arXiv:1205.0145v1 [physics.gen-ph] 1 May 2012

Tachyonic Field Theory and Neutrino Mass Running

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In this paper three things are done. (i) We investigate the analogues of Cerenkov radiation for the decay of a superluminal neutrino and calculate the Cerenkov angles for the emission of a photon through a W loop, and for a collinear electron-positron pair, assuming the tachyonic dispersion relation for the superluminal neutrino. The decay rate of a freely propagating neutrino is found to depend on the shape of the assumed dispersion relation, and is found to decrease with decreasing tachyonic mass of the neutrino. (ii) We discuss a few properties of the tachyonic Dirac equation (symmetries and plane-wave solutions), which may be relevant for the description of superluminal neutrinos seen by the OPERA experiment, and discuss the calculation of the tachyonic propagator. (iii) In the absence of a commonly accepted tachyonic field theory, and in view of an apparent "running" of the observed neutrino mass with the energy, we write down a model Lagrangian, which describes a Yukawa-type interaction of a neutrino coupling to a scalar background field via a scalar-minus-pseudoscalar interaction. This constitutes an extension of the standard model. If the interaction is strong, then it leads to a substantial renormalization-group "running" of the neutrino mass and could potentially explain the experimental observations.

PACS numbers: 95.85.Ry, 11.15.-q, 03.70.+k, 05.10.Cc, 03.65.Pm

I. INTRODUCTION

For subluminal particles ("tardyons"), the dispersion relation for the energy E in terms of the velocity v is given by $E = m/\sqrt{1-v^2}$ (with $v < 1$), and for superluminal particles ("tachyons"), it reads as $E = m/\sqrt{v^2-1}$ with $v > 1$. Therefore, the "light barrier" at $v = 1$ (we set the speed of light equal to one) looks like an (infinitely) elevated mountain in terms of the energy of a relativistic particle. Recami [1] quotes Sudarshan with reference to an imaginary demographer who studies population patterns on the Indian subcontinent: "Suppose a demographer calmly asserts that there are no people North of the Himalayas, since none could climb over the mountain ranges! That would be an absurd conclusion. People of central Asia are born there and live there: they did not have to be born in India and cross the mountain range. So with faster-than-light particles."

In the early morning hours of 23 February 1987 (at 2'52'36"), an unexpected neutrino bunch arrived at the LSD detector under the Mont Blanc roughly 4.5 hours before the rest of the neutrinos from SN1987A, and before the supernova became visible [2]. We are currently facing mounting evidence that neutrinos may be genuinely superluminal particles ("tachyons"). The MINOS experiment [3] has measured superluminal neutrino propagation velocities which differ from the speed of light by a relative factor of $(5.1 \pm 2.9) \times 10^{-5}$ at an energy of about $E_\nu \approx 3\,\mathrm{GeV}$, supporting an earlier FERMILAB experiment where the trend of the data also pointed toward superluminal neutrinos [4]. This result has recently been confirmed by OPERA [5] with better statistics and in a wider energy interval, as detailed below. One of the prime candidates for a genuinely superluminal particle is the neutrino, which has never been observed at rest. A number of experimental groups have measured negative mass squares for the electron neutrino from tritium beta decay endpoints [6–9] with mean values in the interval $-147\,\mathrm{eV}^2 < m_\nu^2 < 0$ for the electron neutrino mass square, at an energy of the order of $E_\nu \approx 18\,\mathrm{keV}$. While some recent measurements indicate values consistent with a vanishing neutrino mass [10–12] at even lower energies, the mean value of the experimental data for m_ν^2 (electron neutrino) still is negative and of the order of a few negative eV^2 (for an excellent overview, see Ref. [13]). The idea that neutrinos might be of tachyonic character is not new [14–17]. Tachyonic neutrinos fulfill the dispersion relation $E_\nu^2 - \vec{p}^{\,2} = -m_\nu^2$ with an (initially) constant parameter m_ν. The quantity $-m_\nu^2$ can be interpreted as the negative mass square of the neutrino. The current situation indicates the need for a convenient descriptions of tachyonic fermions.

Ever since the early days of relativity, the notion of superluminal propagation has intrigued physicists [18], and the name "tachyon" was eventually coined in Ref. [19]. The main problem in the description of a quantum field theory with superluminal propagation is not the superluminal velocity itself [20], but the construction of field operators and the time ordering, which is in disarray because the time ordering of two space-time points which are separated by a space-like interval is not invariant under (subluminal) Lorentz boosts. Generally, it has been assumed that any particle in relativistic quantum theory should be described by a unitary irreducible representation of the Poincaré algebra or its supersymmetric generalization. It may be necessary to relax this restriction somewhat in order to accommodate for a field theory of supersymmetric tachyons [21–23]. Three recent review articles [1, 24, 25] provide rather detailed background information on the development of the theory of superluminal particles.

The recent OPERA experiment [5] uses a baseline of $L = (731278.0 \pm 0.2)\,\mathrm{m}$. Two clocks used in the measurement are accurately synchronized by a technique used to compare atomic clocks [26, 27]. It is of particular importance that the synchronization of the two systems was calibrated by the Federal Swiss Metrology Institute METAS (Bundesamt für Metrologie) in 2008 and verified in 2011 by the Federal German Metrology Institute PTB (Physikalisch-Technische Bundesanstalt). As reported in Ref. [5], the difference between the time base of the CERN and OPERA receivers was measured to be $(2.3 \pm 0.9)\,\mathrm{ns}$ and is taken into account in the evaluation of the measurement. The four data bins are

$$E_\nu = 13.8\,\mathrm{GeV}\,, \qquad \delta t = (54.7 \pm 18.4 \pm 7.1)\,\mathrm{ns}\,, \qquad \Delta = \frac{v-c}{c} = (2.24 \pm 0.75 \pm 0.29) \times 10^{-5}\,, \tag{1a}$$

$$E_\nu = 28.2\,\mathrm{GeV}\,, \qquad \delta t = (61.1 \pm 13.2 \pm 7.1)\,\mathrm{ns}\,, \qquad \Delta = \frac{v-c}{c} = (2.50 \pm 0.54 \pm 0.29) \times 10^{-5}\,, \tag{1b}$$

$$E_\nu = 40.7\,\mathrm{GeV}\,, \qquad \delta t = (68.1 \pm 19.1 \pm 7.1)\,\mathrm{ns}\,, \qquad \Delta = \frac{v-c}{c} = (2.53 \pm 0.78 \pm 0.29) \times 10^{-5}\,, \tag{1c}$$

and the overall average is

$$E_\nu = 17\,\mathrm{GeV}\,, \qquad \delta t = (57.8 \pm 7.2 \pm 7.1)\,\mathrm{ns}\,, \qquad \Delta = \frac{v-c}{c} = (2.37 \pm 0.32 \pm 0.29) \times 10^{-5}\,. \tag{1d}$$

While the OPERA data rather point to a slight increase in the ratio $\Delta = (v-c)/c$ with the neutrino energy, than to a trend in the opposite direction, the data are generally consistent with a constant ratio $\Delta = (v-c)/c$ in the entire energy interval $13.8\,\mathrm{GeV} < E_\nu < 40.7\,\mathrm{GeV}$.

Tachyonic neutrinos fulfill the space-like dispersion relation $E_\nu^2 - \vec{p}^{\,2} = -m_\nu^2$ and travel faster than light. Superluminality is conserved under Lorentz boosts (see Ref. [20] and Fig. 2 below). It has been argued that neutrinos traveling at velocities consistent with the recent OPERA data should decay by neutral massive analogues of Cerenkov radiation [28]. The noncovariant dispersion relation $E_\nu = |\vec{p}|\,v_\nu$ has been used in recent work on the subject [28] (here, v_ν denotes the neutrino velocity). Freely propagating subluminal relativistic particles as well as tachyons [1, 24, 25] fulfill the "opposite" relation $|\vec{p}| = E_\nu\,v_\nu$. Both relations $E_\nu = |\vec{p}|\,v_\nu$ and $|\vec{p}| = E_\nu\,v_\nu$ lead to a large virtuality $|E_\nu^2 - \vec{p}^{\,2}|$ on the order of $(117\,\mathrm{MeV})^2$ when applied to the recently measured OPERA data [see Eqs. (12) and (13) below]. These observations are inconsistent with beta decay end point measurements [6–12] which have led to values of a few eV^2, for neutrinos in the keV energy range. This confusing situation raises a number of questions. Starting from the tachyonic Dirac equation, we conclude that additional interactions, hitherto not accounted for, are required in order to explain the OPERA data which exhibit a larger-than-expected virtuality at higher energies, or, expressed differently, an energy-dependent mass.

