arXiv:1205.0334v1 [math.DG] 2 May 2012

A NO BREATHERS THEOREM FOR SOME NONCOMPACT RICCI FLOWS

QI S. ZHANG

Abstract. Under suitable conditions near infinity and assuming boundedness of curvature tensor, we prove a no breathers theorem in the spirit of Ivey-Perelman for some noncompact Ricci flows. These include Ricci flows on asymptotically flat (AF) manifolds with positive scalar curvature. Since the method for the compact case faces a difficulty, the proof involves solving a new non-local elliptic equation which is the Euler-Lagrange equation of a scaling invariant log Sobolev inequality.

1. Statement of result

A basic question in the study of the Ricci flow is: Are periodic orbits called breathers trivial? Here triviality means that metrics only move by diffeomorphisms and scaling through out the period. A Ricci flow $(M, g(t))$, $t \in [t_1, t_2]$, is called a breather if there is a positive constant c and a diffeomophism Ψ on M so that $g(t_2) = c\Psi^*(g(t_1))$. Perelman's no breathers theorems ([P] Sections 2, 3) say that all periodical solutions of compact Ricci flows are gradient Ricci solitons, and hence trivial in certain sense. See also earlier proof of this result by Ivey [I] in three dimension case, and [Ca] and [L] for further development on compact breathers.

The purpose of this paper is to prove a no breathers theorem for some noncompact Ricci flows. Some times an extension of a theorem from the compact case to a noncompact one merely involves some technical improvements of the method, plus some extra conditions near infinity. However the no breathers theorem is different for two reasons. First, noncompact Ricci flows arise naturally as the blow up limits of finite time singularity of compact Ricci flows. In fact, most of the essential singularity models for compact Ricci flows are noncompact. This includes the well known cylinder $S^2 \times \mathbf{R}$ in the 3 dimensional case. Thus even if one is only interested in compact Ricci flows, one still needs to study noncompact Ricci flows. Second, the method of proof by Perelman for the no breathers theorem does not seem to work for the noncompact case, especially for the steady breather case. Recall that Perelman introduces the F functional which is defined as $F(v) = \int_M (4|\nabla v|^2 + Rv^2) dg$ where R is the scalar curvature of the manifold and $v \in W^{1,2}(M)$ and $\|v\|_{L^2(M,g)} = 1$. He proved that the infimum of F is a nondecreasing function of time along a Ricci flow $(M, g(t))$; moreover it is a constant if and only if the Ricci flow is a steady gradient soliton. Using the fact that the infimum is reached by a minimizer when M is compact, Perelman proved that there is no nontrivial steady breathers for compact Ricci flows, i.e. a steady breather is necessarily a steady gradient soliton. If one attempts to extend this argument to noncompact Ricci flow, one faces an immediate difficulty. Namely, the infimum of the F functional is not reached by a function on a typical noncompact manifold such as $\mathbf{R^n}$ or

Date: May 2012.

$S^2 \times \mathbf{R}$. In fact, on $\mathbf{R^n}$, the F functional is nothing but the Dirichlet energy (multiplied by 4) and it is well known that there is no L^2 minimizer. For this reason, we need to look for a different method.

In this paper, we consider the functional (1.1). When the parameter $\alpha = 1$, it is the limiting case of Perelman's W entropy and which can be regarded as a scaling invariant version of the Log Sobolev inequality introduced by Weissler [W]. The corresponding Euler-Lagrange equation is a nonlocal, nonlinear elliptic equation. Unlike the F functional, the minimizer of (1.1) exists on many typical noncompact manifolds. Using this we prove a no breathers theorem on some noncompact Ricci flows. The study of the functional (1.1) and its minimizer equation potentially has further applications.

Let's introduce notations and definitions to be used in the paper. We use M to denote a $n(\ge 3)$ dimensional Riemann manifold and $g(t)$ to denote the metric at time t; $d(x, y, t)$ is the geodesic distance under $g(t)$; Unless stated otherwise, we assume the curvature tensor is bounded at each time t. $B(x, r, g(t)) = \{y \in \mathbf{M} \mid d(x, y, t) < r\}$ is the geodesic ball of radius r, under metric $g(t)$, centered at x, and $|B(x, r, t)|_{g(t)}$ is the volume of $B(x, r, t)$ under $g(t)$; when no confusion arises we may also use $B(x, r)$ or $B(x, r, t)$ to denote $B(x, r, g(t))$; $dg(t)$ is the volume element; x_0 is a reference point on M. We also reserve $R = R(x, t)$ as the scalar curvature under $g(t)$. A generic positive constant is denoted by C or c whose value may change from line to line.

Definition 1.1. *(Log Sobolev functionals, infimum, infimum at infinity) Let (M, g) be a n dimensional Riemann manifold with metric g and $D \subset M$ be a domain.*

(a). Given functions $v \in W_0^{1,2}(D, g)$ with $\|v\|_{L^2(D)} = 1$, and a number $\alpha \ge 1$, the log Sobolev functionals with parameter α is defined by

$$
\begin{aligned}
L(v, g, \alpha, D) &= -\int_D v^2 \ln v^2 dg + \alpha \frac{n}{2} \ln \left(\int_D (4|\nabla v|^2 + R v^2) dg \right) + s_n \\
&\equiv -N(v) + \alpha \frac{n}{2} \ln F(v) + s_n.
\end{aligned}
\tag{1.1}
$$

Here R is the scalar curvature; $s_n = -\frac{n}{2} \ln(2\pi n) - \frac{n}{2}$.

(b). The infimum of the log Sobolev functional is denoted by

$$\lambda(g, \alpha, D) = \inf\{L(v, g, \alpha, D) \mid v \in W_0^{1,2}(D, g), \quad \|v\|_{L^2(D)} = 1\}.$$

(c). When $\alpha = 1$ and $D = M$, the infimum of the log Sobolev functional at infinity is

$$\lambda_\infty(g, 1, M) = \lim_{r \to \infty} \lambda(g, 1, M - B(x_0, r))$$

where x_0 is a reference point in M.

If $D = M$, then for simplicity we write

$$L(v, g, \alpha) = L(v, g, \alpha, M), \qquad \lambda(g, \alpha) = \lambda(g, \alpha, M).$$

If $\alpha = 1$, we may suppress α and write

$$L(v, g) = L(v, g, 1), \qquad \lambda = \lambda(g) = \lambda(g, 1) = \lambda(g, 1, M) \qquad \lambda_\infty = \lambda_\infty(g) = \lambda_\infty(g, 1, M).$$

Remark 1.1. *When $M = \mathbf{R^n}$ and $\alpha = 1$, then $L(v, g)$ is the log Sobolev functional introduced by Weissler* [W]*, which is a scaling invariant version of the log Sobolev functional originally introduced by Gross* [G] *and Federbush* [F]*. Observe that $\lambda(g)$ is invariant under scaling and diffeomorphism. See the beginning of proof of Theorem 1.1 below.*

$\lambda(g)$ is related to Perelman's ν invariant in Section 3 of [P]*. We are not sure if they are the same.*

Definition 1.2. *(gradient Ricci solitons) A Riemann manifold (M, g) is called a gradient Ricci soliton if there exists a smooth function f on M and a constant ϵ such that*

$$Ric + Hess f + \frac{\epsilon}{2} g = 0.$$

(M, g) is called a expanding, steady and shrinking gradient Ricci soliton if $\epsilon > 0, \epsilon = 0$ and $\epsilon < 0$ respectively.

The following is the main result of the paper.

Theorem 1.1. *Let $(M, g(t))$, $\partial_t g_{ij} = -2R_{ij}$, $t \in [0, T]$ be a complete, noncompact Ricci flow with bounded curvature tensor and nonnegative scalar curvature. Suppose $(M, g(t))$ is a breather, i.e. for two moments $t_1, t_2 \in [0, T]$, $t_1 < t_2$, there is a positive constant c such that $(M, cg(t_1))$ and $(M, g(t_2))$ differ only by diffeomorphism.*

Suppose also the following conditions hold.

(a) $-\infty < \lambda(g(t_1)) < \lambda_\infty(g(t_1))$.

(b) Either $|B(x_0, r, t_1)|_{g(t_1)} \le Cr^n$, for some $C > 0$ and all $r > 0$, or $R(x, t_1) \ge \frac{C}{1+d(x,x_0,t_1)^2}$ for some constant $C > 0$.

Then $(M, g(t))$ is a gradient Ricci soliton.

Naturally one is obliged to present some examples of Ricci flows where the conditions of the theorem is met. Condition (a) is easy to be met since one can modify the metric on a compact domain of a manifold so that $\lambda(g)$ becomes arbitrarily negative, while $\lambda_\infty(g)$ remains the same. Let x_0 be a reference point, we can construct a metric $g(t_1)$ such that the volume of the unit ball $B(x_0, 1)$ is very small but the scalar curvature is bounded by 1. A flat cylinder with small aperture is such an example. So given a positive number κ, the manifold is κ collapsed at scale 1. Hence $\lambda(g(t_1))$ is very negative. Indeed, by Proposition 2.4, if $\lambda(g(t_1)) > -C > -\infty$, then $(M, g(t_1))$ is κ non-collapsed below scale 1. Here C depends on κ. But $\lambda_\infty(g(t_1))$ is totally independent of $\lambda(g(t_1))$.

Condition (b) is satisfied automatically by ancient κ solutions of 3 dimensional Ricci flow, which include gradient shrinking solitons with nonnegative sectional curvature. See [P] and [P2].

Another type of examples is the Ricci flow on asymptotically flat (AF) manifolds (c.f. Definition 2.1), which is interesting due to connections to general relativity. Useful properties of these kind of Ricci flows have bee proven in [DM], [OW]. For example, they proved that the AF property is preserved under Ricci flow.

Corollary 1.1. *Let $(M, g(t))$ be a Ricci flow on an asymptotically flat manifold with positive scalar curvature. If $(M, g(t))$ is a breather then it is a gradient Ricci soliton.*

Proof. By Proposition 2.1 (a), we know $\lambda(M, g(t)) > -\infty$. If $(M, g(t))$ is a gradient Ricci soliton, then the proof is done. Otherwise, applying Proposition 2.1 (b) and Proposition 2.2, we find that $\lambda(M, g(t)) < 0 \le \lambda_\infty(M, g(t))$. By Definition of AF manifolds, we also have $|B(x_0, r, t)|_{g(t)} \le Cr^n$, for some $C = C(t) > 0$. Therefore, all the conditions of the theorem are satisfied and the conclusion follows. □

Remark 1.2. *In a recent paper* [Ha]*, Haslhofer considered Ricci flows on some AF manifolds with positive scalar curvature. Under the extra assumption that the scalar curvature is integrable, he modified the domain of Perelman's F entropy to include only smooth functions converging to* 1 *sufficiently fast at infinity. Using the monotonicity of this modified F entropy, one can also prove that steady breathers are steady solitons in this case, under further assumptions near infinity on the diffeomorphism in the definition of breathers. Also a no breather theorem for some noncompact Ricci flows in the case of shrinking solitons is proven in* [Z2].

Remark 1.3. *One may wonder if a no breathers theorem still holds when the scalar curvature changes sign. When the operator $-\Delta + R$ has a negative eigenvalue, under mild assumptions near infinity, one can prove that the eigenfunction decays to zero exponentially fast. Then one can use Perelman's original method described earlier to prove that steady breathers are steady gradient solitons. However, steady gradient solitons are ancient solutions. According to* [Ch]*, the scalar curvature is nonnegative. So the operator $-\Delta + R$ can not have negative eigenvalue. This contradiction shows that no steady breathers exist in this case.*

Let us outline the proof of the theorem. The main hurdle is to prove the following theorem which states that the infimum of the functional $L(v, g(t_2), 1, M)$ is reached by a smooth function in $W^{1,2}(M, g(t_2))$.

