A non-conservative kinetic exchange model of opinion dynamics with randomness and bounded confidence

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The concept of a bounded confidence level is incorporated in a nonconservative kinetic exchange model of opinion dynamics model where opinions have continuous values $\in [-1, 1]$. The characteristics of the unrestricted model, which has one parameter λ representing conviction, undergo drastic changes with the introduction of bounded confidence parametrised by δ. Three distinct regions are identified in the phase diagram in the $\delta - \lambda$ plane and the evidences of a first order phase transition for $\delta \geq 0.3$ are presented. A neutral state with all opinions equal to zero occurs for $\lambda \leq \lambda_{c_1} \simeq 2/3$, independent of δ, while for $\lambda_{c_1} \leq \lambda \leq \lambda_{c_2}(\delta)$, an ordered region is seen to exist where opinions of only one sign prevail. At $\lambda_{c_2}(\delta)$, a transition to a disordered state is observed, where individual opinions of both signs coexist and move closer to the extreme values (± 1) as λ is increased. For confidence level $\delta < 0.3$, the ordered phase exists for a narrow range of λ only. The line $\delta = 0$ is apparently a line of discontinuity and this limit is discussed in some detail.

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Recently, several opinion dynamics models with continuous opinion have been proposed in which the opinions are updated after a pair of individuals interact in a manner similar to gas molecules in kinetic theory. Deffuant et al [1] introduced a simple model (DNAW model hereafter) in which opinion exchanges between two agents take place only when the difference in the original opinions is less than or equal to a preassigned quantity δ. If $o_i(t)$ is the opinion of the ith agent interacting with the jth agent at time t (with $|o_i - o_j| \leq \delta$), then in this model the opinions evolve according to:

$$\begin{aligned} o_i(t+1) &= o_i(t) + \gamma(o_j(t) - o_i(t)) \\ o_j(t+1) &= o_j(t) + \gamma(o_i(t) - o_j(t)). \end{aligned} \quad (1)$$

Here γ is a constant ($0 \leq \gamma \leq 0.5$) called the convergence parameter and o_i lies in the interval [0,1]. The dynamics is such that the opinions tend to come closer after interaction. Hence as the dynamics proceeds, convergence to a finite number of opinions happens; opinions cluster around a few values and individuals belonging to different clusters no longer interact. The initial distribution of the individual opinions is uniform and therefore symmetric. This symmetry is broken as the time evolved distribution has a multipeaked delta function form. When there is only one peak in the final distribution, it is said to be a case of *consensus*, two peaks imply *polarisation* and the existence of a larger (finite) number of peaks signifies *fragmentation* in the society.

The model is conservative as total opinion is conserved in each interaction. Obviously, in this conservative model, consensus would imply that opinions converge to the value 1/2. Several models have been formulated incorporating the idea of bounded confidence later [2–4] and a general form of kinetic exchange type model proposed in [5].

More recently, a model in which kinetic exchanges take place with randomness, and where there is no conservation, has been introduced by Lallouache et al [6] (LCCC model hereafter). Any two agents can interact in this unrestricted model. The opinion evolution here follows the rule:

$$\begin{aligned} o_i(t+1) &= \lambda[o_i(t) + \epsilon o_j(t)] \\ o_j(t+1) &= \lambda[o_j(t) + \epsilon' o_i(t)]; \end{aligned} \quad (2)$$

where ϵ, ϵ' are drawn randomly from uniform distributions in $[0, 1]$. In this model λ is a parameter which is interpreted as 'conviction'. The opinions are bounded, i.e., $-1 \leq o_i(t) \leq 1$; in case o_i exceeds 1 or becomes less than -1, it is set to 1 and -1 in the respective cases.

It is possible to rescale the opinions in the DNAW model so that they lie in the interval $[-1, 1]$. Continuous opinions are relevant in cases like supporting a issue, rating a movie etc. Thus setting the interval as $[-1, 1]$ appears to be more meaningful since a positive (negative) value of the opinion will mean liking (disliking) the motion. The magnitude of the opinion would then simply correspond to the amount of liking or disliking. For the rest of the paper, we thus consider opinions $\in [-1 : 1]$.