At the current, early stage in the development of theoretical models describing superluminal particles, a certain degree of speculation cannot be avoided. For completeness, we should note that we neither consider models based on deformed special relativity [29–32] nor kinematic constraints resulting from such models [28, 33–35] in any greater detail. Lorentz symmetry is conserved in our approach.

We start with a digression on the kinematic constraints to the observation of neutrinos along the OPERA baseline in Sec. II. The tachyonic Dirac equation and its solutions are being reviewed in Sec. III. Chiral Yukawa interactions, which induce a neutrino mass running via the renormalization group (RG), are studied in Sec. IV. Conclusions are reserved for Sec. V. We always carefully distinguish between $|\vec{p}|$ and the four-vector p, and we use natural units with $\hbar = c = \epsilon_0 = 1$.

II. KINEMATIC CONSTRAINTS

The recent OPERA experiment has analyzed the propagation of muon neutrinos. If neutrinos propagate faster than the speed of light, then a number of decay processes are kinematically allowed which are otherwise forbidden. These include the following decays (see Fig. 1),

$$\nu_\mu \to \nu_\mu + \gamma\,, \tag{2a}$$

$$\nu_\mu \to \nu_\mu + e^+ + e^-\,, \tag{2b}$$

$$\nu_\mu \to \nu_\mu + \nu_e + \bar{\nu}_e\,. \tag{2c}$$

In Ref. [28], these decay processes are analyzed under the assumption of the Lorentz-violating dispersion relation

$$\frac{\mathrm{d}E_\nu}{\mathrm{d}|\vec{p}_\nu|} = \mathrm{const.}\,, \quad E_\nu = |\vec{p}_\nu|\,v_\nu\,, \quad v_\nu \approx 1 + \Delta\,, \tag{3}$$

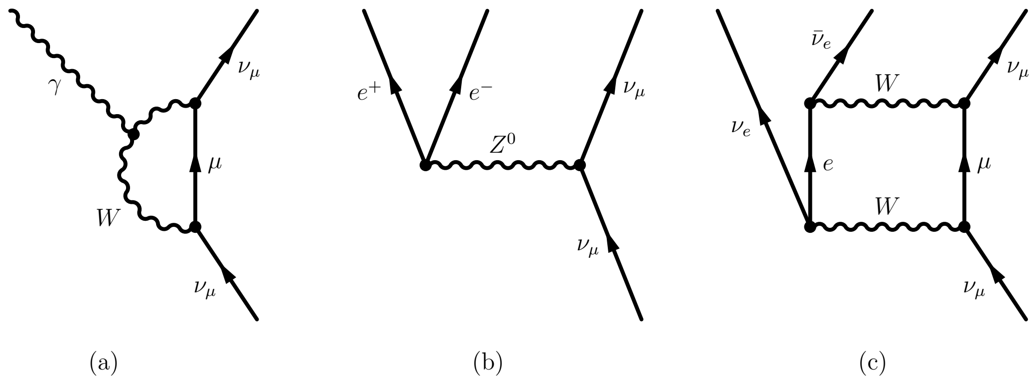

FIG. 1: Feynman diagrams for the decay processes of a tachyonic superluminal neutrino, as given in Eq. (2). The tachyonic neutrino may emit of photon via a W loop [Fig. (a)], or an electron-positron pair, [Fig. (b)], or a neutrino-antineutrino pair [Fig. (c)]. The processes scale with the quantum electrodynamic (QED) coupling constant α and the weak coupling constant G_F as follows, (a) is proportional to $\alpha\, G_F^2$, (b) is proportional to $\alpha\, G_F$, and (c) is proportional to $\alpha^2\, G_F^2$.

where $\Delta = 2.37 \times 10^{-5}$ corresponds to the value given in Ref. [5]. Processes (a) and (c) are parametrically suppressed with respect to process (b), and therefore process (b) is deemed to constitute the dominant decay channel.

One may observe that the dispersion relation $E_\nu = |\vec{p}_\nu|\, v_\nu$ is at variance with both the subluminal (also called tardyonic, see Ref. [19]) dispersion relation for freely propagating massive neutrinos,

$$E_\nu = \frac{m_\nu}{\sqrt{1-v_\nu^2}}\,, \qquad |\vec{p}_\nu| = \frac{m\, v_\nu}{\sqrt{1-v_\nu^2}} = E_\nu\, v_\nu\,, \qquad v_\nu < 1\,, \tag{4}$$

as well as with the dispersion relation for superluminal (tachyonic) particles [1, 24, 25, 36–38], which reads

$$E_\nu = \frac{m_\nu}{\sqrt{v_\nu^2-1}}\,, \qquad |\vec{p}_\nu| = \frac{m_\nu\, v_\nu}{\sqrt{v_\nu^2-1}} = E_\nu\, v_\nu\,, \qquad v_\nu > 1\,. \tag{5}$$

In both cases (4) and (5), one obtains $|\vec{p}_\nu| = E_\nu\, v_\nu$, not the opposite relation $E_\nu = |\vec{p}_\nu|\, v_\nu$ used in Ref. [28]. Under Lorentz transformations, superluminality of tachyonic particles is conserved (see Fig. 2). In two recent papers [39, 40], it has been observed that the conclusions of [28] would change if the dispersion relation were different. Here, we are concerned with a more general question: Namely, to investigate how the kinematic constraints change when we assume a tachyonic dispersion relation for the neutrino, and whether the process (2) is still kinematically allowed when $E_\nu^2 - \vec{p}_\nu^{\,2} < 0$.

For the process (2a), an easy calculation based on the energy and momentum conservation conditions reveals that

$$E_\nu = E'_\nu + E_\gamma\,, \qquad \vec{p}_\nu = \vec{p}^{\,\prime}_\nu + \vec{k}_\gamma\,, \qquad E_\nu = \sqrt{\vec{p}_\nu^{\,2} - m_\nu^2}\,, \qquad E'_\nu = \sqrt{\vec{p}^{\,\prime 2}_\nu - m_\nu^2}\,, \qquad E_\gamma = |\vec{k}_\gamma|\,. \tag{6}$$

Squaring the energy conservation condition, one obtains

$$E_\nu^2 = \left(\vec{p}^{\,\prime}_\nu + \vec{k}_\gamma\right)^2 - m_\nu^2 = \vec{p}^{\,\prime 2}_\nu + \vec{k}_\gamma^{\,2} - m_\nu^2 + 2\,\vec{p}^{\,\prime}_\nu \cdot \vec{k}_\gamma\,, \tag{7a}$$

$$E_\nu^2 = (E'_\nu + E_\gamma)^2 = \vec{p}^{\,\prime 2}_\nu + \vec{k}_\gamma^{\,2} - m_\nu^2 + 2\,|\vec{k}_\gamma|\,\sqrt{\vec{p}^{\,\prime 2}_\nu - m_\nu^2}\,, \tag{7b}$$

$$\vec{p}^{\,\prime}_\nu \cdot \vec{k}_\gamma = |\vec{k}_\gamma|\,\sqrt{\vec{p}^{\,\prime 2}_\nu - m_\nu^2}\,. \tag{7c}$$