Theorem 1.2. *Let (M, g) be a noncompact manifold with bounded curvature and nonnegative scalar curvature, which also satisfies*

(a) $-\infty < \lambda(g) < \lambda_\infty(g)$.

(b) Either $|B(x_0, r, t_1)|_{g(t_1)} \le C r^n$, for some $C > 0$ and all $r > 0$, or $R(x, t_1) \ge \frac{C}{1+d(x,x_0,t_1)^2}$ for some constant $C > 0$.

Then there exists a minimizer v for the Log Sobolev functional $L(\cdot, g, 1, M)$, which satisfies the equation

(1.2)
$$\frac{n}{2}\frac{4\Delta v - Rv}{\int (4|\nabla v|^2 + Rv^2) dg} + 2v \ln v + \left(\lambda(g, 1, M) + \frac{n}{2} - \frac{n}{2} \ln \int (4|\nabla v|^2 + Rv^2) dg - s_n \right) v = 0.$$

The proof is done by an approximation process that involves a priori estimates and a blow up analysis. This strategy has been used to study variational problems involving critical functionals. Recently in [DE] Dolbeault and Esteban treated a similar functional on the cylinder $S^n \times \mathbf{R}$. We benefitted from the ideas in that paper. However, we are facing new difficulties since our functional is scaling invariant and its component $\ln F(v)$ may not be bounded from below. These make it difficult to apply P. L. Lions' concentrated compactness method near infinity directly. However under the extra assumption $\lambda(g(t_2)) < \lambda_\infty(g(t_2))$, we can show that the Lions' method [Lio] works on special regions where the L^2 norm of v has faster than usual decays. We also use a fact that that a sequence of Boltzmann entropy $N(v_k)$ satisfies the reverse Fatou lemma when $\{v_k\}$ is a sequence of bounded functions with the same L^2 norm. Once a minimizer is found, we can use Perelman's monotonicity formula to show that $(M, g(t))$ is a gradient Ricci soliton since $\lambda(g(t_1)) = \lambda(g(t_2))$.

2. PRELIMINARIES

In this section, we present a number of elementary results to be used in the proof of the theorems and the corollary.

Definition 2.1. *A complete, noncompact Riemann manifold M is called Asymptotically Flat of order τ if there is a partition $M = M_0 \cup M_\infty$, which satisfies the following properties.*
(i). M_0 is compact and it contains a reference point 0.
(ii). M_∞ is diffeomorphic to $(\mathbf{R^n} - B(0, r_0))$ for some $r_0 > 0$.
(iii). Under the coordinates induced by the diffeomorphism, the metric g_{ij} satisfies, for $x \in M_\infty$,

$$g_{ij}(x) = \delta_{ij}(x) + O(|x|^{-\tau}), \quad \partial_k g_{ij}(x) = \delta_{ij}(x) + O(|x|^{-\tau-1}), \quad \partial_k \partial_l g_{ij}(x) = \delta_{ij}(x) + O(|x|^{-\tau-2}).$$

According to Theorem (1.1) in [BKN], if M has one end, the curvature tensor decays sufficiently fast near infinity and $|B(0, r)| \geq cr^n$ when r is large, then M is AF. Here n is the dimension.

Proposition 2.1. *Let (M, g) be an AF manifold of dimension $n \geq 3$. Suppose the scalar curvature R is positive everywhere.*
(a). Then there exists a constant $A > 0$, such that

$$\left(\int_M v^{2n/(n-2)} dg \right)^{(n-2)/n} \leq A \int_M (4|\nabla v|^2 + Rv^2) dg, \quad \forall v \in W^{1,2}(M, g); \tag{2.1}$$

moreover $\lambda(g)$ is bounded from below i.e.

$$\int_M v^2 \ln v^2 dg \leq \frac{n}{2} \ln \left(A \int_M (4|\nabla v|^2 + Rv^2) dg \right), \tag{2.2}$$

$\forall v \in W^{1,2}(M, g), \|v\|_{L^2(M,g)} = 1.$
(b). $\lambda_\infty(g) \geq 0$.

Proof. (a). We just need to prove (2.1) since (2.2) follows from Jensen inequality.

Pick and fix $r_0 > 0$ sufficiently large, so that a coordinate system on $M - B(x_0, r_0)$ exists, which satisfies the defining condition of AF manifolds. Let $\phi \in C_0^\infty(M)$ be a cut-off function such that $\phi = 1$ on $B(0, r_0)$, $\quad \phi = 0$ on $M - B(0, 2r_0)$, $\quad 0 \leq \phi \leq 1$ and $\nabla\phi| \leq C/r_0$. For any $v \in C_0^\infty(M)$, the function $v(1-\phi)$ is supported in $M - B(0, r_0)$.

Let $J : M - B(0, r_0) \to \mathbf{R^n}$ be the coordinate map. Then the function

$$f \equiv [v(1-\phi)] \circ J^{-1}$$

is a smooth, compactly supported function in $\mathbf{R^n}$, after extending by zero value. By the Euclidean Sobolev inequality, it holds

$$\left(\int_{\mathbf{R^n}} f^{2n/(n-2)} dx \right)^{(n-2)/n} \leq S_0 \int_{\mathbf{R^n}} |\nabla_{\mathbf{R^n}} f|^2 dx$$

where dx is the Euclidean volume element and $\nabla_{\mathbf{R^n}}$ is the Euclidean gradient. According to the definition of AF manifolds, there exists a positive constant c such that

$$c^{-1} dx \leq dg(x) \leq c dx, \quad c^{-1} |\nabla_{\mathbf{R^n}} f| \leq |\nabla[v(1-\phi)]| \leq c |\nabla_{\mathbf{R^n}} f|.$$

Here $|\nabla[v(1-\phi)]|$ is the length of the gradient of $v(1-\phi)$, both with respect to g. Therefore, there exists a positive constant C such that

$$\left(\int_M |v(1-\phi)|^{2n/(n-2)} dg\right)^{(n-2)/n} \le C \int_M |\nabla[v(1-\phi)]^2 dg.$$

By this and Minkowski inequality, together with the standard Sobolev inequality in the ball $B(0,2r_0)$, we deduce

$$\begin{aligned}&\left(\int_M v^{2n/(n-2)} dg\right)^{(n-2)/n}\\ &\le 2\left(\int_M |v(1-\phi)|^{2n/(n-2)} dg\right)^{(n-2)/n} + 2\left(\int_M (v\phi)|^{2n/(n-2)} dg\right)^{(n-2)/n}\\ &\le C\int_M |\nabla[v(1-\phi)]^2 dg + C\int_M |\nabla(v\phi)|^2 dg + C\int_M (v\phi)^2 dg.\end{aligned}$$

Hence

$$\left(\int_M v^{2n/(n-2)} dg\right)^{(n-2)/n} \le C\int_M |\nabla v|^2 dg + C\sup|\nabla\phi|^2 \int_{B(0,2r_0)} v^2 dg + C\int_{B(0,2r_0)} (v\phi)^2 dg.$$

Since $R(x) > 0$ for every $x \in \mathbf{R^n}$ be assumption, this implies, for some constant $A > 0$, that

$$\left(\int_M v^{2n/(n-2)} dg\right)^{(n-2)/n} \le A\int_M (4|\nabla v|^2 + Rv^2) dg.$$

This is (2.1), i.e. part (a).

Now we prove part (b).

First we prove the following assertion.

When the radius r is sufficiently large, it holds

$$\lambda(g,1,M-B(0,r)) \ge \lambda(g_E,1,\mathbf{R}^n - J(B(0,r)) + o(1).$$

Here J is the coordinate map near infinity in the definition of AF manifold; $o(1)$ is a quantity whose absolute value goes to 0 when $r \to \infty$; g_E is the Euclidean metric.

Pick a function $v \in C_0^\infty(M - B(0,r))$ with $\|v\|_{L^2} = 1$. Given any $\epsilon > 0$, by definition of AF manifolds, for $x \in M - B(0,r)$ with r sufficiently large, there hold

$$(1-\epsilon)dx \le dg(x) = \sqrt{detg(x)}dx \le (1+\epsilon)dx,$$

$$(1-\epsilon)|\nabla_{\mathbf{R^n}} f| \le |\nabla v| \le (1+\epsilon)|\nabla_{\mathbf{R^n}} f|$$

where $f = v \circ J^{-1}$ and J is the coordinate map. Also $\nabla_{\mathbf{R^n}}$ is the Euclidean gradient. Hence

$$\int_M (4|\nabla v|^2 + Rv^2)dg \ge (1-\epsilon)^2 \int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} f|^2 \sqrt{detg(x)}dx$$

Write $\sqrt{detg(x)} = w^2$, a routine calculation shows

$$\begin{aligned}\int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} f|^2 \sqrt{detg(x)}dx &= \int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} f|^2 w^2 dx\\ &= \int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}}(fw)|^2 dx + 4\int_{\mathbf{R^n}} (fw)^2 \frac{\Delta w}{w} dx.\end{aligned}$$

By definition of AF manifolds, we know that $\frac{|\Delta w(x)|}{w(x)} \le \frac{c}{|x|^{2+\tau}}$ with $\tau > 0$. Hence, by the Hardy's inequality in the Euclidean space, we have

$$\int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} f|^2 \sqrt{detg(x)} dx \ge (1+o(1)) \int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} (fw)|^2 dx,$$

which implies

$$\int_M (4|\nabla v|^2 + Rv^2) dg \ge (1-\epsilon)^2 (1+o(1)) \int_{\mathbf{R^n}} 4|\nabla_{\mathbf{R^n}} (fw)|^2 dx. \tag{2.3}$$

Also

$$\begin{aligned}\int_M v^2 \ln v^2 dg &= \int_{\mathbf{R^n}} (fw)^2 \ln f^2 dx = \int_{\mathbf{R^n}} (fw)^2 \ln (fw)^2 dx - \int_{\mathbf{R^n}} (fw)^2 \ln w^2 dx \\ &= \int_{\mathbf{R^n}} (fw)^2 \ln (fw)^2 dx + o(1).\end{aligned}$$

This and (2.3) imply that

$$L(v, g, 1, M - B(0,r)) \ge L(f, g_E, 1, \mathbf{R}^n - J(B(0,r))) + o(1) - n\epsilon.$$

Since $\|fw\|_{L^2(\mathbf{R^n})} = 1$, by taking the infimum of this inequality, it is easy to see that

$$\lambda(g, 1, M - B(0,r)) \ge \lambda(g_E, 1, \mathbf{R}^n - J(B(0,r)) + o(1) - n\epsilon.$$

Since ϵ is arbitrary, the assertion is proven.

Using $\lambda(g_E, 1, \mathbf{R}^n - J(B(0,r)) \ge \lambda(g_E, 1, \mathbf{R}^n) = 0$, we see that

$$\lambda_\infty(g) = \lim_{r\to\infty} \lambda(g, 1, M - B(0,r)) \ge 0.$$

This proves part (b). □

Proposition 2.2. *Let $(M, g(t))$ be a noncompact Ricci flow on the time interval (A, B) such that the curvature tensor is bounded for each time $t \in (A, B)$. Suppose also $(M, g(t))$ is κ noncollapsed below scale 1 and the scalar curvature is nonnegative. If $(M, g(t))$ is not a gradient shrinking soliton, then*

$$\lambda(g(t_0)) \equiv \lambda(g(t_0), 1, M) < 0, \quad t_0 \in (A, B).$$

Moreover, for any $x_0 \in M$, when r_0 is sufficiently large, it holds

$$\lambda(g(t_0), 1, B(x_0, r_0)) < 0.$$

Here $B(x_0, r_0) = B(x_0, r_0, g(t_0))$.

