \frac{n+1}{2}$. In this case, Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k + 2 - \sqrt{(n-2r)^2 + 4\mu^2} \mp (n - 2r + 2)\right) s \;, \tag{57}$$

with multiplicity one and corresponding normalized eigenfunctions $\rho^a\,\phi_{2k,+}$; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (57) is < 0 for $k = 0$. For Δ_s^-, (57) is > 0 for all k.

When (56) is satisfied, we have got two different essentially self-adjoint operators defined by the two different choices of a in (52).

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *third type*.

10.4. **Domains of fourth type.** Let μ, α and β be like in Section 10.3. By Corollary 9.7,

$$\begin{aligned}\Delta_s^\pm \equiv{}& -\frac{d^2}{d\rho^2} + s^2\rho^2 - (n - 2r - 1)\rho^{-1}\,\frac{d}{d\rho} + (\mu^2 + n - 2r - 1)\rho^{-2}\\ &\mp (n - 2r - 2)s\end{aligned}$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,d\rho \wedge \alpha \subset \Omega^{r+1}(M)$. This is another operator of the type of P in (25), which satisfies (28) because

$$(1 - (n - 2r - 1))^2 + 4(\mu^2 + n - 2r - 1) = (n - 2r)^2 + 4\mu^2 > 0 \;.$$

Moreover (26) becomes

$$a = \frac{-n + 2r + 2 \pm \sqrt{(n-2r)^2 + 4\mu^2}}{2} \;. \tag{58}$$

These two possibilities for a are different because $\mu > 0$.

With the choice of positive square root in (58) and according to (27), σ is also given by (53), which is $> 1/2$. Then Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\Big(4k + 2 + \sqrt{(n-2r)^2 + 4\mu^2} \mp (n - 2r - 2)\Big)\, s\ , \tag{59}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. Observe that (59) is > 0 for all k.

With the choice of negative square root in (58) and according to (27), σ is also given by (55), which is $> -1/2$ if and only if (56) is satisfied. In this case, Proposition 6.1 asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\Big(4k + 2 - \sqrt{(n-2r)^2 + 4\mu^2} \mp (n - 2r - 2)\Big)\, s\ , \tag{60}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (60) is > 0 for all k. For Δ_s^-, (60) is < 0 for $k = 0$.

When (56) is satisfied, we have got two different essentially self-adjoint operators defined by the two different choices of a in (58).

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *fourth type*.

10.5. **Domains of fifth type.** Let μ, α and β be like in Sections 10.3 and 10.4. By Corollary 9.7,

$$\Delta_s^\pm \equiv \begin{pmatrix} P_{\mu,s}^\pm & -2\rho^{-1}\mu \\ -2\rho^{-3}\mu & Q_{\mu,s}^\pm \end{pmatrix}$$

on

$$C^\infty(\mathbb{R}_+) \oplus C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\alpha + C^\infty(\mathbb{R}_+)\,d\rho \wedge \beta \subset \Omega^r(M)\ ,$$

where

$$\begin{aligned} P_{\mu,s}^\pm &= H - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + \mu^2\rho^{-2} \mp (n-2r)s\ , \\ Q_{\mu,s}^\pm &= H - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + (\mu^2 + n - 2r + 1)\rho^{-2} \mp (n-2r)s\ . \end{aligned}$$

We will conjugate this matrix expression of $\Delta_s^\pm$ by some non-singular matrix Θ, whose entries are functions of ρ, to get a diagonal matrix whose diagonal entries are operators of the type of P in (25). This matrix will be of the form $\Theta = BC$ with

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}\ , \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\ ,$$

where c_{ij} are constants to be determined. Let $P^{\pm}_{\mu,s}$ and $Q^{\pm}_{\mu,s}$ be simply denoted by P and Q. A key observation here is that, by (24),

$$\begin{aligned} Q-\rho^{-1}\,P\,\rho &= -\frac{d^2}{d\rho^2}-(n-2r+1)\rho^{-1}\,\frac{d}{d\rho}+(n-2r+1)\rho^{-2} \\ &\quad +\rho^{-1}\,\frac{d^2}{d\rho^2}\,\rho+(n-2r-1)\rho^{-2}\,\frac{d}{d\rho}\,\rho \\ &= -\frac{d^2}{d\rho^2}-(n-2r+1)\rho^{-1}\,\frac{d}{d\rho}+(n-2r+1)\rho^{-2} \\ &\quad +\frac{d^2}{d\rho^2}+2\frac{d}{d\rho}+(n-2r-1)\rho^{-1}\,\frac{d}{d\rho}+(n-2r-1)\rho^{-2} \\ &= 2(n-2r)\rho^{-2}\ , \end{aligned}$$

obtaining

$$\begin{aligned} B^{-1}\,\Delta^{\pm}_s\,B &= \begin{pmatrix}1&0\\0&\rho\end{pmatrix}\begin{pmatrix}P&-2\mu\,\rho^{-1}\\-2\mu\,\rho^{-3}&Q\end{pmatrix}\begin{pmatrix}1&0\\0&\rho^{-1}\end{pmatrix} \\ &= \begin{pmatrix}P&-2\mu\rho^{-2}\\-2\mu\rho^{-2}&\rho\,Q\,\rho^{-1}\end{pmatrix} \\ &= \begin{pmatrix}P&-2\mu\rho^{-2}\\-2\mu\rho^{-2}&P+2(n-2r)\rho^{-2}\end{pmatrix}\ . \end{aligned}$$

On the other hand, C must be non-singular and

$$C^{-1}=\frac{1}{\det C}\begin{pmatrix}c_{22}&-c_{12}\\-c_{21}&c_{11}\end{pmatrix}\ .$$

Therefore $\Theta^{-1}\Delta^{\pm}_s\Theta=(X_{ij})$ with

$$\begin{aligned} X_{11} &= P+\frac{2}{\det C}\,\left(\mu\,(-c_{22}c_{21}+c_{12}c_{11})-(n-2r)c_{12}c_{21}\right)\rho^{-2}\ , \\ X_{12} &= \frac{2}{\det C}\,\left(\mu\,(-c_{22}^2+c_{12}^2)-(n-2r)c_{12}c_{22}\right)\rho^{-2}\ , \\ X_{21} &= \frac{2}{\det C}\,\left(\mu\,(c_{21}^2-c_{11}^2)+(n-2r)c_{11}c_{21}\right)\rho^{-2}\ , \\ X_{22} &= P+\frac{2}{\det C}\,\left(\mu(c_{21}c_{22}-c_{11}c_{12})+(n-2r)c_{11}c_{22}\right)\rho^{-2}\ . \end{aligned}$$

We want (X_{ij}) to be diagonal, so we require

$$\mu(c_{12}^2-c_{22}^2)-(n-2r)c_{12}c_{22}=\mu(c_{11}^2-c_{21}^2)-(n-2r)c_{11}c_{21}=0\ .$$

Both of these equations are of the form

$$\mu(x^2-y^2)-(n-2r)xy=0\ , \tag{61}$$

with $x=c_{12}$ and $y=c_{22}$ in the first equation, and $x=c_{11}$ and $y=c_{21}$ in the second one. There is some $c\in\mathbb{R}\setminus\{0\}$ such that

$$x^2-y^2-\frac{n-2r}{\mu}\,xy=(x+cy)\left(x-\frac{y}{c}\right)\ . \tag{62}$$

In fact, since

$$(x+cy)\left(x-\frac{y}{c}\right)=x^2-y^2+\left(c-\frac{1}{c}\right)xy\ ,$$

we need

$$c-\frac{1}{c}=-\frac{n-2r}{\mu}\;,$$

giving

$$\mu c^2+(n-2r)c-\mu=0\;, \tag{63}$$

whose solutions are

$$c_\pm=\frac{-n+2r\pm\sqrt{(n-2r)^2+4\mu^2}}{2\mu}\;. \tag{64}$$

Observe that $c_+c_-=-1$. Let $c=c_+>0$, and therefore $-1/c=c_-$. By (62), the solutions of (61) are given by $x+cy=0$ and $cx-y=0$. Then we can take

$$C=\begin{pmatrix}1 & -c\\ c & 1\end{pmatrix}\;,$$

with $\det C=1+c^2>0$. So, for

$$\Theta=\begin{pmatrix}1 & 0\\ 0 & \rho^{-1}\end{pmatrix}\begin{pmatrix}1 & -c\\ c & 1\end{pmatrix}=\begin{pmatrix}1 & -c\\ c\rho^{-1} & \rho^{-1}\end{pmatrix}\;,$$

we get $X_{12}=X_{21}=0$, and

$$X_{11}=P+\frac{2(-2\mu c+(n-2r)c^2)}{1+c^2}\,\rho^{-2}\;,$$
$$X_{22}=P+\frac{2(2\mu c+n-2r)}{1+c^2}\,\rho^{-2}\;.$$

The notation $X=X_{11}$ and $Y=X_{22}$ will be used; thus $\Theta^{-1}\Delta_s^{\pm}\Theta=X\oplus Y$. The above expressions of X and Y can be simplified as follows. We have

$$1+c^2=2-\frac{n-2r}{\mu}\,c=\frac{2\mu-(n-2r)c}{\mu}$$

by (63), obtaining

$$\frac{2(-2\mu c+(n-2r)c^2)}{1+c^2}=\frac{2\mu c(-2\mu+(n-2r)c)}{2\mu-(n-2r)c}=-2\mu c\;,$$
$$\frac{2(2\mu c+n-2r)}{1+c^2}=\frac{2\mu(2\mu c+n-2r)}{2\mu-(n-2r)c}\;.$$

Moreover

$$(2\mu c+n-2r)^2=(n-2r)^2+4\mu^2>0$$

by (64), and

$$\begin{aligned}(2\mu-(n-2r)c)&(2\mu c+n-2r)\\ &=4\mu^2c+2\mu(n-2r)-(n-2r)2\mu c^2-(n-2r)^2c\\ &=4\mu^2c+2\mu(n-2r)-(n-2r)2\mu(1-\frac{n-2r}{\mu}\,c)-(n-2r)^2c\\ &=c(4\mu^2c+(n-2r)2)\end{aligned}$$

by (63). Therefore

$$\frac{2(2\mu c+n-2r)}{1+c^2}=\frac{2\mu(2\mu c+n-2r)^2}{(2\mu-(n-2r)c)(2\mu c+n-2r)}=\frac{2\mu((n-2r)^2+4\mu^2)}{c(4\mu^2c+(n-2r)2)}=\frac{2\mu}{c}\;.$$

It follows that

$$\begin{aligned}X&=P-2\mu c\rho^{-2}\\&=H+s^2\rho^2-(n-2r-1)\rho^{-1}\,\frac{d}{d\rho}+(\mu^2-2\mu c)\rho^{-2}\mp(n-2r)s\;,\\Y&=P+\frac{2\mu}{c}\rho^{-2}\\&=H+s^2\rho^2-(n-2r-1)\rho^{-1}\,\frac{d}{d\rho}+(\mu^2+\frac{2\mu}{c})\rho^{-2}\mp(n-2r)s\;.\end{aligned}$$

These operators are of the type of P in (25), and satisfy (28) because

$$\begin{aligned}(1-(n-2r-1))^2+4(\mu^2-2\mu c)&\\=4+(n-2r)^2+4\mu^2-4\sqrt{(n-2r)^2+4\mu^2}&\\=(2-\sqrt{(n-2r)^2+4\mu^2)})^2\geq 0&\end{aligned}$$

and

$$\begin{aligned}(1-(n-2r-1))^2+4(\mu^2+\frac{2\mu}{c})&\\=4+(n-2r)^2+4\mu^2+4\sqrt{(n-2r)^2+4\mu^2)}&\\=(2+\sqrt{(n-2r)^2+4\mu^2)})^2>0\;.&\end{aligned}$$

So, for X and Y, the constants (26) and (27) become

$$a=\frac{2-n+2r\pm(2-\sqrt{(n-2r)^2+4\mu^2)})}{2}\;,\tag{65}$$

$$b=\frac{2-n+2r\pm(2+\sqrt{(n-2r)^2+4\mu^2)})}{2}\;,\tag{66}$$

$$\sigma=\frac{1\pm(2-\sqrt{(n-2r)^2+4\mu^2)})}{2}\;,\tag{67}$$

$$\tau=\frac{1\pm(2+\sqrt{(n-2r)^2+4\mu^2)})}{2}\;.\tag{68}$$

Suppose that $\sigma,\tau>-1/2$. By Proposition 6.1, X and Y, with respective domains $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$ and $\rho^b\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)$; the spectra of their closures consist of the eigenvalues

$$(4k+2a+(1\mp 1)(n-2r))s\;,\tag{69}$$

$$(4k+2+2b+(1\mp 1)(n-2r))s\;,\tag{70}$$

with multiplicity one and corresponding normalized eigenfunctions $\chi_{s,a,\sigma,k}$ and $\chi_{s,b,\tau,k}$, respectively, and the smooth cores of their closures are $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$ and $\rho^b\,\mathcal{S}_{\mathrm{ev},+}$.

