Superluminal Spin-$\frac{1}{2}$ Particles are Left–Handed: From the Gordon Decomposition to the Suppression of Right–Handed States

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Superluminal spin-1/2 particles are analyzed under the assumption that the equation of motion is compatible with Lorentz invariance (tachyonic particles). It is found that tachyonic spin-1/2 particles can only be observed in left-handed helicity states, and that tachyonic spin-1/2 antiparticles are always right-handed. This result is independent of the numerical value of the tachyonic mass term, holds even for a tiny tachyonic mass of a few eV and may eventually be verified or falsified by experiments in the long-term future. We propose a superluminal character of the neutrino as an alternative explanation for the lack of a right-handed helicity state. This conclusion is connected with the superluminal Dirac algebra (Gordon identities) for spin-1/2 tachyonic particles. In particular, we derive the decomposition of the superluminal vector and axial vector current into convective and spin parts. Finally, we complement the discussion by giving bispinor solutions for generalized Dirac equations with mixed tachyonic and tardyonic mass terms, of the form $m_1 + \gamma^5 m_2$, and $m_1 + \mathrm{i}\,\gamma^5 m_2$. These solutions and corresponding sums over the fundamental spinor solutions may be useful in a wider context.

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I. INTRODUCTION AND OVERVIEW

A. Name of the Game

Over the last three decades, measurements of the neutrino mass square [1–7] have consistently resulted in negative values for m_ν^2, but within their error bars, all of the measurements are consistent with the hypothesis $m_\nu^2 = 0$. Likewise, a number of direct measurements of the neutrino velocity have resulted in experimental results with $v_\nu > c$ (best estimate), but again, within error bars [8–11], the result have been consistent with the hypothesis $v_\nu = c$. It is likely that the issue of the superluminal character of the neutrino will not be conclusively settled in the near future.

Simultaneously, the question lingers why neutrinos are only observed in left-handed helicity states, whereas antineutrinos are exclusively observed in right-handed helicity states. We here show that superluminal spin-1/2 particles must always be left-handed, under very natural assumptions, and show that this conclusion is connected with superluminal Gordon identities that are shown to have a very natural representation in terms of convective and spin currents. Furthermore, the suppression of the right-handed component holds independent of the concrete form of the Hamiltonian used to describe the particles (the tachyonic mass term can either be inserted as an explicit imaginary mass or via a matrix representation of the imaginary unit).

B. Experimental Sensitivity and Neutrino Mass

Recently, the OPERA experiment [10] has indicated a possible deviation of the (muon) neutrino velocity v_ν from c on the level of $\delta = 2.37 \times 10^{-5}$ for neutrinos in the 17 GeV range. It is noteworthy that such a large deviation from the speed of light at such a large energy seems very surprising, irrespective of the sign of the deviation, and recent statements of the OPERA collaboration come very close to a retraction of the result. Assuming that Lorentz invariance holds, we would have to conclude from the tentative OPERA data that $m_\nu^2 = -(117\,\mathrm{MeV})^2$ for neutrinos in the 17 GeV energy range [12]. The minus sign holds for the OPERA tentative result of $v_\nu = c + \delta$, where $\delta > 0$ is the deviation from the speed of light (units with $\hbar = c = \epsilon_0 = 1$ are used throughout this article). The mass square m_ν^2 would shift to a positive sign (same magnitude) for a hypothetical result of $v_\nu = c - \delta$. The reported OPERA result of $v = c + \delta$ requires the effective neutrino mass to run from a few eV for neutrinos in the keV range [1–7] to a mass in the MeV range for neutrinos in the GeV range. It has been argued in Refs. [12–14] that such a running would require a rather sophisticated fine-tuning of the parameters that enter a conceivable renormalization-group running of the neutrino mass.

Let us assume for a moment that the neutrino mass does not run (this needs to be confirmed), that the OPERA [10] result will soon be conclusively falsified, and that the neutrino mass square is in the range of a few positive or negative eV^2. In that case, the sensitivity of the OPERA [10] and ICARUS experiments [11] is too low to see any significant deviation from $v_\nu = c$ using current technology. The recent ICARUS result [11] is consistent with the trend [1–7] reported above: the best estimate for the neutrino velocity is superluminal, but the deviation from $v_\nu = c$ is insignificant on the level of current technology. Again, the final answer to the question of whether the neutrino is superluminal might not be given in the short-term future.

In any case, recent claims [10] have triggered a lot of theoretical interest in superluminal neutrino physics, and a number of interesting mechanisms have been proposed to incorporate a superluminal neutrino into the field-theoretical formalism without breaking any of the fundamental symmetries of nature. These proposed mechanisms include a dynamically created tachyonic neutrino mass [15], neutrino interactions with background fields [16, 17] and neutrino mass running [12–14]. As a byproduct of these investigations, a closer look at the spin-1/2 representation of tachyonic particles [18–20] has revealed that on a very fundamental level, the tachyonic spin-1/2 field theory is perhaps not quite as problematic as previously thought, and leads to rather interesting conclusions regarding the suppression of helicity components, independent of the magnitude of the tachyonic mass term.

C. Relativity Theory and Superluminality

Contrary to other, somewhat catchy, claims, the existence of superluminal particles would *not* falsify Einstein's theory of special relativity, which according to common wisdom is based on the following postulates: (i) The principle of relativity states that the laws of physics are the same for all observers in uniform motion relative to one another. (ii) The speed of light in a vacuum is the same for all observers, regardless of their relative motion or of the motion of the source of the light. Predictions of relativity theory regarding the relativity of simultaneity, time dilation and length contraction would not change if superluminal particles did exist. Furthermore, as shown by Sudarshan *et al.* (Refs. [21–24]) and Feinberg (Refs. [25, 26]), the existence of tachyons, which are superluminal particles fulfilling a Lorentz-invariant dispersion relation $E^2 = \vec{p}^2 - m_\nu^2$, is fully compatible with special relativity and Lorentz invariance. According to special relativity, it is forbidden to accelerate a particle "through" the light barrier (because $E = m/\sqrt{1-v^2} \to \infty$ for $v \to 1$), but a genuinely superluminal particle remains superluminal upon Lorentz transformation. Significant problems are encountered when one attempts to quantize the tachyonic theories, but again, as shown in Ref. [18], these problems may not be as serious as previously thought. In particular, the so-called reinterpretation of solutions propagating into the past according to the Feynman prescription [24] is a cornerstone of modern field theory. Furthermore, it has been shown in Ref. [18] that tachyonic particles can be localized, and equal-time anticommutators of the spin-1/2 tachyonic field involve an unfiltered Dirac-δ [see Eq. (37) of Ref. [18]]. The spectrum of the tachyonic Hamiltonian involves resonances and anti-resonances [18] which give rise to evanescent waves. Otherwise, these evanescent waves correspond to exponentially suppressed tardyonic components of a genuinely tachyonic wave packet (i.e., components from the "wrong side" of the "light barrier"). Interestingly, superluminal propagation of electromagnetic waves is claimed to have been observed in double-prism experiments [27–29], at the cost of introducing non-unitary time evolution (evanescent waves). The quantum tunneling regime is not described by classical special relativity.

The paper is organized in the following sense: The tachyonic Lagrangian is discussed in Sec. II, and the chiral adjoint is introduced. Gordon identities are given in Sec. III for both vector and axial vector currents, and the suppression mechanism of the "wrong" helicity component is illustrated in Sec. IV in a number of example cases and physically relevant limits. Finally, conclusions are drawn in Sec. V.

II. TACHYONIC LAGRANGIAN AND CHIRAL ADJOINT

A. Notation and Reference to Previous Works

Throughout this paper, let us denote by $\psi(x)$ or $\phi(x)$ a solution to a tachyonic equation. If $\psi(x)$ is a plane-wave solution for positive energy, and $\phi(x)$ goes for negative energy, then we have the structures

$$\psi(x) = U_\sigma(\vec{k}) \, \exp(-\mathrm{i}k \cdot x) \,, \quad \phi(x) = V_\sigma(\vec{k}) \, \exp(\mathrm{i}k \cdot x) \,, \tag{1}$$

where $U_\sigma(k)$ and $V_\sigma(k)$ are general bispinors characterizing positive-energy and negative-energy solutions, respectively. Furthermore, $x = (t, \vec{x})$ is a space-time vector and $k = (E, \vec{k})$ is the energy-momentum four-vector with $E = \sqrt{\vec{k}^2 - m^2}$ being equal to the (absolute value of the) energy of the superluminal particle. Here, $k \cdot x = E\, t - \vec{k} \cdot \vec{x}$ is the scalar product in space-time. The subscript σ in $U_\sigma(k)$ denotes the helicity (positive energy) and the negative of the helicity in $V_\sigma(k)$ (negative energy). The states are normalized according to Eq. (15) of Ref. [18],

$$U_\sigma^+(\vec{k}) \, U_\sigma(\vec{k}) = 1 \,, \qquad V_\sigma^+(\vec{k}) \, V_\sigma(\vec{k}) = 1 \,. \tag{2}$$