Proof. For compact Ricci flows, Perelman ([P] Section 3) already proved a similar inequality for his ν invariant. The following proof for the noncompact case is similar, except that one needs to justify integration by parts near infinity.

Without loss of generality we assume $t_0 < 0 \in (A, B)$. Let $u = u(x,t) = G(x, t; x_0, 0)$ be the fundamental solution of the conjugate heat equation

$$\Delta u - Ru + \partial_t u = 0, \qquad t < t_0.$$

Let $s = -t$ and

$$W(g(t), u(\cdot, t), t) = \int_M \left[s(\frac{|\nabla u|^2}{u} + Ru) - u \ln u - \frac{n}{2} \ln(4\pi s) u - nu \right] dg(t) \tag{2.4}$$

be Perelman's W entropy corresponding to $u = u(x,t)$. According to [P] Section 3,

$$\frac{d}{dt} W(g(t), u(\cdot, t), t) = 2s \int |Ric_{g(t)} - Hess_{g(t)} \ln u - \frac{1}{2s} g(t)|^2 u dg(t) \geq 0$$

with strict inequality holding unless $(M, g(t))$ is a gradient shrinking soliton. Moreover $\lim_{t \to 0} W(g(t), u(\cdot, t), t) = 0$. We comment that Perelman proved the result for compact Ricci flows. In the noncompact case one needs to justify the integrability of the quantities involved. Since $(M, g(t))$ has bounded geometry within any finite time interval and u, as fundamental solution has Gaussian decay near infinity, the integrability issue has been worked out in [CTY] and [C++] Chapter 19 e.g..

Since $(M, g(t))$ is not a gradient shrinking soliton, $\frac{d}{dt} W(g(t), u(\cdot, t), t)$ is strictly positive. From the assumption $t_0 < 0$, we obtain

$$W(g(t_0), u(\cdot, t_0), t_0) < \lim_{t \to 0} W(g(t), u(\cdot, t), t) = 0.$$

Observe that with $\rho > 0$ regarded as a free parameter and taking $v = \sqrt{u(\cdot, t_0)}$, we have

$$\begin{aligned} L(\sqrt{u(\cdot, t_0)}, g(t_0), 1) &= -\int_M v^2 \ln v^2 dg(t_0) + \frac{n}{2} \ln \left(\int_M (4|\nabla v|^2 + R v^2) dg(t_0) \right) + s_n \\ &= \inf_{\rho > 0} \int_M \left[\rho(\frac{|\nabla u|^2}{u} + Ru) - u \ln u - \frac{n}{2} \ln(4\pi\rho) u - nu \right] dg(t_0) \\ &\leq \int_M \left[|t_0|(\frac{|\nabla u|^2}{u} + Ru) - u \ln u - \frac{n}{2} \ln(4\pi|t_0|) u - nu \right] dg(t_0) \\ &= W(g(t_0), u(\cdot, t_0), t_0) < 0. \end{aligned}$$

Here $u = u(\cdot, t_0)$ and $R = R(\cdot, x_0)$. This shows, since $\lambda(g(t_0))$ is the infimum of the log Sobolev functional L, that $\lambda(g(t_0), 1) < 0$.

The second statement of the lemma is an easy consequence of the fact that $\lambda(g(t_0)) = \lim_{r_0 \to \infty} \lambda(g(t_0), 1, B(x_0, r_0))$. □

Proposition 2.3. *Let (M, g) be a noncompact manifold such that $\lambda(g) > -\infty$.*

(a). For any $x_0 \in M$, $r_0 > 0$, and for all $\alpha \geq 1$, the infimum of the log Sobolev functionals $L(\cdot, g, \alpha, B(x_0, r_0))$ satisfy:

$$\lambda(g, \alpha, B(x_0, r_0)) \geq -C$$

where C is a constant depending only on α, n, the constant $\lambda(g)$ and $|B(x_0, r_0)|$.

(b). $\lim_{\alpha \to 1^+} \lambda(g, \alpha, B(x_0, r_0)) = \lambda(g, 1, B(x_0, r_0))$.

Proof. For simplicity we use B to denote $B(x_0, r_0)$ in the proof. Pick a function $v \in C_0^\infty(B)$ such that $\|v\|_{L^2(B)} = 1$. Then

$$L(v, g, \alpha, B) = L(v, g, 1, B) + (\alpha - 1)\frac{n}{2} \ln \left(\int_B (4|\nabla v|^2 + R v^2) dg \right),$$

and hence

$$L(v, g, \alpha, B) \geq \lambda(g) + (\alpha - 1)\frac{n}{2} \ln \left(\int_B (4|\nabla v|^2 + R v^2) dg \right).$$

This shows,

$$L(v, g, \alpha, B) \geq \lambda(g) + (\alpha - 1)\frac{n}{2} \ln \left(A^{-1} \|v\|^2_{L^{2n/(n-2)}(B)} \right),$$

which implies, via Hölder inequality,

$$L(v,g,\alpha,B) \geq -\frac{n}{2} + (\alpha-1)\frac{n}{2}\ln\left(A^{-1}\|v\|^2_{L^2(B)}/|B|^{2/n}\right).$$

Thus

$$L(v,g,\alpha,B) \geq -\frac{n}{2} - (\alpha-1)\frac{n}{2}\ln\left(A\,|B|^{2/n}\right),$$

proving part (a) of the proposition.

In order to prove part (b), we notice that in the last paragraph we actually showed that

$$L(v,g,\alpha,B) \geq L(v,g,1,B) - (\alpha-1)\frac{n}{2}\ln\left(A\,|B|^{2/n}\right).$$

On the other hand, since $v^2\ln v^2 \geq -e^{-1}$, we have

$$L(v,g,1,B) = \alpha L(v,g,\alpha,B) + (\alpha-1)\int_B v^2\ln v^2 dg \geq \alpha L(v,g,\alpha,B) - (\alpha-1)e^{-1}|B|.$$

Part (b) of the proposition follows from the last 2 expressions when $\alpha \to 1$. □

Proposition 2.4. *Let (M,g) be a noncompact manifold with bounded curvature such that $\lambda(g) > -\infty$. If also the scalar curvature $R \geq 0$, then there exists a positive constant A depending only on $\lambda(g)$ and n such that*

$$\left(\int_M v^{2n/(n-2)}dg\right)^{(n-2)/n} \leq A\int_M (4|\nabla v|^2 + Rv^2)dg, \quad \forall v \in W^{1,2}(M,g).$$

Moreover, M is κ non-collapsed below scale 1.

Proof. This statement is nothing but the well known equivalence of the Sobolev inequality and log Sobolev inequality, which is proved via an upper bound for the heat kernel $e^{(4\Delta-R)t}$. When $R=0$ one can find a proof in Davies [Da] Chapter 2. When $R \geq 0$, then the L^1 to L^1 norm of the heat kernel is less than or equal to 1. The same proof still goes through as written in [Z] Section 6.2.

It is well known that the above Sobolev inequality implies that M is κ non-collapsed. □

3. Proof of Theorems

We will prove a number of lemmas first and proceed to prove Theorems 1.2 and 1.1. First we show that a minimizer for the functional $L(\cdot,g,\alpha,B)$ exists when $\alpha > 1$ and B is a ball.

Lemma 3.1. *Let (M,g) be a noncompact manifold such that $\lambda(g) > -\infty$ and the scalar curvature $R \geq 0$.*

(a). For any $x_0 \in M$, and $r_0 > 0$, write $B = B(x_0,r_0)$. Then for all $\alpha > 1$, the infimum of the log Sobolev functionals $L(\cdot,g,\alpha,B)$ is reached. Namely, there exists a function $v \in C_0^\infty(B)$ with unit L^2 norm such that

$$L(v,g,\alpha,B) = \lambda(g,\alpha,B).$$

(b). The function v, called the minimizer, satisfies the equation

$$\alpha\frac{n}{2}\frac{4\Delta v - Rv}{\int_B(4|\nabla v|^2+Rv^2)dg} + 2v\ln v + \beta v = 0, \tag{3.1}$$

where

$$\beta = \lambda(g, \alpha, B) + \alpha\frac{n}{2} - \alpha\frac{n}{2}\ln\left[\int_B (4|\nabla v|^2 + Rv^2)dg\right] - s_n. \tag{3.2}$$

Here s_n is the number given in Definition 1.1.

Proof. By Proposition 2.3, the log Sobolev functional is bounded from below. Hence there exists a sequence of functions $\{v_k\} \subset W_0^{1,2}(B)$ with unit L^2 norm such that

$$L(v_k, g, \alpha, B) \to \lambda(g, \alpha, B) > -\infty, \quad k \to \infty. \tag{3.3}$$

So, for all large k, there hold

$$-\int_B v_k^2 \ln v_k^2 dg + \alpha\frac{n}{2}\ln\left[\int_B (4|\nabla v_k|^2 + Rv_k^2)dg\right] + s_n \le \lambda(g, \alpha, B) + 1.$$

By the assumption $\lambda(g) > \infty$ and Proposition 2.3,

$$-\int_B v_k^2 \ln v_k^2 dg + \frac{n}{2}\ln\left[\int_B (4|\nabla v_k|^2 + Rv_k^2)dg\right] \ge \lambda(g, 1, B) \ge -\lambda(g, 1) \le -C > -\infty.$$

Substituting this to the previous inequality, we obtain

$$(\alpha - 1)\frac{n}{2}\ln\left[\int_B (4|\nabla v_k|^2 + Rv_k^2)dg\right] \le \lambda(g, \alpha, B) + C - s_n + 1. \tag{3.4}$$

By Proposition 2.4

$$\begin{aligned} A^{-1}\left(\int_B v_k^{2n/(n-2)} dg\right)^{(n-2)/n} &\le \int_B (4|\nabla v_k|^2 + Rv_k^2)dg \\ &\le \exp\left[(\alpha - 1)^{-1}\left(\lambda(g, \alpha, B) + C - s_n + 1\right)\right]. \end{aligned}$$

Pick a number $q \in (2, 2n/(n-2))$. Since the embedding to $L^q(B)$ is compact, we can find a subsequence, still denoted by $\{v_k\}$, which converges strongly to a function v in $L^q(B)$ norm. By (3.4), clearly $v \in W_0^{1,2}(B)$.

