Uniquely dimensional graphs

Behrooz Bagheri Gh., Mohsen Jannesari, Behnaz Omoomi

Department of Mathematical Sciences

Isfahan University of Technology

84156-83111, Isfahan, Iran

Abstract

A set $W \subseteq V(G)$ is called a resolving set, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$, where $d(x, y)$ is the distance between the vertices x and y. A resolving set for G with minimum cardinality is called a metric basis. A graph with a unique metric basis is called a uniquely dimensional graph. In this paper, we study some properties of uniquely dimensional graphs.

Keywords: Resolving set; Metric basis; Uniquely dimensional.

1 Introduction

Throughout the paper, $G = (V, E)$ is a finite, simple, and connected graph of order n. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. For a vertex $v \in V(G)$, $\Gamma_i(v) = \{u \mid d(u, v) = i\}$. The diameter of G is $\text{diam(G)} = \max\{d(u, v) \mid u, v \in V(G)\}$. The girth of G is the length of a shortest cycle in G. The set of all adjacent vertices to a vertex v is denoted by $N(v)$ and $|N(v)|$ is the degree of a vertex v, $\deg(v)$. The maximum degree and the minimum degree of a graph G, are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relations between u and v, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the *metric representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different metric representations. A resolving set for G with minimum cardinality is called a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$. If $\beta(G) = k$, then G is said to be k-dimensional.

In [14], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for what we call a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [7] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [3, 4, 6, 11]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [13], network discovery and verification [1], robot navigation [11], mastermind game [3], problems of pattern recognition and image processing [12], and combinatorial search and optimization [13].

It is obvious that to see whether a given set W is a resolving set, it is sufficient to consider the vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex in G for which $d(w,w) = 0$. When W is a resolving set for G, we say that W *resolves* G. In general, we say an ordered set W resolves a set $T \subseteq V(G)$, if for each two distinct vertices $u, v \in T$, $r(u|W) \neq r(v|W)$.

The following bound is the known upper bound for the metric dimension.

Theorem A. [5] *If G is a connected graph of order n and diameter d, then $\beta(G) \leq n - d$.*

In [9, 10], the properties of k-dimensional graphs in which every k subset of vertices is a metric basis are studied. Such graphs are called randomly k-dimensional graphs. In the opposite point there are graphs which have a unique metric basis.

Definition. *A graph G is called* uniquely dimensional *if G has a unique metric basis. A uniquely dimensional graph G with $\beta(G) = k$ is called a* uniquely k-dimensional *graph.*

In this paper, we first obtain some upper bounds for the metric dimension of uniquely dimensional graphs. Then, we give some construction for uniquely k-dimensional graphs of the given order. Finally, we obtain a lower bound and an upper bound for the minimum order of uniquely k-dimensional graphs in terms of k.

2 Some upper bounds

In this section we obtain some upper bounds for the metric dimension of uniquely dimensional graphs.

Two vertices $u, v \in V(G)$ are called *twin* vertices if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. It is known that, if u and v are twin vertices, then every resolving set W for G contains at least one of the vertices u and v. Moreover, if $u \notin W$ then $(W \setminus v) \cup \{u\}$ is also a resolving set for G. [8]

For a uniquely dimensional graph we have the following fact.

Lemma 1. *If G is a uniquely dimensional graph, then G contains no twin vertices.*

Proof. Let B be the unique metric basis of G. If $u, v \in V(G)$ are twin vertices, then $u, v \in B$; otherwise we can replace the one in B with the other one. Now, since $B \setminus \{u\}$ is not a basis of G, there is exactly one vertex $w \in V(G) \setminus B$ such that $r(u|B \setminus \{u\}) = r(w|B \setminus \{u\})$. Consequently, $(B \setminus \{u\}) \cup \{w\}$ is a metric basis of G different from B, which is a contradiction. ■

Theorem 1. *If G is a uniquely dimensional graph of order n and diameter d, then $\beta(G) \leq n-d-2$.*

Proof. Let $(v_0, v_1, \ldots, v_d)$ be a path of length d in G. Two sets $V(G) \setminus \{v_1, v_2, \ldots, v_d\}$ and $V(G) \setminus \{v_0, v_1, \ldots, v_{d-1}\}$ are two resolving set of G of size $n-d$. Hence, if G is uniquely dimensional, then $\beta(G) \leq n-d-1$. To complete the proof we show that $\beta(G) \neq n-d-1$.