Since $\frac{1}{\sqrt{1+c^2}}C$ is an orthogonal matrix, it defines a unitary isomorphism

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,$$

and we already know that

$$B=1\oplus\rho^{-1}:L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)$$

is a unitary isomorphism too. So $\frac{1}{\sqrt{1+c^2}}\Theta$ is a unitary isomorphism

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\;.$$

Therefore, when $\sigma,\tau>-1/2$, the operator $\Delta_s^{\pm}$, with domain

$$\Theta(\rho^a\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^b\,\mathcal{S}_{\mathrm{ev},+})=\{(\rho^a\phi-c\rho^b\psi,c\rho^{a-1}\phi+\rho^{b-1}\psi)\mid\phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\}\;, \tag{71}$$

is essentially self-adjoint in

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho) \equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,\alpha+L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,d\rho\wedge\beta\;, \tag{72}$$

which is a Hilbert subspace of $L^2\Omega^r(M,g)$; the spectrum of its closure consists of the eigenvalues (69) and (70), with multiplicity one and corresponding normalized eigenvectors

$$\frac{1}{\sqrt{1+c^2}}\,\Theta(\chi_{s,a,\sigma,k},0)\;,\quad \frac{1}{\sqrt{1+c^2}}\,\Theta(0,\chi_{s,b,\tau,k})\;,$$

respectively; and the smooth cores of its closure is (71).

The condition $\tau>-1/2$ only holds with the choice

$$\tau=\frac{3+\sqrt{(n-2r)^2+4\mu^2}}{2}$$

in (68), which corresponds to the choice

$$b=\frac{4-n+2r+\sqrt{(n-2r)^2+4\mu^2}}{2} \tag{73}$$

in (66). With this choice, the eigenvalues (70) become

$$\left(4k+6\mp(n-2r)+\sqrt{(n-2r)^2+4\mu^2}\right)s\;, \tag{74}$$

which are >0 for all k.

Consider the choice

$$a=\frac{-n+2r+\sqrt{(n-2r)^2+4\mu^2}}{2} \tag{75}$$

in (65), and, correspondingly,

$$\sigma=\frac{-1+\sqrt{(n-2r)^2+4\mu^2}}{2}>-\frac{1}{2}$$

in (67). Then the eigenvalues (69) become

$$\left(4k\mp(n-2r)+\sqrt{(n-2r)^2+4\mu^2}\right)s\;, \tag{76}$$

which are > 0 for all k.

Now, consider the choice

$$a = \frac{4 - n + 2r - \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{77}$$

in (65), and therefore

$$\sigma = \frac{3 - \sqrt{(n-2r)^2 + 4\mu^2}}{2}$$

in (67). In this case, the condition $\sigma > -1/2$ means that

$$\mu < 2 \quad \text{and} \quad |n - 2r| < 2\sqrt{4 - \mu^2} \ . \tag{78}$$

The eigenvalues (69) become

$$\left(4k + 4 \mp (n - 2r) - \sqrt{(n-2r)^2 + 4\mu^2}\right) s \ . \tag{79}$$

For Δ_s^+, (79) is:

- ≥ 0 for all k if and only if $n - 2r \leq 2 - \mu^2/2$, and
- $= 0$ just when $k = 0$ and $n - 2r = 2 - \mu^2/2$.

For Δ_s^-, (79) is:

- ≥ 0 for all k if and only if $n - 2r \geq \mu^2/2 - 2$, and
- $= 0$ just when $k = 0$ and $n - 2r = \mu^2/2 - 2$.

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *fifth type*.

11. Splitting of the Witten complex on a cone

11.1. **Subcomplexes defined by domains of first and second types.** Consider the notation of Sections 10.1 and 10.2. The following result follows from Corollary 9.5.

Lemma 11.1. *For $s \geq 0$, $d_s^\pm$ and $\delta_s^\pm$ define maps*

$$0 \underset{\delta^\pm_{s,r-1}}{\overset{d^\pm_{s,r-1}}{\rightleftarrows}} C^\infty(\mathbb{R}_+)\,\gamma \underset{\delta^\pm_{s,r}}{\overset{d^\pm_{s,r}}{\rightleftarrows}} C^\infty(\mathbb{R}_+)\,d\rho\wedge\gamma \underset{\delta^\pm_{s,r+1}}{\overset{d^\pm_{s,r+1}}{\rightleftarrows}} 0\ ,$$

which are given by

$$d^\pm_{s,r} = \frac{d}{d\rho} \pm s\rho\ , \quad \delta^\pm_{s,r} = -\frac{d}{d\rho} - (n - 2r - 1)\rho^{-1} \pm s\rho\ ,$$

using to the canonical identities

$$C^\infty(\mathbb{R}_+)\,\gamma \equiv C^\infty(\mathbb{R}_+)\,d\rho\wedge\gamma \equiv C^\infty(\mathbb{R}_+)\ .$$

According to Sections 10.1 and 10.2, γ can be used to define the following domains of first and second type:

$$\begin{aligned}
\mathcal{E}^r_{\gamma,1} &= \mathcal{S}_{\mathrm{ev},+}\,\gamma && \text{for} \quad r \leq \frac{n-1}{2}\ ,\\
\mathcal{E}^r_{\gamma,2} &= \rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma && \text{for} \quad r \geq \frac{n-3}{2}\ ,\\
\mathcal{E}^{r+1}_{\gamma,1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma && \text{for} \quad r \leq \frac{n+1}{2}\ ,\\
\mathcal{E}^{r+1}_{\gamma,2} &= \rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma && \text{for} \quad r \geq \frac{n-1}{2}\ .
\end{aligned}$$

The following is a direct consequence of Lemma 11.1.

Lemma 11.2. *For any $s \geq 0$, $d_s^\pm$ and $\delta_s^\pm$ define maps*

$$0 \underset{\delta^{\pm}_{s,r-1}}{\overset{d^{\pm}_{s,r-1}}{\rightleftarrows}} \mathcal{E}^r_{\gamma,i} \underset{\delta^{\pm}_{s,r}}{\overset{d^{\pm}_{s,r}}{\rightleftarrows}} \mathcal{E}^{r+1}_{\gamma,i} \underset{\delta^{\pm}_{s,r+1}}{\overset{d^{\pm}_{s,r+1}}{\rightleftarrows}} 0\;,$$

where $i = 1$ if $r \leq \frac{n-1}{2}$, and $i = 2$ if $r \geq \frac{n-1}{2}$.

Remark 18. If n is odd, by Lemma 11.1 and (45), and since $\mathcal{S}_{\mathrm{ev},+} \subset L^2(\mathbb{R}_+, \rho^{2\sigma}\, d\rho)$ if and only if $\sigma > -1/2$, we get

$$\begin{aligned} d_s^\pm(\mathcal{E}^r_{\gamma,2}) &\not\subset L^2\Omega^r(M) \qquad \text{for} \quad r = \frac{n-3}{2}\;,\\ \delta_s^\pm(\mathcal{E}^{r+1}_{\gamma,1}) &\not\subset L^2\Omega^{r+1}(M) \quad \text{for} \quad r = \frac{n+1}{2}\;. \end{aligned}$$

This is compatible with $\Delta_s^+ \not\geq 0$ on $\mathcal{E}^r_{\gamma,2}$ when $r = \frac{n-3}{2}$ (Section 10.1), and $\Delta_s^- \not\geq 0$ on $\mathcal{E}^{r+1}_{\gamma,1}$ when $r = \frac{n+1}{2}$ (Section 10.2).

Remark 19. If n is even, notice that

$$\begin{aligned} \mathcal{E}^r_{\gamma,1} = \mathcal{E}^r_{\gamma,2} &= \mathcal{S}_{\mathrm{ev},+}\,\gamma \qquad\qquad \text{for} \quad r = \frac{n}{2} - 1\;,\\ \mathcal{E}^{r+1}_{\gamma,1} = \mathcal{E}^{r+1}_{\gamma,2} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \quad \text{for} \quad r = \frac{n}{2}\;. \end{aligned}$$

By Lemma 11.2, $\mathcal{E}_{\gamma,i} = \mathcal{E}^r_{\gamma,i} \oplus \mathcal{E}^{r+1}_{\gamma,i}$ is a subcomplex of length two of $\Omega(M)$ with $d_s^\pm$ and $\delta_s^\pm$, even for $s = 0$, where $i = 1$ for $r \leq \frac{n-1}{2}$, and $i = 2$ for $r \geq \frac{n-1}{2}$. Moreover let $\mathcal{E}_{\gamma,0}$ denote the dense subcomplex of $\mathcal{E}_{\gamma,i}$ defined by

$$\begin{aligned} \mathcal{E}^r_{\gamma,0} &= C_0^\infty(\mathbb{R}_+)\,\gamma \equiv C_0^\infty(\mathbb{R}_+)\;,\\ \mathcal{E}^{r+1}_{\gamma,0} &= C_0^\infty(\mathbb{R}_+)\,d\rho\wedge\gamma \equiv C_0^\infty(\mathbb{R}_+)\;. \end{aligned}$$

The closure of $\mathcal{E}_{\gamma,i}$ (and $\mathcal{E}_{\gamma,0}$) in $L^2\Omega(M)$ is denoted by $L^2\mathcal{E}_\gamma$. We have

$$\begin{aligned} L^2\mathcal{E}^r_\gamma &= L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)\,\gamma \equiv L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)\;,\\ L^2\mathcal{E}^{r+1}_\gamma &= L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)\,d\rho\wedge\gamma \equiv L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)\;. \end{aligned}$$

Assume now that $s > 0$. With the notation of Section 7.2, consider the real version of the elliptic complex (E, d) determined by the constants s and

$$\kappa = \frac{n-2r-1}{2}\;, \tag{80}$$

and also its subcomplexes $\mathcal{E}_i$, where $i = 1$ if $\kappa > -1/2$ ($r \leq \frac{n-1}{2}$), and $i = 2$ if $\kappa < 1/2$ ($r \geq \frac{n-1}{2}$).