Here, σ is a quantum number which is equal to the eigenvalue of the helicity operator for positive-energy solutions, and equal to the negative of the helicity for negative-energy solutions. Furthermore, by $\mathcal{U}_\sigma(k)$ and $\mathcal{V}_\sigma(k)$ we denote solutions normalized according to Eq. (25) of Ref. [18],

$$\overline{\mathcal{U}}_\sigma(\vec{k}) \, \mathcal{U}_\sigma(\vec{k}) = \sigma \,, \qquad \overline{\mathcal{V}}_\sigma(\vec{k}) \, \mathcal{V}_\sigma(\vec{k}) = -\sigma \,. \tag{3}$$

The explicit form of the solutions is recalled here in Appendix A. Under a Lorentz transformation Λ, a bispinor $U(k)$ transforms according to Eq. (2.14) of Ref. [30],

$$\psi'(x') = S(\Lambda) \, \psi(x) \,, \tag{4}$$

where $S(\Lambda)$ is the bispinor Lorentz transformation that fulfills Eq. (2.15) of Ref. [30],

$$S(\Lambda) \, \gamma^\mu \, S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^\mu{}_\nu \, \gamma^\nu \,. \tag{5}$$

In view of the identity $\gamma^0 \, S^{+}(\Lambda) \, \gamma^0 = S^{-1}(\Lambda)$, the Dirac adjoint which has been used in Eqs. (2) and (3),

$$\overline{\psi}(x) = \psi^{+}(x) \, \gamma^0 \tag{6}$$

transforms with the inverse bispinor transformation $S^{-1}(\Lambda)$,

$$\overline{\psi}'(x') = \overline{\psi}(x) \, S^{-1}(\Lambda) \, . \tag{7}$$

In the following, we anticipate that the chiral adjoint

$$\widetilde{\psi}(x) = \psi^{+}(x) \, \gamma^0 \, \gamma^5 = \overline{\psi}(x) \, \gamma^5 \tag{8}$$

will be a useful concept for the description of tachyonic spin-1/2 particles. It is rather straightforward to show that the chiral adjoint transforms as

$$\widetilde{\psi}'(x') = \widetilde{\psi}(x) \, S^{-1}(\Lambda) \, \det(\Lambda) \, , \tag{9}$$

which differs from the transformation property of the Dirac adjoint by a sign if the determinant of the Lorentz transformation is negative (e.g., for a parity transformation).

Dirac matrices are used in the standard representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} ,$$
$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} , \tag{10}$$

as it was done in Ref. [18].

B. Tachyonic Lagrangian

When using the standard representation of the Dirac matrices as in Ref. [18], it is natural to induce superluminality of the spin-1/2 particle by a multiplication of the mass term in the Dirac equation by γ^5,

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - \gamma^5 m \right) \psi(x) = 0 \, , \tag{11}$$

and the solutions can be found in Ref. [18]. So, for positive-energy plane waves of the form

$$\psi(x) = U_{\pm}(\vec{k}) \, \exp(-\mathrm{i} k \cdot x), \tag{12}$$

the bispinor solution fulfills

$$\left(\not{k} - \gamma^5 m \right) U_{\pm}(\vec{k}) = 0 \, , \tag{13}$$

where we use the common notation $\not{k} = \gamma^\mu k_\mu$. The corresponding equation for the Dirac adjoint follows from the above using the identities $\gamma^0 (\gamma^\mu)^{+} \gamma^0 = \gamma^\mu$ and $\gamma^0 \left(\gamma^5\right)^{+} \gamma^0 = -\gamma^5$ and reads

$$\overline{U}_{\pm}(\vec{k}) \left(\not{k} + \gamma^5 m \right) = 0 \, . \tag{14}$$

By analogy with the ordinary Dirac equation, one might assume that

$$L(x) \overset{?}{=} \overline{\psi}(x) \left(\mathrm{i}\gamma^\mu \partial_\mu - \gamma^5 m \right) \psi(x) \tag{15}$$

should be a convenient ansatz for the Lagrangian. However, varying $L(x)$ with respect to $\psi(x)$, we find the variational equation for the Dirac adjoint

$$\partial_\mu \frac{\delta L}{\delta \partial_\mu \psi} - \frac{\delta L}{\delta \psi} \overset{?}{=} \mathrm{i} \left(\partial_\mu \overline{\psi}(x) \right) \gamma^\mu + m \overline{\psi}(x) \gamma^5 = 0 \, . \tag{16}$$

For a positive energy solution of the form $\psi(x) = U_{\pm}(\vec{k}) \, \exp(-\mathrm{i} k \cdot x)$, the resulting bispinor equation is

$$\overline{U}_{\pm}(\vec{k}) \left(-\not{k} + \gamma^5 m \right) \overset{?}{=} 0 \, , \tag{17}$$

which is incompatible with Eq. (14). Indeed, there is an additional problem with $L(x)$: For solutions $\psi(x)$ of the tachyonic Dirac equation, each of the its two terms reduces to zero. A similar problem has already been encountered by Feinberg for the tachyonic scalar particle, as explained in the text in between Eqs. (4.19) and (4.20) of Ref. [25].

With Ref. [31], we thus investigate the Lagrangian

$$\mathcal{L}(x) = \overline{\psi}(x) \left(\mathrm{i}\gamma^5 \gamma^\mu \partial_\mu - m \right) \psi(x) \, , \tag{18}$$

which upon variation with respect to $\overline{\psi}(x)$ yields a tachyonic Dirac equation which follows from Eq. (11) by multiplication with γ^5 from the left,

$$\left(\mathrm{i}\gamma^5 \gamma^\mu \partial_\mu - m \right) \psi(x) = 0 \, . \tag{19}$$

The equation of motion for $\overline{\psi}(x)$ then is

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} - \frac{\delta \mathcal{L}}{\delta \psi} = \mathrm{i} \left(\partial_\mu \overline{\psi}(x) \right) \gamma^5 \gamma^\mu + m \overline{\psi}(x) = 0 \, , \tag{20}$$

which for a positive-energy solution of the form (12) reduces to the equation

$$\overline{U}_{\pm}(\vec{k}) \left(-\gamma^5 \not{k} + m \right) = 0 \, . \tag{21}$$

We multiply with γ^5 from the *right* and find

$$\overline{U}_{\pm}(\vec{k}) \left(\not{k} + \gamma^5 m \right) = 0 \, , \tag{22}$$

which is equivalent to Eq. (14) and thus consistent. Additionally, it is interesting to note that both the kinetic energy as well as the mass term are nonvanishing for plane-wave solutions of the tachyonic Dirac equation. The latter point is fully clarified below in the context of tachyonic Gordon identities (see Sec. III).

For completeness, we also indicate the plane-wave solution for negative energy in the form

$$\phi(x) = V_{\pm}(k) \, \exp(\mathrm{i} k \cdot x) \, , \tag{23}$$

where $V_{\pm}(k)$ solves the equation

$$\left(\not{k} + \gamma^5 m \right) V_{\pm}(\vec{k}) = 0 \, , \tag{24}$$

with the adjoint

$$\overline{V}_{\pm}(\vec{k}) \left(\not{k} - \gamma^5 m \right) = 0 \, . \tag{25}$$

C. Chiral Adjoint

With the definition of the chiral adjoint given in Eq. (8), the Lagrangian (18) can trivially be rewritten as

$$\mathcal{L}(x) = \widetilde{\psi}(x)\,\left(\mathrm{i}\,\gamma^\mu \partial_\mu - \gamma^5\, m\right)\psi(x)\,. \tag{26}$$

Variation with respect to $\widetilde{\psi}(x)$ and $\psi(x)$ then leads to consistent dynamical equations for the two independent variables. Specifically, the resulting variational equation for $\widetilde{\psi}(x)$ is identical to the one obtained by taking the conjugate of Eq. (11) and multiplying with γ^5 from the right. In the form (26), the γ^5 term stands with the mass of the particle as suggested by intuition. Here, one could rightfully ask the question about any further benefits from the introduction of the chiral adjoint.