Let $\beta(G) = n-d-1$ and for each i, $1 \leq i \leq d$, $\Gamma_i = \Gamma_i(v_0)$. We claim that for each i, $1 \leq i \leq d$, Γ_i is an independent set or a clique; otherwise there exists an i for which Γ_i contains vertices x, y, z such that $x \sim y$ and $x \nsim z$. Therefore, $V(G) \setminus \{y, z, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is a metric basis of G. Now, if $y \nsim z$, then $V(G) \setminus \{x, z, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ and if $y \sim z$, then $V(G) \setminus \{x, y, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is another metric basis of G, respectively, which both are contradictions. Consequently, for each i, $1 \leq i \leq d$, Γ_i is an independent set or a clique.

Now let for some i, $1 \leq i \leq d$, $|\Gamma_i| \geq 2$. Then, all vertices in Γ_i are adjacent to all vertices in Γ_{i-1}; otherwise there exist $a \in \Gamma_{i-1}$ and $x \in \Gamma_i$ such that $a \nsim x$. Therefore, x has a neighbor in Γ_{i-1}, say b. Assume that $y \in \Gamma_i$ and $y \neq x$. Clearly $i \geq 2$. Thus, $V(G) \setminus \{a, b, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is a metric basis of G. Now, if $y \sim a$, then $V(G) \setminus \{b, x, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$, and if $y \nsim b$, then $V(G) \setminus \{a, x, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is another metric basis of G, respectively. These contradictions imply that $y \nsim a$ and $y \sim b$. Hence, $V(G) \setminus \{a, b, x, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is a metric basis of G, which is also a contradiction. Consequently, all vertices in Γ_i are adjacent to all vertices in Γ_{i-1}.

The above two facts imply that, if $|\Gamma_i| \geq 2$ and $|\Gamma_{i+1}| \geq 2$, then all vertices in Γ_i have the same neighbors in $\Gamma_{i-1} \cup \Gamma_i \cup \Gamma_{i+1}$. Therefore, all vertices $u, v \in \Gamma_i$ are twin vertices, which by Lemma 1 this is impossible. Thus, $|\Gamma_i| \geq 2$ implies that $|\Gamma_{i+1}| = 1$ and $|\Gamma_{i-1}| = 1$. Hence, if $|\Gamma_i| > 2$, then since $\Gamma_{i+1} = \{v_{i+1}\}$, by the Pigenhole principle there are two vertices $u, v \in \Gamma_i$ with the same adjacency relation with v_{i+1} . Therefore, u and v are twin vertices, which is impossible. That is, for each i, $1 \leq i \leq d$, $|\Gamma_i| \leq 2$. Now let j be the largest integer in $\{1, 2, \ldots, d\}$ with $|\Gamma_j| = 2$ and $\Gamma_j = \{v_j, y_j\}$, where y_j is the vertex with no neighbor in Γ_{j+1}. Therefore, the sets $\{v_0, v_d\}$ and $\{v_0, y_j\}$ are two metric bases of G. This contradiction implies that $\beta(G) \neq n-d-1$. ■

Theorem 2. *If G is a uniquely dimensional graph of order n and girth g, then $\beta(G) \leq n-g+1$.*

Proof. Let $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in G. Then $V(G) \setminus \{v_3, v_4, \ldots, v_g\}$ and $V(G) \setminus \{v_2, v_3, \ldots, v_{g-1}\}$ are two resolving set for G of size $n-g+2$. Since G has a unique basis, none of these two sets is a metric basis of G. Therefore, $\beta(G) \leq n-g+1$. ■

Theorem 3. *If G is a uniquely dimensional graph of order n, then $\beta(G) < \frac{n}{2}$.*

Proof. By the contrary assume that G has a unique metric basis $B = \{v_1, v_2, \ldots, v_k\}$ and $n \leq 2k$. Since $k \leq n-1$, $W = (V(G) \setminus B) \cup \{v_1, v_2, \ldots, v_{2k-n}\} \neq B$ with $|W| = k$. Therefore, W is not a basis of G and there exist vertices $x, y \in V(G) \setminus W \subseteq B$ such that $r(x|W) = r(y|W)$. Say $x = v_i$ and $y = v_j$. Hence, for each $v \in V(G) \setminus B$, $d(v, v_i) = d(v, v_j)$. By this reason, $B \setminus \{v_i\}$ resolves $V(G) \setminus B$. Therefore, there is exactly one vertex $u \in V(G) \setminus B$ such that $r(u|B \setminus \{v_i\}) = r(v_i|B \setminus \{v_i\})$. Consequently, $(B \setminus \{v_i\}) \cup \{u\}$ is a metric basis of G, which is a contradiction. Thus, $2\beta(G) < n$. ■

3 Construction of uniquely k-dimensional graphs

In this section, we provide some construction for uniquely k-dimensional graphs of given order. Then we end with giving a lower bound and an upper bound for the minimum number of vertices in such graphs in terms of k.