Now we show that v is a minimizer for $L(\cdot, g, \alpha, B)$. By Fatou's lemma

$$\int_B (4|\nabla v|^2 + Rv^2)dg \le \lim_{k\to\infty}\int_B (4|\nabla v_k|^2 + Rv_k^2)dg. \tag{3.5}$$

According to Theorem 2 in [BL],

$$\int_B v^2 \ln v^2 dg = \lim_{k\to\infty}\int_B v_k^2 \ln v_k^2 dg + \lim_{k\to\infty}\int_B (v_k - v)^2 \ln(v_k - v)^2 dg.$$

Write $B_k = \{x \,|\, |v_k(x) - v(x)| \le 1\}$. Then

$$\int_B (v_k - v)^2 \ln(v_k - v)^2 dg = \int_{B_k} (v_k - v)^2 \ln(v_k - v)^2 dg + \int_{B - B_k} (v_k - v)^2 \ln(v_k - v)^2 dg,$$

and therefore

$$\left|\int_B (v_k - v)^2 \ln(v_k - v)^2 dg\right| \le \left|\int_{B_k} (v_k - v)^2 \ln(v_k - v)^2 dg\right| + C_q \int_{B - B_k} |v_k - v|^q dg.$$

Applying dominated convergence theorem on the first term of the right hand side, we obtain, since also $v_k \to v$ in $L^q(B)$ norm, that

$$\lim_{k\to\infty} \int_B (v_k - v)^2 \ln(v_k - v)^2 dg = 0.$$

Consequently

$$\int_B v^2 \ln v^2 dg = \lim_{k\to\infty} \int_B v_k^2 \ln v_k^2 dg.$$

By this and (3.5), we find that

$$L(v, g, \alpha, B) \le lim_{k\to\infty} L(v_k, g, \alpha, B) = \lambda(g, \alpha, B) \le L(v, g, \alpha, B).$$

Hence v is a minimizer. By the Lagrange multiplier method, there is a constant β such that

$$\alpha\frac{n}{2}\frac{4\Delta v - Rv}{\int_B (4|\nabla v|^2 + Rv^2) dg} + 2v \ln v + \beta v = 0. \tag{3.6}$$

Since $F \equiv \int_B (4|\nabla v|^2 + Rv^2) dg$ is a finite number, we can multiply it on both sides of the equation to obtain

$$\alpha\frac{n}{2} 4\Delta v - Rv + F 2v \ln v + F\beta v = 0.$$

Since the nonlinear term $v \ln v$ is very mild, it is known that $v \in C_0^\infty(B)$. See [Rot] e.g.

Multiplying (3.6) by v and integrating, we deduce

$$-\alpha\frac{n}{2} + \int_B v^2 \ln v^2 dg + \beta = 0.$$

Since we have proven that v is a minimizer for $L(\cdot, g, \alpha, B)$, it holds

$$\lambda(g, \alpha, B) = -\int_B v^2 \ln v^2 dg + \alpha\frac{n}{2} \ln F + s_n.$$

Combining the last two identity, we see that

$$\beta = \lambda(g, \alpha, B) + \alpha\frac{n}{2} - \alpha\frac{n}{2} \ln F - s_n,$$

which is just (3.2). □

The next lemma deals with the case $\alpha = 1$.

Lemma 3.2. *Let (M, g) be a noncompact manifold such that $\lambda(g) > -\infty$ and that the scalar curvature $R \ge 0$.*

(a). For any $x_0 \in M$ and $r_0 > 0$, let $B = B(x_0, r_0)$. If $\lambda(g, 1, B) < 0$, then, the infimum of the log Sobolev functionals $L(\cdot, g, 1, B)$ is reached. Namely, there exists a function $v \in C_0^\infty(B)$ with unit L^2 norm such that

$$L(v, g, 1, B) = \lambda(g, 1, B).$$

(b). The function v, called the minimizer, satisfies the equation

$$\frac{n}{2}\frac{4\Delta v - Rv}{\int_B (4|\nabla v|^2 + Rv^2) dg} + 2v \ln v + \beta v = 0, \tag{3.7}$$

where

$$\beta = \lambda(g, 1, B) + \frac{n}{2} - \frac{n}{2} \ln \left[\int_B (4|\nabla v|^2 + Rv^2) dg\right] - s_n. \tag{3.8}$$

Here s_n is the number given in Definition 1.1.

Proof. The proof is consisted of a number of steps.

step 1. constructing an approximating sequence.

Pick a sequence $\alpha_k \to 1^+$, as $k \to \infty$. Let v_k be a minimizer for $L(\cdot, g, \alpha_k, B)$, which exists according to Lemma 3.1, and which satisfies

$$\alpha_k \frac{n}{2} \frac{4\Delta v_k - Rv_k}{\int_B (4|\nabla v_k|^2 + Rv_k^2) dg} + 2v_k \ln v_k + \beta_k v_k = 0, \tag{3.9}$$

where

$$\beta_k = \lambda(g, \alpha_k, B) + \alpha_k \frac{n}{2} - \alpha_k \frac{n}{2} \ln \left[\int_B (4|\nabla v_k|^2 + Rv_k^2) dg \right] - s_n. \tag{3.10}$$

Write

$$F_k \equiv \int_B (4|\nabla v_k|^2 + Rv_k^2) dg, \qquad m_k = \max\{v_k(x) \,|\, x \in B\}. \tag{3.11}$$

Since $v_k = 0$ on ∂B, we know $\Delta v_k \leq 0$ at the maximum point of v_k. Hence (3.9) implies, at the maximum point,

$$2v_k \ln v_k \geq -\beta_k v_k + \alpha_k \frac{n}{2} R v_k F_k^{-1} \geq -\beta_k v_k.$$

By Lemma 2.3

$$\lim_{k\to\infty} \lambda(g, \alpha_k, B) = \lambda(g, 1, B) < 0.$$

Therefore, for sufficiently large k, we also have $\lambda(g, \alpha_k, B) < 0$. This fact and (3.10) infer that

$$m_k = \max v_k \geq e^{-\alpha_k n/4} F_k^{\alpha_k n/4} e^{s_n/2}. \tag{3.12}$$

Next we perform the scaling

$$g_k = m_k^{4/n} g, \quad R_k = m_k^{-4/n} R, \quad \tilde{v}_k = m_k^{-1} v_k.$$

Notice that $0 \leq \tilde{v}_k \leq 1$ and that

$$\|\tilde{v}_k\|_{L^2(M, g_k)} = 1.$$

By (3.9), $\tilde{v}_k$ satisfies the equation

$$\begin{aligned} &\alpha_k \frac{n}{2} F_k^{-1} m_k^{4/n} (4\Delta_{g_k} - m_k^{-4/n} R)(m_k \tilde{v}_k) + 2 m_k \tilde{v}_k \ln(m_k \tilde{v}_k) \\ &\qquad + (\lambda(g, \alpha_k, B) + \alpha_k \frac{n}{2} - \alpha_k \frac{n}{2} \ln F_k - s_n)(m_k \tilde{v}_k) = 0 \end{aligned}$$

which becomes, after simplification,

$$\begin{aligned} &\alpha_k \frac{n}{2} (4\Delta_{g_k} - R_k) \tilde{v}_k + (2\tilde{v}_k \ln \tilde{v}_k + \lambda(g, \alpha_k, B) \tilde{v}_k + \alpha_k \frac{n}{2} \tilde{v}_k - s_n \tilde{v}_k) \, F_k m_k^{-4/n} \\ &\quad - \alpha_k \frac{n}{2} F_k m_k^{-4/n} \ln(F_k m_k^{-4/(n\alpha_k)}) \, \tilde{v}_k = 0. \end{aligned} \tag{3.13}$$

Here $B = B(x_0, r_0, g)$ again.

step 2. We prove that for all sequences $\{\alpha_k\} \subset (1, 2]$ such that $\alpha_k \to 1$, and fixed r_0 sufficiently large, there exists a uniform constant C_0 such that

$$\limsup_{k\to\infty} F_k \leq C_0 = C_0(r_0). \tag{3.14}$$

Suppose for contradiction that there exists a sequence of numbers $\{\alpha_k\} \subset (1, 2]$ such that $\alpha_k \to 1$, and that v_k is a minimizer of $L(\cdot, g, \alpha_k, B(x_0, r_0))$ but

$$\lim_{k\to\infty} F_k = \lim_{k\to\infty} \int_{B(x_0,r_0)} (4|\nabla v_k|^2 + R v_k^2) dg = \infty. \tag{3.15}$$

Then (3.12) shows that $m_k \to \infty$ as $k \to \infty$ and that there exists a constant C such that

$$F_k m_k^{-4/(n\alpha_k)} \leq C,$$

and when k is large

$$F_k m_k^{-4/n} \leq F_k m_k^{-4/(n\alpha_k)} \leq C; \qquad a_k \frac{n}{2} F_k m_k^{-4/n} |\ln(F_k m_k^{-4/(n\alpha_k)})| \leq C.$$

Therefore the coefficients of equation (3.13) are uniformly bounded. Moreover the manifold (M, g_k) has uniformly bounded geometry since $g_k = m_k^{4/n} g$ and $m_k \to \infty$. Now we extend $\tilde{v}_k$ to be a function on the whole manifold M by setting $\tilde{v}_k = 0$ outside of $B(x_0, r_0, g) = B(x_0, m_k^{2/n} r_0, g_k)$. The extended function, still denoted by $\tilde{v}_k$, is a subsolution of the equation in (3.13); further more $0 \leq \tilde{v}_k \leq 1$ and $\|\tilde{v}_k\|_{L^2(M,g_k)} = 1$.

Let x_k be a maximum point of $\tilde{v}_k$ and $r > 0$ be a large number. Construct a standard cut-off function ϕ such that $\phi = 1$ on $B(x_k, r, g_k)$, $\phi = 0$ outside of $B(x_k, 2r, g_k)$, $0 \leq \phi \leq 1$ and $|\nabla_{g_k} \phi| \leq C/r$. Since the extended function $\tilde{v}_k$ is a sub-solution of (3.13), we can use $\tilde{v}_k \phi^2$ as a test function to conclude, using the bounds in the previous paragraph, that

$$\begin{aligned} &\int_{B(x_k,r,g_k)} |\nabla_{g_k} \tilde{v}_k|^2 dg_k \\ &\leq \frac{C}{r^2} \int_{B(x_k,2r,g_k)} \tilde{v}_k^2 dg_k + C(1 + |\lambda(g, \alpha_k, B)|) F_k m_k^{-4/n} \int_{B(x_k,2r,g_k)} \tilde{v}_k^2 dg_k \\ &\leq \frac{C}{r^2} + C(1 + |\lambda(g, \alpha_k, B)|) F_k m_k^{-4/n}. \end{aligned} \tag{3.16}$$

Here $B = B(x_0, r_0, g)$ again.

We consider 2 cases.

Case 1. A subsequence of $\{F_k m_k^{-4/n}\}$, denoted by the same symbol, converges to 0.

Let x_k be a maximum point of v_k. Since $m_k \to \infty$ and $g_k = m_k^{-4/n} g$, we know that a subsequence of the pointed manifolds $\{(M, g_k, x_k)\}$, converges in C^∞_{loc} topology, to the pointed Euclidean space $(\mathbf{R^n}, 0)$. This is due to the Cheeger-Gromov compactness theorem. By the bound (3.16) and the fact $R_k \to 0$, $\lambda(g, 1, B_k) \to \lambda(g, 1, M) = \lambda(g)$, we know that a subsequence of $\tilde{v}_k$ converges pointwise, modulo composition with diffeomorphisms, to a function v_∞ on $\mathbf{R^n}$, which is a sub-solution of the Laplacian. Furthermore $\|v_\infty\|_{L^2(\mathbf{R^n})} \leq 1$ and $v_\infty(0) = 1$. By (3.16) again

$$\int_{B(0,r)} |\nabla v_\infty|^2 dx \leq \frac{C}{r^2}.$$

Here all expressions are in the Euclidean setting. Letting $r \to \infty$, we see that $\nabla v_\infty = 0$ and therefore $v_\infty \equiv 1$. But this is impossible since $\|v_\infty\|_{L^2(\mathbf{R^n})} \leq 1$.