As there is no conservation in the LCCC model, the average opinion given by $m = |\sum_i o_i|/N$ for a population of N agents, evolves in time and m can play the role of an order parameter, and is analogous to magnetisation in magnetic systems. One can say that there is order/disorder when $m(t \to \infty)$ converges to a nonzero/zero value. The model shows a unique behaviour: below a critical value of $\lambda \simeq 2/3$, all opinions identically turn out to be zero while above it, there is a nonzero value of the average opinion. Thus for $\lambda > 2/3$, an ordered phase exists. Interestingly, the opinions in the ordered phase have either all positive or all negative values. Generalisation and variations of the LCCC model

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have been considered in some subsequent works [7–9].

Symmetry breaking has different connotations in the conserved DNAW model and the nonconserved LCCC model. In the former, if opinions are initially in the interval $[-1, 1]$, a consensus implies convergence of all opinions to zero value and this is regarded as symmetry breaking as mentioned earlier. In LCCC, the identical state of all zeros is also obtained below $\lambda \simeq 2/3$ even without putting any restriction on the interactions. However, this state has been interpreted as a symmetric state [6]. This is following the idea that as $m = 0$ here, it is like a paramagnetic state (which is a symmetric state in magnetic systems). But obviously, this is a very special paramagnetic state which also has zero fluctuation.

To avoid confusion, we adopt the following terminology for nonconserved systems: when $m = 0$ and also the individual opinion distribution is a delta function peaked at zero, we call it a *neutral* state. If it is not a delta function at zero, the state will be termed a *disordered* state. When $m \neq 0$, it is an *ordered* state; further if all individuals have identical opinion (which is perhaps only ideological as opinions are continuously distributed), it is a *consensus* state. Hence consensus is not merely an agreement in this terminology. Obviously, in the conserved system, the nomenclature of order and disorder is irrelevant.

Having conservation in the opinions is rather unrealistic but the concept of having a bounded confidence level is relevant in many cases. We thus combine the LCCC model and the DNAW model by putting the restriction of the bounded confidence in the former.

Hence in the model proposed in the present paper, we follow eq (2) for the evolution of opinions but put the restriction that agents interact only when $|o_i - o_j| \leq 2\delta$. δ is once again the parameter representing the confidence level and can vary from zero to 1.

We therefore have two parameters in the model, λ and δ. $\delta = 1$ is identical to the original LCCC model. $\delta = 0$ is an interesting limit. Here, agents interact only when their opinions are exactly same. We will discuss this limit in greater detail later.

Results and numerical analysis:

We take a population of N agents and let it evolve according to the dynamical rules defined above (i.e., eq 2 with a bounded confidence level δ). The behaviour of the order parameter after the system reaches equilibrium is presented in Fig. 1. As a function of λ, we find that the order parameter first assumes non-zero values at a threshold value of $\lambda = \lambda_{c_1}$ which is independent of δ; $\lambda_{c_1} \simeq 2/3$ as in the original LCCC model. The order parameter increases with λ beyond λ_{c_1} up to a certain value of λ and decreases to zero as λ is increased further. The decrease becomes steeper with δ and more so when the system size is increased. The results indicate that there are three distinct regions: one ordered region for intermediate values of λ and two regions at low and high values of λ where the order parameter vanishes. These two regions may be either disordered or neutral.

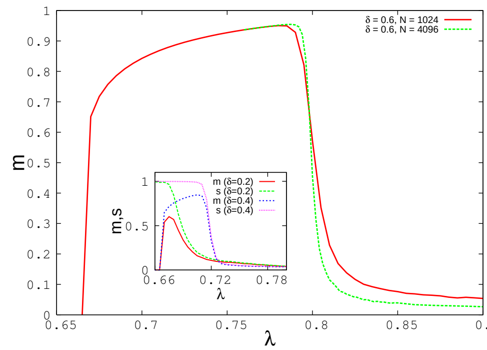

FIG. 1: (Color online) Variation of the order parameter m with λ for $\delta = 0.6$ for $N = 1048$ and 4096. Inset shows m and s for $\delta = 0.2$ and 0.4 against λ for $N = 1048$.