Remark 1. *Note that, if G is a graph of diameter d, then every $W \subseteq V(G)$ can resolve at most $d^{|W|}$ vertices of $V(G) \setminus W$. Hence, every k-dimensional graph of diameter d has at most $k + d^k$ vertices.*

In [2], Buczkowski et al. constructed a uniquely k-dimensional graph with diameter 2 and order $k + 2^k$.

Theorem B. [2] *For $k \geq 2$, there exists a uniquely k-dimensional graph of order $n = k + 2^k$, diameter 2, and maximum degree $n - 1$.*

In the following theorem regarding to constructing uniquely k-dimensional graphs with diameter d, we obtain two necessary conditions for the existence of k-dimensional graphs with diameter d and order $k + d^k$.

Theorem 4. *If G is a k-dimensional graph with diameter d and order $k + d^k$, then*
(i) $d \leq 3$.
(ii) *For a basis B and every $v \in B$, $|\Gamma_d(v)| \geq d^{k-1}$.*

Proof. (i) Let G be a k-dimensional graph of diameter $d \geq 4$ and order $k + d^k$. Thus, $V(G) = U \cup B$, where $U = \{u_1, u_2, \ldots, u_{d^k}\}$ and the ordered set $B = \{v_1, v_2, \ldots, v_k\}$ is a basis of G. Clearly, $\{r(u_i|B) \mid 1 \leq i \leq d^k\} = [d]^k$, where $[d]^k$ denotes the set of all k-tuples with entries in $\{1, 2, \ldots, d\}$. Without loss of generality, suppose that $r(u_1|B) = (1, 1, \ldots, 1)$ and $r(u_2|B) = (4, 1, \ldots, 1)$. Therefore, $d(v_1, v_2) \leq 2$ and $d(u_2, v_1) \leq d(u_2, v_2) + d(v_2, v_1) \leq 3$, a contradiction. Thus, $d \leq 3$.

(ii) Let $B = \{v_1, v_2, \ldots, v_k\}$. By the order and diameter of G, each k-vector with coordinates in $\{1, 2, \ldots, d\}$ is the metric representation of a vertex $u \in V(G) \setminus B$ with respect to B. Therefore, for each $v \in B$, there are d^{k-1} vertices of G that the i-th coordinate of their metric representations is d. Thus, $|\Gamma_d(v)| \geq d^{k-1}$. ■

In the following, we give a construction for uniquely k-dimensional graphs of diameter 3 and order $k + 3^k$.

Theorem 5. *For every integer $k \geq 2$, there exists a uniquely k-dimensional graph of diameter 3 and order $k + 3^k$.*

Proof. Let G be a graph with vertex set $U \cup W$, where $U = \{u_1, u_2, \dots, u_k\}$ is an independent set and W is the set of all k-tuples with entries in $\{1, 2, 3\}$ and two vertices $x, y \in W$ are adjacent if they are different in exactly one coordinate and this difference is one. Moreover, the vertex $(2, 2, \dots, 2)$ is adjacent to all vertices in W. Also, $w \in W$ is adjacent to $u_i \in U$ if the i-th coordinate of w is 1.

The vertex $(2, 2, \dots, 2)$ is adjacent to all vertices in W and $(1, 1, \dots, 1)$ is adjacent to all vertices in U, thus $\mathrm{diam}(G) \leq 3$. On the other hand, $d((3, 3, \dots, 3), u_1) = 3$. Therefore, $\mathrm{diam}(G) = 3$. Since $\mathrm{diam}(G) = 3$ and the order of G is $k + 3^k$, by Remark 1, $\beta(G) \geq k$. For each $w \in W$, $r(w|U) = w$, thus, U is a resolving set for G of size k. Hence, U is a metric basis of G.

Now since $\mathrm{diam}(\langle W \rangle) = 2$, for each $w \in W$, $|\Gamma_1(w) \cup \Gamma_2(w)| \geq 3^k - 1$ and hence $|\Gamma_3(w)| \leq k < 3^{k-1}$. Therefore, by Theorem 4(ii), no vertex of W is in a metric basis of G. Consequently, U is the unique metric basis of G. ■

By Theorems 1 and 3, if G is a uniquely k-dimensional graph of order n, then $n \geq k + d + 2$ and $n \geq 2k + 1$. Let

$$n_0(k) = \min\{n \mid \text{ there exists a uniquely } k\text{-dimensional graph of order } n\}.$$

Hence, we have $\max\{2k + 1, k + d + 2\} \leq n_0(k)$.