Case 2. $\{F_k m_k^{-4/n}\}$ is bounded away from 0.

Then we can find a subsequence of $\{F_k m_k^{-4/n}\}$, denoted by the same symbol, which converges to a number $A > 0$. As in the previous paragraph, $\{(M, g_k, x_k)\}$, converges in C^∞_{loc} topology, to the pointed Euclidean space $(\mathbf{R^n}, 0)$. Also a subsequence of the extended function $\tilde{v}_k$ converges pointwise, modulo composition with diffeomorphisms, to a function v_∞ on $\mathbf{R^n}$. Furthermore $\|v_\infty\|_{L^2(\mathbf{R^n})} \le 1$, $v_\infty(0) = 1$ and, in the weak sense,

$$\frac{n}{2} 4\Delta v_\infty + A(2v_\infty \ln v_\infty + \lambda(g,1,B)v_\infty + \frac{n}{2} v_\infty - s_n v_\infty) - (\frac{n}{2} A \ln A)\, v_\infty \ge 0. \tag{3.17}$$

Dividing both sides by A and recalling from Definition 1.1 that $s_n = -\frac{n}{2}\ln(2\pi n) - \frac{n}{2}$, we obtain

$$\lambda(g,1,B) v_\infty \ge -\frac{n}{2A} 4\Delta v_\infty - 2v_\infty \ln v_\infty - n v_\infty + \frac{n}{2}\ln(2\pi n A)\, v_\infty.$$

We multiply the last inequality by v_∞. By Moser's iteration, it is easy to prove that v_∞ has Gaussian decay near infinity. See [Rot] or Lemma 2.3 in [Z2] e.g. Therefore, we can carry out integration by parts to deduce

$$\begin{aligned}\lambda(g,1,B)\|v_\infty\|^2_{L^2(\mathbf{R^n})} &\ge \int_{\mathbf{R^n}} \Big(\frac{n}{2A} 4|\nabla v_\infty|^2 - v^2_\infty \ln v^2_\infty - n v^2_\infty - \frac{n}{2}\ln(2\pi n/A)\, v^2_\infty \Big)\, dx \\ &= \int_{\mathbf{R^n}} \Big(s4|\nabla v_\infty|^2 - v^2_\infty \ln v^2_\infty - \frac{n}{2}\ln(4\pi s)\, v^2_\infty - n v^2_\infty \Big)\, dx,\end{aligned}$$

where $s = \frac{n}{2A}$. Write $\hat{v} = \frac{v_\infty}{\|v_\infty\|_{L^2(\mathbf{R^n})}}$. Then, by $\|v_\infty\|^2_{L^2(\mathbf{R^n})} \le 1$, we have

$$\begin{aligned}\lambda(g,1,B)\|v_\infty\|^2_{L^2(\mathbf{R^n})} \ge \|v_\infty\|^2_{L^2(\mathbf{R^n})} \int_{\mathbf{R^n}} \Big(s4|\nabla \hat{v}|^2 - \hat{v}^2 \ln \hat{v}^2_\infty - \frac{n}{2}\ln(4\pi s)\, \hat{v}^2_\infty - n\hat{v}^2_\infty \Big)\, dx \\ - \|v_\infty\|^2_{L^2(\mathbf{R^n})} \ln \|v_\infty\|^2_{L^2(\mathbf{R^n})} \ge 0.\end{aligned}$$

Here we just used the fact that the best constant for the log Sobolev inequality for functions with unit L^2 norms in $\mathbf{R^n}$ is 0. This is a contradiction with the assumption that $\lambda(g,1,B) < 0$. This proves (3.14), i.e.

$$F_k = \int_{B(x_0,r_0)} (4|\nabla v_k|^2 + R v_k^2) dg \le C_0.$$

step 3. We prove v_k converges to a minimizer of $L(\cdot, g, 1, B)$.

By (3.9), we know that v_k satisfies

$$\frac{n}{2}(4\Delta v_k - R v_k) + \alpha_k^{-1} F_k 2 v_k \ln v_k + \alpha_k^{-1} F_k \beta_k v_k = 0,$$

where

$$\beta_k = \lambda(g, \alpha_k, B) + \alpha_k \frac{n}{2} - \alpha_k \frac{n}{2} \ln F_k - s_n.$$

Since, by Step 2, F_k is uniformly bounded, we know that the coefficients in the above equation are uniformly bounded. Since the nonlinear term $v_k \ln v_k$ is only mildly nonlinear, it is easy to prove that $\|v_k\|_{L^\infty}$ is also uniformly bounded. See Lemma 2.1 in [Z2] e.g. Now, since the ball B is bounded,a routine argument shows that a subsequence of v_k converges to a minimizer v of $L(\cdot, g, 1, B)$. Using the same argument near the end of the proof of Lemma 3.1, we see that v satisfies equation (3.7). This proves the lemma . □

The next lemma shows that the minimizers of $L(\cdot, g, 1, B)$ are uniformly bounded even if the radius of B tends to ∞.

Lemma 3.3. *Under the same assumption as in Lemma 3.2, let v be a minimizer for $L(\cdot, g, 1, B)$, where $B = B(x_0, r_0)$. Then the quantity*

$$F = \int_B (4|\nabla v|^2 + Rv^2) dg$$

is uniformly bounded for all large r_0. Furthermore $\|v\|_{L^\infty(B)}$ is uniformly bounded for all large r_0.

Proof. The idea of the proof is similar to that for the previous lemma. Suppose for contradiction that there exists a sequence of radii $\{r_{0k}\}$ and that v_k is a minimizer of $L(\cdot, g, 1, B(x_0, r_{0k}))$ but

$$\lim_{k\to\infty} F_k = \lim_{k\to\infty} \int_{B(x_0, r_{0k})} (4|\nabla v_k|^2 + Rv_k^2) dg = \infty. \tag{3.18}$$

From the previous lemma v_k satisfies

$$\frac{n}{2} \frac{4\Delta v_k - Rv_k}{F_k} + 2v_k \ln v_k + \beta_k v_k = 0, \tag{3.19}$$

where

$$\beta_k = \lambda(g, 1, B_k) + \frac{n}{2} - \frac{n}{2} \ln F_k - s_n. \tag{3.20}$$

Here and later $B_k = B(x_0, r_{0k}) = B(x_0, r_{0k}, g)$. Since $v_k = 0$ on ∂B_k, we know $\Delta v_k \le 0$ at the maximum point of v_k. Hence (3.19) implies, at the maximum point,

$$2v_k \ln v_k \ge -\beta_k v_k + \frac{n}{2} R v_k F_k^{-1} \ge -\beta_k v_k.$$

Since by definition

$$\lim_{k\to\infty} \lambda(g, 1, B_k) = \lambda(g, 1, M) < 0,$$

for sufficiently large k, we also have $\lambda(g, 1, B_k) < 0$. This fact and (3.20) infer that

$$m_k \equiv \max v_k \ge e^{-n/4} F_k^{n/4} e^{s_n/2}. \tag{3.21}$$

Next we do the scaling

$$g_k = m_k^{4/n} g, \quad R_k = m_k^{-4/n} R, \quad \tilde{v}_k = m_k^{-1} v_k.$$

Notice that $0 \le \tilde{v}_k \le 1$ and that

$$\|\tilde{v}_k\|_{L^2(M, g_k)} = 1.$$

By (3.19), $\tilde{v}_k$ satisfies the equation

$$\begin{aligned} &\frac{n}{2} F_k^{-1} m_k^{4/n} (4\Delta_{g_k} - m_k^{-4/n} R)(m_k \tilde{v}_k) + 2m_k \tilde{v}_k \ln(m_k \tilde{v}_k) \\ &\qquad + (\lambda(g, 1, B_k) + \frac{n}{2} - \frac{n}{2} \ln F_k - s_n)(m_k \tilde{v}_k) = 0 \end{aligned}$$

which becomes, after simplification,

$$\begin{aligned} &\frac{n}{2}(4\Delta_{g_k} - R_k)\tilde{v}_k + (2\tilde{v}_k \ln \tilde{v}_k + \lambda(g, 1, B_k)\tilde{v}_k + \frac{n}{2}\tilde{v}_k - s_n \tilde{v}_k)\, F_k m_k^{-4/n} \\ &- \frac{n}{2} F_k m_k^{-4/n} \ln(F_k m_k^{-4/n})\, \tilde{v}_k = 0. \end{aligned} \tag{3.22}$$

Since $F_k \to \infty$ by assumption, (3.21) shows that $m_k \to \infty$ as $k \to \infty$ and that there exists a constant C such that

$$F_k m_k^{-4/n} \le C,$$

Therefore the coefficients of equation (3.22) are uniformly bounded. Moreover the manifold (M, g_k) has uniformly bounded geometry since $g_k = m_k^{4/n} g$ and $m_k \to \infty$. Now we extend $\tilde{v}_k$ to be a function on the whole manifold M by setting $\tilde{v}_k = 0$ outside of $B_k = B(x_0, r_{0k}, g) = B(x_0, m_k^{2/n} r_{0k}, g_k)$. The extended function, still denoted by $\tilde{v}_k$, is a subsolution of (3.22); further more $0 \le \tilde{v}_k \le 1$ and $\|\tilde{v}_k\|_{L^2(M,g_k)} = 1$.

Let x_k be a maximum point of $\tilde{v}_k$ and $r > 0$ be a large number. Construct a standard cut-off function ϕ such that $\phi = 1$ on $B(x_k, r, g_k)$, $\phi = 0$ outside of $B(x_k, 2r, g_k)$, $0 \le \phi \le 1$ and $|\nabla_{g_k} \phi| \le C/r$. Since the extended function $\tilde{v}_k$ is a sub-solution of (3.22), we can use $\tilde{v}_k \phi^2$ as a test function to conclude, using the bounds in the previous paragraph, that

$$\begin{aligned} &\int_{B(x_k,r,g_k)} |\nabla_{g_k} \tilde{v}_k|^2 dg_k \\ &\le \frac{C}{r^2} \int_{B(x_k,2r,g_k)} \tilde{v}_k^2 dg_k + C(1 + |\lambda(g,1,B_k)|) F_k m_k^{-4/n} \int_{B(x_k,2r,g_k)} \tilde{v}_k^2 dg_k \\ &\le \frac{C}{r^2} + C(1 + |\lambda(g,1,B_k)|) F_k m_k^{-4/n}. \end{aligned} \tag{3.23}$$

We consider 2 cases.

Case 1. A subsequence of $\{F_k m_k^{-4/n}\}$, denoted by the same symbol, converges to 0.

Let x_k be a maximum point of v_k again. Since $m_k \to \infty$ and $g_k = m_k^{-4/n} g$, by Cheeger-Gromov compactness theorem, we know that a subsequence of the pointed manifolds $\{(M, g_k, x_k)\}$, converges in C^∞_{loc} topology, to the pointed Euclidean space $(\mathbf{R^n}, 0)$. By the bound (3.23) and the fact $R_k \to 0$, $\lambda(g,1,B_k) \to \lambda(g,1,M)$, we know that a subsequence of $\tilde{v}_k$ converges pointwise, modulo composition with diffeomorphisms, to a function v_∞ on $\mathbf{R^n}$, which is a sub-solution of the Laplacian. Furthermore $\|v_\infty\|_{L^2(\mathbf{R^n})} \le 1$ and $v_\infty(0) = 1$. By (3.23) again

$$\int_{B(0,r)} |\nabla v_\infty|^2 dx \le \frac{C}{r^2}.$$

Here all expressions are in the Euclidean setting. Letting $r \to \infty$, we see that $\nabla v_\infty = 0$ and therefore $v_\infty \equiv 1$. But this is impossible since $\|v_\infty\|_{L^2(\mathbf{R^n})} \le 1$.