A part of the answer is provided by the seemingly natural emergence of the chiral adjoint in equations describing superluminal spin-1/2 particles. Let us consider the imaginary-mass Dirac equation

$$(\mathrm{i}\,\gamma^\mu\,\partial_\mu - \mathrm{i}m)\,\psi(x) = 0\,, \tag{27}$$

which is obtained from the ordinary Dirac equation by the substitution $m \to \mathrm{i}\,m$. Complex conjugation and insertion of γ^0 leads to

$$\psi^+(x)\,\gamma^0\left(-\mathrm{i}\,\gamma^0\,(\gamma^\mu)^+\,\gamma^0\overleftarrow{\partial}_\mu + \mathrm{i}m\right) = 0\,, \tag{28}$$

which is equivalent to

$$\overline{\psi}(x)\,\left(-\mathrm{i}\,\gamma^\mu\overleftarrow{\partial}_\mu + \mathrm{i}m\right) = 0\,. \tag{29}$$

The result is not consistent with the Lagrangian

$$L'(x) \stackrel{?}{=} \overline{\psi}(x)\,\left(\mathrm{i}\,\gamma^\mu\partial_\mu - \mathrm{i}m\right)\psi(x)\,. \tag{30}$$

because variation with respect to ψ would lead to

$$\overline{\psi}(x)\,\left(-\mathrm{i}\,\gamma^\mu\overleftarrow{\partial}_\mu - \mathrm{i}m\right) \stackrel{?}{=} 0\,. \tag{31}$$

However, if we insert a γ^5 matrix in Eq. (29),

$$\overline{\psi}(x)\,\gamma^5\left(-\mathrm{i}\,\gamma^5\,\gamma^\mu\,\gamma^5\overleftarrow{\partial}_\mu + \mathrm{i}(\gamma^5)^2 m\right) = 0\,, \tag{32}$$

this leads to

$$\widetilde{\psi}(x)\,\left(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + \mathrm{i}m\right) = 0\,, \tag{33}$$

consistent with the Lagrangian

$$\mathcal{L}'(x) = \widetilde{\psi}(x)\,\left(\mathrm{i}\,\gamma^\mu\partial_\mu - \mathrm{i}m\right)\psi(x) \tag{34}$$

for the imaginary-mass Dirac equation. For superluminal particles, the chiral adjoint naturally takes the role of the Dirac adjoint in the Lagrangian, and in the Gordon decomposition, as well as the spin sums over eigenspinors, which will be studied next.

D. Relations for the Tachyonic Dirac Equation

We study the emergence of the chiral adjoint in a number of expressions that pertain to the tachyonic Dirac equation (11). These are related to Eqs. (33)–(38) of Ref. [18]. Indeed, with the chiral adjoint, the field anti-commutator for the tachyonic Dirac equation takes the form

$$\begin{aligned}\{\psi(x), \widetilde{\psi}(y)\} &= \left\langle 0 \left| \{\psi(x), \widetilde{\psi}(y)\} \right| 0 \right\rangle \qquad (35)\\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{m}{E}\sum_{\sigma=\pm}\Big\{\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,(-\sigma)\,\mathcal{U}_\sigma(\vec{k})\otimes\widetilde{\mathcal{U}}_\sigma(\vec{k})\\ &\qquad + \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\,(-\sigma)\,\mathcal{V}_\sigma(\vec{k})\otimes\widetilde{\mathcal{V}}_\sigma(\vec{k})\Big\}\,,\end{aligned}$$

where σ is the chirality, $\otimes$ denotes the tensor product in bispinor space, and we use the definitions for the field $\psi(x)$, the bispinors $\mathcal{U}_\sigma$, $\mathcal{V}_\sigma$ as well as the creation and annihilation operators as defined Ref. [18]. The spin sum relations for the bispinor and its chiral adjoint are given as

$$\sum_\sigma (-\sigma)\,\mathcal{U}_\sigma(\vec{k})\otimes\widetilde{\mathcal{U}}_\sigma(\vec{k}) = \frac{\not{k} - \gamma^5\, m}{2m}\,, \tag{36a}$$

$$\sum_\sigma (-\sigma)\,\mathcal{V}_\sigma(\vec{k})\otimes\widetilde{\mathcal{V}}_\sigma(\vec{k}) = \frac{\not{k} + \gamma^5\, m}{2m}\,. \tag{36b}$$

The presence of the factor $(-\sigma)$ is explained in Ref. [18]. Using Eq. (36), the field anti-commutator can be brought into a compact form,

$$\{\psi(x), \widetilde{\psi}(y)\} = \left(\mathrm{i}\not{\partial} - \gamma^5\, m\right)\mathrm{i}\,\Delta(x-y)\,, \tag{37}$$

where $\Delta(x-y)$ is given by

$$\mathrm{i}\,\Delta(x-y) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\,\frac{1}{2E}\,\left[\mathrm{e}^{-\mathrm{i}k\cdot(x-y)} - \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\right]\,, \tag{38}$$

which is centered on the tachyonic mass shell.

E. Relations for the Imaginary–Mass Dirac Equation

We now study solutions of the imaginary-mass Dirac equation (27) using the definitions of Ref. [20], in agreement with the normalizations given in Eqs. (2) and (3). The spin sums reduce to

$$\sum_\sigma (-\sigma)\,\mathcal{U}'_\sigma(\vec{k})\otimes\widetilde{\mathcal{U}}'_\sigma(\vec{k}) = \frac{\not{k} + \mathrm{i}\,m}{2m}\,, \tag{39a}$$

$$\sum_\sigma (-\sigma)\,\mathcal{V}'_\sigma(\vec{k})\otimes\widetilde{\mathcal{V}}'_\sigma(\vec{k}) = \frac{\not{k} - \mathrm{i}\,m}{2m}\,, \tag{39b}$$

where the $\mathcal{U}'_\sigma(\vec{k})$ and $\mathcal{V}'_\sigma(\vec{k})$ are the solutions of the bispinors in the plane-wave solutions of the imaginary-mass Dirac equation as defined in Ref. [20]. These are the desired projectors onto positive- and negative-energy solutions for the imaginary-mass Dirac equation (27).

III. TACHYONIC GORDON IDENTITIES

A. Vector Current

We study the tachyonic Dirac equation (11) and investigate the matrix element of the vector current

$$\mathcal{J}^\mu = \overline{U}_\pm(\vec{k}')\,\gamma^\mu\,U_\pm(\vec{k})\,. \tag{40}$$

It is instructive to consider the derivation of the Gordon decomposition in detail,

$$\begin{aligned}\mathcal{J}^\mu &= \frac{1}{2m}\left(\overline{U}_\pm(\vec{k}')m\gamma^\mu U_\pm(\vec{k}) + \overline{U}_\pm(\vec{k}')\gamma^\mu m U_\pm(\vec{k})\right)\\ &= \frac{1}{2m}\left(\overline{U}_\pm(\vec{k}')\gamma^5 \not{k}'\gamma^\mu U_\pm(\vec{k}) + \overline{U}_\pm(\vec{k}')\gamma^\mu\gamma^5\not{k} U_\pm(\vec{k})\right)\\ &= \frac{1}{2m}\overline{U}_\pm(\vec{k}')\gamma^5\left(\gamma^\nu\gamma^\mu k'_\nu - \gamma^\mu\gamma^\nu k_\nu\right)U_\pm(\vec{k})\,.\end{aligned}$$

Using the familiar identity

$$\gamma^\nu\gamma^\mu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} - \frac{1}{2}[\gamma^\mu,\gamma^\nu] = g^{\mu\nu} + \mathrm{i}\sigma^{\mu\nu} \tag{41}$$

with $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu] = -\sigma^{\nu\mu}$, we find

$$\mathcal{J}^\mu = \frac{1}{2m}\widetilde{U}_\pm(\vec{k}')\left[(k'^\mu - k^\mu) + \mathrm{i}\sigma^{\mu\nu}\left(k'_\nu + k_\nu\right)\right]U_\pm(\vec{k})\,, \tag{42}$$

where we use the chiral adjoint in order to simplify the notation. The vector current has been decomposed into convective and spin parts. For $k' = k$, the current $\mathcal{J}^\mu$ specializes to

$$j^\mu = \overline{U}_\pm(\vec{k})\gamma^\mu U_\pm(\vec{k}) = \frac{\mathrm{i}}{m}\widetilde{U}_\pm(\vec{k})\,\sigma^{\mu\nu}k_\nu\,U_\pm(\vec{k})\,, \tag{43}$$

which is divergence-free, i.e., $k_\mu\, j^\mu = 0$. For negative-energy solutions, the identity reads as

$$\begin{aligned}\bar{\mathcal{J}}^\mu &= \overline{V}_\pm(\vec{k}')\gamma^\mu V_\pm(\vec{k}) \qquad (44)\\ &= -\frac{1}{2m}\widetilde{V}_\pm(\vec{k}')\,\left[(k'^\mu - k^\mu) + \mathrm{i}\sigma^{\mu\nu}\left(k'_\nu + k_\nu\right)\right]\,V_\pm(\vec{k})\end{aligned}$$

which differs from Eq. (42) by a minus sign.