We conclude that under the assumption of the Lorentz-covariant, tachyonic dispersion relation (5), weak-interaction Cerenkov radiation is allowed. In view of Eq. (7c), the photon is radiated off at a Cerenkov angle

$$\cos\theta_\gamma = \frac{\vec{p}^{\,\prime}_\nu \cdot \vec{k}_\gamma}{|\vec{k}_\gamma|\,|\vec{p}^{\,\prime}_\nu|} = \frac{\sqrt{\vec{p}^{\,\prime 2}_\nu - m_\nu^2}}{|\vec{p}^{\,\prime}_\nu|} = \frac{E'_\nu}{|\vec{p}^{\,\prime}_\nu|} = \frac{1}{v'_\nu} < 1\,, \tag{8}$$

under the assumption of a tachyonic neutrino with dispersion (7a). One may add that the kinematic consideration is somewhat analogous to that for the emission of ordinary Cerenkov radiation. The important observation is that

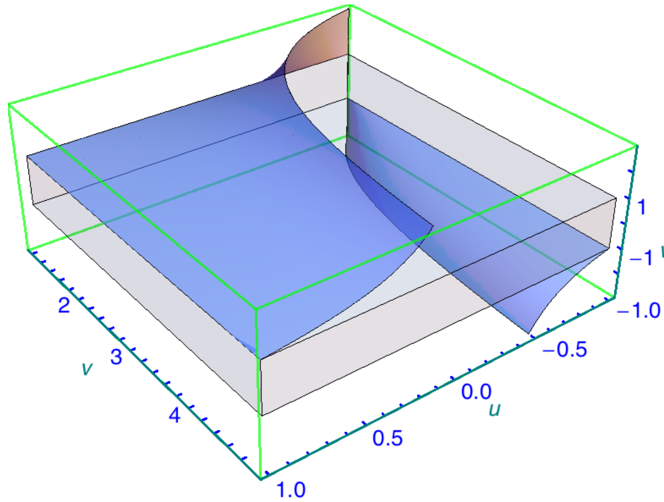

FIG. 2: (Color online.) Illustration of the Einstein velocity addition theorem $w = (u+v)/(1+u\,v)$, in the superluminal domain with $u \in (-1,1)$ and $v \in (1,3)$. For superluminal v, the range $w \in [-1,1]$ of values is excluded, as shown by the rectangular box.

under the tachyonic dispersion relation (5), the emission of a photon by the neutrino is always allowed, i.e., there is no threshold energy for the neutrino and there is no threshold for the tachyonic mass $-m_\nu^2$. Once the particle becomes tachyonic, weak Cerenkov radiation is kinematically allowed, but the Cerenkov cone narrows as $-m_\nu^2 \to 0$. For a particle fulfilling the noncovariant dispersion relation $E_\nu = |\vec{p}\,'_\nu|\, v_\nu$, with $v_\nu > 1$, the modified Cerenkov angle $\cos\theta'_\gamma$ is easily computed as

$$\cos\theta'_\gamma = \frac{1}{v'_\nu} + \frac{(v'^2_\nu - 1)|\vec{k}_\gamma|}{2v'_\nu\, E'_\nu} \approx \frac{1}{v'_\nu} < 1\,, \tag{9}$$

assuming a neutrino with the dispersion $E'_\nu = p'_\nu\, v'_\nu$ and $v'_\nu > 1$. This is very well approximated by $\cos\theta'_\gamma \approx 1/v'_\nu$ for $v'_\nu \approx 1$.

As a second step, let us consider a process in which a tachyonic neutrino fulfilling Eq. (5) emits a massive neutral vector meson of mass m_0. This is not depicted in Fig. (1) but still instructive. The kinematic conditions change,

$$E_\nu = E'_\nu + E_0\,, \qquad \vec{p}_\nu = \vec{p}\,'_\nu + \vec{k}_0\,, \qquad E_\nu = \sqrt{\vec{p}_\nu^{\,2} - m_\nu^2}\,, \qquad E'_\nu = \sqrt{\vec{p}\,'^2_\nu - m_\nu^2}\,, \qquad E_0 = \sqrt{\vec{k}_0^2 + m_0^2}\,. \tag{10}$$

The Cerenkov angle then becomes

$$\cos\theta_0 = \frac{m_0^2 + 2\sqrt{\vec{k}_0^2 + m_0^2}\,\sqrt{\vec{p}\,'^2_\nu - m_\nu^2}}{2\,|\vec{k}_0|\,|\vec{p}\,'_\nu|} \approx \frac{\sqrt{\vec{k}_0^2 + m_0^2}\,\sqrt{\vec{p}\,'^2_\nu - m_\nu^2}}{|\vec{k}_0|\,|\vec{p}\,'_\nu|}\,, \tag{11}$$

where the last expression is valid in the high-energy limit, i.e, for $|\vec{k}_0| \gg m_0$, and $|\vec{p}\,'_\nu| \gg m_\nu$. If the vector meson carries away the bulk of the energy, i.e. $E_0 = x\,E_\nu$ and $E'_\nu = (1-x)\,E_\nu$, with $x \lesssim 1$, then for highly energetic incoming superluminal neutrinos, one can always find a narrow cone near $\theta_0 \approx 0$ in which vector meson emission is possible. Again, for highly energetic tachyonic superluminal neutrinos, we conclude that there is no kinematic constraint on the size of the tachyonic mass term $-m_\nu^2$ which would restrict massive vector meson emission. Once the particle becomes tachyonic and the energy of the tachyonic particle is large enough, massive vector emission becomes kinematically allowed in a narrow angular region. By contrast, if we replace in Eq. (11) $-m_\nu^2 \to +m_\nu^2$, we would have $\cos\theta_0 > 1$, forbidding vector meson emission. Also, the Cerenkov angle θ_0 vanishes in the limit $m_\nu \to 0$. Using more extensive calculations, we have checked that the same statement applies to the light fermion pair emission given in Eq. (2b) and depicted in Fig. 1(b). Cerenkov-type pair emission becomes kinematically possible for highly energetic neutrinos, in a narrow angular region.

In the application of the tachyonic dispersion relation (5) to the OPERA data, we face a dilemma which also plagues the application of the Lorentz-noncovariant dispersion relation (3). Namely, we have for the OPERA data according to Eq. (1d),

$$-m_\nu^2 = E_\nu^2 - \vec{p}_\nu^{\,2} = E_\nu^2\,\left[1 - (1+\Delta)^2\right] = -\,(117\,\mathrm{MeV})^2 \qquad \text{[dispersion relation (5)]} \tag{12}$$

which is at least six orders of magnitude larger than the neutrino masses at low energy [6–12]. Likewise, assuming the dispersion relation (3) implies that

$$E_\nu^2 - \vec{p}_\nu^{\,2} = \vec{p}_\nu^{\,2} \left[(1+\Delta)^2 - 1\right] \approx E_\nu^2 \left[(1+\Delta)^2 - 1\right] = (117\,\mathrm{MeV})^2 \quad \text{[dispersion relation (3)]} . \tag{13}$$

If we define the expression $|E_\nu^2 - \vec{p}_\nu^{\,2}|$ as the "virtuality" of the neutrino which measures the deviation of the neutrino propagation velocity from the speed of light, then we can say that at the high OPERA energies, the neutrino velocity was not expected to deviate so much from the speed of light, neither in the superluminal nor in the subluminal direction. For example, if OPERA had hypothetically found a result of

$$\widetilde{\Delta} = -2.37 \times 10^{-5} \qquad \text{[opposite sign as compared to Eq. (1d)]} , \tag{14}$$

then this would have been equally surprising. In the latter case, one would probably have concluded immediately that the neutrino must be subject to a hitherto unknown interaction at high energy, modifying its effective (running) mass. We here advocate the viewpoint that the same conclusion should be drawn from the OPERA data: namely, the neutrino is genuinely tachyonic and subject to an unknown interaction at high energy which modifies its mass and its decay channels. Otherwise, it seems that the high-energy OPERA data [5] (in the GeV range) cannot be reconciled with the low-energy experimental results (in the keV range) of Refs. [6–12]. Of course, this statement holds provided the OPERA data are not subject to a hitherto undiscovered systematic error.