Case 2. $\{F_k m_k^{-4/n}\}$ is bounded away from 0.

Then we can find a subsequence of $\{F_k m_k^{-4/n}\}$, denoted by the same symbol, which converges to a number $A > 0$. As in the previous paragraph, $\{(M, g_k, x_k)\}$, converges in C^∞_{loc} topology, to the pointed Euclidean space $(\mathbf{R^n}, 0)$. Also a subsequence of the extended function $\tilde{v}_k$ converges pointwise, modulo composition with diffeomorphisms, to a function v_∞ on $\mathbf{R^n}$. Furthermore $\|v_\infty\|_{L^2(\mathbf{R^n})} \le 1$, $v_\infty(0) = 1$ and, in the weak sense,

$$\frac{n}{2} 4\Delta v_\infty + A(2 v_\infty \ln v_\infty + \lambda(g,1,M) v_\infty + \frac{n}{2} v_\infty - s_n v_\infty) - (\frac{n}{2} A \ln A)\, v_\infty \ge 0. \tag{3.24}$$

Dividing both sides by A and recalling from Definition 1.1 that $s_n = -\frac{n}{2} \ln(2\pi n) - \frac{n}{2}$, we obtain

$$\lambda(g,1,M) v_\infty \ge -\frac{n}{2A} 4\Delta v_\infty - 2 v_\infty \ln v_\infty - n v_\infty + \frac{n}{2} \ln(2\pi n A)\, v_\infty.$$

We multiply the last inequality by v_∞. By Moser's iteration, it is easy to prove, as in Lemma 2.3 in [Z2], v_∞ has Gaussian decay near infinity. Therefore, we can carry out integration by parts to deduce

$$\begin{aligned}\lambda(g,1,M)\|v_\infty\|^2_{L^2(\mathbf{R^n})} &\ge \int_{\mathbf{R^n}} \left(\frac{n}{2A}4|\nabla v_\infty|^2 - v_\infty^2 \ln v_\infty^2 - n v_\infty^2 - \frac{n}{2}\ln(2\pi n/A)\, v_\infty^2\right) dx\\ &= \int_{\mathbf{R^n}} \left(s4|\nabla v_\infty|^2 - v_\infty^2 \ln v_\infty^2 - \frac{n}{2}\ln(4\pi s)\, v_\infty^2 - n v_\infty^2\right) dx,\end{aligned}$$

where $s = \frac{n}{2A}$. Write $\hat v = \frac{v_\infty}{\|v_\infty\|_{L^2(\mathbf{R^n})}}$. Then, by $\|v_\infty\|^2_{L^2(\mathbf{R^n})} \le 1$, we have

$$\begin{aligned}\lambda(g,1,M)\|v_\infty\|^2_{L^2(\mathbf{R^n})} \ge \|v_\infty\|^2_{L^2(\mathbf{R^n})} &\int_{\mathbf{R^n}} \left(s4|\nabla \hat v|^2 - \hat v^2 \ln \hat v^2 - \frac{n}{2}\ln(4\pi s)\, \hat v^2 - n\hat v^2\right) dx\\ &- \|v_\infty\|^2_{L^2(\mathbf{R^n})} \ln \|v_\infty\|^2_{L^2(\mathbf{R^n})} \ge 0.\end{aligned}$$

Here we just used the fact that the best constant for the log Sobolev inequality for functions with unit L^2 norms in $\mathbf{R^n}$ is 0. This is a contradiction with the assumption that $\lambda(g,1,M) < 0$. This proves that F_k is uniformly bounded.

The uniform boundedness of v_k comes from the following arguments. By (3.19), we know that v_k satisfies

$$\frac{n}{2}(4\Delta v_k - R v_k) + F_k 2 v_k \ln v_k + F_k \beta_k v_k = 0,$$

where

$$\beta_k = \lambda(g,1,B_k) + \frac{n}{2} - \frac{n}{2}\ln F_k - s_n.$$

Since F_k is uniformly bounded, we know that the coefficients in the above equation are uniformly bounded. As explained at the end of the proof of Lemma 3.2, it is easy to show that $\|v_k\|_{L^\infty}$ is also uniformly bounded. This proves the lemma. □

Now we are ready to give

Proof. **of Theorem 1.2.**

We will use the minimizers v_k on balls of radius r_k to construct a minimizer on the whole manifold. The core argument is to show that v_k has a non vanishing limit.

Step 1.

Pick $r_k \to \infty$ and let v_k be a minimizer for $L(\cdot, g, 1, B(x_0, r_k))$ whose infimum is λ_k. Then

$$\begin{aligned}\lambda_k &= L(v_k, g, 1, B(x_0, r_k))\\ &= -\int_{B(x_0,r_k)} v_k^2 \ln v_k^2 dg + \frac{n}{2}\ln\left(\int_{B(x_0,r_k)} (4|\nabla v_k|^2 + R v_k^2) dg\right) + s_n.\end{aligned} \tag{3.25}$$

According to the previous 2 lemmas, v_k exists and is uniformly bounded. By standard elliptic theory, a subsequence of $\{v_k\}$, still denoted by the same symbol, converges in C^∞_{loc} sense, to a limit function $v_\infty \in C^\infty(M)$. In this step, we prove that v_∞ is not 0. We will use P. L. Lion's concentrated compactness method at infinity. But a new twist occurs. That is, even though λ_k is bounded, the components on the right hand side of (3.25) may not be bounded from below uniformly.

Suppose for contradiction that $v_\infty = 0$. Then $v_k \to 0$ a.e. as $k \to \infty$. Then there exists a sequence of positive integers $\{i_k\}$ and a subsequence of $\{v_k\}$, denoted by the same symbol, such that $i_k \to \infty$ as $k \to \infty$ and that

$$\int_{B(x_0, 2^{2i_k})} v_k^2 dg \to 0, \qquad k \to \infty. \tag{3.26}$$

For any positive integer i we introduce the following notations

$$\Omega_i = B(x_0, 2^i) - B(x_0, 2^{i-1}),$$
$$F(v_k) = \int_M (4|\nabla v_k|^2 + R v_k^2) dg, \quad N(v_k) = \int_M v_k^2 \ln v_k^2 dg.$$

Here v_k is considered 0 outside of the ball $B(x_0, r_k)$.

By $\lambda \equiv \lambda(g) = \lambda(g, 1) > -\infty$ in assumption (a) of the theorem and Proposition 2.4, there exists a positive constant A such that

$$\left(\int_{B(x_0, r_k)} v_k^{2n/(n-2)} dg \right)^{(n-2)/n} \le A F(v_k).$$

Hence

(3.27)
$$\left(\Sigma_{i=i_k}^{2i_k} \int_{\Omega_i} v_k^{2n/(n-2)} dg \right)^{(n-2)/n} e^{-N(v_k)2/n} \le \left(\int_{B(x_0, r_k)} v_k^{2n/(n-2)} dg \right)^{(n-2)/n} e^{-N(v_k)2/n}$$
$$\le C F(v_k) e^{-N(v_k)2/n} = C e^{(\lambda_k - s_n)2/n} \le C,$$

where we also used (3.25) and the fact that λ_k is uniformly bounded. Thus, there exists an integer $j_k \in [i_k, 2i_k]$ such that

$$\left(\int_{\Omega_{j_k}} v_k^{2n/(n-2)} dg \right)^{(n-2)/n} \le C i_k^{-(n-2)/n} e^{N(v_k)2/n} \tag{3.28}$$

By partition of unity, we can choose a sequence of cut-off functions ϕ_k, η_k on M such that $\phi_k = 1$ on $B(x_0, 2^{j_k-1})$, $supp\, \phi_k \subset B(x_0, 2^{j_k})$; $\eta_k = 1$ on $M - B(x_0, 2^{j_k})$, $supp\, \eta_k \subset M - B(x_0, 2^{j_k-1})$; $|\nabla \phi_k| + |\nabla \phi_k| \le C/2^{j_k}$; $\phi_k^2 + \eta_k^2 = 1$. We introduce the notations

$$a_k \equiv \|v_k \phi_k\|_{L^2}^2, \quad b_k \equiv \|v_k \eta_k\|_{L^2}^2;$$

$$A_k \equiv \exp(\frac{2}{n} N(v_k \phi_k)), \quad B_k \equiv \exp(\frac{2}{n} N(v_k \eta_k)).$$

By (3.26), we know that

$$a_k \to 0, \quad b_k \to 1, \quad \text{as} \quad k \to \infty. \tag{3.29}$$

Now we will split the terms in the log Sobolev functional into terms involving $v_k\phi_k$ and $v_k\eta_k$. By direct computation

$$
\begin{aligned}
&\int (4|\nabla v_k|^2 + R v_k^2) dg \\
&= \int (4|\nabla (v_k\phi_k)|^2 + R(v_k\phi_k)^2) dg + \int (4|\nabla (v_k\eta_k)|^2 + R(v_k\eta_k)^2) dg \\
&\quad - 4\int (|\nabla \phi_k|^2 + |\nabla \eta_k|^2) v_k^2 dg,
\end{aligned} \tag{3.30}
$$

where we have used the identity

$$0 = \Delta(\phi_k^2 + \eta_k^2) = 2|\nabla \phi_k|^2 + 2\phi_k \Delta \phi_k + 2|\nabla \eta_k|^2 + 2\eta_k \Delta \eta_k.$$

Suppose Condition (b) on volume of geodesic balls holds, namely $|B(x_0, r)| \le C r^n$. Using Hölder's inequality we deduce

$$
\begin{aligned}
4\int (|\nabla \phi_k|^2 + |\nabla \eta_k|^2) v_k^2 dg &\le C 2^{-2j_k} \int_{\Omega_{j_k}} v_k^2 dg \le C 2^{-2j_k} |\Omega_{j_k}|^{2/n} \left(\int_{\Omega_{j_k}} v_k^{2n/(n-2)} dg \right)^{(n-2)/n} \\
&\le C \left(\int_{\Omega_{j_k}} v_k^{2n/(n-2)} dg \right)^{(n-2)/n}.
\end{aligned}
$$

Using (3.28), we know that

$$4\int (|\nabla \phi_k|^2 + |\nabla \eta_k|^2) v_k^2 dg = o(1) e^{N(v_k)2/n}. \tag{3.31}$$

Here $o(1)$ is a quantity that goes to 0 when $k \to \infty$. This and (3.30) imply

$$F(v_k) = F(v_k\phi_k) + F(v_k\eta_k) - o(1) e^{N(v_k)2/n}. \tag{3.32}$$

Now, suppose Condition (b) on the scalar curvature holds, namely $R(x) \ge \frac{c}{1+d^2(x_0,x)}$. Then

$$4\int (|\nabla \phi_k|^2 + |\nabla \eta_k|^2) v_k^2 dg \le C 2^{-2j_k} \int_{\Omega_{j_k}} v_k^2 dg \le C \int_{\Omega_{j_k}} R v_k^2 dg.$$

By the second line of (3.27), we have

$$\Sigma_{i=i_k}^{2i_k} \int_{\Omega_i} (4|\nabla v_k|^2 + R v_k^2) dg \le C e^{2N(v_k)/n}.$$

Therefore one can also find a $j_k \in [i_k, 2i_k]$ such that (3.31) and (3.32) hold.