Proposition 11.3. *There is a unitary isomorphism $L^2\mathcal{E}_\gamma \to L^2(E)$, which restricts to isomorphisms of complexes up to a shift of degree, $(\mathcal{E}_{\gamma,0}, d_s^\pm) \to (C_0^\infty(E), d)$ and $(\mathcal{E}_{\gamma,i}, d_s^\pm) \to (\mathcal{E}_i, d)$, where $i = 1$ if $r \leq \frac{n-1}{2}$, and $i = 2$ if $r \geq \frac{n-1}{2}$.*

Proof. The unitary isomorphism

$$\rho^\kappa : L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+, d\rho)$$

induces a unitary isomorphism $L^2\mathcal{E}_\gamma \to L^2(E)$, which restricts to an isomorphism $\mathcal{E}_{\gamma,0} \to C_0^\infty(E)$. Furthermore

$$\rho^\kappa \mathcal{E}^r_{\gamma,1} = \rho^\kappa \mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^\kappa \mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^0_1\ ,$$
$$\rho^\kappa \mathcal{E}^{r+1}_{\gamma,1} = \rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \equiv \rho^{1+\kappa}\mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^1_1$$

if $r \le \frac{n-1}{2}$, and

$$\rho^\kappa \mathcal{E}^r_{\gamma,2} = \rho^{\kappa-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^0_2\ ,$$
$$\rho^\kappa \mathcal{E}^{r+1}_{\gamma,2} = \rho^{\kappa-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^1_2$$

if $r \ge \frac{n-1}{2}$. By Lemma 11.1 and (24), we also have

$$\rho^\kappa\, d^\pm_{s,r}\,\rho^{-\kappa} = \rho^\kappa\left(\frac{d}{d\rho} \pm s\rho\right)\rho^{-\kappa} = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\ ,$$

which is the operator d of Section 7.2. □

Corollary 11.4. (i) *If $r \ne \frac{n-1}{2}$, then $(\mathcal{E}_{\gamma,0}, d^\pm_s)$ has a unique Hilbert complex extension in $L^2\mathcal{E}_\gamma$, whose smooth core is $\mathcal{E}_{\gamma,i}$, where $i = 1$ if $r < \frac{n-1}{2}$, and $i = 2$ if $r > \frac{n-1}{2}$.*

(ii) *If $r = \frac{n-1}{2}$, then $(\mathcal{E}_{\gamma,0}, d^\pm_s)$ has minimum and maximum Hilbert complex extensions in $L^2\mathcal{E}_\gamma$, whose smooth cores are $\mathcal{E}_{\gamma,2}$ and $\mathcal{E}_{\gamma,1}$, respectively.*

Proof. This follows from Propositions 7.4 and 11.3. □

For each degree r, we will choose one of the possible domains of first and second type defined by γ, denoted by $\mathcal{E}^r_\gamma$ and $\mathcal{E}^{r+1}_\gamma$, so that $\mathcal{E}_\gamma = \mathcal{E}^r_\gamma \oplus \mathcal{E}^{r+1}_\gamma$ is a subcomplex of $(\Omega(U), d^\pm_s)$ according to Lemma 11.2.

If n is even, there is only one choice of domains of first and second types by Remark 19. Thus $\mathcal{E}^r_\gamma$ and $\mathcal{E}^{r+1}_\gamma$ have only one possible definition in this case.

If n is odd, there are two possible choices of domains of first and second types just for the following values of r:

$$\left.\begin{aligned}\mathcal{E}^r_{\gamma,1} &= \mathcal{S}_{\mathrm{ev},+}\,\gamma\\ \mathcal{E}^r_{\gamma,2} &= \rho^{-1}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma\end{aligned}\right\}\quad \text{for}\quad r = \frac{n-3}{2}\ ,$$
$$\left.\begin{aligned}\mathcal{E}^r_{\gamma,1} &= \mathcal{S}_{\mathrm{ev},+}\,\gamma\\ \mathcal{E}^r_{\gamma,2} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,\gamma\\ \mathcal{E}^{r+1}_{\gamma,1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma\\ \mathcal{E}^{r+1}_{\gamma,2} &= \mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma\end{aligned}\right\}\quad \text{for}\quad r = \frac{n-1}{2}\ ,$$
$$\left.\begin{aligned}\mathcal{E}^{r+1}_{\gamma,1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma\\ \mathcal{E}^{r+1}_{\gamma,2} &= \rho^2\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma\end{aligned}\right\}\quad \text{for}\quad r = \frac{n+1}{2}\ .$$

By Remark 18 and Corollary 11.4, we choose

$$\mathcal{E}^r_\gamma = \mathcal{E}^r_{\gamma,1}\quad \text{for}\quad r = \frac{n-3}{2}\ ,$$
$$\mathcal{E}^{r+1}_\gamma = \mathcal{E}^{r+1}_{\gamma,2}\quad \text{for}\quad r = \frac{n+1}{2}\ .$$

In order to get the minimum and maximum i.b.c. of $(\bigwedge TM^*, d)$, according to Corollary 11.3, we choose

$$\left.\begin{array}{r}\left.\begin{array}{r}\mathcal{E}_\gamma^r=\mathcal{E}_{\gamma,2}^r\\ \mathcal{E}_\gamma^{r+1}=\mathcal{E}_{\gamma,2}^{r+1}\end{array}\right\}\quad \text{if}\quad \gamma\in\widetilde{\mathcal{H}}^r_{\min}\\ \left.\begin{array}{r}\mathcal{E}_\gamma^r=\mathcal{E}_{\gamma,1}^r\\ \mathcal{E}_\gamma^{r+1}=\mathcal{E}_{\gamma,1}^{r+1}\end{array}\right\}\quad \text{if}\quad \gamma\in\widetilde{\mathcal{H}}^r_{\max}\end{array}\right\}\quad \text{for}\quad r=\frac{n-1}{2}\;.$$

According to Corollary 11.4, the above choices of $\mathcal{E}_\gamma$ satisfy the following.

Corollary 11.5. (i) *If $r\neq\frac{n-1}{2}$, then $(\mathcal{E}_{\gamma,0},d_s^\pm)$ has a unique Hilbert complex extension in $L^2\mathcal{E}_\gamma$, whose smooth core is $\mathcal{E}_\gamma$.*

(ii) *If $r=\frac{n-1}{2}$, then $(\mathcal{E}_{\gamma,0},d_s^\pm)$ has different minimum and maximum Hilbert complex extensions in $L^2\mathcal{E}_\gamma$. If $\gamma\in\widetilde{\mathcal{H}}_{\min/\max}$, then $\mathcal{E}_\gamma$ is the smooth core of the minimum/maximum Hilbert complex extension of $(\mathcal{E}_{\gamma,0},d_s^\pm)$.*

Let $(\mathcal{D}_\gamma,\mathbf{d}_{s,\gamma}^\pm)$ denote the Hilbert complex extension of $(\mathcal{E}_{\gamma,0},d_s^\pm)$ with core $\mathcal{E}_\gamma$, let $\mathbf{\Delta}_{s,\gamma}^\pm$ be the corresponding Laplacian, and let $\mathcal{H}_{s,\gamma}^\pm=\mathcal{H}_{s,\gamma}^{\pm,r}\oplus\mathcal{H}_{s,\gamma}^{\pm,r+1}=\ker\mathbf{\Delta}_{s,\gamma}^\pm$. The following result follows from Sections 10.1 and 10.2, Lemma 6.2 and the choices made to define $\mathcal{E}_\gamma$.

Proposition 11.6. (i) *$(\mathcal{D}_\gamma,\mathbf{d}_{s,\gamma}^\pm)$ is discrete.*

(ii) *$\mathcal{H}_{s,\gamma}^{+,r+1}=0$, $\dim\mathcal{H}_{s,\gamma}^{+,r}=1$ if*

$$r\le\begin{cases}\frac{n}{2}-1 & \text{if } n \text{ is even}\\ \frac{n-3}{2} & \text{if } n \text{ is odd and } \gamma\in\widetilde{\mathcal{H}}^r_{\min}\\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \gamma\in\widetilde{\mathcal{H}}^r_{\max}\;,\end{cases}$$

and $\mathcal{H}_{s,\gamma}^{+,r}=0$ otherwise.

(iii) *$\mathcal{H}_{s,\gamma}^{-,r}=0$, $\dim\mathcal{H}_{s,\gamma}^{-,r+1}=1$ if*

$$r\ge\begin{cases}\frac{n}{2} & \text{if } n \text{ is even}\\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \gamma\in\widetilde{\mathcal{H}}^r_{\min}\\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma\in\widetilde{\mathcal{H}}^r_{\max}\;,\end{cases}$$

and $\mathcal{H}_{s,\gamma}^{-,r+1}=0$ otherwise.

(iv) *If $e_s^\pm\in\mathcal{H}_{s,\gamma}^\pm$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho)\to1$ as $\rho\to0$, then $\langle he_s^\pm,e_s^\pm\rangle\to1$ as $s\to\infty$.*

(v) *All non-zero eigenvalues of $\mathbf{\Delta}_{s,\gamma}^\pm$ are in $O(s)$ as $s\to\infty$.*

11.2. **Subomplexes defined by domains of third, fourth and fifth types.** Consider the notation of Sections 10.3–10.5. The following result follows from Corollary 9.5.

Lemma 11.7. *For $s\ge0$, $d_s^\pm$ and $\delta_s^\pm$ define maps*

$$0\;\underset{\delta^\pm_{s,r-2}}{\overset{d^\pm_{s,r-2}}{\rightleftarrows}}\;C^\infty(\mathbb{R}_+)\,\beta\;\underset{\delta^\pm_{s,r-1}}{\overset{d^\pm_{s,r-1}}{\rightleftarrows}}\;C^\infty(\mathbb{R}_+)\,\alpha+C^\infty(\mathbb{R}_+)\,d\rho\wedge\beta$$
$$\underset{\delta^\pm_{s,r}}{\overset{d^\pm_{s,r}}{\rightleftarrows}}\;C^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha\;\underset{\delta^\pm_{s,r+1}}{\overset{d^\pm_{s,r+1}}{\rightleftarrows}}\;0\;,$$

which are given by

$$\begin{aligned} d^{\pm}_{s,r-1} &= \begin{pmatrix} \mu \\ \frac{d}{d\rho} \pm s\rho \end{pmatrix} , \\ \delta^{\pm}_{s,r-1} &= \begin{pmatrix} \mu\rho^{-2} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \pm s\rho \end{pmatrix} , \\ d^{\pm}_{s,r} &= \begin{pmatrix} \frac{d}{d\rho} \pm s\rho & -\mu \end{pmatrix} , \\ \delta^{\pm}_{s,r} &= \begin{pmatrix} -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \pm s\rho \\ -\mu\rho^{-2} \end{pmatrix} , \end{aligned}$$

according to the canonical identities

$$\begin{aligned} C^\infty(\mathbb{R}_+)\,\beta \equiv C^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha \equiv C^\infty(\mathbb{R}_+) , \\ C^\infty(\mathbb{R}_+)\,\alpha + C^\infty(\mathbb{R}_+)\,d\rho\wedge\beta \equiv C^\infty(\mathbb{R}_+)\oplus C^\infty(\mathbb{R}_+) . \end{aligned}$$

Consider only the choices of a given by the positive square roots in (52) and (58) for domains of third and fourth types, and (75) for domains of fifth type; the other choices of a are rejected because they are very restrictive on μ and r, and give rise to some negative eigenvalues. If these values of a are denoted by a_3, a_4 and a_5 according to the types of domains, then $a_5 = a_3 = a_4 - 1$, and therefore the notation $a_5 = a_3 = a$ and $a_4 = a+1$ will be used. Recall also that we only have the choice (73) for b, which equals $a+2$. So we only consider the following domains of third, fourth and fifth types defined by α and β:

$$\begin{aligned} \mathcal{F}^{r-1}_{\alpha,\beta} &= \rho^a\,\mathcal{S}_{\mathrm{ev},+}\,\beta \equiv \rho^a\,\mathcal{S}_{\mathrm{ev},+} , \\ \mathcal{F}^{r+1}_{\alpha,\beta} &= \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\alpha \equiv \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+} , \\ \mathcal{F}^{r}_{\alpha,\beta} &= \rho^a\,\left\{\,(\phi - c\rho^2\psi)\,\alpha + (c\rho^{-1}\phi + \rho\psi)\,d\rho\wedge\beta \mid \phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\,\right\} \\ &\equiv \rho^a\,\left\{\,(\phi - c\rho^2\psi, c\rho^{-1}\phi + \rho\psi) \mid \phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\,\right\} . \end{aligned}$$

Lemma 11.8. *For any $s\ge 0$, $d^{\pm}_s$ and $\delta^{\pm}_s$ define maps*

$$0 \underset{\delta^{\pm}_{s,r-2}}{\overset{d^{\pm}_{s,r-2}}{\rightleftarrows}} \mathcal{F}^{r-1}_{\alpha,\beta} \underset{\delta^{\pm}_{s,r-1}}{\overset{d^{\pm}_{s,r-1}}{\rightleftarrows}} \mathcal{F}^{r}_{\alpha,\beta} \underset{\delta^{\pm}_{s,r}}{\overset{d^{\pm}_{s,r}}{\rightleftarrows}} \mathcal{F}^{r+1}_{\alpha,\beta} \underset{\delta^{\pm}_{s,r+1}}{\overset{d^{\pm}_{s,r+1}}{\rightleftarrows}} 0$$

Proof. Lemma 11.7 gives $\delta^{\pm}_s(\mathcal{F}^{r-1}_{\alpha,\beta}) = d^{\pm}_s(\mathcal{F}^{r+1}_{\alpha,\beta}) = 0$.