B. Axial Current

For the tachyonic Dirac equation (11), the matrix element of the axial current reads

$$\mathcal{J}^{5,\mu} = \overline{U}_\pm(\vec{k}')\,\gamma^5\,\gamma^\mu\,U_\pm(\vec{k}) = \widetilde{U}_\pm(\vec{k}')\,\gamma^\mu\,U_\pm(\vec{k})\,. \tag{45}$$

A straightforward calculation reveals that

$$\begin{aligned}\mathcal{J}^{5,\mu} &= \overline{U}_\pm(\vec{k}')\gamma^5\gamma^\mu U_\pm(\vec{k}) \qquad (46)\\ &= \frac{1}{2m}\overline{U}_\pm(\vec{k}')\left[\gamma^5\not{k}'\gamma^5\gamma^\mu + \gamma^5\gamma^\mu\gamma^5\not{k}\right]U_\pm(\vec{k})\\ &= -\frac{1}{2m}\overline{U}_\pm(\vec{k}')\left(\gamma^\nu\gamma^\mu k'_\nu + \gamma^\mu\gamma^\nu k_\nu\right)U_\pm(\vec{k})\\ &= -\frac{1}{2m}\overline{U}_\pm(\vec{k}')\left[(k'^\mu + k^\mu) + \mathrm{i}\sigma^{\mu\nu}\left(k'_\nu - k_\nu\right)\right]U_\pm(\vec{k})\,.\end{aligned}$$

For $k' = k$, this simplifies to

$$j^{5,\mu} = \overline{U}_\pm(\vec{k})\gamma^5\gamma^\mu U_\pm(\vec{k}) = -\frac{1}{m}\overline{U}_\pm(\vec{k})\,k^\mu\,U_\pm(\vec{k})\,. \tag{47}$$

The results (46) and (47) for the tachyonic *axial vector* current have a similar structure as the Gordon decomposition for the tardyonic *vector* current obtained with the ordinary Dirac equation [see Eq. (2.54) of Ref. [30]]. This finding illustrates once more that to a certain degree, the role of the Dirac adjoint for the tardyonic case is taken over by the chiral adjoint for the tachyonic particle.

From Eqs. (43) and (47), together with the identity $\sigma^{\mu\nu}\,k^\mu\,k^\nu = 0$ and the result $\overline{U}_\pm(\vec{p})\gamma^5 U_\pm(\vec{p}) = 0$, imply that for a plane-wave solution $\psi(x) = U_\pm(\vec{p})\exp(-\mathrm{i}p\cdot x)$, each term in the trial Lagrangian L [see Eq. (15)] vanishes separately. For the correct choice of the Lagrangian given in Eq. (26), this is not the case.

For negative-energy solutions, we have

$$\begin{aligned}\bar{\mathcal{J}}^{5,\mu} &= \overline{V}_\pm(\vec{k}')\,\gamma^5\gamma^\mu\,V_\pm(\vec{k}) = \widetilde{V}_\pm(\vec{k}')\,\gamma^\mu\,V_\pm(\vec{k}) \qquad (48)\\ &= \frac{1}{2m}\overline{V}_\pm(\vec{k}')\left[(k'^\mu + k^\mu) + \mathrm{i}\sigma^{\mu\nu}\left(k'_\nu - k_\nu\right)\right]V_\pm(\vec{k})\,.\end{aligned}$$

The structure differs from Eq. (46) by an overall minus sign.

C. Parity in Gordon Identities

We recall Eqs. (43) and (47) in the form

$$j^\mu = \overline{U}_\pm(\vec{k})\gamma^\mu U_\pm(\vec{k}) = \frac{\mathrm{i}}{m}\overline{U}_\pm(\vec{k}')\gamma^5\,\sigma^{\mu\nu}k_\nu U_\pm(\vec{k}) \tag{49}$$

and

$$j^{5,\mu} = \overline{U}_\pm(\vec{k})\gamma^5\gamma^\mu U_\pm(\vec{k}) = -\frac{1}{m}\overline{U}_\pm(\vec{k})k^\mu U_\pm(\vec{k})\,. \tag{50}$$

There seems to be something peculiar concerning the transformation features in these identities. In Eq. (49), an apparent vector current j^μ on the left-hand side appears to transform into an axial current on the right-hand side, whereas in Eq. (50), an apparent axial vector on the left-hand side of the equation becomes what appears to be a vector on the right-hand side. The reason lies in the more complicated behavior of the tachyonic Dirac equation under parity as investigated in Ref. [19]. Namely, the tachyonic Dirac equation (11) contains a term which transforms as a scalar under parity,

$$\mathrm{i}\,\gamma^\nu\,\partial_\mu \xrightarrow{\mathcal{P}} \gamma^0\left(\mathrm{i}\,\gamma^0\,\partial_0 + \mathrm{i}\,\gamma^i\,(-\partial_i)\right)\,\gamma^0 = \mathrm{i}\,\gamma^\nu\,\partial_\mu \tag{51}$$

as well as a term which transforms as a pseudoscalar,

$$\gamma^5\,m \xrightarrow{\mathcal{P}} \gamma^0\left(\gamma^5\,m\right)\,\gamma^0 = -\gamma^5\,m\,. \tag{52}$$

The mass term in the tachyonic Dirac equation is not parity invariant, and indeed, in Ref. [19], the tachyonic

Dirac equation has been shown to be separately $\mathcal{CP}$ invariant, and $\mathcal{T}$ invariant, but not $\mathcal{P}$ invariant, due to the change in the mass term.

The transformation (52) can be interpreted as a transformation $m \to -m$ under parity. Thus, if we interpret the mass m as a pseudoscalar quantity, then the right-hand sides of (49) and (50) transform as a vector and an axial vector, respectively. It is the parity non-invariance of the mass term in the tachyonic Dirac equation which leads to the somewhat peculiar structure of Eqs. (49) and (50).

IV. SUPPRESSION OF RIGHT-HANDED STATES

A. Spin Sums and Tachyonic Gordon Identities

Let us explore the connection of the bispinor sums given in Eq. (36) with the tachyonic Gordon identities. We start with the left-hand side of Eq. (36a). The bispinor trace (with a γ^0 multiplied from the right) is given as

$$\begin{aligned}
&\mathrm{tr}\left(\sum_\sigma (-\sigma)\, \mathcal{U}_\sigma(\vec{k}) \otimes \widetilde{\mathcal{U}}_\sigma(\vec{k}) \gamma^0\right) \\
&= \sum_\sigma (-\sigma)\, \overline{\mathcal{U}}_\sigma(\vec{k})\, \gamma^5\, \gamma^{\mu=0}\, \mathcal{U}_\sigma(\vec{k}) \\
&= \sum_\sigma (-\sigma)\, \left(-\frac{k^0}{m}\right) \overline{\mathcal{U}}_\sigma(\vec{k}) \mathcal{U}_\sigma(\vec{k}) = 2\,\frac{E}{m}\,. \qquad (53a)
\end{aligned}$$

Here, $E = k^0$ is the energy and the tachyonic Gordon decomposition (47) as well as the normalization (3) have been used. The bispinor trace of the right-hand side of Eq. (36a) is

$$\mathrm{tr}\left(\gamma^0\, \frac{\not{k} - \gamma^5\, m}{2m}\right) = \mathrm{tr}\left(\gamma^0\, \frac{\not{k}}{2m}\right) = 4\,\frac{E}{2m} = 2\,\frac{E}{m}\,, \quad (54)$$

which shows the consistency of the bispinor sum (36a) with the Gordon decomposition (47).

The bispinor trace of the left-hand side of Eq. (36b) is

$$\begin{aligned}
&\mathrm{tr}\left(\sum_\sigma (-\sigma)\, \mathcal{V}_\sigma(\vec{k}) \otimes \widetilde{\mathcal{V}}_\sigma(\vec{k}) \gamma^0\right) \\
&= \sum_\sigma (-\sigma)\, \overline{\mathcal{V}}_\sigma(\vec{k})\, \gamma^5\, \gamma^{\mu=0}\, \mathcal{V}_\sigma(\vec{k}) \\
&= \sum_\sigma (-\sigma)\, \left(\frac{k^0}{m}\right) \overline{\mathcal{V}}_\sigma(\vec{k})\, \mathcal{V}_\sigma(\vec{k}) = 2\,\frac{E}{m}\,. \qquad (55a)
\end{aligned}$$

We have used the Gordon decomposition for negative-energy states as given in Eq. (48) as well as the normalization (3). From the right-hand side of Eq. (36b), we have

$$\mathrm{tr}\left(\gamma^0\, \frac{\not{k} + \gamma^5\, m}{2m}\right) = \mathrm{tr}\left(\gamma^0\, \frac{\not{k}}{2m}\right) = 2\,\frac{E}{m}\,, \qquad (56)$$

which again is fully consistent. From these considerations it is obvious that the appearance of the factor $(-\sigma)$ in the spin sums in Eqs. (36a) and (36b) is consistent with the tachyonic Gordon decomposition.