The data bins given in Eq. (1) are consistent with an energy-independent propagation velocity. While the quantity Δ need not be energy independent over large energy intervals, it appears to be so in in the energy interval $13.8\,\mathrm{GeV} < E_\nu < 40.7\,\mathrm{GeV}$. Therefore, in this energy interval observed by OPERA [5], the dispersion relation is assumed to be close to a linear relationship

$$m_\nu = m(E_\nu) \approx \eta\, E_\nu , \qquad 13.8\,\mathrm{GeV} < E_\nu < 40.7\,\mathrm{GeV} , \qquad \eta = \sqrt{(1+\Delta)^2 - 1} = 6.88 \times 10^{-3} \approx \frac{1}{145} . \tag{15}$$

The unknown interaction leading to the energy-dependent mass must now be investigated. When calculating decay rates, the existence of the additional interaction implies that one should use eigenstates of the neutrino in the additional, hitherto unknown interaction potential (i.e., taking into account the running mass) rather than freely propagating tachyonic states, with an effective energy-dependent tachyonic mass $m_\nu = m_\nu(E_\nu) \propto E_\nu$.

III. TACHYONIC DIRAC EQUATION

Given the obvious inconsistency of the OPERA data [5] with low-energy neutrino data [6–12], as manifest in the energy-dependent effective mass (15), one may ask why an equation that describes a genuinely tachyonic neutrino with an energy-independent, fixed tachyonic mass m_ν should be considered at all in the following. The reason is that if the neutrino is genuinely tachyonic, then one has to start from an equation which describes a genuinely tachyonic particle, with the possibility to describe additional perturbative interactions that modify the high-energy behaviour. Expressed differently, we would expect the tachyonic Dirac equation given below to describe the low-energy behaviour of neutrinos [6–12], while the large deviation from the light cone seen at high energies [3, 5] should be ascribed to additional interactions. We briefly recall here that the Lorentz-covariant tachyonic Dirac equation reads

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - \gamma^5\, m_\nu\right) \psi(x) = 0 , \qquad \gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix} , \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} . \tag{16}$$

Here, $x = (t, \vec{x})$, and we use the Dirac matrices in the Dirac representation [17]. The partial derivatives are $\partial_\mu = \partial/\partial x^\mu$, while $\gamma^5 = \mathrm{i}\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$ is the fifth current matrix. The tachyonic Dirac equation has been briefly discussed in Ref. [14–16]. It has recently been verified that this equation is $\mathcal{CP}$, as well as $\mathcal{T}$ invariant [17]. These symmetry properties apply to neutrinos. The positive-energy plane-wave solutions [17] of the tachyonic Dirac equation have the properties

$$\Psi(x) = \frac{1}{\sqrt{V}} U_\pm(\vec{k}_\nu)\, \mathrm{e}^{-\mathrm{i}k_\nu\cdot x} , \quad k_\nu = (E_\nu, \vec{k}_\nu) , \quad E_\nu = \sqrt{\vec{k}_\nu^2 - m_\nu^2} , \quad |\vec{k}_\nu| \geq m_\nu . \tag{17}$$

The negative-energy solutions [17] are given by

$$\Phi(x) = \frac{1}{\sqrt{V}} V_\pm(\vec{k}_\nu)\, \mathrm{e}^{\mathrm{i}k_\nu\cdot x} , \quad k_\nu = (E_\nu, \vec{k}_\nu) , \quad E_\nu = \sqrt{\vec{k}_\nu^2 - m_\nu^2} , \quad |\vec{k}_\nu| \geq m_\nu , \tag{18}$$

where V is the normalization volume. These states are normalized, with $U_+^+(\vec{k}_\nu)\,U_+(\vec{k}_\nu) = U_-^+(\vec{k}_\nu)\,U_-(\vec{k}_\nu) = V_+^+(\vec{k}_\nu)\,V_+(\vec{k}_\nu) = V_-^+(\vec{k}_\nu)\,V_-(\vec{k}_\nu) = 1$. The spinors entering these expressions read as

$$U_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu - E_\nu + |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_+(\vec{k}_\nu) \\ \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_+(\vec{k}_\nu) \end{pmatrix}, \qquad U_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_-(\vec{k}_\nu) \\ \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_-(\vec{k}_\nu) \end{pmatrix}, \tag{19}$$

where the helicity spinors $a_\pm(\vec{k}_\nu)$ are given below in Eq. (21). If we are interested in the massless limit, then we should first take into account the fact that massless particles propagate at velocities very close to the light cone. For $v = 1 + \Delta$, we have $E - |\vec{k}_\nu| \approx -m\,\Delta/2 \ll m$. Therefore, letting $\Delta \to 0$, the dominant term for the massless limit actually is the mass $m \gg E - |\vec{k}_\nu|$. This implies, e.g., that $U_+(\vec{k}_\nu) \to \frac{1}{\sqrt{2}} \begin{pmatrix} a_+(\vec{k}_\nu) \\ a_+(\vec{k}_\nu) \end{pmatrix}$ for the massless case. The negative-energy eigenstates are given by

$$V_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu - E_\nu + |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_+(\vec{k}_\nu) \\ \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_+(\vec{k}_\nu) \end{pmatrix}, \qquad V_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_-(\vec{k}_\nu) \\ \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{(E_\nu - |\vec{k}_\nu|)^2 + m_\nu^2}}\, a_-(\vec{k}_\nu) \end{pmatrix}. \tag{20}$$

The helicity spinors entering these expressions are given in terms of the polar and azimuthal angles θ and φ of the three-vector $\vec{k}_\nu$,

$$a_+(\vec{k}_\nu) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\, \mathrm{e}^{\mathrm{i}\,\varphi} \end{pmatrix}, \qquad a_-(\vec{k}_\nu) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, \mathrm{e}^{-\mathrm{i}\,\varphi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{21}$$

and fulfill

$$\frac{\vec{\sigma}\cdot\vec{k}_\nu}{|\vec{k}_\nu|}\, a_+(\vec{k}_\nu) = a_+(\vec{k}_\nu)\,, \qquad \frac{\vec{\sigma}\cdot\vec{k}_\nu}{|\vec{k}_\nu|}\, a_-(\vec{k}_\nu) = -a_+(\vec{k}_\nu)\,. \tag{22}$$

For plane waves, $E_\nu = \sqrt{\vec{k}_\nu^{\,2} - m_\nu^2}$ and $\vec{p}_\nu = \vec{k}_\nu$ fulfill the tachyonic dispersion relation (5), which we recall for convenience,

$$E_\nu = \frac{m_\nu}{\sqrt{v_\nu^2 - 1}}\,, \qquad |\vec{k}_\nu| = \frac{m\, v_\nu}{\sqrt{v_\nu^2 - 1}} = E_\nu\, v_\nu\,, \qquad v_\nu > 1\,, \tag{23}$$

so that $\sqrt{\vec{k}_\nu^{\,2} - m_\nu^2}$ never becomes imaginary. For $\vec{k}_\nu^{\,2} < m_\nu^2$, we have resonance and antiresonance energies. We start with the resonances, whose energies have a negative imaginary part,

$$R_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu + \frac{\mathrm{i}}{2}\Gamma_\nu + |\vec{k}_\nu|}{\sqrt{2}\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\, a_+(\vec{k}_\nu) \\ \dfrac{m_\nu - \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\, a_+(\vec{k}_\nu) \end{pmatrix}, \qquad R_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu - \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\, a_-(\vec{k}_\nu) \\ \dfrac{-m_\nu - \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\, a_-(\vec{k}_\nu) \end{pmatrix}, \tag{24a}$$

$$E_\nu = -\,\tfrac{\mathrm{i}}{2}\,\Gamma_\nu = -\mathrm{i}\sqrt{m_\nu^2 - \vec{k}_\nu^{\,2}}\,, \qquad \vec{k}_\nu^{\,2} < m_\nu^2\,. \tag{24b}$$