Observe that

$$a = c\mu , \tag{81}$$

obtaining

$$c(a+n-2r) = \mu \tag{82}$$

by (63). By Lemma 11.7, (81) and (82), for $h\in\mathcal{S}_{\mathrm{ev},+}$,

$$d^{\pm}_s(\rho^a h\,\beta) = \rho^a\left(\mu h\,\alpha + \left(\frac{d}{d\rho} + c\mu\rho^{-1} \pm s\rho\right)(h)\,d\rho\wedge\beta\right) , \tag{83}$$

$$\delta^{\pm}_s(\rho^{a+1} h\,d\rho\wedge\alpha) = \rho^a\left(\left(-\rho\frac{d}{d\rho} - \frac{\mu}{c} \pm s\rho^2\right)(h)\,\alpha - \mu\rho^{-1} h\,d\rho\wedge\beta\right) . \tag{84}$$

The inclusion $d_s^{\pm}(\mathcal{F}^{r-1}_{\alpha,\beta}) \subset \mathcal{F}^r_{\alpha,\beta}$ follows from (83) if we can find $\phi,\psi \in \mathcal{S}_{\mathrm{ev},+}$ such that

$$\phi - c\rho^2\psi = \mu h \;, \tag{85}$$

$$c\rho^{-1}\phi + \rho\psi = \left(\frac{d}{d\rho} + c\mu\rho^{-1} \pm s\rho\right)(h) \;. \tag{86}$$

Subtract $c\rho^{-2}$ times (85) from ρ^{-1} times (86) to get

$$\psi = \frac{1}{1+c^2}\left(\rho^{-1}\frac{d}{d\rho} \pm s\right)(h) \;,$$

which is well defined in $\mathcal{S}_{\mathrm{ev},+}$. Then

$$\phi = \mu h + c\rho^2\psi$$

by (85). These functions ϕ and ψ satisfy (85) and (86).

The inclusion $\delta_s^{\pm}(\mathcal{F}^{r+1}_{\alpha,\beta}) \subset \mathcal{F}^r_{\alpha,\beta}$ follows from (84) if we can find $\phi,\psi \in \mathcal{S}_{\mathrm{ev},+}$ such that

$$\phi - c\rho^2\psi = \left(-\rho\frac{d}{d\rho} - \frac{\mu}{c} \pm s\rho^2\right)(h) \;, \tag{87}$$

$$c\rho^{-1}\phi + \rho\psi = -\mu\rho^{-1}h \;. \tag{88}$$

The sum (87) and $c\rho$ times (88) gives

$$\phi = \frac{1}{1+c^2}\left(-\rho\frac{d}{d\rho} - \frac{1+c^2}{c}\,\mu \pm s\rho^2\right)(h) \;,$$

which belongs to $\mathcal{S}_{\mathrm{ev},+}$. The even extensions of h and ϕ to $\mathbb{R}$, also denoted by h and ϕ, satisfy $c\phi(0) = -\mu h(0)$, and therefore $\mu h + c\phi \in \rho^2\,\mathcal{S}_{\mathrm{ev}}$. It follows that

$$\psi = \rho^{-2}(\mu h + c\phi) \;,$$

obtained from (88), is well defined in $\mathcal{S}_{\mathrm{ev},+}$. These functions ϕ and ψ satisfy (87) and (88).

For arbitrary $\phi,\psi \in \mathcal{S}_{\mathrm{ev},+}$, let

$$\zeta = \rho^a\left(\left(\phi - c\rho^2\psi\right)\,\alpha + \left(c\rho^{-1}\phi + \rho\psi\right)\,d\rho\wedge\beta\right) \;. \tag{89}$$

By Corollary 9.5, (81) and (82),

$$\begin{aligned} d_s^{\pm}(\zeta) = \rho^{a+1}\Bigg(&\left(\rho^{-1}\frac{d}{d\rho} \pm s\right)(\phi) \\ &+ c\left(-\rho^{-1}\frac{d}{d\rho} - \left(\frac{c^2+1}{c}\,\mu + 2\right) \pm s\rho^2\right)(\psi)\Bigg)\,d\rho\wedge\alpha \;, \\ \delta_s^{\pm}(\zeta) = \rho^{a}\Bigg(&c\left(-\rho^{-1}\frac{d}{d\rho} \pm s\right)(\phi) \\ &+ \left(-\rho\frac{d}{d\rho} - \left(\frac{c^2+1}{c}\,\mu + 2\right) \pm s\rho^2\right)(\psi)\Bigg)\,\beta \;, \end{aligned}$$

showing $d_s^{\pm}(\mathcal{F}^r_{\alpha,\beta}) \subset \mathcal{F}^{r+1}_{\alpha,\beta}$ and $\delta_s^{\pm}(\mathcal{F}^r_{\alpha,\beta}) \subset \mathcal{F}^{r-1}_{\alpha,\beta}$. □

By Lemma 11.8,

$$\mathcal{F}_{\alpha,\beta} = \mathcal{F}^{r-1}_{\alpha,\beta} \oplus \mathcal{F}^{r}_{\alpha,\beta} \oplus \mathcal{F}^{r+1}_{\alpha,\beta}$$

is a subcomplex of length three of $\Omega(M)$ with $d_s^\pm$ and $\delta_s^\pm$. Moreover let $\mathcal{F}_{\alpha,\beta,0}$ denote the dense subcomplex of $\mathcal{F}_{\alpha,\beta}$ defined by

$$\begin{aligned}\mathcal{F}^{r-1}_{\alpha,\beta,0} &= C^\infty_0(\mathbb{R}_+)\,\beta \equiv C^\infty_0(\mathbb{R}_+)\;,\\ \mathcal{F}^{r+1}_{\alpha,\beta,0} &= C^\infty_0(\mathbb{R}_+)\,d\rho\wedge\alpha \equiv C^\infty_0(\mathbb{R}_+)\;,\\ \mathcal{F}^{r}_{\alpha,\beta,0} &= C^\infty_0(\mathbb{R}_+)\,\alpha + C^\infty_0(\mathbb{R}_+)\,d\rho\wedge\beta \equiv C^\infty_0(\mathbb{R}_+)\oplus C^\infty_0(\mathbb{R}_+)\;.\end{aligned}$$

The closure of $\mathcal{F}_{\alpha,\beta}$ (and $\mathcal{F}_{\alpha,\beta,0}$) in $L^2\Omega(M)$ is denoted by $L^2\mathcal{F}_{\alpha,\beta}$. We have

$$\begin{aligned}L^2\mathcal{F}^{r-1}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,\beta \equiv L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\;,\\ L^2\mathcal{F}^{r+1}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,d\rho\wedge\alpha \equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,\\ L^2\mathcal{F}^{r}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,\alpha + L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,d\rho\wedge\beta\\ &\equiv L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;.\end{aligned}$$

Assume now that $s>0$. With the notation of Section 7.3, consider the real version of the elliptic complex (F,d), as well as its subcomplex $\mathcal{F}_1$, determined by the constants s, c and

$$\kappa = \frac{-1+\sqrt{(n-2r)^2+4\mu^2}}{2} > -\frac{1}{2}\;. \tag{90}$$

By (64),

$$\kappa = c\mu + \frac{n-2r-1}{2} = \frac{\mu}{c} - \frac{n-2r+1}{2}\;. \tag{91}$$

Proposition 11.9. *There is a unitary isomorphism $L^2\mathcal{F}_{\alpha,\beta}\to L^2(F)$, which restricts to isomorphisms of complexes up to a shift of degree, $(\mathcal{F}_{\alpha,\beta},d_s^\pm)\to(\mathcal{F}_1,d)$ and $(\mathcal{F}_{\alpha,\beta,0},d_s^\pm)\to(C^\infty_0(F),d)$.*

Proof. As an intermediate step, let

$$\begin{gathered}\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta} = \rho\,\mathcal{F}^{r-1}_{\alpha,\beta} = \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \widehat{\mathcal{F}}^{r+1}_{\alpha,\beta} = \mathcal{F}^{r+1}_{\alpha,\beta} = \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\;,\\ \widehat{\mathcal{F}}^{r}_{\alpha,\beta} = \Theta^{-1}(\mathcal{F}^{r}_{\alpha,\beta}) = \rho^{a}\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^{a+2}\,\mathcal{S}_{\mathrm{ev},+}\;,\\ \widehat{\mathcal{F}}_{\alpha,\beta} = \widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\oplus\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\oplus\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}\;,\quad \widehat{\mathcal{F}}_{\alpha,\beta,0} = \mathcal{F}_{\alpha,\beta,0}\;,\\ L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta} = L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta} = L^2\mathcal{F}^{r+1}_{\alpha,\beta} = L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,\\ L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta} \equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,\\ L^2\widehat{\mathcal{F}}_{\alpha,\beta} = L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\oplus L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\oplus L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}\;.\end{gathered}$$

Moreover let $\Xi: L^2\mathcal{F}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}_{\alpha,\beta}$ be the unitary isomorphism defined by

$$\rho: L^2\mathcal{F}^{r-1}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\;,\qquad \frac{1}{\sqrt{1+c^2}}\,\Theta^{-1}: L^2\mathcal{F}^{r}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta}$$

and the identity map $L^2\mathcal{F}^{r+1}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}$. It restricts to isomorphisms $\mathcal{F}_{\alpha,\beta}\to\widehat{\mathcal{F}}_{\alpha,\beta}$ and $\mathcal{F}_{\alpha,\beta,0}\to\widehat{\mathcal{F}}_{\alpha,\beta,0}$. Thus, by Lemma 11.8, $(\mathcal{F}_{\alpha,\beta},d_s^\pm)$ induces via Ξ a complex

$$0 \xrightarrow{\hat d^\pm_{s,r-2}} \widehat{\mathcal{F}}^{r-1}_{\alpha,\beta} \xrightarrow{\hat d^\pm_{s,r-1}} \widehat{\mathcal{F}}^{r}_{\alpha,\beta} \xrightarrow{\hat d^\pm_{s,r}} \widehat{\mathcal{F}}^{r+1}_{\alpha,\beta} \xrightarrow{\hat d^\pm_{s,r+2}} 0\;.$$

By Lemma 11.7 and (24),

$$\begin{aligned}
\hat{d}^{\pm}_{s,r-1} &= \frac{1}{\sqrt{1+c^2}}\,\Theta^{-1}\,d^{\pm}_{s,r-1}\,\rho^{-1}\\
&= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}1 & c\rho\\ -c & \rho\end{pmatrix}\begin{pmatrix}\mu\\ \frac{d}{d\rho}\pm s\rho\end{pmatrix}\rho^{-1}\\
&= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}c\,\frac{d}{d\rho}+(\mu-c)\rho^{-1}\pm cs\rho\\ \frac{d}{d\rho}+(c\mu+1)\rho^{-1}\pm s\rho\end{pmatrix}\;, \qquad (92)\\
\hat{d}^{\pm}_{s,r} &= \frac{1}{\sqrt{1+c^2}}\,\Theta\,d^{\pm}_{s,r}\\
&= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}\frac{d}{d\rho}\pm s\rho & -\mu\end{pmatrix}\begin{pmatrix}1 & -c\\ c\rho^{-1} & \rho^{-1}\end{pmatrix}\\
&= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}\frac{d}{d\rho}-c\mu\rho^{-1}\pm s\rho & -c\,\frac{d}{d\rho}-\mu\rho^{-1}\mp cs\rho\end{pmatrix}\;. \qquad (93)
\end{aligned}$$