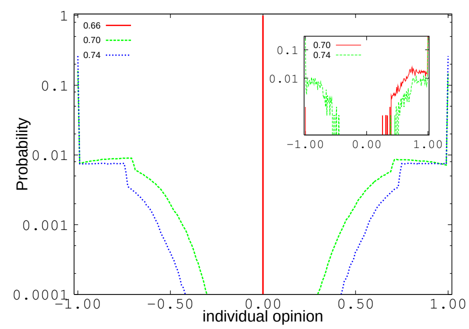

FIG. 2: (Color online) The distribution of individual opinions shown for $N = 2048$ different values of λ for $\delta = 0.4$ averaged over different configurations. Inset shows the same for a single realisation of the system. For $\lambda < \lambda_{c_1} \simeq 2/3$, the average distribution is a delta function at zero. For $\lambda_{c_1} < \lambda < \lambda_{c_2} \simeq 0.7183$ (for $\delta = 0.4$), for a single configuration, opinions are all of one particular sign, while for averages over all configuration, the distribution is symmetric. For $\lambda > \lambda_{c_2}$, the distribution is symmetric even for the single configuration.

To understand the nature of the phases, the distribution of individual opinions may be studied. Such studies are known to lead to a correct speculation about phase transitions [10]. This study shows (Fig. 2) that the probability for zero opinion is nearly equal to 1 below $\lambda_{c_1} \simeq 2/3$ as in the LCCC model for all δ. Hence a neutral state exists here as well and the confidence level is absolutely irrelevant as λ_{c_1} is independent of δ.

As λ is increased beyond λ_{c_1}, in a single configuration, only all positive or all negative values are obtained as $N \to \infty$, while the average over all configurations is symmetric about zero as expected. However, as λ is increased further, the opinions, even in a single configuration, assume both negative and positive values symmetrically (Fig. 2 inset). Hence we infer that an order-disorder transition is taking place at a value $\lambda_{c_2} > \lambda_{c_1}$ which is later confirmed from more detailed analysis. For $\delta = 1$, the LCCC model, λ_{c_2} is equal to 1 as expected.

Consistent with the above observation is the behaviour of another quantity

$$s = \langle |f_+ - f_-| \rangle, \tag{3}$$

where f_+ denotes the fraction of population with opinion greater than zero and $f_- = 1 - f_+$ in a particular configuration. $\langle \cdots \rangle$ denotes average over all configurations. It can be easily seen that s is equal to unity both in the neutral state and ordered state of LCCC. Deviation of s from unity will indicate that opinions with both signs are present in general. We notice that s remains close to unity as λ is increased from zero before showing a sharp fall close to a value of λ where the order parameter also starts to fall (Fig. 1 inset). Evidently, as the system enters the disordered state, individual opinions are > 0 and ≤ 0 with equal probability. Comparison of s and m shows that these two measures become closer and tend to merge as λ is increased further. This indicates that as one moves deeper into the disordered region, opinions become more and more close to the extreme values ± 1, leading to a polarisation tendency in the opinions. This is also evident from the data shown in Fig. 2.

The ordered, disordered and neutral regions may be identified in a phase diagram in the $\delta - \lambda$ plane. To obtain the phase boundaries in this plane, we estimate the phase transition points by traditional methods, i.e., attempt finite size scaling for the relevant physical variables, if possible. Among these variables is the fourth order Binder cumulant (BC) defined as

$$U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}. \tag{4}$$

Here we discuss the case for $\delta < 0.3$ and $\delta \geq 0.3$ separately for reasons which will be clear later.

$\delta \geq 0.3$: Plotting U against λ, we find that there is indeed a crossing point but interestingly, the BC shows a negative dip (Fig. 3) for $\delta < 0.6$ for the system sizes considered. In fact it becomes more negative as the system size N is increased and the location of the negative dip approaches the crossing point as well. These are typical indications of a first order phase transition [11]. To confirm whether a first order transition is really taking place, we also plot the distribution of the order parameter very close to λ_{c_2}. One expects the distribution to have peaks at nonzero values of m below the critical point (usually the distribution is a double gaussian). For a continuous phase transition, as the critical point is approached from below, the peaks occur at smaller and smaller values of m, finally merging at $m = 0$ continuously at the critical point. For a first order phase transition, on the other hand, the peaks at nonzero values of the order parameter remain at more or less the same positions up to the transition point [11–15]. Here we find exactly this behaviour (Fig. 4); note that for finite systems, weak peaks will still show at nonzero values of m just above the transition point (instead of a perfect gaussian with mean zero).