The factor $(-\sigma)$ also occurs in the nonvanishing anti-commutators of the particle and antiparticle annihilation and creation operators $b_\sigma(k)$, $b^+_\sigma(k)$ $d_\sigma(k)$, and $d^+_\sigma(k)$,

$$\left\{b_\sigma(k), b^+_\rho(k')\right\} = (-\sigma)\,(2\pi)^3\,\frac{E}{m}\delta^3(\vec{k}-\vec{k}')\,\delta_{\sigma\rho}\,, \quad (57a)$$

$$\left\{d_\sigma(k), d^+_\rho(k')\right\} = (-\sigma)\,(2\pi)^3\,\frac{E}{m}\delta^3(\vec{k}-\vec{k}')\,\delta_{\sigma\rho}\,. \quad (57b)$$

As shown in Ref. [18], the factor $(-\sigma)$ induces a negative norm for right-handed particle and left-handed antiparticle states (σ is equal to the helicity for positive-energy states and equal to minus the helicity for negative-energy states). The suppression of the right-handed particle states is thus consistent with the bispinor sums and the tachyonic Gordon decompositions.

B. Spin Sums and Massless Limit

It is very instructive to study the massless limit of the spin sums (36). In the limit $m \to 0$, the denominator of the spin sums vanishes, and a finite limit is obtained as

$$\lim_{m\to 0} \sum_\sigma 2m\,(-\sigma)\,\mathcal{U}_\sigma(\vec{k}) \otimes \widetilde{\mathcal{U}}_\sigma(\vec{k}) = \lim_{m\to 0} \not{k} - \gamma^5\, m = \not{k}\,, \qquad (58a)$$

$$\lim_{m\to 0} \sum_\sigma 2m\,(-\sigma)\,\mathcal{V}_\sigma(\vec{k}) \otimes \widetilde{\mathcal{V}}_\sigma(\vec{k}) = \lim_{m\to 0} \not{k} + \gamma^5\, m = \not{k}\,. \qquad (58b)$$

In the massless limit, the solutions to the Dirac equation are given as (see Chap. 2 of Ref. [30])

$$u_+(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} a_+(\vec{k}) \\ a_+(\vec{k}) \end{pmatrix}\,, \quad u_-(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} a_-(\vec{k}) \\ -a_-(\vec{k}) \end{pmatrix}\,, \qquad (59a)$$

$$v_+(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -a_+(\vec{k}) \\ -a_+(\vec{k}) \end{pmatrix}\,, \quad v_-(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -a_-(\vec{k}) \\ a_-(\vec{k}) \end{pmatrix}\,, \qquad (59b)$$

The helicity bispinors are given as

$$a_+(\vec{k}) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\,\mathrm{e}^{\mathrm{i}\,\varphi} \end{pmatrix}\,, \quad a_-(\vec{k}) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\,\mathrm{e}^{-\mathrm{i}\,\varphi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}\,. \qquad (60)$$

These fulfill the fundamental relations $(\vec{\sigma}\cdot\hat{\vec{k}})\,a_\sigma(\vec{k}) = \sigma\, a_\sigma(\vec{k})$, as well as $\sum_\sigma a_\sigma(\vec{k}) \otimes a^+_\sigma(\vec{k}) = \mathbb{1}_2$, and $\sum_\sigma \sigma\; a_\sigma(\vec{k}) \otimes a^+_\sigma(\vec{k}) = \vec{\sigma}\cdot\hat{\vec{k}}$, where $\hat{\vec{k}} = \vec{k}/|\vec{k}|$ and the sum over σ is over the values ± 1. The u and v bispinors fulfill the relations

$$\sum_\sigma 2\,|\vec{k}|\,u_\sigma(\vec{k}) \otimes \overline{u}_\sigma(\vec{k}) = \not{k}\,, \tag{61a}$$

$$\sum_\sigma 2\,|\vec{k}|\,v_\sigma(\vec{k}) \otimes \overline{v}_\sigma(\vec{k}) = \not{k}\,, \tag{61b}$$

A quick calculation also shows that in the massless limit, we have

$$\overline{u}_\sigma(\vec{k})\gamma^5 = \left(\gamma^5\,\gamma^0\,u_\sigma(\vec{k})\right)^+ = (-\sigma)\,\overline{u}_\sigma(\vec{k})\,, \tag{62a}$$

$$\overline{v}_\sigma(\vec{k})\gamma^5 = \left(\gamma^5\,\gamma^0\,v_\sigma(\vec{k})\right)^+ = (-\sigma)\,\overline{v}_\sigma(\vec{k})\,, \tag{62b}$$

We can thus introduce a factor $(-\sigma)^2 = 1$ in the sum over spins in Eq. (61) and replace one of the factors $(-\sigma)$ by a multiplication from the right by the fifth current, which turns Eq. (61) into

$$\sum_\sigma 2\,|\vec{k}|\,(-\sigma)\,u_\sigma(\vec{k}) \otimes \overline{u}_\sigma(\vec{k})\,\gamma^5 = \not{k}\,, \tag{63a}$$

$$\sum_\sigma 2\,|\vec{k}|\,(-\sigma)\,v_\sigma(\vec{k}) \otimes \overline{v}_\sigma(\vec{k})\,\gamma^5 = \not{k}\,. \tag{63b}$$

We can thus verify the consistency of Eq. (58) with Eq. (63) upon the following identification of the normalization in the massless limit. Namely, an elementary calculation yields the results

$$\overline{u}_\sigma(\vec{k})\,\gamma^5\,\gamma^0\,u_\sigma(\vec{k}) = \sigma\,, \tag{64a}$$

$$\overline{v}_\sigma(\vec{k})\,\gamma^5\,\gamma^0\,v_\sigma(\vec{k}) = -\,\sigma\,, \tag{64b}$$

and we also have

$$\overline{\mathcal{U}}_\sigma(\vec{k})\,\gamma^5\,\gamma^0\,\mathcal{U}_\sigma(\vec{k}) = \sigma\,\frac{E}{m}\,, \tag{64c}$$

$$\overline{\mathcal{V}}_\sigma(\vec{k})\,\gamma^5\,\gamma^0\,\mathcal{V}_\sigma(\vec{k}) = -\,\sigma\,\frac{E}{m}\,, \tag{64d}$$

by virtue of the tachyonic Gordon decomposition (47) for the axial current and the normalization (3). Observing that $E = |\vec{k}|$ in the massless limit, the identifications $\sqrt{m}\,\mathcal{U}_\sigma(\vec{k}) \to \sqrt{|\vec{k}|}\,u_\sigma(\vec{k})$ and $\sqrt{m}\,\mathcal{V}_\sigma(\vec{k}) \to \sqrt{|\vec{k}|}\,v_\sigma(\vec{k})$ in the massless limit follow immediately. The massless limit of the tachyonic bispinor sums, which connects the tachyonic spin-1/2 equation with the luminal (massless) and tardyonic Dirac equations, is thus verified.

C. Why Particles are Left–Handed

The appearance of the factor $(-\sigma)$ in the spin sums in the massless case [see Eqs. (63)], as well as the tachyonic Dirac case [see Eqs. (36)] and the imaginary-mass Dirac equation [Eq. (39)] is fully consistent with the anticommutators of the tachyonic field operators in Eq. (57). These in turn induce negative norm for particles with right-handed helicity. The factor $(-\sigma)$ also appears under the spin sum if one reverses the sign of the tachyonic mass term in Eq. (11) or Eq. (27) and constructs bispinor solutions according to the procedure outlined in Ref. [18]. For the imaginary-mass Dirac equation, this is explicitly shown in Sec. 4 of Ref. [20]. The factor $(-\sigma)$ needs to appear because any consistent massive tachyonic equation has to reproduce the massless limit (63). This raises the pertinent question why the right-handed component of the tachyonic Dirac field is suppressed rather than the left-handed component.

Because a reversal of the sign of the mass term in the tachyonic equation (tachyonic Dirac Hamiltonian) does not change the suppression mechanism, the origin of the suppression of states of definite helicity has to be searched somewhere else. Indeed, the reason is to be found in the fundamental properties of the massless Dirac Hamiltonian. This can be seen as follows. First, in view of Eq. (59), the wave functions

$$u_+(\vec{k})\,\exp(\mathrm{i}\vec{k}\cdot\vec{r})\,, \qquad u_-(\vec{k})\,\exp(\mathrm{i}\vec{k}\cdot\vec{r})\,, \tag{65}$$

are eigenstates of the massless Hamiltonian $H_0 = \vec{\alpha}\cdot\vec{p}$ with the helicity being equal to the chirality. The corresponding eigenvalue is indicated in the subscript. The wave functions

$$v_+(\vec{k})\,\exp(-\mathrm{i}\vec{k}\cdot\vec{r})\,, \qquad v_-(\vec{k})\,\exp(-\mathrm{i}\vec{k}\cdot\vec{r})\,, \tag{66}$$

are eigenstates of the massless Hamiltonian with the helicity being equal to the negative of the chirality. The subscript indicates the eigenvalue of the chirality operator γ^5.