Now, the unitary isomorphism

$$\rho^{\frac{n-2r-1}{2}} : L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\to L^2(\mathbb{R}_+,d\rho)$$

induces a unitary isomorphism $L^2\widehat{\mathcal{F}}_{\alpha,\beta}\to L^2(F)$, which restricts to isomorphisms $\widehat{\mathcal{F}}_{\alpha,\beta}\to\mathcal{F}_1$ and $\widehat{\mathcal{F}}_{\alpha,\beta,0}\to C^\infty_0(F)$. Moreover, by (92), (93), (24) and (91),

$$\begin{aligned}
&\rho^{\frac{n-2r-1}{2}}\,\hat{d}^{\pm}_{s,r-1}\,\rho^{-\frac{n-2r-1}{2}}\\
&\qquad= \frac{1}{\sqrt{1+c^2}}\,\rho^{\frac{n-2r-1}{2}}\begin{pmatrix}c\,\frac{d}{d\rho}+(\mu-c)\rho^{-1}\pm cs\rho\\ \frac{d}{d\rho}+(c\mu+1)\rho^{-1}\pm s\rho\end{pmatrix}\rho^{-\frac{n-2r-1}{2}}\\
&\qquad= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}c\left(\frac{d}{d\rho}+\kappa\rho^{-1}\pm s\rho\right)\\ \frac{d}{d\rho}-(\kappa+1)\rho^{-1}\pm s\rho\end{pmatrix}\;,
\end{aligned}$$

$$\begin{aligned}
&\rho^{\frac{n-2r-1}{2}}\,\hat{d}^{\pm}_{s,r}\,\rho^{-\frac{n-2r-1}{2}}\\
&\quad= \frac{1}{\sqrt{1+c^2}}\,\rho^{\frac{n-2r-1}{2}}\begin{pmatrix}\frac{d}{d\rho}-c\mu\rho^{-1}\pm s\rho & -c\,\frac{d}{d\rho}-\mu\rho^{-1}\mp cs\rho\end{pmatrix}\rho^{-\frac{n-2r-1}{2}}\\
&\quad= \frac{1}{\sqrt{1+c^2}}\begin{pmatrix}\frac{d}{d\rho}-\kappa\rho^{-1}\pm s\rho & c\left(-\frac{d}{d\rho}-(\kappa+1)\rho^{-1}\mp s\rho\right)\end{pmatrix}\;,
\end{aligned}$$

which are the operators d_0 and d_1 of Section 7.3. □

Corollary 11.10. $(\mathcal{F}_{\alpha,\beta,0},d^{\pm}_s)$ *has a unique Hilbert complex extension in* $L^2\mathcal{F}_{\alpha,\beta}$*, whose smooth core is* $\mathcal{F}_{\alpha,\beta}$*.*

Proof. This follows from Propositions 7.6 and 11.9. □

Let $(\mathcal{D}_{\alpha,\beta},\mathbf{d}^{\pm}_{s,\alpha,\beta})$ denote the unique Hilbert complex extension of $(\mathcal{F}_{\alpha,\beta,0},d^{\pm}_s)$, according to Corollary 11.10, and let $\boldsymbol{\Delta}^{\pm}_{s,\alpha,\beta}$ denote the corresponding Laplacian. The following result follows from Sections 10.3–10.5.

Proposition 11.11. (i) $(\mathcal{D}_{\alpha,\beta},\mathbf{d}^{\pm}_{s,\alpha,\beta})$ *is discrete.*

(ii) *The eigenvalues of* $\boldsymbol{\Delta}^{\pm}_{s,\alpha,\beta}$ *are positive and in* $O(s)$ *as* $s\to\infty$*.*

11.3. **Splitting into subcomplexes.** Let $\mathcal{B}_{\min/\max,0}$ denote an orthonormal frame of $\widetilde{\mathcal{H}}_{\min/\max}$ consisting of homogeneous differential forms. For each positive eigenvalue μ of $\widetilde{D}_{\min/\max}$, let $\mathcal{B}_{\min/\max,\mu}$ be an orthonormal frame of $E_\mu(\widetilde{D}_{\min/\max})$ consisting of differential forms $\alpha+\beta$ like in Section 11.2. Then let

$$\mathbf{d}^{\pm}_{s,\min/\max} = \bigoplus_{\gamma} \mathbf{d}^{\pm}_{s,\gamma} \oplus \widehat{\bigoplus_{\mu}} \bigoplus_{\alpha+\beta} \mathbf{d}^{\pm}_{s,\alpha,\beta} \;,$$

where γ runs in $\mathcal{B}_{\min/\max,0}$, μ runs in the positive spectrum of $\widetilde{D}_{\min/\max}$, and $\alpha+\beta$ runs in $\mathcal{B}_{\min/\max,\mu}$. Observe that the domain of $\mathbf{d}^{\pm}_{s,\min/\max}$ is independent of s, and therefore it is denoted by $\mathcal{D}_{\min/\max}$. Let also

$$\mathcal{G}_{\min/\max} = \bigoplus_{\gamma} \mathcal{E}_{\gamma,0} \oplus \bigoplus_{\mu} \bigoplus_{\alpha+\beta} \mathcal{F}_{\alpha,\beta,0} \;.$$

Proposition 11.12. *$\mathcal{D}(d^{\pm}_{s,\min/\max}) = \mathcal{D}_{\min/\max}$ and $d^{\pm}_{s,\min/\max} = \mathbf{d}^{\pm}_{s,\min/\max}$.*

Proof. By Corollaries 11.5 and 11.10, Lemma 4.2 and (47), $(\mathcal{D}_{\min/\max}, \mathbf{d}^{\pm}_{s,\min/\max})$ is the minimum/maximum Hilbert complex extension of $(\mathcal{G}_{\min/\max}, d^{\pm}_s)$. Then the result easily follows from the following assertions.

Claim 14. $\mathcal{G}_{\min/\max} \subset \mathcal{D}(d^{\pm}_{s,\min/\max})$.

Claim 15. $\Omega_0(M) \subset \mathcal{D}_{\min/\max}$.

Let $\hat{d}^{\pm}_{s,\min/\max}$ denote the minimum/maximum Hilbert complex extension of $(\Omega_0(M), d^{\pm}_s)$ with respect to the product metric $\hat{g} = \tilde{g} + (d\rho)^2$ on $M = N \times \mathbb{R}_+$. With the terminology of [6, p. 110], observe that $(\Omega(M), d^{\pm}_s)$ is the product complex of the de Rham complex of N, $(\Omega(N), \tilde{d})$, and the Witten deformation of the de Rham complex of $\mathbb{R}_+$, defined by the function $\frac{1}{2}\rho^2$. Then, by [6, Lemma 3.6 and (2.38b)],

$$\begin{aligned}\mathcal{D}(\hat{d}^{\pm}_{s,\min/\max}) &\supset C^\infty_0(\mathbb{R}_+)\,\mathcal{D}(\tilde{d}_{\min/\max}) + C^\infty_0(\mathbb{R}_+)\,d\rho\wedge\mathcal{D}(\tilde{d}_{\min/\max})\\ &\supset \mathcal{G}_{\min/\max}\;. \end{aligned} \tag{94}$$

On the other hand, for $0<a<b<\infty$, let $L^2_{a,b}\Omega(M,g)$ and $L^2_{a,b}\Omega(M,\hat{g})$ denote the Hilbert subspaces of $L^2\Omega(M,g)$ and $L^2\Omega(M,\hat{g})$, respectively, consisting of L^2 differential forms supported in $N\times[a,b]$. Since g and $\hat{g}$ are quasi-isometric on $N\times(a',b')$ for $0<a'<a$ and $b<b'<\infty$, it follows that

$$\mathcal{D}(d^{\pm}_{s,\min/\max}) \cap L^2_{a,b}\Omega(M,g) = \mathcal{D}(\hat{d}^{\pm}_{s,\min/\max}) \cap L^2_{a,b}\Omega(M,\hat{g})\;. \tag{95}$$

Moreover

$$\mathcal{G}_{\min/\max} \subset \bigcup_{0<a<b<\infty} L^2_{a,b}\Omega(M,g)\;. \tag{96}$$

Now Claim 14 follows from (94)–(96).

Finally, Claim 15 follows from

$$\Omega_0(M) \subset \bigoplus_{\gamma} \mathcal{E}_{\gamma,0} \oplus \widehat{\bigoplus_{\mu}} \bigoplus_{\alpha+\beta} \mathcal{F}_{\alpha,\beta,0}\;, \tag{97}$$

where γ, μ and $\alpha+\beta$ vary as above. The inclusion (97) can be proved as follows. According to (41), any $\xi\in\Omega_0(M)$ can be written as $\xi = \xi_0 + d\rho\wedge\xi_1$, where

$\xi_0, \xi_1 \in C_0^\infty(\mathbb{R}_+, \Omega_0(N))$. Then, by (47), we get functions $f_{k,\gamma}, f_{k,\ell,\alpha,\beta} \in C_0^\infty(\mathbb{R}_+)$, for $k, \ell \in \{0,1\}$, defined by

$$f_{k,\gamma}(\rho) = \langle \xi_k(\rho), \gamma \rangle_{\tilde{g}} \ ,$$
$$f_{k,0,\alpha,\beta}(\rho) = \langle \xi_k(\rho), \beta \rangle_{\tilde{g}} \ , \quad f_{k,1,\alpha,\beta}(\rho) = \langle \xi_k(\rho), \alpha \rangle_{\tilde{g}} \ ,$$

and moreover

$$\begin{aligned}\alpha = \sum_\gamma (f_{0,\gamma}\, \gamma + f_{1,\gamma}\, d\rho \wedge \gamma) \\ + \sum_\mu \sum_{\alpha+\beta} (f_{0,0,\alpha,\beta}\, \beta + f_{1,0,\alpha,\beta}\, \alpha + f_{1,0,\alpha,\beta}\, d\rho \wedge \beta + f_{1,1,\alpha,\beta}\, d\rho \wedge \alpha)\end{aligned}$$

in $L^2\Omega(M, g)$, where γ, μ and $\alpha + \beta$ vary as above. Thus ξ belongs to the space in the right hand side of (97). □

Remark 20. From (9), Remark 13, and Propositions 7.4, 7.6 and 11.12, it follows that, with the notation of Example 3.2, $h(\rho)\, \mathcal{D}^\infty(d^\pm_{s,\min/\max}) \subset \mathcal{D}^\infty(d^\pm_{s,\min/\max})$ for all $h \in C^\infty(\mathbb{R}_+)$ such that $h' \in C_0^\infty(\mathbb{R}_+)$.

Let $\mathcal{H}^\pm_{s,\min/\max} = \bigoplus_r \mathcal{H}^{\pm,r}_{s,\min/\max} = \ker \Delta^\pm_{s,\min/\max}$.

Corollary 11.13. (i) *$d^\pm_{s,\min/\max}$ is discrete.*

(ii) *$\mathcal{H}^{+,r}_{\min} \cong H^r_{\min}(N)$ if*

$$r \leq \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-3}{2} & \text{if } n \text{ is odd} \ , \end{cases}$$

and $\mathcal{H}^{+,r}_{\min} = 0$ otherwise.

(iii) *$\mathcal{H}^{+,r}_{\max} \cong H^r_{\max}(N)$ if*

$$r \leq \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \ , \end{cases}$$

and $\mathcal{H}^{+,r}_{\max} = 0$ otherwise.

(iv) *$\mathcal{H}^{-,r+1}_{\min} \cong H^r_{\min}(N)$ if*

$$r \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \ , \end{cases}$$

and $\mathcal{H}^{+,r+1}_{\min} = 0$ otherwise.

(v) *$\mathcal{H}^{-,r+1}_{\max} \cong H^r_{\max}(N)$ if*

$$r \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \ , \end{cases}$$

and $\mathcal{H}^{+,r+1}_{\max} = 0$ otherwise.

(vi) *If $e^\pm_s \in \mathcal{H}^\pm_{s,\min/\max}$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle h e^\pm_s, e^\pm_s \rangle \to 1$ as $s \to \infty$.*

(vii) *Let $0 \leq \lambda^\pm_{s,\min/\max,0} \leq \lambda^\pm_{s,\min/\max,1} \leq \cdots$ be the eigenvalues of $\Delta_{s,\min/\max}$, repeated according to their multiplicities. Given $k \in \mathbb{N}$, if $\lambda^\pm_{s,\min/\max,k} > 0$ for some s, then $\lambda^\pm_{s,\min/\max,k} \in O(s)$ as $s \to \infty$.*

(viii) *There is some $\theta > 0$ such that $\liminf_k \lambda^\pm_{s,\min/\max,k} k^{-\theta} > 0$.*

Proof. For γ, μ and $\alpha+\beta$ as above, the spectra of $\Delta_s^\pm$ on $\mathcal{E}_\gamma$ and $\mathcal{F}_{\alpha,\beta}$ is discrete by Propositions 11.6-(i) and 11.11-(i). Moreover the union of all of these spectra has no accumulation points according to Sections 10.1–10.5 and since $\widetilde{\Delta}_{\mathrm{min/max}}$ has a discrete spectrum. Then (i) follows by Proposition 11.12.