The following theorem shows that if a uniquely k-dimensional graph of order n_0 exists, then for every $n \geq n_0$, a uniquely k-dimensional graph of order n exists.

Theorem 6. *If G is a uniquely k-dimensional graph of order n_0, then for every $n \geq n_0$, there exists a uniquely k-dimensional graph of order n.*

Proof. Let G be a given uniquely k-dimensional graph of order n_0 and u be a vertex in the basis B. Assume that $v_0 \in V(G) \setminus B$ is a vertex that $d(v_0, u) = \max\{d(v, u) \mid v \in V(G) \setminus B\}$. We construct a graph G' by identifying an end vertex of a path P of length $n - n_0$ by v_0. By the property of v_0, B is also a resolving set for G'. Thus, $\beta(G') \leq k$. On the other hand, since every basis of G' contains at most one vertex of the path P, by replacing that vertex by v_0, we obtain a basis for G. Thus, G' is also a uniquely k-dimensional graph. ■

In the following theorem we give a recursive construction for uniquely dimensional graphs to obtain an upper bound for $n_0(G)$.

Theorem 7. *If G_i, $i = 1, 2$, is a uniquely k_i-dimensional graph of order n_i with $\Delta(G_i) = n_i - 1$, then there exists a uniquely $(k_1 + k_2)$-dimensional graph G of order $n_1 + n_2 - 1$ with $\Delta(G) = n_1 + n_2 - 2$.*

Proof. Let G_i be a uniquely k_i-dimensional graph of order n_i with the basis B_i and $v_i \in V(G_i)$ such that $\deg(v_i) = n_i - 1$, for $i = 1, 2$. Let G be a graph that obtained from joining G_1 and G_2, and then identifying v_1 and v_2, say v_0. Thus, $\deg(v_0) = n_1 + n_2 - 2$. Since for every $u \in V(G_1) \setminus \{v_1\}$ and $v \in V(G_2) \setminus \{v_2\}$, $d(u, v) = 1$, if B is a basis of G, then $B \cap V(G_i)$ is a basis of G_i, for $i = 1, 2$. Therefore, B is the unique basis of G. ■

Proposition 1. *There exists a uniquely* 3*-dimensional graph of order* 9 *and maximum degree* 8*.*

Proof. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_6\}$. Also let G be graph with $V(G) = U \cup W$ and $E(G) = \{w_i w_j \mid 1 \leq i \neq j \leq 6\} \cup \{u_i w_j \mid 1 \leq i \leq 3, j = i, i+1, 6\}$. We show that U is the unique basis of G.

Clearly, $\text{diam}(G) = 2$. Since $|V(G)| = 9$, by Remark 1, $\beta(G) \geq 3$. It is easy to see that U is resolving set and consequently is a basis of G. Now let B be another basis of G. Since $\langle W \rangle$ is a complete graph, $B \nsubseteq W$. Therefore, $|B \cap W| = 1$ or 2. If $|B \cap W| = 1$, then five vertices of W have the same representation with respect to $B \cap W$ while since $\text{diam}(G) = 2$, $B \setminus W$ can not resolve five vertices. If $|B \cap W| = 2$, then four vertices of W have the same representation with respect to $B \cap W$ while $B \setminus W$ can not resolve 4 vertices. These contradictions imply that U is the unique basis of G. ■

In the following theorem, based on the recursive construction in Theorem 7, we obtain an upper bound for $n_0(k)$.

Theorem 8. *For every k, $k \geq 2$, there exists a uniquely k-dimensional graph of order $\lceil \frac{5k}{2} + 1 \rceil$.*

Proof. Let k be a positive integer. If $k = 2k'$, then the graph G obtained by the recursive construction given in Theorm 7 from k' copies of the uniquely 2-dimensional graph of order 6, constructed in Theorem B is a uniquely k-dimensional graph of order $6k' - (k' - 1) = 5k' + 1 = \frac{5k}{2} + 1$.

If $k = 2k' + 1$, then the graph G obtained by the recursive construction given in Theorem 7 from $k' - 1$ copies of the uniquely 2-dimensional graph of order 6, constructed in Theorem B and one copy of the uniquely 3-dimensional graph of order 9 given in Proposition 1, is a uniquely k-dimensional graph of order $6(k' - 1) - (k' - 2) + 8 = 5k' + 4 = \lceil \frac{5k}{2} + 1 \rceil$. ■

Although the above theorem provides the recursive construction for uniquely k-dimentional graphs of order $\lceil \frac{5k}{2} + 1 \rceil$, to get the more explicit construction, we construct uniquely k-dimensional graphs of order $3k$, in the following theorem.