Eigenstates of the massless Hamiltonian $H_0 = \vec{\alpha}\cdot\vec{p}$ have to be eigenstates of the chirality operator γ^5 because the chirality commutes with the Hamiltonian, in the sense that $\left[\gamma^5, H_0\right] = 0$. Furthermore,

$$H_0 = \vec{\alpha}\cdot\vec{p} = |\vec{p}|\,\gamma^5\,\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\,, \tag{67}$$

where $\vec{\Sigma} = \gamma^5\,\vec{\alpha}$ is the vector of 4×4 spin matrices, and the helicity operator is identified as $\vec{\Sigma}\cdot\vec{p}/|\vec{p}|$. Let λ_1 be the eigenvalue of chirality and λ_2 be the eigenvalue of the helicity operator. Then, the eigenvalue of the Hamiltonian is $E_0 = |\vec{p}|\,\lambda_1\,\lambda_2$. Since $\lambda_1 = \pm 1$ and $\lambda_2 = \pm 1$, we conclude that helicity equals chirality for positive energy, whereas the relation is reversed for negative-energy states. So far, we have recalled common wisdom [30].

All tachyonic spin-1/2 equations must correctly reproduce the spin sums in the massless limit (63). In this limit, the $\gamma^5\,\gamma^0$ factor from Eq. (62) must necessarily be equal to $(-\sigma)$ because of the discussed eigenvalue properties of the states $u_\sigma(\vec{k})$ and $v_\sigma(\vec{k})$ with respect to chirality and helicity. E.g., for positive energy, helicity must equal chirality, and eigenstates of chirality with eigenvalue unity have the bispinor form $\chi = (a, a)$ where a is a spinor. But for these states, $\gamma^5\,\gamma^0\,\chi = -\chi$, etc.

Indeed, γ^0 reverses the eigenvalue of γ^5, as an elementary calculations immediately shows. The factor $(-\sigma)$ also occurs in Eq. (57) and is necessary for a consistent formulation of the propagator; it is responsible for the suppression of the right-handed particle states [18]. We conclude that indeed, superluminal particles always have to be left-handed, whereas antiparticles are right-handed.

V. CONCLUSIONS

In this paper, we continue the analysis [18] of tachyonic spin-1/2 particles which are described by superluminal extensions of the Dirac equation. After recalling basic definitions in Sec. II A, we investigate the structure of the tachyonic Lagrangian in Sec. II B. The chiral adjoint $\overline{\psi}(x)\,\gamma^5$ is defined in Sec. II C and leads to a consistent structure for the spin-1/2 Lagrangians of both the tachyonic [18] as well as the imaginary-mass Dirac equation [20]. In Secs. II D and II E, we find that a number of bispinor sums relevant to the analysis of the tachyonic Dirac states find a particularly simple form if the chiral adjoint is used.

We then continue with the analysis of the tachyonic Gordon identities in Secs. III A (vector current) and III B (axial vector current). Parity transformations and the mixing of scalar and pseudoscalar quantities in the tachyonic identities are analyzed in Sec. III C. The main results are as follows: The tachyonic vector current for positive-energy bispinors is decomposed into convective and spin parts in Eq. (42), and for negative-energy bispinors we refer to Eq. (44). The tachyonic axial vector current for positive-energy bispinors is found in Eq. (46) and for negative-energy bispinors in Eq. (48).

The suppression of right-handed states is analyzed in Sec. IV. The structure of the bispinor sum (36) together with the fundamental anticommutators (57) imply the suppression of right-handed particle states. However, extra scrutiny is indicated in verifying certain properties and limits of the fundamental relations (36a) and (36b). In Sec. IV A, we verify that the relations (36a) and (36b) are consistent with the tachyonic Gordon identities. We then continue to verify in Sec. IV B that the massless limit is correctly reproduced. Indeed, the fundamental relations of parity and chirality for the massless Dirac Hamiltonian (chirality equals helicity for positive energy, etc.) imply that the massless limit involves factors $(-\sigma)$ and γ^5 matrices as indicated in Eq. (63). As shown in Sec. IV C, these properties imply the suppression of the right-handed particle and left-handed antiparticle states. The appearance of the factor $(-\sigma)$ in the spin sums in the massless case [see Eqs. (63)], as well as the tachyonic Dirac case [see Eqs. (36)] and the imaginary-mass Dirac equation [Eq. (39)] is thus fully clarified.

Two slightly different equations have been discussed in the literature for the description of superluminal spin-1/2 particles, namely, the tachyonic Dirac equation [18, 31] and the imaginary-mass Dirac equation [20, 32]. One may generalize the Dirac equation even further. In Appendix A, we investigate both tachyonic as well as tardyonic mass terms of the form m_1, $\gamma^5\,m_2$, and $\mathrm{i}\gamma^5 m_2$. The plane-wave solutions of the mixed equations are derived in Appendix A. These may be useful in a more general context because the unitarity of the S matrix implies the existence of useful relations [33] for the even powers $(m_2)^{2n}$ obtained upon expanding a one-loop amplitude, formulated with a mass term $m_1 + \mathrm{i}\,\gamma^5\,m_2$, in powers of m_2. Connections to the classical-force mechanism for baryogenesis [34] and to lattice gauge theory [35, 36] also motivate an investigation of generalized mass terms.

To conclude, let us recall that our arguments lead to the conclusion that superluminal spin-1/2 particles always are left-handed, and that superluminal antiparticles are right-handed, no matter whether the tachyonic Dirac equation or the imaginary-mass Dirac equations is used in the description of the superluminal particles, and no matter how small the tachyonic mass term is. An experimental confirmation or refutation of the theoretical considerations reported here therefore remains to be completed in the long-term future, when experimental sensitivity will conclusively allow us to distinguish between a conceivable tachyonic and tardyonic nature of the neutrino.

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Appendix A: (Appendix:) Dirac Equation with Tachyonic and Tardyonic Mass Terms

We wish to discuss solutions for generalized Dirac equations. In the helicity basis, we start from the massless spinors given in Eq. (59) and search for solutions of the form

$$\psi(x) = U_\pm(\vec{k})\,\exp(-\mathrm{i}k\cdot x)\,, \quad \phi(x) = V_\pm(\vec{k})\,\exp(\mathrm{i}k\cdot x)\,, \tag{A1}$$

The algebraic relations that have to be fulfilled by the bispinor amplitudes $U_\pm(\vec{k})$ and $V_\pm(\vec{k})$ reads as follows,

$$\left(\not{k} - m_1\right)\,U^{(1)}_\pm(\vec{k}) = 0\,, \quad \left(\not{k} + m_1\right)\,V^{(1)}_\pm(\vec{k}) = 0\,, \tag{A2}$$

for the tardyonic Dirac equation with mass m_1. For the tachyonic equation,

$$\left(\not{k} - \gamma^5 m_2\right)\,U_\pm(\vec{k}) = 0\,, \quad \left(\not{k} + \gamma^5 m_2\right)\,V_\pm(\vec{k}) = 0\,. \tag{A3}$$

In previous sections of this article, we have denoted m_2 by m. For a mixed tachyonic/tardyonic Dirac equation with masses m_1 and m_2, we have

$$\left(\not{k} - m_1 - \gamma^5 m_2\right) U^{(m)}_{\pm}(\vec{k}) = 0\,, \tag{A4a}$$

$$\left(-\not{k} - m_1 - \gamma^5 m_2\right) V^{(m)}_{\pm}(\vec{k}) = 0\,, \tag{A4b}$$

for the positive- and negative-energy solutions of a mixed equation with tardyonic and tachyonic mass terms,

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m_1 - \gamma^5 m_2\right) \psi^{(m)}(x) = 0\,. \tag{A5}$$

Using the algebraic identity

$$\left(\not{k} - m_1 - \gamma^5 m_2\right) \left(\not{k} + m_1 - \gamma^5 m_2\right) = k^2 - m_1^2 + m_2^2\,, \tag{A6}$$

we find the dispersion relation

$$E^{(m)} = \sqrt{\vec{k}^2 + m_1^2 - m_2^2}\,. \tag{A7}$$

For $m_1 = m_2$, the particle travels with the speed of light. The solutions of the mixed equation are thus given as

$$U^{(m)}_{+}(\vec{k}) = \frac{\gamma^5 m_2 - m_1 - \not{k}}{\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, u_+(\vec{k}) \tag{A8a}$$

$$= \left(\begin{array}{c} \dfrac{m_2 - m_1 - E^{(m)} + |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, a_+(\vec{k}) \\ \dfrac{m_2 - m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, a_+(\vec{k}) \end{array} \right),$$

$$U^{(m)}_{-}(\vec{k}) = \frac{\not{k} + m_1 - \gamma^5 m_2}{\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 + m_1)^2}}\, u_-(\vec{k}) \tag{A8b}$$

$$= \left(\begin{array}{c} \dfrac{m_2 + m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 + m_1)^2}}\, a_-(\vec{k}) \\ \dfrac{-m_2 - m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 + m_1)^2}}\, a_-(\vec{k}) \end{array} \right).$$