Now, properties (ii)–(vii) follow directly from Propositions 11.6, 11.11 and 11.12.

To prove (viii), let $0 \le \tilde\lambda_{\mathrm{min/max},0} \le \tilde\lambda_{\mathrm{min/max},1} \le \cdots$ denote the eigenvalues of $\widetilde{\Delta}_{s,\mathrm{min/max}}$, repeated according to their multiplicities, and let $\mu_{\mathrm{min/max},\ell} = \sqrt{\tilde\lambda_{\mathrm{min/max},\ell}}$ for each $\ell \in \mathbb{N}$. Since N satisfies Theorem 1.1-(ii) with $\tilde g$, there is some $C_0, \tilde\theta > 0$ such that

$$\tilde\lambda_{\mathrm{min/max},\ell} \ge C_0^2 \ell^{\tilde\theta} \tag{98}$$

for all ℓ. Consider the counting function

$$\mathfrak{N}^\pm_{s,\mathrm{min/max}}(\lambda) = \#\left\{\, k \in \mathbb{N} \mid \lambda^\pm_{s,\mathrm{min/max},k} < \lambda \,\right\}$$

for $\lambda > 0$. From (48)–(51), (54), (59), (74), (76) and (98), and the choices made in Section 11, it follows that there are some $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
\mathfrak{N}^\pm_{s,\mathrm{min/max}}(\lambda) &\le \#\left\{\, (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2\, \mu_{\mathrm{min/max},\ell} + C_3 \le \lambda \,\right\}\\
&\le \#\{\, (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2 C_0 \ell^{\tilde\theta/2} + C_3 \le \lambda \,\}\\
&\le \#\left\{\, (k,\ell) \in \mathbb{N}^2 \;\middle|\; \ell \le \left(\frac{\lambda - C_3}{C_2C_0} - \frac{C_1 k}{C_2 C_0}\right)^{2/\tilde\theta} \,\right\}\\
&\le \int_0^{\frac{\lambda - C_3}{C_1}} \left(\frac{\lambda - C_3}{C_2C_0} - \frac{C_1 x}{C_2 C_0}\right)^{2/\tilde\theta} dx\\
&= \frac{\tilde\theta(\lambda - C_3)^{(2+\tilde\theta)/\tilde\theta}}{(2+\tilde\theta)(C_2C_0)^{2/\tilde\theta} C_1}\,.
\end{aligned}$$

So $\mathfrak{N}^\pm_{s,\mathrm{min/max}}(\lambda) \le C\lambda^{(2+\tilde\theta)/\tilde\theta}$ for some $C > 0$ and all large enough λ, giving (viii) with $\theta = \frac{\tilde\theta}{2+\tilde\theta}$. □

Example 11.14. Consider the notation of Examples 2.6, 2.12 and 8.1. On the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$, the model rel-Morse function $\pm\frac{1}{2}\rho^2$ and the metric g_1 define the Witten's perturbed operators $d_s^\pm$, $\delta_s^\pm$, $D_s^\pm$ and $\Delta_s^\pm$. Since ρ_0 and g_0 respectively correspond to ρ and g_1 by $\mathrm{can} : \mathbb{S}^{m-1} \times \mathbb{R}_+ \to \mathbb{R}^m \setminus \{0\}$, it follows that $d_s^\pm$, $\delta_s^\pm$, $D_s^\pm$ and $\Delta_s^\pm$ respectively correspond to $d_{0,s}^\pm$, $\delta_{0,s}^\pm$, $D_{0,s}^\pm$, $\Delta_{0,s}^\pm$ by $\mathrm{can}^* : \Omega(\mathbb{R}^m \setminus \{0\}) \to \Omega(\mathbb{S}^{m-1} \times \mathbb{R}_+)$, and moreover

$$L^2\Omega(\mathbb{R}^m, g_0) \equiv L^2\Omega(\mathbb{R}^m \setminus \{0\}, g_0) \xrightarrow{\ \mathrm{can}^*\ } L^2\Omega(\mathbb{S}^{m-1} \times \mathbb{R}_+, g_1) \tag{99}$$

is a unitary isomorphism. The extension by zero defines a canonical injection $\Omega_0(\mathbb{R}^m \setminus \{0\}) \to \Omega_0(\mathbb{R}^m)$, whose composite with $(\mathrm{can}^*)^{-1}$ is an injective homomorphism of complexes, $(\Omega_0(\mathbb{S}^{m-1} \times \mathbb{R}_+), d_s^\pm) \to (\Omega_0(\mathbb{R}^m), d_{0,s}^\pm)$. Thus the unique i.b.c. of $(\bigwedge T\mathbb{R}^{m*}, d_{0,s}^\pm)$ in $L^2\Omega(\mathbb{R}^m, g_0)$ corresponds to $d_{s,\max}^\pm$ via (99).

If $m \ge 2$, then $H^{\frac{m-1}{2}}(\mathbb{S}^{m-1}) = 0$ for odd m. So $(\bigwedge T(\mathbb{S}^{m-1} \times \mathbb{R}_+)^*, d_s^\pm)$ has a unique i.b.c. by Corollaries 11.5 and 11.10, and Proposition 11.12.

If $m = 1$, then $\Omega(\mathbb{S}^0) = \Omega^0(\mathbb{S}^0) \equiv \mathbb{R}^2$, and therefore, according to (41), (42) and Corollary 9.5,

$$\Omega^0(\mathbb{S}^0 \times \mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+, \mathbb{R}^2) ,$$
$$\Omega^1(\mathbb{S}^0 \times \mathbb{R}_+) \equiv d\rho \wedge C^\infty(\mathbb{R}_+, \mathbb{R}^2) \equiv C^\infty(\mathbb{R}_+, \mathbb{R}^2) ,$$
$$d_s^\pm \equiv \frac{d}{d\rho} \pm s\rho , \quad \delta_s^\pm \equiv -\frac{d}{d\rho} \pm s\rho ,$$

giving $d_{s,\min}^\pm \neq d_{s,\max}^\pm$ by Proposition 7.4-(i).

12. Local model of the Witten's perturbation

The local model of our version of Morse functions around their critical points will be as follows. Let $m_\pm \in \mathbb{N}$, let $L_\pm$ be a compact Thom-Mather stratification, and let $M_\pm$ be a stratum in $c(L_\pm)$. Thus, either $M_\pm = N_\pm \times \mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$, or $M_\pm$ is the vertex stratum of $c(L_\pm)$. On the stratum $M = \mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$ of the Thom-Mather stratification $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times c(L_+) \times c(L_-)$ (for any choice of product Thom-Mather structure on $c(L_+) \times c(L_-)$), consider an adapted metric given as product of standard metrics on the Euclidean spaces $\mathbb{R}^{m_\pm}$ and model adapted metrics on the strata $M_\pm$. Let d_s denote the Witten's perturbed differential map on $\Omega(M)$ induced by the model rel-Morse function $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ (Remark 12-(iii)). Let $\Delta_{s,\min/\max}$ be the Laplacian defined by $d_{s,\min/\max}$, and $\mathcal{H}_{s,\min/\max} = \bigoplus_r \mathcal{H}_{s,\min/\max}^r = \ker \Delta_{s,\min/\max}$. The following result is a direct consequence of Example 8.1, Corollary 11.13 and Lemma 4.1.

Corollary 12.1. (i) *$d_{s,\min/\max}$ is discrete.*

(ii) *If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = N_- \times \mathbb{R}_+$, then*

$$\mathcal{H}_{s,\min/\max}^r \cong \bigoplus_{r_+,r_-} H_{\min/\max}^{r_+}(N_+) \otimes H_{\min/\max}^{r_-}(N_-) ,$$

where (r_+, r_-) runs in the subset of $\mathbb{Z}^2$ defined by (1)–(3).

(iii) *If M_+ is the vertex stratum of $c(L_+)$ and $M_- = N_- \times \mathbb{R}_+$, then*

$$\mathcal{H}_{s,\min/\max}^r \cong \bigoplus_{r_-} H_{\min/\max}^{r_-}(N_-) ,$$

where r_- runs in the subset of $\mathbb{Z}$ defined by $r = m_- + r_- + 1$ and (3).

(iv) *If $M_+ = N_+ \times \mathbb{R}_+$ and M_- is the vertex stratum of $c(L_+)$, then*

$$\mathcal{H}_{s,\min/\max}^r \cong \bigoplus_{r_+} H_{\min/\max}^{r_+}(N_+) ,$$

where r_+ runs in the subset of $\mathbb{Z}$ defined by $r = m_- + r_+$ and (2).

(v) *If M_+ and M_- are the vertex strata of $c(L_+)$ and $c(L_-)$, then we have $\dim \mathcal{H}_{s,\min/\max}^r = \delta_{r,m_-}$.*

(vi) *If $e_s \in \mathcal{H}_{s,\min/\max}$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle he_s^\pm, e_s^\pm \rangle \to 1$ as $s \to \infty$.*

(vii) *Let $0 \leq \lambda_{s,\min/\max,0} \leq \lambda_{s,\min/\max,1} \leq \cdots$ be the eigenvalues of $\Delta_{s,\min/\max}$, repeated according to their multiplicities.Given $k \in \mathbb{N}$, if $\lambda_{s,\min/\max,k} > 0$ for some s, then $\lambda_{s,\min/\max,k} \in O(s)$ as $s \to \infty$.*

(viii) *There is some $\theta > 0$ such that $\liminf_k \lambda_{s,\min/\max,k}\, k^{-\theta} > 0$.*

Remark 21. According to Example 11.14, except for the case $m = 1$ and $d_{s,\min}$, the above local study of $d_{s,\min/\max}$ could be simplified by using the homeomorphism $\mathrm{can} \times \mathrm{id} : \mathbb{R}^m \times c(L) \to c(\mathbb{S}^{m-1}) \times c(L)$ and an isomorphism $c(\mathbb{S}^{m-1}) \times c(L) \to c(L')$ for some compact Thom-Mather stratification L' (Section 2.1.3). This would allow to consider only a quasi-isometry $c(L') \to c(L'_+) \times c(L'_-)$ and the model rel-Morse function on $M'_+ \times M'_-$ for strata $M'_\pm$ of $c(L'_\pm)$. The factors $\mathbb{R}^{m_\pm}$ could be forgotten in this way.

13. Proof of Theorem 1.1

Consider the notation of Theorem 1.1: M be a stratum with compact closure of a Thom-Mather stratification A, and g is an adapted metric on M. Let $\{(O_a, \xi_a)\}$ be a finite covering of $\overline{M}$ by charts of A. For each a, we have $\xi_a(O_a) = B_a \times c_{\epsilon_a}(L_a)$, where B_a is an open subset of $\mathbb{R}^{m_a}$ for some $m_a \in \mathbb{N}$, L_a is a compact Thom-Mather stratification, and $\epsilon_a > 0$. Then each ξ_a defines an open embedding $M \cap O_a$ into $\mathbb{R}^{m_a} \times M_a$ for some stratum M_a of L_a. We have, either $M_a = N_a \times \mathbb{R}_+$ for some stratum N_a of L_a, or $M_a = \{*_a\}$, where $*_a$ is the vertex of $c(L_a)$. If $M_a = N_a \times \mathbb{R}_+$, then $\xi_a(M \cap O_a) = B_a \times N_a \times (0, \epsilon_a)$. If $M_a = \{*_a\}$, then $\xi_a(M \cap O_a) = B_a \times \{*_a\} \equiv B_a$. Thus every $\xi_a(M \cap O_a)$ is, either open in $\mathbb{R}^{m_a}$, or open in $\mathbb{R}^{m_a} \times N_a \times \mathbb{R}_+$. By shrinking $\{(O_a, \xi_a)\}$ if necessary, we can assume that each diffeomorphism $\xi_a : M \cap O_a \to \xi_a(M \cap O_a)$ is quasi-isometric with respect to a model adapted metric on $\mathbb{R}^{m_a} \times M_a$.