We attempt to obtain scaling forms for the BC (U), order parameter (m) and a quantity analogous to susceptibility per spin (in magnetic systems) given by $\chi = \frac{1}{N}[\langle M^2 \rangle - \langle M \rangle^2]$ where M is the total opinion, $M = |\sum o_i|$. The expected behaviour are given by

$$\begin{aligned} U &= f_1((\lambda - \lambda_{c2})N^{\mu}) \\ m &= N^{-a} f_2((\lambda - \lambda_{c2})N^{\mu}) \\ \chi &= N^{b} f_3((\lambda - \lambda_{c2})N^{\mu}). \end{aligned} \tag{5}$$

For first order phase transitions in finite systems, one expects that instead of a delta function behaviour at the transition point, there will be a peak in the susceptibility which will diverge with the system size. The order parameter exponent a is expected to be close to zero. We find that the above scaling forms are indeed appropriate in the present case, the data collapse to a single curve for specific values of a, b and μ (shown for m and χ in Fig. 5). The value of a is indeed very close to zero and $b \simeq 1$ for all values of $\delta \geq 0.3$. However, the value of μ appears to have a systematic variation with δ. Since δ effectively puts a restriction on the number of compatible neighbours, it is not surprising that μ, which is associated with N, shows a dependence on δ. The values of

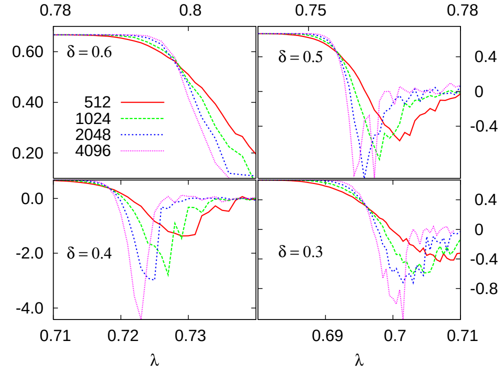

FIG. 3: (Color online) The Binder cumulant is shown for different values of δ for $N = 512, 1024, 2048$ and 4096. Colour code is same for all the figures.

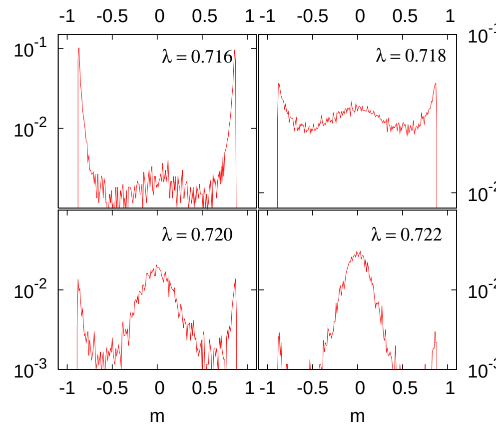

FIG. 4: (Color online) Distribution of the order parameter close to the transition point $\lambda_{c_2} \simeq 0.7183$ for $\delta = 0.4$ ($N = 2048$).

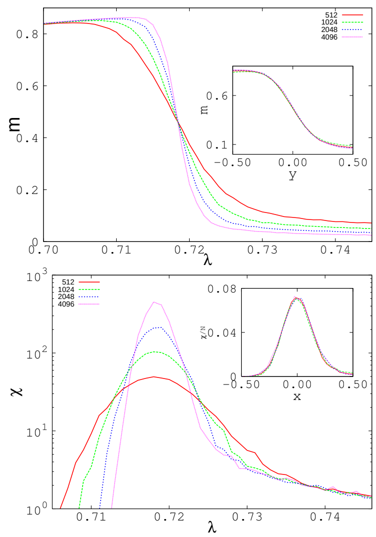

FIG. 5: (Color online) The variation of m and χ are shown for different system sizes for $\delta = 0.4$. The insets show the scaling plots with m and χ/N plotted against $x = (\lambda - 0.7183)N^{0.53}$.