The antiresonance energies have a positive imaginary part,

$$S_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu - \frac{\mathrm{i}}{2}\Gamma_\nu + |\vec{k}_\nu|}{\sqrt{2}\,\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\; a_+(\vec{k}_\nu) \\ \dfrac{-m_\nu + \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\,\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\; a_+(\vec{k}_\nu) \end{pmatrix}, \qquad S_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu + \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\,\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\; a_-(\vec{k}_\nu) \\ \dfrac{m_\nu + \frac{\mathrm{i}}{2}\Gamma_\nu - |\vec{k}_\nu|}{\sqrt{2}\,\sqrt{\vec{k}_\nu^{\,2} + m_\nu^2 + \frac{1}{4}\,\Gamma_\nu^2}}\; a_-(\vec{k}_\nu) \end{pmatrix}, \tag{25a}$$

$$E_\nu = \tfrac{\mathrm{i}}{2}\,\Gamma_\nu = \mathrm{i}\,\sqrt{m_\nu^2 - \vec{k}_\nu^{\,2}}\,, \qquad \vec{k}_\nu^{\,2} < m_\nu^2\,. \tag{25b}$$

These states are also normalized, with $R_+^+(\vec{k}_\nu)\,R_+(\vec{k}_\nu) = R_-^+(\vec{k}_\nu)\,R_-(\vec{k}_\nu) = S_+^+(\vec{k}_\nu)\,S_+(\vec{k}_\nu) = S_-^+(\vec{k}_\nu)\,S_-(\vec{k}_\nu) = 1$. The term "resonances" is used in the physics literature in two contexts: (i) in order to designate the complex energy eigenvalue of a Hamiltonian, and (ii) in order to designate the peak in a cross section or a quantum state which can decay into a final state with a different particle content. In the current case, the interpretation (i) is relevant. The resonances have complex resonance energies; the waves are evanescent (exponentially damped) just like the diffracted wave under total reflection, or a wave in a waveguide below the minimum frequency for the $\mathrm{TE}_{1,0}$ mode necessary for propagation, and the resonance energies are complex just as in the case of a resonance energy of the Stark effect [41]. Resonances are damped for propagation forward in time, antiresonances for propagation backward in time, in accordance with the Feynman prescription. The wavelength of the resonance states is too long to be supported in a genuinely superluminal wave packet of tachyonic mass m_ν^2.

The noncovariant, Hamiltonian form of Eq. (16) reads as

$$H_5\psi(\vec{x}) = \left(\vec{\alpha}\cdot\vec{p} + \beta\,\gamma^5\,m_\nu\right)\,\psi(\vec{x}) = E_\nu\,\psi(\vec{x})\,, \tag{26}$$

where $\beta = \gamma^0$, and $\vec{\alpha} = \gamma^0\,\vec{\gamma}$. The Hamiltonian H_5 has the pseudo-Hermitian [42–50] property

$$H = \mathcal{P}\,H_5^+(\vec{x})\,\mathcal{P}^{-1} = P\,H_5^+(-\vec{x})\,P^{-1}\,, \tag{27}$$

where $\mathcal{P}$ is the full parity transformation and P is the parity matrix $P = \gamma^0$. The eigenvalues of a pseudo-Hermitian operator come in complex-conjugate pairs and are real if the tachyonic dispersion relations (5) are fulfilled. This can be seen as follows. Because the spectrum of a Hermitian adjoint operator consists of the complex conjugate eigenvalues, we have an eigenvector $\phi(\vec{x})$ with eigenvalue E^* provided there exists an eigenvector $\psi(\vec{x})$ with eigenvalue E,

$$H_5(\vec{x})\,\psi(\vec{x}) = E\,\psi(\vec{x})\,, \qquad H_5^+(\vec{x})\,\phi(\vec{x}) = E^*\,\phi(\vec{x})\,. \tag{28}$$

Then, the transformation $\vec{x} \to -\vec{x}$ and the introduction of the $P = \gamma^0$ parity matrix leads to

$$H_5^+(-\vec{x})\,\phi(-\vec{x}) = E^*\,\phi(-\vec{x})\,, \qquad PH_5^+(-\vec{x})P^{-1}\,\left(P\phi(-\vec{x})\right) = E^*\,P\phi(-\vec{x})\,. \tag{29}$$

By assumption, $PH_5^+(-\vec{x})P^{-1} = H_5(\vec{x})$ and thus

$$H_5(\vec{x})\,P\phi(-\vec{x}) = E^*\,P\phi(-\vec{x})\,, \qquad H_5(\vec{x})\,\widetilde{\psi}(\vec{x}) = E^*\widetilde{\psi}(\vec{x})\,, \qquad \widetilde{\psi}(\vec{x}) = P\phi(-\vec{x})\,. \tag{30}$$

This implies that $\widetilde{\psi}(\vec{x}) = P\psi(-\vec{x})$ is an eigenvector with eigenvalue E^*. The eigenvalues of H_5 thus come in complex-conjugate pairs, and furthermore, they are real for plane waves fulfilling the dispersion relation (5).

The covariant Green function corresponding to the Hamiltonian H_5 thus reads as

$$S_T(p) = \gamma^0\,\frac{1}{E - H_5} = \frac{\not{p} + \gamma^5\,m_\nu}{p^2 + m_\nu^2}\,. \tag{31}$$

The tachyonic poles at $E_\nu^2 - \vec{p}^{\,2} = -m_\nu^2$ have to be encircled in a way consistent with the boundary conditions imposed on the Green function. Eigenvalues with $E_\nu^2 = \vec{p}^{\,2} - m_\nu^2 < 0$ represent evanescent waves. If one encircles the poles of the Green function according to the Feynman prescription,

$$S_T(p) = \frac{1}{\not{p} - \gamma^5\,(m_\nu + \mathrm{i}\,\epsilon)} = \frac{\not{p} - \gamma^5\,m_\nu}{p^2 + m_\nu^2 + \mathrm{i}\,\epsilon}\,, \tag{32}$$

then the energy-momentum dispersion relation is infinitesimally displaced to read $E_\nu = \pm\sqrt{\vec{p}^{\,2} - m_\nu^2 - \mathrm{i}\,\epsilon}$. This is consistent with the evanescent wave picture because positive-energy solutions have the form $E_\nu = \epsilon - \mathrm{i}\,\sqrt{|\vec{p}^{\,2} - m_\nu^2|}$

and are thus exponentially damped for the propagation into the future, whereas negative-energy solutions have the form $E_\nu = -\epsilon + \mathrm{i}\sqrt{|\vec{p}^{\,2} - m_\nu^2|}$ and are thus exponentially damped for the propagation into the past. In general, the Feynman prescription assigns an infinitesimal negative imaginary part to energies whose real part is positive, and vice versa.

Thus, while the time propagation of strictly tachyonic wave packets (superpositions of the tachyonic plane-wave solutions) is fully unitary (they have real eigenvalues), a slight violation of unitarity cannot be avoided if one allows eigenstates with $E_\nu^2 = \vec{p}^{\,2} - m_\nu^2 < 0$. The complex resonance energies (the real part is only infinitesimal)

$$E_\nu = \epsilon - \mathrm{i}\sqrt{|\vec{p}^{\,2} - m_\nu^2|}\,, \qquad E_\nu = -\epsilon + \mathrm{i}\sqrt{|\vec{p}^{\,2} - m_\nu^2|}\,, \qquad \vec{p}^{\,2} < m_\nu^2\,, \tag{33}$$

describe the suppression of subluminal components of a superluminal wave packet under time evolution. One has to allow these solutions in the propagator (32) if one would like to carry out the Fourier transformation consistently, i.e., over the entire range $p^\mu \in \mathbb{R}^4$, or describe the time evolution of a general wave packet under the Green function (32). It seems that a slight violation of unitarity, relevant to the small sector $\vec{p}^{\,2} < m_\nu^2$, where m_ν initially is on the order of a few eV, is a price for the introduction of tachyonic particles [23]. Note that full unitarity cannot be preserved anyway in a tachyonic theory if one goes beyond tree-level amplitudes, as shown in Ref. [51]. The time propagation of wave packets in potentials with manifestly complex resonance energies has been described in Refs. [52, 53]. The evanescence of the subluminal neutrino wave function components, which are excluded from the real neutrino plane-wave eigenstates but included in the propagator, is somewhat analogous to the photon propagator, where one includes the so-called scalar and longitudinal photons in the photon propagator but leaves them out from the real, physical states of the photon field, which are composed of transverse photons.