By Lemma 3.4, there is a smooth partition of unity $\{\lambda_a\}$ on M subordinated to the open covering $\{M \cap O_a\}$ such that each function $|d\lambda_a|$ is rel-locally bounded; indeed, by shrinking $\{(O_a, \xi_a)\}$ again if necessary, we can assume that each $|d_a\lambda_a|$ is bounded. Also, by using Example 3.2, it is easy to construct another family $\{\tilde\lambda_a\} \subset C^\infty(M)$ such that $\tilde\lambda_a$ and $|d\tilde\lambda_a|$ are bounded, $\tilde\lambda_a = 1$ on $\operatorname{supp}\lambda_a$, and $\operatorname{supp}\tilde\lambda_a \subset M \cap O_a$. The existence of such families $\{\lambda_a\}$ and $\{\tilde\lambda_a\}$ is required to apply Propositions 5.5 and 5.6.

Let $d_{a,s}$ be the Witten's perturbation of d_a induced by the function $f_a = \frac{1}{2}\rho_a^2$ on $\mathbb{R}^{m_a} \times M_a$, where ρ_a is the canonical function of $\mathbb{R}^{m_a} \times c(L_a)$. According to Corollary 12.1-(i),(viii), each $d_{a,s,\min/\max}$ satisfies the properties stated in Theorem 1.1, and let $\Delta_{a,s,\min/\max}$ denote the corresponding Laplacian.

By using Example 3.2 again, it is easy to see that there is some rel-admissible function h_a on $\mathbb{R}^{m_a} \times M_a$ such that $h_a = 0$ on $\xi(M \cap O_a)$ and $h_a = 1$ on the complement of some rel-compact neighborhood of $\xi(M \cap O_a)$ in $\mathbb{R}^m \times M_a$. Let $\hat d_{a,s}$ and $\widehat\Delta_{a,s}$ be the Witten's perturbation of d_a and Δ_a induced by the function $\hat f_a = h_a f_a$. The functions $|d_a \hat f_a - d_a f_a|$ and $|\operatorname{Hess} \hat f_a - \operatorname{Hess} f_a|$ are uniformly bounded, and therefore $\widehat\Delta_{a,s} - \Delta_{a,s}$ is a homomorphism with uniformly bounded norm by (39). By the min-max principle (see e.g. [32, Theorem XIII.1]), we get that $\hat d_{a,s,\min/\max}$ satisfies the properties stated in Theorem 1.1. Then Theorem 1.1 follows by Propositions 5.5 and 5.6.

14. Functions of the perturbed Laplacian on strata

The first ingredient of Theorem 1.2 is the following properties of the functional calculus of the perturbed Laplacian on strata.

Let M be a stratum of a compact Thom-Mather stratification endowed with an adapted metric, and let d and Δ be the de Rham derivative and Laplacian on M.

Let f be any rel-admissible function on M, and let d_s and Δ_s be the corresponding Witten's perturbations of d and Δ. Since f is rel-admissible, for each s, $\Delta_s - \Delta$ is a homomorphism with uniformly bounded norm by (39). Hence $d_{s,\min/\max}$ defines the same Sobolev spaces as $d_{\min/\max}$. Moreover the properties stated in Theorem 1.1 can be extended to the perturbation $d_{s,\min/\max}$ by (39) and the min-max principle.

For any rapidly decreasing function ϕ on $\mathbb{R}$, we easily get that $\phi(\Delta_{s,\min/\max})$ is a Hilbert-Schmidt operator on $L^2\Omega(M)$ by the version of Theorem 1.1-(ii) for $d_{s,\min/\max}$. In fact, $\phi(\Delta_{s,\min/\max})$ is a trace class operator because ϕ can be given as the product of two rapidly decreasing functions, $|\phi|^{1/2}$ and $\operatorname{sign}(\phi)\,|\phi|^{1/2}$, where $\operatorname{sign}(\phi)(x) = \operatorname{sign}(\phi(x)) \in \{\pm 1\}$ if $\phi(x) \neq 0$.

The extension of Theorem 1.1-(ii) to $d_{s,\min/\max}$ also shows that $\phi(\Delta_{s,\min/\max})$ is valued in $W^\infty(d_{\min/\max})$. However we do not have a "rel-Sobolev embedding theorem" describing $W^\infty(d_{\min/\max})$; for instance, we do not know whether the elements of $W^m(d_{\min/\max})$ are uniformly bounded for m large enough (see Section 17). We can only assert that $W^\infty(d_{\min/\max}) \subset \Omega(M)$ by the usual elliptic regularity.

Like in the case of closed manifolds (see e.g. [33, Chapters 5 and 8]), it can be easily proved that $\phi(\Delta_{s,\min/\max})$ can be given by a Schwartz kernel K, and $\operatorname{Tr}\phi(\Delta_{s,\min/\max})$ equals the integral of the pointwise trace of K on the diagonal. But we do not know whether K is uniformly bounded by the indicated lack of a "rel-Sobolev embedding theorem".

15. Finite propagation speed of the wave equation on strata

Let M be a stratum of a compact Thom-Mather stratification, g an adapted metric on M, and f a rel-Morse function on M. Let d_s, δ_s, D_s and Δ_s $(s \geq 0)$ be the corresponding Witten's perturbed operators on $\Omega(M)$, defined by f and g. These operators make sense on complex valued differential forms as well as real valued ones. Complex coefficients are needed to consider the induced wave equation

$$\frac{d\alpha_t}{dt} - iD_s\alpha_t = 0 , \tag{100}$$

where $i = \sqrt{-1}$ and $\alpha_t \in \Omega(M)$ depends smoothly on $t \in \mathbb{R}$. We may also consider that (100) is satisfied only on some open subset of M.

If (100) holds on the whole of M, then, given $\alpha \in \mathcal{D}^\infty(d_{s,\min/\max})$, a usual energy estimate shows the uniqueness of the solution of (100) with the initial conditions $\alpha_0 = \alpha$ (see e.g. [33, Proposition 7.4]). In this case the solution is given by

$$\alpha_t = \exp(itD_{s,\min/\max})\alpha ,$$

which belongs to $\mathcal{D}^\infty(d_{s,\min/\max})$ for all t.

It is known that compactly supported smooth solutions of (100) propagate at finite speed (see e.g. [33, Proposition 7.20]). To prove Theorem 1.2, we need a version of that result for strata, stating this finite propagation speed towards/from the rel-critical points of f with forms in $\mathcal{D}^\infty(d_{s,\min/\max})$. For that purpose, we show first the corresponding result for the simple elliptic complexes of Sections 7.2 and 7.3.

Take a rel-Morse chart around each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, like in Definition 3.7, with values in a stratum $M'_x = \mathbb{R}^{m_{x,+}} \times \mathbb{R}^{m_{x,-}} \times M_{x,+} \times M_{x,-}$ of a product $\mathbb{R}^{m_{x,+}} \times \mathbb{R}^{m_{x,-}} \times c(L_{x,+}) \times c(L_{x,-})$, where either $M_{x,\pm} = N_{x,\pm} \times \mathbb{R}_+$, or $M_{x,\pm}$ is the vertex stratum of $c(L_{x,\pm})$. We can assume that the domains of these rel-Morse charts are disjoint one another. Consider a model metric g_x on each M'_x. For each $\rho > 0$, let $B_{x,\pm,\rho}$ be the standard ball of radius ρ in $\mathbb{R}^{m_{x,\pm}}$. If $M_{x,+} = N_{x,+} \times \mathbb{R}_+$ and $M_{x,\pm} = N_{x,-} \times \mathbb{R}_+$, let

$$U_{x,\rho} = B_{x,+,\rho} \times B_{x,-,\rho} \times N_{x,+} \times (0,\rho) \times N_{x,-} \times (0,\rho) \subset M'_x .$$

If $M_{x,\pm}$ is the vertex stratum, remove the factor $N_{x,\pm} \times (0,\rho)$ from the definition of $U_{x,\rho}$ (or change it by the corresponding vertex stratum). Let $d'_{x,s}$, $\delta'_{x,s}$, $D'_{x,s}$ and $\Delta'_{x,s}$ denote Witten's perturbed operators on $\Omega(M'_x)$ defined by g_x and the model rel-Morse function (Section 12). The corresponding wave equation is

$$\frac{d\alpha_t}{dt} - iD'_{x,s}\alpha_t = 0 , \tag{101}$$

with $\alpha_t \in \Omega(M'_x)$ depending smoothly on $t \in \mathbb{R}$. By Propositions 11.3, 11.9 and 11.12, the following result clearly boils down to the case of Proposition 7.7.

Proposition 15.1. *For $0 < a < b$, suppose that $\alpha_t \in \mathcal{D}^\infty(d'_{x,s,\min/\max})$, depending smoothly on $t \in \mathbb{R}$, satisfies (101) on $U_{x,b}$. The following properties hold:*

(i) *If $\operatorname{supp}\alpha_0 \subset M'_x \setminus U_{x,a}$, then $\operatorname{supp}\alpha_t \subset M'_x \setminus U_{x,a-|t|}$ for $0 < |t| \le a$.*
(ii) *If $\operatorname{supp}\alpha_0 \subset \overline{U_{x,a}}$, then $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$.*

There is some $\rho_0 > 0$ such that each $\overline{U_{x,\rho_0}}$ is contained in the image of the rel-Morse chart centered at x, and moreover these charts are disjoint one another. We will identify each U_{x,ρ_0} with an open subset of M via the rel-Morse chart. According to Example 2.11, we can choose g so that its restriction to each U_{x,ρ_0} is identified to the restriction of g_x.

Proposition 15.2. *Let $0 < a < b < \rho_0$ and $\alpha \in L^2\Omega(M)$. The following properties hold for $\alpha_t = \exp(itD_{s,\min/\max})\alpha$:*

(i) *If $\operatorname{supp}\alpha \subset M \setminus U_{x,a}$, then $\operatorname{supp}\alpha_t \subset M \setminus U_{x,a-|t|}$ for $0 < |t| \le a$.*
(ii) *If $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$, then $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$.*

Proof. Since $\exp(itD_{s,\min/\max})$ is bounded, we can assume that $\alpha \in \mathcal{D}^\infty(d_{s,\min/\max})$, and therefore $\alpha_t \in \mathcal{D}^\infty(d_{s,\min/\max})$ for all t. According to Remark 20, there is some $h \in C^\infty(M)$ such that $\operatorname{supp} h \subset U_{x,\rho_0}$, $h = 1$ on $U_{x,b}$, and $h\,\mathcal{D}^\infty(d_{s,\min/\max}) \subset \mathcal{D}^\infty(d_{s,\min/\max})$. Then $h\alpha_t$ satisfies (101) on $U_{x,b}$ and belongs to $\mathcal{D}^\infty(d'_{s,\min/\max})$. So, by Proposition 15.1,

- $h\alpha_t = 0$ on $\overline{U_{x,a-|t|}}$ for $0 < |t| \le a$ if $\operatorname{supp}\alpha \subset M \setminus U_{x,a}$, and
- $\operatorname{supp} h\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$ if $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$.

Thus the result follows because $h = 1$ on $U_{x,b}$. □

16. Proof of Theorem 1.2

Consider the notation of Section 15.

16.1. **Analytic inequalities.** By (36), we have the isomorphism of complexes $e^{sf} : (\Omega_0(M), d_s) \to (\Omega_0(M), d)$. Since f is bounded, we also have the quasi-isometric isomorphism $e^{sf} : L^2\Omega(M) \to L^2\Omega(M)$. So we obtain the isomorphism of Hilbert complexes

$$e^{sf} : (\mathcal{D}(d_{s,\min/\max}), d_{s,\min/\max}) \to (\mathcal{D}(d_{\min/\max}), d_{\min/\max}) ,$$

and therefore

$$\beta^r_{\min/\max} = \dim H^r(\mathcal{D}(d_{s,\min/\max}), d_{s,\min/\max}) \tag{102}$$

for all $s \geq 0$. In fact, since $|df|$ is bounded, it also follows from (36) that

$$\mathcal{D}(d_{s,\min/\max}) = \mathcal{D}(d_{\min/\max}) , \quad d_{s,\min/\max} = d_{\min/\max} + s\, df \wedge \ .$$

Thus

$$e^{sf}\, \mathcal{D}(d_{\min/\max}) = \mathcal{D}(d_{\min/\max}) \ .$$

Let ϕ be a smooth rapidly decreasing function on $\mathbb{R}$ with $\phi(0) = 1$. Then the operator $\phi(\Delta_{s,\min/\max})$ is of trace class (Section 14), and set

$$\mu^r_{s,\min/\max} = \mathrm{Tr}(\phi(\Delta_{s,\min/\max,r})) \ .$$

By (102), the following result follows with the obvious adaptation of the proof of [33, Proposition 14.3].