TABLE I: The transition point and the values of the exponents for $\lambda > 0.3$. The typical errors in the data are $\pm 1.0 \times 10^{-4}$ for λ_{c_2}; ± 0.01 for μ, $O(10^{-3})$ for a and $O(10^{-2})$ for b.

δ	λ_{c_2}	μ	a	b
0.30	0.6958	0.56	0.00	1.04
0.40	0.7183	0.53	0.00	1.05
0.50	0.7555	0.50	0.00	1.02
0.60	0.7980	0.43	0.00	0.90
0.70	0.8415	0.34	0.035	0.90
0.80	0.8850	0.26	0.00	1.00
0.90	0.9530	0.20	0.025	0.90

the transition point λ_{c_2} and the exponents are presented in Table I.

All the above discussions are however, valid for $\delta > 0.3$ only. The first order phase transition is most strongly observed close to $\delta = 0.4$. As for the negative dip, it is not observed for $\delta \geq 0.6$ with $N \leq 4096$, but the values of the exponents indicate that the transition is first order-like. The negative dip for $\delta \geq 0.6$ is thus expected to be observed for even higher values of N [11].

$\delta < 0.3$: When δ is decreased below 0.3, the results do not give any clear indications about the nature of the phase transition and shows some anomalous behaviour. A rather uncharacteristic behaviour of the order parameter and the Binder cumulant is observed. The order parameter m shows a nonmonotonic behaviour when plotted as a function of δ with fixed λ or vice versa (Fig. 6). A hump appears in the m versus λ plots for $\delta < 0.3$ and $\lambda \geq 0.7$ showing the existence of a local maximum value. A closer examination reveals that this hump disappears, albeit very slowly, when N is increased for $\lambda > 0.71$, For $\delta < 0.3$, there is large fluctuations and irregularities in the BC as well which does not allow one to do a finite size scaling analysis and get the exponents. The irregular behaviour of the BC and the order parameter may be because of the fact that the interactions become less likely to occur as the confidence level is decreased. For this reason we restrict our study to $\delta \geq 0.1$. In the inset of Fig. 6, the m versus λ plot consistently shows a hump occurring at large values of λ. This is most prominently observed close to $\delta \sim 0.25$ where the local maxima of the order parameter appear in the m versus δ plots. This, however, turns out to be a finite size behaviour as expected.

We have estimated, somewhat approximately, λ_{c_2} for $\delta < 0.3$ from the crossing point of the order parameter curves for different sizes. The complete phase diagram in the $\delta - \lambda$ plane is shown in Fig. 7.

Discussions: Let us try to understand the results by analysing the role of the confidence level δ. Let the ith agent with opinion o_i interact with another agent with opinion $o_i + x$ where $|x| \leq 2\delta$. Then,

$$o_i(t+1) = \lambda(1+\epsilon)o_i(t) + \lambda\epsilon x. \qquad (6)$$

Consider the case when δ is small. On an average, when λ is smaller than $2/3$, the first term will make o_i smaller in magnitude while the contribution of the second term can be neglected. Then we find that $o_i(t) \to 0$ as $t \to \infty$ implying the convergence to a neutral state. Since for $\delta = 1$, it is already known that there is a transition to the neutral state at $\lambda \simeq 2/3$, we conclude that for any δ this is the case as is confirmed by the numerical results. Actually, when individual opinions decrease towards zero because of the effect of the first term, the difference in opinions automatically becomes less ($x \to 0$) so that large and small δ values have the same effect; the second term does not contribute eventually. Thus λ_{c_1} is independent of δ.

What happens at higher values of λ? First consider small values of δ again which means x is small too. Hence the second term still contributes less compared to the first and o_i will retain its original sign in most cases if λ is sufficiently large. So it is expected that there will be a region where opinions of both signs are present and $m = 0$, as originally opinions are uniformly distributed with positive and negative signs.