In Ref. [17], the tachyonic propagator (32) is derived not by inversion of the Hamiltonian, but by a quantization of the tachyonic field operators. We briefly sketch the essential elements of the derivation. The field operator is written as

$$\begin{aligned}\hat{\psi}(x) &= \int \frac{\mathrm{d}^3 k_\nu}{(2\pi)^3}\,\frac{m_\nu}{E_\nu}\sum_{\sigma=\pm}\left[b_\sigma(k_\nu)\,\mathcal{U}_\sigma(\vec{k}_\nu)\,\mathrm{e}^{-\mathrm{i}\,k_\nu\cdot x} + b_\sigma(-k_\nu)\,\mathcal{V}_\sigma(\vec{k}_\nu)\,\mathrm{e}^{\mathrm{i}\,k_\nu\cdot x}\right] \\ &= \int \frac{\mathrm{d}^3 k_\nu}{(2\pi)^3}\,\frac{m}{E_\nu}\sum_{\sigma=\pm}\left[b_\sigma(k_\nu)\,\mathcal{U}_\sigma(\vec{k}_\nu)\,\mathrm{e}^{-\mathrm{i}\,k_\nu\cdot x} + d^+_\sigma(k_\nu)\,\mathcal{V}_\sigma(\vec{k}_\nu)\,\mathrm{e}^{\mathrm{i}\,k_\nu\cdot x}\right]\,, \end{aligned} \tag{34}$$

where $E_\nu = \sqrt{\vec{k}_\nu^2 - m_\nu^2 - \mathrm{i}\,\epsilon}$ is the tachyonic energy and the four-vector k_ν equals $k_\nu = (E_\nu, \vec{k}_\nu)$. Here, the b operators annihilate particle, whereas the d^+ create antiparticles. We here explicitly accept a Lorentz-covariant vacuum state, which transforms according to Ref. [19]. The Lorentz-transformed vacuum is filled with all particle and antiparticle states whose energy changes sign under a Lorentz transformation (Lorentz boost), as outlined in Eqs. (5.6) and (5.7) of Ref. [19]. The spinors $\mathcal{U}$ and $\mathcal{V}$ are given by

$$\mathcal{U}_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu - E_\nu + |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_+(\vec{k}_\nu) \\ \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_+(\vec{k}_\nu) \end{pmatrix}, \qquad \mathcal{U}_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_-(\vec{k}_\nu) \\ \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_-(\vec{k}_\nu) \end{pmatrix} \tag{35a}$$

for positive energy, and by

$$\mathcal{V}_+(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu - E_\nu + |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_+(\vec{k}_\nu) \\ \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_+(\vec{k}_\nu) \end{pmatrix}, \qquad \mathcal{V}_-(\vec{k}_\nu) = \begin{pmatrix} \dfrac{-m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_-(\vec{k}_\nu) \\ \dfrac{m_\nu + E_\nu - |\vec{k}_\nu|}{2\sqrt{m_\nu}\,\sqrt{|\vec{k}_\nu| - m_\nu}}\,a_-(\vec{k}_\nu) \end{pmatrix} \tag{35b}$$

for negative energy (in both cases, we assume that $|\vec{k}_\nu| > m$). The normalization conditions are given by

$$\overline{\mathcal{U}}_\sigma(\vec{k}_\nu)\,\mathcal{U}_\sigma(\vec{k}_\nu) = \mathcal{U}^+_\sigma(\vec{k}_\nu)\gamma^0\mathcal{U}_\sigma(\vec{k}_\nu) = \sigma\,, \qquad \overline{\mathcal{V}}_\sigma(\vec{k}_\nu)\,\mathcal{V}_\sigma(\vec{k}_\nu) = \mathcal{V}^+_\sigma(\vec{k}_\nu)\gamma^0\mathcal{V}_\sigma(\vec{k}_\nu) = -\sigma\,. \tag{35c}$$

Quantizing the theory according to Fermi–Dirac statistics,

$$\{b_\sigma(k_\nu), b_\rho(k'_\nu)\} = \left\{a^+_\sigma(k_\nu), a^+_\rho(k'_\nu)\right\} = \{d_\sigma(k_\nu), d_\rho(k'_\nu)\} = \left\{d^+_\sigma(k_\nu), d^+_\rho(k'_\nu)\right\} = 0\,, \tag{36a}$$

$$\left\{b_\sigma(k_\nu), b^+_\rho(k'_\nu)\right\} = (-\sigma)\,(2\pi)^3\,\frac{E}{m}\delta^3(\vec{k}_\nu - \vec{k}'_\nu)\,\delta_{\sigma\rho}\,, \qquad \left\{d_\sigma(k_\nu), d^+_\rho(k'_\nu)\right\} = (-\sigma)\,(2\pi)^3\,\frac{E}{m}\delta^3(\vec{k}_\nu - \vec{k}'_\nu)\,\delta_{\sigma\rho}\,, \tag{36b}$$

one can easily show that

$$\sum_\sigma (-\sigma)\, \mathcal{U}_\sigma(\vec{k}_\nu) \otimes \overline{\mathcal{U}}_\sigma(\vec{k}_\nu)\, \gamma^5 = \frac{\not{k}_\nu - \gamma^5\, m}{2m}\,, \qquad \sum_\sigma (-\sigma)\, \mathcal{V}_\sigma(\vec{k}_\nu) \otimes \overline{\mathcal{V}}_\sigma(\vec{k}_\nu)\, \gamma^5 = \frac{\not{k}_\nu + \gamma^5\, m}{2m}\,. \tag{37}$$

The field anticommutator is

$$\begin{aligned} \{\hat{\psi}_\xi(x), \overline{\hat{\psi}}_{\xi'}(y)\} &= \left\langle 0 \left| \{\hat{\psi}_\xi(x), \overline{\hat{\psi}}_{\xi'}(y)\} \right| 0 \right\rangle \\ &= \int \frac{\mathrm{d}^3 k_\nu}{(2\pi)^3} \frac{m_\nu}{E_\nu} \sum_{\sigma=\pm} \left\{ \mathrm{e}^{-\mathrm{i} k_\nu \cdot (x-y)}\, (-\sigma)\, \left[\mathcal{U}_\sigma(\vec{k}_\nu)\right]_\xi \left[\overline{\mathcal{U}}_\sigma(\vec{k}_\nu)\right]_{\xi'} + \mathrm{e}^{\mathrm{i} k_\nu \cdot (x-y)}\, (-\sigma)\, \left[\mathcal{V}_\sigma(\vec{k}_\nu)\right]_\xi \left[\overline{\mathcal{V}}_\sigma(\vec{k}_\nu)\right]_{\xi'} \right\}\,, \end{aligned} \tag{38}$$

where ξ denotes the spinor index. It follows that

$$\{\hat{\psi}_\xi(x), \overline{\hat{\psi}}_{\xi'}(y)\}\, \gamma^5 = \left(\mathrm{i}\,\not{\partial} - \gamma^5\, m_\nu\right)_{\xi\xi'} \mathrm{i}\, \Delta(x-y)\,, \qquad \Delta(x-y) = -\mathrm{i} \int \frac{\mathrm{d}^3 k_\nu}{(2\pi)^3} \frac{1}{2E_\nu} \left(\mathrm{e}^{-\mathrm{i} k_\nu \cdot (x-y)} - \mathrm{e}^{\mathrm{i} k_\nu \cdot (x-y)}\right)\,, \tag{39}$$

where $\Delta(x-y)$ is introduced as in Chap. 3 of Ref. [54]. Furthermore, Eq. (3.170) of [54] finds the generalization

$$\{\hat{\psi}_\xi(x), \overline{\hat{\psi}}_{\xi'}(y)\}\, \gamma^5 \Big|_{x_0=y_0} = -\left(\gamma^0\right)_{\xi\xi'}\, \partial_0\, \Delta(x-y) \Big|_{x_0=y_0} = \left(\gamma^0\right)_{\xi\xi'}\, \delta^3(\vec{x}-\vec{y})\,. \tag{40}$$