The negative-energy eigenstates of the mixed equation are given as

$$V^{(m)}_{+}(\vec{k}) = \frac{\gamma^5 m_2 - m_1 + \not{k}}{\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, v_+(\vec{k}) \tag{A9a}$$

$$= \left(\begin{array}{c} \dfrac{-m_2 + m_1 - E^{(m)} + |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, a_+(\vec{k}) \\ \dfrac{-m_2 + m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_2 - m_1)^2}}\, a_+(\vec{k}) \end{array} \right),$$

$$V^{(m)}_{-}(\vec{k}) = \frac{-\not{k} - \gamma^5 m_2 + m_1}{\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_1 + m_2)^2}}\, v_-(\vec{k}) \tag{A9b}$$

$$= \left(\begin{array}{c} \dfrac{-m_2 - m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_1 + m_2)^2}}\, a_-(\vec{k}) \\ \dfrac{m_2 + m_1 + E^{(m)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_1 + m_2)^2}}\, a_-(\vec{k}) \end{array} \right).$$

In the massless limit (first $E^{(m)} \to |\vec{k}|$, then $m_1 \to 0$, and then $m_2 \to 0$), we have $U^{(m)}_{+}(\vec{k}) \to u_+(\vec{k})$, $U^{(m)}_{-}(\vec{k}) \to u_-(\vec{k})$, $V^{(m)}_{+}(\vec{k}) \to v_+(\vec{k})$ and $V^{(m)}_{-}(\vec{k}) \to v_-(\vec{k})$. The states are normalized with respect to the condition

$$U^{(m)+}_{\pm}(\vec{k})\, U^{(m)}_{\pm}(\vec{k}) = V^{(m)+}_{\pm}(\vec{k})\, V^{(m)}_{\pm}(\vec{k}) = 1\,. \tag{A10}$$

The normalization denominators in Eqs. (A8) and (A9) can be written as

$$\mathcal{N}_\sigma = \sqrt{(E^{(m)} - |\vec{k}|)^2 + (m_1 - \sigma\, m_2)^2} \tag{A11}$$

where $\sigma = \pm$ is the helicity subscript in the solution. The normalization of the solutions with respect to the Dirac adjoint is given by

$$\overline{U}^{(m)}_{\pm}(\vec{k})\, U^{(m)}_{\pm}(\vec{k}) = \frac{\pm|\vec{k}|\, m_2 + E^{(m)}\, m_1}{\vec{k}^2 + m_1^2}\,, \tag{A12a}$$

$$\overline{V}^{(m)}_{\pm}(\vec{k})\, V^{(m)}_{\pm}(\vec{k}) = -\frac{\pm|\vec{k}|\, m_2 + E^{(m)}\, m_1}{\vec{k}^2 + m_1^2}\,, \tag{A12b}$$

where $E^{(m)}$ is given in Eq. (A7). For the purely tachyonic case, one sets $m_2 \to m$ and $m_1 \to 0$. We have not been able to find compact representations for the sums over bispinors generalizing Eq. (39) to the mixed tachyonic-tardyonic case, indicating that a fully consistent formulation of a mixture of tachyonic and tardyonic mass terms may be problematic.

A further mixed equation with two tardyonic mass terms can be written down as

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m_1 - \mathrm{i}\,\gamma^5 m_2\right) \psi^{(t)}(x) = 0\,. \tag{A13}$$

For the corresponding bispinor solutions, this implies that

$$\left(\not{k} - m_1 - \mathrm{i}\gamma^5 m_2\right) U^{(t)}_{\pm}(\vec{k}) = 0\,, \tag{A14a}$$

$$\left(-\not{k} - m_1 - \mathrm{i}\gamma^5 m_2\right) V^{(t)}_{\pm}(\vec{k}) = 0\,. \tag{A14b}$$

The appropriate normalization factor now changes to

$$\mathcal{N} = \sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}\,, \tag{A15}$$

and the energy is

$$E^{(t)} = \sqrt{\vec{k}^2 + m_1^2 + m_2^2}\,. \tag{A16}$$

The positive-energy bispinors for the equation with two tardyonic mass terms read as follows,

$$U^{(t)}_{+}(\vec{k}) = \frac{\mathrm{i}\gamma^5\, m_2 - m_1 - \not{k}}{\mathrm{i}\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, u_{+}(\vec{k}) \tag{A17a}$$

$$= \begin{pmatrix} \dfrac{\mathrm{i}m_2 - m_1 - E^{(t)} + |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{+}(\vec{k}) \\ \dfrac{\mathrm{i}m_2 - m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{+}(\vec{k}) \end{pmatrix},$$

$$U^{(t)}_{-}(\vec{k}) = \frac{\not{k} + m_1 - \mathrm{i}\gamma^5\, m_2}{\mathrm{i}\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, u_{-}(\vec{k}) \tag{A17b}$$

$$= \begin{pmatrix} \dfrac{\mathrm{i}m_2 + m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{-}(\vec{k}) \\ \dfrac{-\mathrm{i}m_2 - m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{-}(\vec{k}) \end{pmatrix}.$$

The negative-energy eigenstates of the mixed equation with two tardyonic mass terms are given as

$$V^{(t)}_{+}(\vec{k}) = \frac{\mathrm{i}\gamma^5\, m_2 - m_1 + \not{k}}{\mathrm{i}\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, v_{+}(\vec{k}) \tag{A18a}$$

$$= \begin{pmatrix} \dfrac{-\mathrm{i}m_2 + m_1 - E^{(t)} + |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{+}(\vec{k}) \\ \dfrac{-\mathrm{i}m_2 + m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{+}(\vec{k}) \end{pmatrix},$$

$$V^{(t)}_{-}(\vec{k}) = \frac{-\not{k} - \mathrm{i}\gamma^5\, m_2 + m_1}{\mathrm{i}\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, v_{-}(\vec{k}) \tag{A18b}$$

$$= \begin{pmatrix} \dfrac{-\mathrm{i}m_2 - m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{-}(\vec{k}) \\ \dfrac{\mathrm{i}m_2 + m_1 + E^{(t)} - |\vec{k}|}{\sqrt{2}\,\mathrm{i}\,\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}}\, a_{-}(\vec{k}) \end{pmatrix}.$$

In the massless limit (first $E^{(t)} \to |\vec{k}|$, then $m_1 \to 0$, and then $m_2 \to 0$), we again reproduce the massless solutions, $U^{(t)}_{+}(\vec{k}) \to u_{+}(\vec{k})$, $U^{(t)}_{-}(\vec{k}) \to u_{-}(\vec{k})$, $V^{(t)}_{+}(\vec{k}) \to v_{+}(\vec{k})$ and $V^{(t)}_{-}(\vec{k}) \to v_{-}(\vec{k})$. The states are normalized with respect to the condition

$$U^{(t)+}_{\pm}(\vec{k})\, U^{(t)}_{\pm}(\vec{k}) = V^{(t)+}_{\pm}(\vec{k})\, V^{(t)}_{\pm}(\vec{k}) = 1\,. \tag{A19}$$

The normalizations with respect to the Dirac adjoint are given as

$$\overline{U}^{(t)}_{\pm}(\vec{k})\, U^{(t)}_{\pm}(\vec{k}) = \frac{m_1}{E^{(t)}}\,, \tag{A20a}$$

$$\overline{V}^{(t)}_{\pm}(\vec{k})\, V^{(t)}_{\pm}(\vec{k}) = -\,\frac{m_1}{E^{(t)}}\,. \tag{A20b}$$

The solutions (A17) and (A18) approach the massless solutions if one replaces $m_1 \to 0$ first; and then lets $m_2 \to 0$. They are thus useful for systems where the m_2 mass is greater than m_1. For $m_1 \gg m_2$, one would like to calculate solutions that approach the massless case for the sequence $m_2 \to 0$, then $m_1 \to 0$. These are almost identical to the solutions (A17) and (A18) but they acquire a nontrivial phase factor. We shall denote them with a prime,

$$U'^{(t)}_{\sigma}(\vec{k}) = -\,\mathrm{i}\,\sigma\, U^{(t)}_{\sigma}(\vec{k})\,, \tag{A21a}$$

$$V'^{(t)}_{\sigma}(\vec{k}) = -\,\mathrm{i}\,\sigma\, V^{(t)}_{\sigma}(\vec{k})\,. \tag{A21b}$$