Next, observe that

$$
\begin{aligned}
&\int v_k^2 \ln v_k^2 dg - \int (v_k\phi_k)^2 \ln (v_k\phi_k)^2 dg - \int (v_k\eta_k)^2 \ln (v_k\eta_k)^2 dg \\
&= \int (v_k\phi_k)^2 \left[\ln((v_k\phi_k)^2 + (v_k\eta_k)^2) - \ln (v_k\phi_k)^2 \right] dg \\
&\quad + \int (v_k\eta_k)^2 \left[\ln((v_k\phi_k)^2 + (v_k\eta_k)^2) - \ln (v_k\eta_k)^2 \right] dg \\
&\le C \int v_k^4 \phi_k^2 \eta_k^2 dg \le C \int_{\Omega_{j_k}} v_k^2 \, dg.
\end{aligned}
$$

Here we just used the uniform boundedness of v_k, proven in Lemma 3.3. This means

$$N(v_k) = N(v_k\phi_k) + N(v_k\eta_k) + o(1). \tag{3.33}$$

Recall that v_k is a minimizer for the log Sobolev functional. By (3.25),

$$e^{\frac{2}{n}(\lambda_k - s_n)} = \frac{F(v_k)}{\exp(\frac{2}{n}N(v_k))}. \tag{3.34}$$

By (3.32) and (3.33), this implies

$$\begin{aligned} e^{\frac{2}{n}(\lambda_k - s_n)} &= \frac{F(v_k\phi_k) + F(v_k\eta_k) + o(1)\exp(\frac{2}{n}N(v_k))}{\exp(\frac{2}{n}N(v_k))} \\ &= \frac{F(v_k\phi_k) + F(v_k\eta_k)}{\exp(\frac{2}{n}N(v_k\phi_k))\ \exp(\frac{2}{n}N(v_k\eta_k))\, e^{o(1)}} + o(1). \end{aligned}$$

On the other hand, by definition of λ_k, it holds

$$F(v_k\phi_k) \ge e^{\frac{2}{n}(\lambda_k - s_n)} \|v_k\phi_k\|^2_{L^2} \exp\left(-\frac{2}{n}\ln \|v_k\phi_k\|^2_{L^2}\right) \exp\left(\frac{2}{n}N(v_k\phi_k)/\|v_k\phi_k\|^2_{L^2}\right).$$

Since the support of η_k is outside of the ball $B(x_0, 2^{j_k-1})$, by definition of $\lambda_\infty \equiv \lambda_\infty(g)$ in Definition 1.1, we know

$$F(v_k\eta_k) \ge e^{\frac{2}{n}(\lambda_\infty - s_n + o(1))} \|v_k\eta_k\|^2_{L^2} \exp\left(-\frac{2}{n}\ln \|v_k\eta_k\|^2_{L^2}\right) \exp\left(\frac{2}{n}N(v_k\eta_k)/\|v_k\eta_k\|^2_{L^2}\right).$$

Write $\lambda = \lambda(g, 1)$. Combining the last three expressions, we deduce, since $\lambda_k = \lambda + o(1)$ that

$$1 \ge \frac{a_k^{-2/n} a_k A_k^{1/a_k} + b_k^{-2/n} b_k B_k^{1/b_k} e^{(\lambda_\infty - \lambda)2/n + o(1)}}{A_k B_k e^{o(1)}} + o(1),$$

where

$$\begin{aligned} a_k &\equiv \|v_k\phi_k\|^2_{L^2}, \quad b_k \equiv \|v_k\eta_k\|^2_{L^2}; \\ A_k &\equiv \exp(\frac{2}{n}N(v_k\phi_k)), \quad B_k \equiv \exp(\frac{2}{n}N(v_k\eta_k)). \end{aligned}$$

Therefore

$$\min\{a_k^{-2/n},\, b_k^{-2/n}\}\, \frac{a_k A_k^{1/a_k} + b_k B_k^{1/b_k} e^{(\lambda_\infty - \lambda)2/n + o(1)}}{A_k B_k e^{o(1)}} + o(1) \le 1,$$

Since a_k and b_k are positive numbers in the interval $(0, 1)$, this shows

$$\ln(a_k A_k^{1/a_k} + b_k B_k^{1/b_k} e^{(\lambda_\infty - \lambda)2/n + o(1)}) \le \ln(A_k B_k) + o(1).$$

Notice that $a_k + b_k = 1$. By concavity of ln function we obtain

$$b_k(\lambda_\infty - \lambda)2/n + o(1) \le o(1).$$

Letting $k \to \infty$ and using the fact that $b_k \to 1$ (from (3.29)), we arrive at

$$0 < \lambda_\infty - \lambda \le 0.$$

This is a contradiction which proves that v_∞ is not identically zero.

Step 2. We prove $\|v_\infty\|_{L^2(M)} = 1$.

This is done by adopting a method by Dolbeault and Esteban [DE], which is in the spirit of P. L. Lions' concentrated compactness.

Suppose for contradiction that $\|v_\infty\|_{L^\infty(M)} = \delta < 1$. Then for all large integer k, there exists $l_k > 0$ such that $l_k \to \infty$ when $k \to \infty$ and

$$\int_{B(x_0,l_k)} v_\infty^2 dg = \delta - \frac{1}{k}, \qquad \int_{B(x_0,4l_k)-B(x_0,l_k)} v_\infty^2 dg \le \frac{1}{k}.$$

Fixing this k for the moment, by C^∞_{loc} convergence of v_k to v_∞ and the fact that the L^2 norm of v_k is 1, we can find a subsequence $\{n_k\}$ of positive integers so that

$$\delta - \frac{2}{k} \le \int_{B(x_0,l_k)} v_{n_k}^2 dg \le \delta - \frac{1}{2k}, \quad \int_{B(x_0,4l_k)-B(x_0,l_k)} v_{n_k}^2 dg \le \frac{2}{k},$$

and that

$$1 - \delta - \frac{2}{k} \le \int_{M-B(x_0,4l_k)} v_{n_k}^2 dg \le 1 - \delta + \frac{2}{k}.$$

Renaming n_k as k, we have found a subsequence of $\{v_k\}$, which is still denoted by $\{v_k\}$, such that

$$\begin{aligned} &\lim_{k\to\infty} \int_{B(x_0,l_k)} v_k^2 dg = \delta, \qquad \lim_{k\to\infty} \int_{B(x_0,4l_k)-B(x_0,l_k)} v_k^2 dg = 0, \\ &\lim_{k\to\infty} \int_{M-B(x_0,4l_k)} v_k^2 dg = 1 - \delta. \end{aligned} \tag{3.35}$$

By partition of unity, we can choose a sequence of cut-off functions ϕ_k, η_k on (M, x_0, g) such that $\phi_k = 1$ on $B(x_0, l_k)$, $\quad supp\,\phi_k \subset B(x_0, 2l_k)$; $\eta_k = 1$ on $M - B(x_0, 2l_k)$, $supp\,\eta_k \subset M - B(x_0, l_k)$; $|\nabla\phi_k| + |\nabla\phi_k| \le C/l_k$; $\phi_k^2 + \eta_k^2 = 1$. Using (3.35), we know that

$$\lim_{k\to\infty} \int_{B(x_0,l_k)} (v_k\phi_k)^2 dg = \delta, \qquad \lim_{k\to\infty} \int_{M-B(x_0,4l_k)} (v_k\eta_k)^2 dg = 1 - \delta. \tag{3.36}$$

Next we will again split the terms in the log Sobolev functional into terms involving $v_k\phi_k$ and $v_k\eta_k$. Since $|\nabla\phi_k| + |\nabla\eta_k| \to 0$ when $k \to \infty$, it is easy to see that

$$\begin{aligned} &\int (4|\nabla v_k|^2 + R v_k^2) dg \\ &= \int (4|\nabla(v_k\phi_k)|^2 + R(v_k\phi_k)^2) dg + \int (4|\nabla(v_k\eta_k)|^2 + R(v_k\eta_k)^2) dg + o(1). \end{aligned} \tag{3.37}$$

Here $o(1)$ is a quantity that goes to 0 when $k \to \infty$. As in Step 1,

$$\begin{aligned} &\int v_k^2 \ln v_k^2 dg - \int (v_k\phi_k)^2 \ln(v_k\phi_k)^2 dg - \int (v_k\eta_k)^2 \ln(v_k\eta_k)^2 dg \\ &= \int (v_k\phi_k)^2 \left[\ln((v_k\phi_k)^2 + (v_k\eta_k)^2) - \ln(v_k\phi_k)^2\right] dg \\ &\quad + \int (v_k\eta_k)^2 \left[\ln((v_k\phi_k)^2 + (v_k\eta_k)^2) - \ln(v_k\eta_k)^2\right] dg \\ &\le C \int v_k^4 \phi_k^2 \eta_k^2 dg \le C \int_{B(x_k,4l_k)-B(x_k,l_k)} v_k^2\, dg. \end{aligned}$$

Here we just used the uniform boundedness of v_k, proven in Lemma 3.3. This and (3.35) shows

$$\int v_k^2 \ln v_k^2 dg = \int (v_k\phi_k)^2 \ln(v_k\phi_k)^2 dg + \int (v_k\eta_k)^2 \ln(v_k\eta_k)^2 dg + o(1). \tag{3.38}$$

Recall that v_k is a minimizer for $\lambda_k \equiv \lambda(g, 1, B(x_0, r_k))$. By (3.25),

$$e^{\frac{2}{n}(\lambda_k - s_n)} = \frac{F(v_k)}{\exp(\frac{2}{n}N(v_k))}.$$

By (3.37) and (3.38), this implies

$$e^{\frac{2}{n}(\lambda_k - s_n)} = \frac{F(v_k\phi_k) + F(v_k\eta_k)}{\exp(\frac{2}{n}N(v_k\phi_k))\ \exp(\frac{2}{n}N(v_k\eta_k))} + o(1). \tag{3.39}$$

Here we just used the fact that $\exp(\frac{2}{n}N(v_k))$ is bounded away from zero. The reason is

$$\liminf_{k\to\infty} \exp(\frac{2}{n}N(v_k)) = \liminf_{k\to\infty} e^{-\frac{2}{n}(\lambda_k - s_n)} F(v_k) \geq e^{-\frac{2}{n}(\lambda - s_n)} F(v_\infty) > 0,$$

which is due to Step 1.

On the other hand, by definition of λ_k, it holds

$$F(v_k\phi_k) \geq e^{\frac{2}{n}(\lambda_k - s_n)} \|v_k\phi_k\|_{L^2}^2 \exp\left(-\frac{2}{n}\ln\|v_k\phi_k\|_{L^2}^2\right) \exp\left(\frac{2}{n}N(v_k\phi_k)/\|v_k\phi_k\|_{L^2}^2\right);$$

$$F(v_k\eta_k) \geq e^{\frac{2}{n}(\lambda_k - s_n)} \|v_k\eta_k\|_{L^2}^2 \exp\left(-\frac{2}{n}\ln\|v_k\eta_k\|_{L^2}^2\right) \exp\left(\frac{2}{n}N(v_k\eta_k)/\|v_k\eta_k\|_{L^2}^2\right).$$

Plugging the last two expressions into (3.39), we deduce

$$\frac{a_k^{-2/n} a_k A_k^{1/a_k} + b_k^{-2/n} b_k B_k^{1/b_k}}{A_k B_k} \leq 1 + o(1),$$

where

$$a_k \equiv \|v_k\phi_k\|_{L^2}^2, \quad b_k \equiv \|v_k\eta_k\|_{L^2}^2;$$

$$A_k \equiv \exp(\frac{2}{n}N(v_k\phi_k)), \quad B_k \equiv \exp(\frac{2}{n}N(v_k\eta_k)).$$

Therefore

$$\min\{a_k^{-2/n}, b_k^{-2/n}\} \frac{a_k A_k^{1/a_k} + b_k B_k^{1/b_k}}{A_k B_k} \leq 1 + o(1),$$

Notice that $a_k + b_k = 1$. Therefore we have the Young's inequality: $\frac{a_k A_k^{1/a_k} + b_k B_k^{1/b_k}}{A_k B_k} \geq 1$. Letting $k \to \infty$ and using (3.36), we arrive at

$$\min\{\delta^{-2/n}, (1-\delta)^{-2/n}\} \leq 1.$$

This is a contradiction with the assumption that $\delta = \|v_\infty\|_{L^2(M_\infty, g_\infty(0))} < 1$.