In full analogy with Eq. (3.174) of Ref. [54] and in agreement with Ref. [55], the tachyonic (T) propagator is then found as

$$\left\langle 0 \left| T\, \hat{\psi}_\xi(x)\, \overline{\hat{\psi}}_{\xi'}(y) \gamma^5 \right| 0 \right\rangle = \mathrm{i}\, S_T(x-y)_{\xi\xi'}\,, \qquad S_T(x-y) = \int \frac{\mathrm{d}^4 k_\nu}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i} k_\nu \cdot (x-y)}\, \frac{\not{k}_\nu - \gamma^5\, m_\nu}{k_\nu^2 + m_\nu^2 + \mathrm{i}\,\epsilon}\,, \tag{41}$$

which confirms Eq. (32). Indeed, the propagator obtained from the quantized theory is equal to the propagator obtained from the inversion of the Hamiltonian in the Lorentz-covariant formulation, as it should. The couplings of the neutrino involve the chirality projector $(1-\gamma^5)/2$, and in view of $\gamma^5\,(1\pm\gamma^5)/2 = \pm(1\pm\gamma^5)/2$, the introduction of the γ^5 matrix in Eq. (32) is reabsorbed into the interaction Lagrangian. The non-unitarity is small because m_ν^2 is very small. For a tachyonic particle, the evanescence of non-tachyonic wave packet components is natural because its tachyonic components remain tachyonic upon Lorentz transformation (see Fig. 2). The tachyonic Dirac equation provides for a convenient framework to describe freely propagating, superluminal, electromagnetically neutral, particles.

IV. NEUTRINO MASS RUNNING

In Secs. II and III, we have seen that a running neutrino mass (with the energy) is able to conceivably suppress Cerenkov-type decay processes, and the quantization of the tachyonic Dirac equation has been discussed as a convenient description for tachyonic spin-1/2 particles; it naturally implies the suppression of the right-handed neutrino. If current experimental data [5] are confirmed, then we now have to explain why the effective mass of the neutrino, which needs to be inserted into the tachyonic Dirac equation, changes from a few eV in the keV neutrino energy range, to a mass on the order of MeV in the GeV energy range. We note that neutrino mass running is usually assumed to initiate on the energy scales of Grand Unification (see-saw mechanism). However, the experimental data [2, 3, 5–9] all point to a neutrino mass running which sets in at much lower energy scales. We assume that the mass term is genuinely tachyonic.

The scenario that we would like to propose is as follows: We conjecture that the neutrino mass running is due to an interaction with a hitherto unknown field that modifies its effective mass with the energy. At low energy, the interaction with the unknown field is weak, so that the apparent neutrino mass is in the eV range, whereas at higher energies, the interaction becomes stronger and leads to the observed [5] large tachyonic masses. We thus assume that the (bulk of the) neutrino mass is created dynamically [56]. Possibly, there is some threshold region where the effective mass of the neutrino intersects with the mass of the field it interacts with, and this might help explain consistency with astrophysical data [2]. In the following, we would like to present a semi-quantitative analysis which supports these conjectures.

We investigate a scalar-minus-pseudoscalar ($S-P$) interaction Lagrangian of the form

$$\mathcal{L}_{\rm int} = G\, \hat{\phi}_X\, \overline{\hat{\psi}}\, (1-\gamma^5)\, \hat{\psi}\,. \tag{42}$$

Here, $\hat{\phi}_X$ is a scalar field operator, G is a dimensionless coupling, and the fermionic field operators for the neutrino are denoted as $\hat{\psi}$. The operator is of dimension 4 and therefore renormalizable; it describes a Yukawa interaction with a chirality projector. The complete Lagrangian of the tachyonic neutrino field, the scalar field plus the $S-P$ interaction reads

$$\mathcal{L}(x) = \frac{\mathrm{i}}{2}\left[\overline{\hat{\psi}}(x)\gamma^\mu\left(\partial_\mu\hat{\psi}(x)\right) - \left(\partial_\mu\overline{\hat{\psi}}(x)\right)\gamma^\mu\hat{\psi}(x)\right] - \overline{\hat{\psi}}(x)\,\gamma^5\,m_\nu\,\hat{\psi}(x)$$
$$- \frac{1}{2}\,\hat{\phi}_X(x)\,\left(\Box + M_X^2\right)\hat{\phi}_X(x) + G\,\hat{\phi}_X(x)\,\overline{\hat{\psi}}(x)\,(1-\gamma^5)\,\hat{\psi}(x)\,. \tag{43}$$

At low energy, from dimensional analysis alone, the induced one-loop neutrino mass running via the renormalization group (RG) can be written down as

$$\frac{\mathrm{d}m_\nu}{\mathrm{d}\ln(\mu)} = \mu\,\frac{\mathrm{d}m_\nu}{\mathrm{d}\mu} \propto [m_\nu(\mu)]^3\;[G_X(\mu)]^2\,, \qquad [G_X(\mu)]^2 = G_X^2\,\ln(\mu) = \frac{G^2}{M_X^2}\,\ln(\mu)\,, \tag{44}$$

where we assume a logarithmic running of the coupling constant with the scale μ. Integrating the RG evolution equation,

$$\int\frac{\mathrm{d}m_\nu}{m_\nu^3} = G_X^2\int\frac{\mathrm{d}\mu}{\mu}\,, \qquad \int\limits_{m_\nu(18\,\mathrm{keV})}^{m_\nu(17\,\mathrm{GeV})}\frac{\mathrm{d}m_\nu}{m_\nu^3} = G_X^2\int_{18\,\mathrm{keV}}^{17\,\mathrm{GeV}}\frac{\mathrm{d}\mu}{\mu}\,\ln(\mu), \tag{45}$$

with $m_\nu(18\,\mathrm{keV}) \approx 100\,\mathrm{eV}$ (see Ref. [6]) and $m_\nu(17\,\mathrm{GeV}) \approx 117\,\mathrm{MeV}$ (see Ref. [5]), we find that an X particle of mass in the range $M_X \approx 1.4\,\mathrm{keV}$ could potentially induce a neutrino mass running from about 100 eV at 18 keV energies [6] to 117 MeV at energies of 17 GeV [5]. Here, we assume that $G \approx 1$ and a universal running of the electron neutrino mass [6] and the muon neutrino mass [5] with the energy. The difference in the observed OPERA neutrino mass [5] of $117\,\mathrm{MeV}$ with low-energy neutrino data Refs. [6–12], where masses in the eV range were observed, suggests that significant neutrino mass running has to set in at energies much below $17\,\mathrm{GeV}$, so that we can safely assume that $M_X \ll 17\,\mathrm{GeV}$. This finding and the interaction (42) is not described by any known particle in the standard model, and thus, our model constitutes a pertinent extension. However, one may object that this treatment amounts to an application of a one-loop running of the mass in a domain which in view of $G \approx 1$ clearly is nonperturbative.

This high-energy limit could be analyzed as follows. We first recall that in the high-energy domain, where the effective neutrino mass is in the MeV range (see Refs. [4, 5]) we assume that the neutrino mass is (almost) exclusively generated by the strong (nonperturbative) self-interaction with the X field. It is interesting to observe that *polynomial* behaviour of RG functions in the *strong-coupling domain* has recently been obtained by a sophisticated analysis of higher-order perturbative terms, for the β functions of ϕ^4 theories and of quantum electrodynamics [57, 58]. If the mass of the X particle is negligible as compared to the mass of the neutrino in the high-energy domain, then the mass scaling must be independent of M_X, and again, from dimensional analysis alone, we may conjecture that in the high-energy, strong-coupling limit,

$$\mu\,\frac{\mathrm{d}m_\nu}{\mathrm{d}\mu} \propto G^2\,m_\nu\,, \qquad \int\frac{\mathrm{d}m_\nu}{m_\nu} = K\,G^2\int\frac{\mathrm{d}\mu}{\mu}\,, \qquad m_\nu(\mu) = m_\nu(\mu_0)\,\left(\frac{\mu}{\mu_0}\right)^{K\,G^2}\,, \tag{46}$$

where K is a constant of order unity. In view of Eq. (15), if we assume that $G \approx 1/\sqrt{K}$, then

$$m_\nu = m_\nu(E_\nu) = \eta\,(E_\nu)^{K\,G^2} \approx \eta\,E_\nu\,, \qquad G \approx \frac{1}{\sqrt{K}}\,, \qquad \eta \approx \frac{1}{145}\,, \tag{47}$$

where the value of η is chosen such as to be consistent with Eq. (15).