Theorem 9. *For each $k \geq 2$, there exists a uniquely k-dimensional graph of order $3k$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_{2k}\}$. Also, let G be a graph with vertex set $V(G) = U \cup W$ such that the induced subgraph $\langle W \rangle$ of G be a complete graph, U be an independent set, u_k be adjacent to w_{2i}, $1 \leq i \leq k$, and for each i, $1 \leq i \leq k-1$, u_i be adjacent to w_{2i-1} and w_{2i}. We prove that G is the desired graph.

Let w_i and w_j be two arbitrary vertices of $V(G) \setminus U = W$. If i and j have different parity, then $d(w_i, u_k) \neq d(w_j, u_k)$. If i and j have the same parity, then $\lfloor \frac{i}{2} \rfloor \neq \lfloor \frac{j}{2} \rfloor$ and hence $d(w_i, u_i) \neq d(w_j, u_i)$. Therefore, U is a resolving set for G of size k and $\beta(G) \leq k$.

Now let B be a metric basis of G. If $u_k \notin B$, then to resolve the set $\{u_1, w_1, w_2, w_{2k-1}, w_{2k}\}$, B should contain at least three vertices from this set, since $\langle W \rangle$ is a complete graph, while replacing these three vertices by u_1 and u_k provides a resolving set with smaller size. This contradiction implies that $u_k \in B$. If for some i, $1 \leq i \leq k-1$, $u_i \notin B$, then to resolve the set $\{u_i, w_{2i-1}, w_{2i}, w_{2k-1}, w_{2k}\}$, B should contain at least two vertices from $\{w_{2i-1}, w_{2i}, w_{2k-1}, w_{2k}\}$, because $\langle W \rangle$ is a complete graph. But replacing these two vertices by u_i provides a resolving set with smaller size. This contradiction implies that $U \subseteq B$. Since U is a resolving set, $U = B$ is the unique metric basis of G. ■

By Theorems 3 and 8, we have the following corollary.

Corollary 1. *Let $k \geq 2$ be an integer. Then $2k+1 \leq n_0(k) \leq \lceil \frac{5k}{2} + 1 \rceil$.*

For $k = 2$, $n \geq 4 + d$ implies $n \geq 6$. Hence, $n_0(2) = 6$. It can be seen, there is no uniquely 3-dimensional graph of order 7. Thus, $8 \leq n_0(3) \leq 9$. The determination of $n_0(k)$, for every integer k could be an nontrivial interesting problem.

References

[1] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L.S. Ram, Network dicovery and verification, *IEEE Journal On Selected Areas in Communications* **24(12)** (2006) 2168-2181.

[2] P. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.* **46(1)** (2003) 9-15.

[3] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara and D.R. Wood, *On the metric dimension of cartesian products of graphs*, SIAM Journal on Discrete Mathematics **21(2)** (2007) 423-441.

[4] G.G. Chappell, J. Gimbel and C. Hartman, *Bounds on the metric and partition dimensions of a graph*, Ars Combinatoria **88** (2008) 349-366.

[5] G. Chartrand, L. Eroh, M.A. Johnson and O.R. Ollermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Applied Mathematics **105** (2000) 99-113.

[6] G. Chartrand and P. Zhang, *The theory and applications of resolvability in graphs.* A survey. In Proc. 34th Southeastern International Conf. on Combinatorics, Graph Theory and Computing **160** (2003) 47-68.

[7] F. Harary and R.A. Melter, *On the metric dimension of a graph*, Ars Combinatoria **2** (1976) 191-195.

[8] C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D.R. Wood, *Extremal graph theory for metric dimension and diameter*, The Electronic Journal of Combinatorics (2010) #R30.

[9] M. Jannesari and B. Omoomi, *On randomly k-dimensional graphs*, Applied Mathematics Letters **24** (2011) 1625-1629.

[10] M. Jannesari and B. Omoomi, *Characterization of Randomly k-dimensional graphs*, arXiv:1103.357v1.

[11] S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Discrete Applied Mathematics **70(3)** (1996) 217-229.

[12] R.A. Melter and I. Tomescu, *Metric bases in digital geometry*, Computer Vision Graphics and Image Processing **25** (1984) 113-121.

[13] A. Sebo and E. Tannier, *On metric generators of graphs*, Mathematics of Operations Research **29(2)** (2004) 383-393.

[14] P.J. Slater, *Leaves of trees*, Congressus Numerantium **14** (1975) 549-559.