Step 3. Finally we prove that v_∞ is a minimizer.

Using Fatou's Lemma, it is clear that $F(v) \leq \lim_{k\to\infty} F(v_k)$. We claim that

$$N(v_\infty) \geq \lim_{k\to\infty} N(v_k),$$

which is a reversed inequality comparing with that in Fatou's lemma. Here goes the proof. Let C be a uniform upper bound for $\|v_k\|_\infty$. Then $\ln(C/v_k)^2 \ge 0$. By Fatou's lemma

$$\int v_\infty^2 \ln(C/v_\infty)^2 dg \le \lim_{k\to\infty} \int v_k^2 \ln(C/v_k)^2 dg,$$

Since $\|v_\infty\|_{L^2} = \|v_k\|_{L^2} = 1$, the above shows

$$N(v_\infty) = \int v_\infty^2 \ln v_\infty^2 dg \ge \lim_{k\to\infty} \int v_k^2 \ln v_k^2 dg = \lim_{k\to\infty} N(v_k),$$

which is the claim.

Taking $k \to \infty$ in (3.34), using the claim and Fatou's lemma on $F(v_k)$, we deduce

$$e^{\frac{2}{n}(\lambda - s_n)} = \lim_{k\to\infty} e^{\frac{2}{n}(\lambda_k - s_n)} = \lim_{k\to\infty} \frac{F(v_k)}{\exp(\frac{2}{n}N(v_k))} \ge \frac{F(v_\infty)}{\exp(\frac{2}{n}N(v_\infty))}.$$

Taking ln on both sides, we see that v_∞ is a minimizer. From here, it is straight forward to see that v_∞ satisfies equation (1.2). □

Now we are ready to give

Proof of Theorem 1.1.

For simplicity, we use the notations $L(v,g) \equiv L(v,g,1,M)$ and $\lambda(g) \equiv \lambda(g,1,M)$ during the proof.

First we claim that $\lambda(g)$ is invariant under scaling and diffeomorphism. The proof is quite easy. But we present it here to stress its independence on the behavior of the diffeomorphism at infinity. Given any positive number a. It is clear that $L(v,g) = L(a^{-n/4}v, ag)$ and $\|v\|_{L^2(g)} = \|a^{-n/4}v\|_{L^2(ag)}$. Hence $\lambda(g)$ is invariant under scaling.

Next, let ψ be a diffeomorphism on M and write $h = \psi^* g$. For any $v \in C_0^\infty(M)$, we have

$$\int_M (4|\nabla v|^2 + Rv^2) dg = \int_M (4|\nabla_h (v \circ \psi^{-1})|^2 + R(v\circ\psi^{-1})^2) dh,$$

$$\int_M v^2 \ln v^2 dg = \int_M (v\circ\psi^{-1})^2 \ln(v\circ\psi^{-1})^2 dh.$$

These imply $L(v,g) = L(v\circ\psi^{-1}, \psi^* g)$. Taking the infimum on both sides, we see that $\lambda(g)$ is also invariant under diffeomorphism.

Hence, we know from the assumption $g(t_2) = c\psi^* g(t_1)$ that

$$\lambda(g(t_1)) - \lambda(g(t_2)) = 0. \tag{3.40}$$

According to Theorem 1.2, there exists a function $v_2 \in W^{1,2}(M, g(t_2))$, which is a minimizer for $\lambda(g(t_2))$, i.e.

$$L(v_2, g(t_2)) = L(v_2, g(t_2), 1, M) = \lambda(g(t_2)). \tag{3.41}$$

Moreover, by Moser's iteration, it is known, as done in Lemma 2.3 in [Z2], v_2 has Gaussian type decay at infinity.

Next, we solve the conjugate heat equation for $t < t_2$, with final value as v_2^2. This solution is denoted by $u = u(x,t)$. Write $v = \sqrt{u}$, then by Definition 1.1

$$L(v, g(t)) = -N(v) + \frac{n}{2}\ln F(v) + s_n,$$

where, due to $v = \sqrt{u}$,

$$N(v) = \int_M u \ln u \, dg(t); \quad F(v) = \int_M (\frac{|\nabla u|^2}{u} + Ru) dg(t) = \int_M (4|\nabla v|^2 + Rv^2) dg(t).$$

According to Perelman [P] Section 1, $\frac{d}{dt} N(v) = F(v)$ and

$$\frac{d}{dt} F(v) = 2 \int_M |Ric - Hess(\ln u)|^2 u dg(t).$$

We mention that although Perelman only proved the formulas for compact manifolds, but his proof also works for noncompact manifolds with bounded geometry when the functions involved have sufficiently fast decay such as the Gaussian function. See [C++] Chapter 19 and [CTY] e.g. for a detailed computation. In our case, the function v has Gaussian type decay at each time level just like the final value $v(t_2)$ does. Hence

$$\frac{d}{dt} L(v, g(t)) = \left(n \int_M |Ric - Hess(\ln u)|^2 u dg(t) - F^2(v) \right) F^{-1}(v). \tag{3.42}$$

Following Perelman's computation,

$$|Ric - Hess(\ln u)|^2 \geq \left| Ric - Hess(\ln u) - \frac{1}{n}(R - \Delta \ln u) g \right|^2 + \frac{1}{n}(R - \Delta \ln u)^2;$$

Using the relation $F(v) = \int_M (R - \Delta \ln u) u \, dg(t)$, we deduce

$$\frac{d}{dt} L(\sqrt{u}, g(t)) \geq \frac{Q(u)}{F(v)} \geq 0 \tag{3.43}$$

where

$$\begin{aligned} Q(u)(t) = n \int_M |Ric - Hess(\ln u) - \frac{1}{n}(R - \Delta \ln u) g|^2 u dg(t) \\ + \int_M (R - \Delta \ln u)^2 u \, dg(t) - \left(\int_M (R - \Delta \ln u) u \, dg(t) \right)^2. \end{aligned} \tag{3.44}$$

Observe that $\sqrt{u(\cdot, t_2)} = v_2(\cdot)$ by definition. So by (3.41) we deduce

$$\begin{aligned} \int_{t_1}^{t_2} \frac{d}{dt} L(\sqrt{u}, g(t)) dt &= L(\sqrt{u(\cdot, t_2)}, g(t_2)) - L(\sqrt{u(\cdot, t_1)}, g(t_1)) \\ &\leq \lambda(g(t_2)) - \lambda(g(t_1)) = 0. \end{aligned}$$

The last line is due to (3.40). By (3.43), we then have

$$F^{-1}(v) Q(u) = 0. \tag{3.45}$$

By (3.44), this shows that $(R - \Delta \ln u)(\cdot, t) = l(t)$, where $l = l(t)$ is a function of t only. Also

$$Ric - Hess(\ln u) - \frac{1}{n} l(t) g = 0.$$

Therefore, $(M, g(t))$ is a gradient Ricci soliton. □

Acknowledgment. We wish to thank Professors Xiaodong Cao and Xiouxiong Chen for very helpful suggestions.

References

[BKN] Bando, Shigetoshi; Kasue, Atsushi; Nakajima, Hiraku, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth.* Invent. Math. 97 (1989), no. 2, 313349.

[BL] Brezis, Haim; Lieb, Elliott, *A relation between pointwise convergence of functions and convergence of functionals.* Proc. Amer. Math. Soc. 88 (1983), no. 3, 486490.

[Ca] Cao, Xiaodong, *Eigenvalues of* $(-\Delta + \frac{R}{2})$ *on manifolds with nonnegative curvature operator.* Math. Ann. 337 (2007), no. 2, 435-441.

[Ch] Chen, Bing-Long, *Strong uniqueness of the Ricci flow.* J. Differential Geom. 82 (2009), no. 2, 363-382.

[CTY] Albert Chau, Luen-Fai Tam and Chengjie Yu. *Pseudolocality for the Ricci flow and applications.* Canad. J. Math. 63 (2011), no. 1, 5585

[C++] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. *The Ricci flow: techniques and applications. Part III.* Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI. Geometric-Analysis aspects.

[Da] Davies, E. B. *Heat kernels and spectral theory.* Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1989.

[DE] Jean Dolbeault and Maria J. Esteban. *Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities,* Preprint Ceremade no. 1005, (2010).

[DM] Dai, Xianzhe; Ma, Li, *Mass under the Ricci flow.* Comm. Math. Phys. 274 (2007), no. 1, 6580.

[F] Federbush, P. *Partially Alternate Derivation of a Result of Nelson* J. Math. Physics, Vol. 10, no 1 Jan. 1969, 50-53.

[G] Gross, Leonard , *Logarithmic Sobolev inequalities.* Amer. J. Math. 97 (1975), no. 4, 1061-1083.

[Ha] Robert Haslhofer, *Title: A mass-decreasing flow in dimension three* arXiv:1107.3220

[I] Ivey, Thomas, *Ricci solitons on compact three-manifolds.* Differential Geom. Appl. 3 (1993), no. 4, 301307.

[L] Li, Jun-Fang, *Eigenvalues and energy functionals with monotonicity formulae under Ricci flow.* Math. Ann. 338 (2007), no. 4, 927-946.

[Lio] Lions, P.-L. *The concentration-compactness principle in the calculus of variations.* The limit case. I. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145-201.

[OW] Oliynyk, Todd A.; Woolgar, Eric, *Rotationally symmetric Ricci flow on asymptotically flat manifolds.* Comm. Anal. Geom. 15 (2007), no. 3, 535-568.

[P] Perelman, Grisha, *The entropy formula for the Ricci flow and its geometric applications*, Math. ArXiv, math.DG/0211159.

[P2] Grisha Perelman, *Ricci flow with surgery on three manifolds*, arXiv.org/ math.DG/0303109.

[Rot] Rothaus, O. S. *Logarithmic Sobolev inequalities and the spectrum of Schrdinger operators.* J. Funct. Anal. 42 (1981), no. 1, 110-120.

[W] Weissler, Fred B. *Logarithmic Sobolev inequalities for the heat-diffusion semigroup.* Trans. Amer. Math. Soc. 237 (1978), 255269.

[Z] Qi S. Zhang. *Sobolev inequalities, heat kernels under Ricci flow and the Poincaré conjecture.* CRC Press, Boca Raton, FL, 2011.

[Z2] Qi S. Zhang *Extremal of Log Sobolev inequality and W entropy on noncompact manifolds*, arXiv:1105.1544.

e-mail: qizhang@math.ucr.edu

Department of Mathematics, University of California, Riverside, CA 92521, USA