However, if δ is large, there is no guarantee that the second term is small and opinions will retain their original signs, unless λ is also very high. This explains why we observe the transition to the disordered state at a higher value of λ as δ is increased. Also, it is not surprising that there will be a ordered region between λ_{c_1} and λ_{c_2} (as already it is known to be present for $\delta = 1.0$) where the LCCC property of all opinions assuming the same sign is still valid.

Although we have restricted to $\delta \geq 0.1$ in the numerical simulations, the case when δ is exactly equal to zero

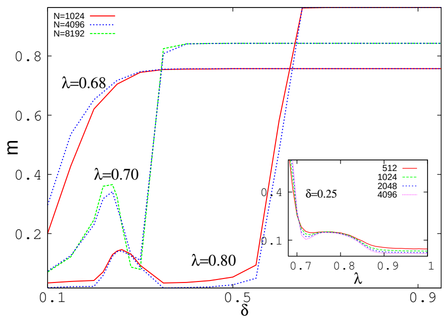

FIG. 6: (Color online) Variation of the order parameter m with δ for fixed λ values. For each value of λ, data for two system sizes shown. Inset shows variation of m with λ at $\delta = 0.25$.

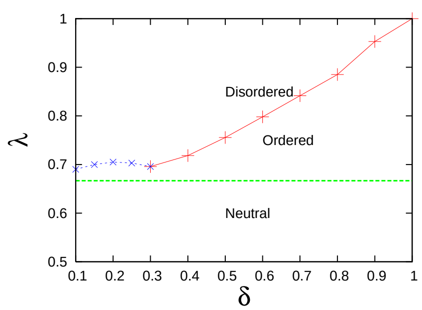

FIG. 7: (Color online) The phase diagram in the $\delta - \lambda$ plane shows the existence of the neutral region (for $\lambda \leq \lambda_{c1} \simeq 2/3$), the ordered region and the disordered region. The ordered and disordered regions are separated by a first order boundary at least for $\lambda \geq 0.3$ (see text).

can be discussed theoretically to some extent. If an interaction takes place, the opinion for the ith agent follows the evolution equation

$$o_i(t+1) = \lambda(1+\epsilon)o_i(t). \quad (7)$$

This equation is nothing but the dynamical equation obtained for the LCCC model in the limit of a single parameter map [6, 16, 17], where the transition to an ordered state occurs at a value of $\lambda = e/4$. However, in the present model the above is valid only when there is a second agent with opinion equal to o_i as well. Since this will be extremely rare, effectively most of the opinions remain frozen and the single parameter map is not representative of all the agents' opinion evolution. In fact, $\delta = 0$ may be regarded as a line of discontinuity in the phase diagram of the model in the $\delta - \lambda$ plane as it will neither have the neutral state nor the ordered state anywhere. The disordered state is also different in nature for $\delta = 0$; here the individual opinion distribution will be flat while for $\delta \neq 0$, however small, it is not so.

In summary, we have studied a model of continuous opinion dynamics with an attempt to merge the concepts of confidence level and conviction. We find the interesting result that with a large value of conviction and with any finite bound on the confidence level (i.e., $\delta < 1$), a disordered state exists with tendency to polarisation. This is indeed justified, if agents are convinced to a large extent in their opinions and interact with like minded people only, the sign of the opinion, (representing liking/disliking) is likely to be maintained, giving rise to a polarised society. The neutral state with all opinions equal to zero remains unperturbed with the introduction of the bounded confidence. It is found that at least for $\delta > 0.3$, the order-disorder transition is first order in nature. For smaller confidence level, when $\delta < 0.3$, the ordered phase shrinks to a narrow region of the phase diagram.

In conclusion, we obtain a phase diagram with many features when the concept of bounded confidence is incorporated in the LCCC model of opinion dynamics. The overall result is that when bounded confidence level is large, there will be order in the society provided people are not too rigid. In the original DNAW model also, it had been shown that above a certain confidence level, society behaves more homogeneously. We show that this tendency remains true but only up to a certain level of conviction. This seems to be a realistic scenario and thus the combination of concepts from two different models in the present model of opinion dynamics is successful in reproducing this desired feature of a society.

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