Of course, in the limit $m_2 \to 0$, one also has to expand the normalization denominators in powers of m_2. In the normalization

$$\mathcal{U}^{(t)}_{\sigma}(\vec{k}) = \left(\frac{E^{(t)}}{m}\right)^{1/2} U^{(t)}_{+}(\vec{k})\,, \tag{A22a}$$

$$\mathcal{V}^{(t)}_{\sigma}(\vec{k}) = \left(\frac{E^{(t)}}{m}\right)^{1/2} V^{(t)}_{+}(\vec{k})\,, \tag{A22b}$$

we can derive the following sums over bispinors,

$$\sum_{\sigma} \mathcal{U}^{(t)}_{\sigma}(\vec{k}) \otimes \overline{\mathcal{U}}^{(t)}_{\sigma}(\vec{k}) = \frac{\not{k} + m_1 - \mathrm{i}\,\gamma^5\, m_2}{2m_1}\,, \tag{A23a}$$

$$\sum_{\sigma} \mathcal{V}^{(t)}_{\sigma}(\vec{k}) \otimes \overline{\mathcal{V}}^{(t)}_{\sigma}(\vec{k}) = \frac{\not{k} - m_1 + \mathrm{i}\,\gamma^5\, m_2}{2m_1}\,. \tag{A23b}$$

In accordance with general wisdom about the tardyonic case, these do not involve helicity-dependent prefactors. The Feynman propagator is then easily found as

$$\begin{aligned} S^{(t)}(k) &= \frac{1}{\not{k} - m_1 + \mathrm{i}\epsilon - \mathrm{i}\,\gamma^5\,(m_2 - \mathrm{i}\,\eta)} \\ &= \frac{\not{k} + m_1 - \mathrm{i}\,\gamma^5\, m_2}{k^2 - m_1^2 - m_2^2 + \mathrm{i}\,\epsilon}\,, \end{aligned} \tag{A24}$$

where ϵ and η are infinitesimal imaginary parts.

There is a connection of the tardyonic γ^5 mass term to lattice calculations. Let us consider the Dirac–Wilson operator $D_W = \tilde{\gamma}^\mu \partial_\mu + m_2$ on a lattice in the limit $a \to 0$ where a is the lattice spacing [36]. The $\tilde{\gamma}^\mu$ fulfill $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\,\delta^{\mu\nu}$ with a Euclidean space-time metric, and therefore $(\tilde{\gamma}^\mu \partial_\mu + m_2)\,(\tilde{\gamma}^\nu \partial_\nu - m_2) = -\vec{p}^2 - \tilde{E}^2 - m_2^2$ where $\tilde{E}^2 = -E^2$ is the Euclidean energy. A possible representation is $\tilde{\gamma}^0 = \gamma^0$ and $\tilde{\gamma}^i = \alpha^i$. Then, with $\tilde{\gamma}^5 = \mathrm{i}\,\tilde{\gamma}^0\,\tilde{\gamma}^1\,\tilde{\gamma}^2\,\tilde{\gamma}^3 = -\mathrm{i}\,\gamma^0\,\gamma^5$, we have $D_W = \tilde{\gamma}^5\, D_W^{+}\,\tilde{\gamma}^5$, which is called γ^5–Hermiticity in lattice theory [35, 36], or pseudo-Hermiticity in Refs. [37–47]. Then, $\tilde{\gamma}^5\, D_W = \mathrm{i}\,\gamma^5\,\gamma^0\,(\gamma^0\,\gamma^i\,\partial_i + \gamma^0\,\tilde{\partial}_0 + m_2) \simeq -\left(\gamma^5\,\gamma^i\right)\,(-\mathrm{i}\partial_i) - \mathrm{i}\,\gamma^0\,\gamma^5\, m_2 = -\tilde{\alpha}^i\, p^i - \mathrm{i}\,\gamma^0\,\gamma^5\, m_2$ where we have neglected the Euclidean time derivative after the $\simeq$ sign. The $\tilde{\alpha}^i = \gamma^5\,\gamma^i$ fulfill the relation $\{\tilde{\alpha}^i, \tilde{\alpha}^j\} = \delta^{ij}$, and we have a mass term of the form $\mathrm{i}\,\gamma^0\,\gamma^5\,m_2$. Furthermore, as pointed out in Ref. [36], the operator $\tilde{\gamma}^5\,D_W$ is Hermitian. Indeed, our tardyonic Hamiltonian operator $H^{(t)} = \vec{\alpha}\cdot\vec{p} + \gamma^0\,m_1 + \mathrm{i}\,\gamma^0\,\gamma^5\,m_2$ is Hermitian, i.e., we have explicitly $H^{(t)+} = H^{(t)}$.

Finally, we notice that imaginary mass terms, and mass terms involving the γ^5 matrix, have been studied in the context of the classical force mechanism (CFM) for baryogenesis [34]. In Ref. [34], it is shown that for a mass term of the form $\gamma^0 m_1 \to \gamma^0 m_1 + \mathrm{i}\gamma^0\gamma^5 m_2$ (in the Hamiltonian), the fermion propagator may get nontrivial gradient corrections already at the first order in derivative expansion. A complex, position-dependent mass (involving γ^5) in the fermion self-energy may contribute to a conceivable explanation for $\mathcal{CP}$-violation during electroweak baryogenesis [34].

The well-known Feynman propagator for a tardyonic spin-1/2 field is given by letting $m_2 \to 0$ in Eq. (A24), and only retaining the mass m_1,

$$S^{(1)}(k) = \frac{1}{\not{k} - m_1 + \mathrm{i}\epsilon} = \frac{\not{k} + m_1}{k^2 - m_1^2 + \mathrm{i}\,\epsilon}\,. \tag{A25}$$

The energy in this case is

$$E^{(1)} = \sqrt{\vec{k}^2 + m_1^2}\,. \tag{A26}$$

In the helicity basis, the solutions of the equation (A2) with a tardyonic m_1 mass term are easily written down. For positive energy, they are given as

$$U_+^{(1)}(\vec{k}) = \frac{\not{k} + m_1}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,u_+(\vec{k}) = \begin{pmatrix} \dfrac{m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_+(\vec{k}) \\ \dfrac{m_1 - E^{(1)} + |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_+(\vec{k}) \end{pmatrix}, \tag{A27a}$$

$$U_-^{(1)}(\vec{k}) = \frac{\not{k} + m_1}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,u_-(\vec{k}) = \begin{pmatrix} \dfrac{m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_-(\vec{k}) \\ \dfrac{-m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + m_1^2}}\,a_-(\vec{k}) \end{pmatrix}. \tag{A27b}$$

The negative-energy eigenstates of the tardyonic equations in the helicity basis are given as

$$V_+^{(1)}(\vec{k}) = \frac{m_1 - \not{k}}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,v_+(\vec{k}) = \begin{pmatrix} \dfrac{-m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_+(\vec{k}) \\ \dfrac{m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(m)} - |\vec{k}|)^2 + m_1^2}}\,a_+(\vec{k}) \end{pmatrix}, \tag{A28a}$$

$$V_-^{(m)}(\vec{k}) = \frac{m_1 - \not{k}}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,v_-(\vec{k}) = \begin{pmatrix} \dfrac{-m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_-(\vec{k}) \\ \dfrac{m_1 + E^{(1)} - |\vec{k}|}{\sqrt{2}\,\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}}\,a_-(\vec{k}) \end{pmatrix}. \tag{A28b}$$

The normalization with respect to the Dirac adjoint involves the factor $m_1/E^{(1)}$ and $-m_1/E^{(1)}$, respectively, in full analogy to Eq. (A20). In the normalization

$$\mathcal{U}_\sigma^{(1)}(\vec{k}) = \left(\frac{E^{(1)}}{m}\right)^{1/2} U_+^{(1)}(\vec{k})\,, \tag{A29a}$$

$$\mathcal{V}_\sigma^{(1)}(\vec{k}) = \left(\frac{E^{(1)}}{m}\right)^{1/2} V_+^{(1)}(\vec{k})\,. \tag{A29b}$$

We reproduce the following known sums over bispinors,

$$\sum_\sigma \mathcal{U}_\sigma^{(1)}(\vec{k}) \otimes \overline{\mathcal{U}}_\sigma^{(1)}(\vec{k}) = \frac{\not{k} + m_1}{2m_1}\,, \tag{A30a}$$

$$\sum_\sigma \mathcal{V}_\sigma^{(1)}(\vec{k}) \otimes \overline{\mathcal{V}}_\sigma^{(1)}(\vec{k}) = \frac{\not{k} - m_1}{2m_1}\,. \tag{A30b}$$

In accordance with general wisdom about the tardyonic case, these do not involve helicity-dependent prefactors. This concludes our discussion of generalized Dirac equations with tardyonic and tachyonic mass terms.

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