Proposition 16.1. *We have the inequalities*

$$\begin{gathered}
\beta^0_{\min/\max} \leq \mu^0_{\min/\max} \ , \\
\beta^1_{\min/\max} - \beta^0_{\min/\max} \leq \mu^1_{s,\min/\max} - \mu^0_{s,\min/\max} \ , \\
\beta^2_{\min/\max} - \beta^1_{\min/\max} + \beta^0_{\min/\max} \leq \mu^2_{s,\min/\max} - \mu^1_{s,\min/\max} + \mu^0_{s,\min/\max} \ ,
\end{gathered}$$

etc., and the equality

$$\chi_{\min/\max} = \sum_r (-1)^r\, \mu^r_{s,\min/\max} \ .$$

16.2. **Null contribution away from the critical points.** By (39) and because $|df|$ and $|\,\mathrm{Hess}\, f|$ are bounded on M, we have

$$\mathcal{D}(\Delta_{s,\min/\max}) = \mathcal{D}(\Delta_{\min/\max}) \ , \tag{103}$$

$$\Delta_{s,\min/\max} = \Delta_{\min/\max} + s\, \mathbf{Hess} f + s^2\, |df|^2 \tag{104}$$

for all $s \geq 0$.

For $\rho \leq \rho_0$, let $U_\rho = \bigcup_x U_{x,\rho}$, with x running in $\mathrm{Crit}_{\mathrm{rel}}(f)$. Fix some $\rho_1 > 0$ such that $3\rho_1 < \rho_0$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Hilbert subspaces of $L^2\Omega(M)$ consisting of forms essentially supported in $M \setminus U_{\rho_1}$ and $M \setminus U_{2\rho_1}$, respectively. It follows from (103) and (104) that there is some $C > 0$ such that[15]

$$\Delta_{s,\min/\max} \geq \Delta_{\min/\max} + Cs^2 \quad \text{on} \quad \mathfrak{G} \cap \mathcal{D}(\Delta_{\min/\max}) \tag{105}$$

if s is large enough.

Let h be a rel-admissible function on M such that $h \leq 0$, $h \equiv 1$ on U_{ρ_0} and $h \equiv 0$ on $M \setminus U_{2\rho_1}$ (see Example 3.2). Then $T_{s,\min/\max} = \Delta_{s,\min/\max} + hCs^2$, with domain $\mathcal{D}(\widetilde{\Delta}_{\min/\max})$, is essentially self-adjoint in $L^2\Omega(M)$ with a discrete spectrum, and moreover

$$T_{s,\min/\max} \geq \Delta_{\min/\max} + Cs^2 \tag{106}$$

for s is large enough by (105).

Fix some[16] $\phi \in \mathcal{S}_{\mathrm{ev}}$ such that $\phi \geq 0$, $\phi(0) = 1$ and $\mathrm{supp}\, \hat{\phi} \subset [-\rho_1, \rho_1]$, and let $\psi \in \mathcal{S}$ such that $\phi(x) = \psi(x^2)$. By using Proposition 15.2-(i), the argument of the

[15] Recall that, for symmetric operators S and T in a Hilbert space, with the same domain $\mathcal{D}$, it is said that $S \leq T$ if $\langle Su, u\rangle \leq \langle Tu, u\rangle$ for all $u \in \mathcal{D}$.

[16] The Schwartz functions with compactly supported Fourier transform are characterized by the Paley-Wiener-Schwartz theorem (see e.g. [20, Theorem 7.3.1]); they form a dense subalgebra of $\mathcal{S}$, which is invariant by linear changes of variables.

first part of the proof of [33, Lemma 14.6] can be obviously adapted to show the following.

Lemma 16.2. $\psi(\Delta_{s,\min/\max}) = \psi(T_{s,\min/\max})$ *on* $\mathfrak{H}$.

Let $\Pi : L^2\Omega(M) \to \mathfrak{H}$ denote the orthogonal projection. According to Section 14, $\psi(\Delta_{s,\min/\max})$ is of trace class for all $s \geq 0$. Then the self-adjoint operator $\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi$ is also of trace class (see e.g. [33, Proposition 8.8]).

Lemma 16.3. $\operatorname{Tr}(\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi) \to 0$ *as* $s \to \infty$.

Proof. Let

$$0 \leq \lambda_{\min/\max,0} \leq \lambda_{\min/\max,1} \leq \cdots\ , \quad 0 \leq \lambda_{s,\min/\max,0} \leq \lambda_{s,\min/\max,1} \leq \cdots$$

be the eigenvalues of $\Delta_{\min/\max}$ and $T_{s,\min/\max}$, respectively, repeated according to their multiplicities. By (106) and the min-max principle, we have

$$\lambda_{s,\min/\max,k} \geq \lambda_{\min/\max,k} + Cs^2$$

for s large enough. So

$$\operatorname{Tr}(\psi(T_{s,\min/\max})) = \sum_k \psi(\lambda_{s,\min/\max,k}) \leq \sum_k \psi(\lambda_{\min/\max,k} + Cs^2)$$

for s large enough, giving $\operatorname{Tr}(\psi(T_{s,\min/\max})) \to 0$ as $s \to \infty$ since ψ is rapidly decreasing. Then the result follows because

$$\operatorname{Tr}(\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi) = \operatorname{Tr}(\Pi\,\psi(T_{s,\min/\max})\,\Pi) \leq \operatorname{Tr}(\psi(T_{s,\min/\max}))$$

by Lemma 16.2. □

16.3. **Contribution from the rel-critical points.** The following is a direct consequence of Corollary 12.1.

Corollary 16.4. *If* h *is a bounded measurable function on* $\mathbb{R}_+$ *such that* $h(\rho) \to 1$ *as* $\rho \to 0$*, then*

$$\lim_{s\to\infty} \operatorname{Tr}(h(\rho)\,\phi(\Delta'_{x,s,\min/\max,r})) = \lim_{s\to\infty} \operatorname{Tr}\phi(\Delta'_{x,s,\min/\max,r}) = \nu^r_{x,\min/\max}\,.$$

For each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\mathfrak{H}}_x \subset L^2\Omega(M)$ be the Hilbert subspace of differential forms supported in $\overline{U_{x,2\rho_1}}$; it can be also considered as a Hilbert subspace of $L^2\Omega(M'_x)$ since g and g_x have identical restrictions to U_{x,ρ_0}. Moreover Δ_s and $\Delta'_{x,s}$ can be identified on differential forms supported in U_{x,ρ_0}. By using Proposition 15.2-(ii), the argument of the first part of the proof of [33, Lemma 14.6] can be obviously adapted to show the following.

Lemma 16.5. $\phi(\Delta_{s,\min/\max}) \equiv \phi(\Delta'_{x,s,\min/\max})$ *on* $\widetilde{\mathfrak{H}}_x$ *for all* $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$.

For each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}}_x$ and $\widetilde{\Pi}'_x : L^2\Omega(M'_x) \to \widetilde{\mathfrak{H}}_x$ denote the orthogonal projections. Since the subspaces $\widetilde{\mathfrak{H}}_x$ are orthogonal to each other, $\Pi := \sum_x \widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}} := \sum_x \widetilde{\mathfrak{H}}_x$ is the orthogonal projection.

Lemma 16.6. $\operatorname{Tr}(\widetilde{\Pi}\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}) \to \nu^r_{\min/\max}$ *as* $s \to \infty$.

Proof. By Corollary 16.4 and Lemma 16.5, and because Π'_x is the multiplication operator by the characteristic function of U_{ρ_1} in M'_x,

$$\begin{aligned}\lim_{s\to\infty}\operatorname{Tr}(\widetilde{\Pi}\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}) &= \lim_{s\to\infty}\sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)}\operatorname{Tr}(\widetilde{\Pi}_x\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}_x)\\ &= \lim_{s\to\infty}\sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)}\operatorname{Tr}(\widetilde{\Pi}'_x\,\phi(\Delta'_{x,s,\min/\max,r})\,\widetilde{\Pi}'_x)\\ &= \sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)}\nu^r_{x,\min/\max} = \nu^r_{\min/\max}\,. \qquad \square\end{aligned}$$

Now,

$$\lim_{s\to\infty}\operatorname{Tr}(\phi(\Delta_{s,\min/\max,r})) = \nu^r_{\min/\max}$$

by Lemmas 16.3 and 16.6, and because $\Pi+\widetilde{\Pi}=1$, showing Theorem 1.2 by Proposition 16.1.

17. Remark on the Sobolev spaces on strata

Our version of the Sobolev spaces on strata, $W^m(d_{\min/\max})$, may depend on the chosen adapted metric; thus there is no "rel-version" of the elliptic estimate. By taking local charts and arguing like in Section 13, it is enough to check this assertion for the perturbed local models $d^{\pm}_{s,\min/\max}$.

With the notation of Section 10.1, consider the case where n is odd, $r=\frac{n-1}{2}$ and $a=0$; thus $\sigma=0$. We have $\chi_0\,\gamma\in W^\infty(d^{\pm}_{s,\min/\max})$ with the metric g. Let $\tilde g'$ be another adapted metric on N such that $\widetilde{\Delta}'\gamma\neq 0$, and consider the corresponding adapted metric $g'=\rho^{-2}\tilde g'+d\rho^2$ on M. Let $\widetilde{\Delta}'$ be the laplacian on $\Omega(N)$ defined by $\tilde g'$, Δ' the Laplacian on $\Omega(M)$ defined by g', and $\Delta'^{\pm}_s$ the Witten's perturbation of Δ' induced by the function $\pm\frac{1}{2}\rho^2$. Let $\langle\ ,\ \rangle^{\sim\prime}$ and $\langle\ ,\ \rangle'$ denote the scalar products of $L^2\Omega(N,\tilde g')$ and $L^2\Omega(M,g')$, respectively, and let $\|\ \|^{\sim\prime}$ denote the norm defined by $\langle\ ,\ \rangle^{\sim\prime}$. By Corollary 9.7, we have $\Delta'^{\pm}_s=\rho^{-2}\widetilde{\Delta}'+H\mp s$ on $C^\infty(\mathbb{R}_+)\,\gamma$. Then

$$\langle\Delta'^{\pm}_s(\chi_0\,\gamma),\chi_0\gamma\rangle' = \langle\widetilde{\Delta}'\gamma,\gamma\rangle^{\sim\prime}\int_0^\infty\rho^{-2}\chi_0^2\,d\rho+\|\gamma\|^{\sim\prime 2}(1\mp 1)s=\infty$$

according to (45) and Section 10.1, and because $\chi_0(\rho)=\sqrt{2}p_0e^{-s\rho^2/2}$ is bounded away from zero for $0<\rho\le 1$. So $\chi_0\,\gamma\notin W^1(d^{\pm}_{s,\min/\max})$ with the metric g', obtaining different spaces $W^1(d^{\pm}_{s,\min/\max})$ by using g and g'.

The above observation is related with the following problem.

Problem 17.1. Let M be a stratum of an arbitrary compact stratification endowed with an adapted metric, and let $L^1\Omega(M)$ denote the Banach space of uniformly bounded measurable differential forms on M. Is there a continuous inclusion of $W^m(d_{\min/\max})$ into $L^1\Omega(M)$ for m large enough?

For the perturbation P of harmonic oscillator indicated in Section 6, the corresponding version of this problem has an affirmative answer [1]. If the spaces $W^m(d_{\min/\max})$ were independent of the adapted metric, we could give an affirmative answer to Problem 17.1 by using the local arguments of this paper and induction. An affirmative solution of Problem 17.1 would allow to adapt the nice arguments of [33, Lemma 14.6] to show a stronger version of Lemma 16.3: the Schwartz kernel of $\psi(\Delta_{s,\min/\max})$ would converge uniformly to zero on $(M\setminus U_{2\rho_1})\times(M\setminus U_{2\rho_1})$.

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