We are now in the position to add some more, somewhat speculative, remarks on the experimental findings of Refs. [3–5]. Based on the numerical entries in Table II and Fig. 3 of Ref. [4], one may investigate the observed neutrino velocities as a function of the propagation energy. The authors of the somewhat inconclusive 1979 paper (Ref. [4]) suggest to ascribe a path length correction of $\Delta_{\mathrm{path}} = -0.5^{+0.2}_{-0.1} \times 10^{-4}$ to their data, because the muons that were "racing" against the neutrinos in the experiment were assumed to be artificially delayed due to multiple scattering events, which extend the muon path length in comparison to the muon neutrino path length. The path length correction was assumed to be constant over the energy range analyzed in Ref. [4], uniformly affecting neutrinos in the energy range of $32\,\mathrm{GeV} < E_\nu < 195\,\mathrm{GeV}$ in an experiment over a relatively short baseline of about 900 m

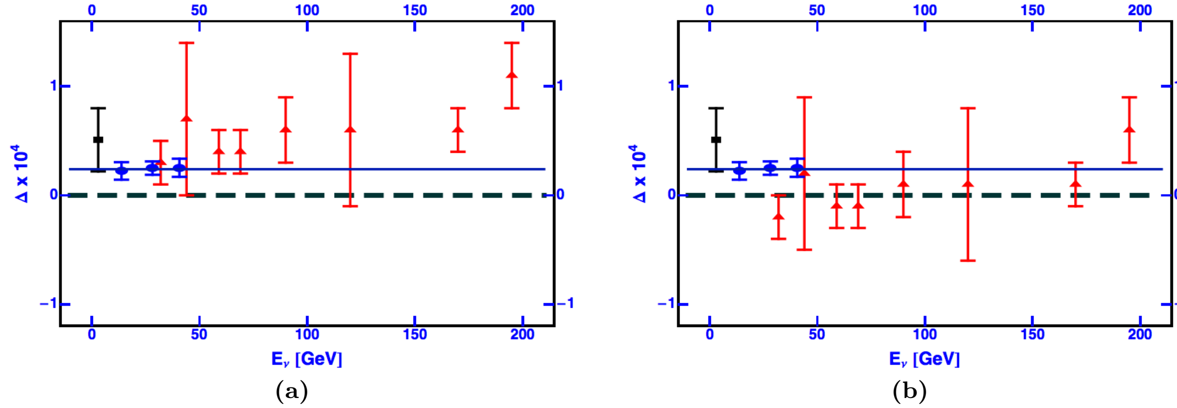

FIG. 3: (Color online.) Measured neutrino velocities in the range $E_\nu = 3\,\mathrm{GeV}$ (Ref. [3]) up to $E_\nu = 195\,\mathrm{GeV}$ (Ref. [4]). The OPERA data are given in Eq. (1) and correspond to the data bins at $E = 13.8\,\mathrm{GeV}$, $E = 28.2\,\mathrm{GeV}$, and $E = 40.7\,\mathrm{GeV}$ (circles). The data point at $E_\nu = 3\,\mathrm{GeV}$ is from Ref. [3] (square). All remaining data points (triangles) are from Ref. [4]. Panel (a) corresponds to the data plotted in Fig. 3 of Ref. [3], while panel (b) applies a path length correction of $\Delta_{\rm path} = -0.5 \times 10^{-4}$ to the data (triangles) of Ref. [3], as discussed near the end of Ref. [3]. Here, Δ is the relative deviation from the speed of light in vacuum, which we multiply by a scaling factor 10^4 on y axis. The solid line at $\Delta = 2.4 \times 10^{-4}$ corresponds to the result (48) based on our model.

(which is smaller than the OPERA baseline by a factor of roughly 10^3). We find that the discussion on the derivation of the path length correction in Ref. [4] is rather short and therefore present data with and without this correction in Fig. 3; the same approach was recently taken in Figs. 1 and 2 of Ref. [31]. The model (47) leads to a constant deviation of the neutrino velocity of

$$\Delta = \frac{v-c}{c} = \sqrt{1+\eta^2} - 1 = 2.4 \times 10^{-5}\,, \tag{48}$$

independent of the neutrino energy. This result is compared to the available experimental data [3–5] in Fig. 3. While our model is somewhat speculative at the current stage, it is intriguing to observe that the solution of the simple-minded RG equation (47) is in good agreement with the observed neutrino velocities over a wide energy interval (see Fig. 3). We also recall that the concomitant significant neutrino mass running will suppress decays because the tachyonic mass in the exit channel is much lower than in the incoming channel (see Sec. II).

V. CONCLUSIONS

Tachyons have a potential of fundamentally altering our view of physical law, but they can be incorporated into the framework of Lorentz transformations, despite obvious problems with the causality principle. In Ref. [36], the authors argue that a "sensible" theory is obtained if one insists that the only physical quantities are transition amplitudes, and a negative-energy in (out) state is understood to be a positive-energy out (in) state. This statement is in need of further explanation. Suppose that observer A sees event E before E', and observer A' sees event E' before E, because the two events are separated by a space-like interval, and the Lorentz transform for the frames A and A' reverses the time-ordering of events E and E'. According to Ref. [20], the reversed time ordering occurs if and only if the energy between the two frames also changes sign. So, provided one reinterprets the negative-energy eigenstates of tachyonic Dirac Hamiltonian propagating backward in time (the antiresonances included) as positive-energy solutions propagating forward in time, the creation and absorption of a particle can be consistently reinterpreted if only the transition amplitude is unaffected by the reinterpretation. This point has also been stressed in Refs. [20, 59].

One problem, though, in the consistency of observations of tachyons lies in conceivable decay processes [28]. In this paper, we investigate threshold conditions for the emission (see Sec. II) of real particles by analogues of Cerenkov radiation emitted by superluminal, tachyonic neutrinos that fulfill the dispersion relation (5). We find that such emissions, as shown in Fig. 1, are possible at high energies for small Cerenkov angles in a narrow cone of emission angles θ [see Eqs. (8) and (11)]. Furthermore, at sufficiently large energy, a nonvanishing emission probability exists for even very small tachyonic mass squares $-m_\nu^2$. However, the calculation of the corresponding decay rates crucially depends on the dispersion relation used in the calculation. The tachyonic relation (5) is Lorentz-invariant, and the effective mass m_ν crucially influences the decay rate. We then investigate, based on the tachyonic Dirac equation (see

Sec. III), how the effective neutrino mass m_ν could possibly change from a few eV at low energies in the keV range to energies of a hundred MeV in the GeV range [see Eqs. (12), (13) and (15)].

We here come to the conclusion that a viable explanation for the large virtuality $E_\nu^2 - \vec{p}^2$ of the OPERA neutrinos could be due to an additional interaction that modifies the neutrino propagation at high energies. At an energy in the GeV range, as measured by OPERA, the propagation velocity of a particle with a rest mass on the order of a few eV is not expected to deviate from the speed of light by a factor on the order of 10^{-5}. It does not really matter in this case that the OPERA experiment has measured a deviation of v_ν from c in the *super*luminal direction. A hypothetical experimental result for $v_\nu - c < 0$ in the *sub*luminal direction, of the same order-of-magnitude, as indicated in Eq. (14), would have been equally surprising. According to previous neutrino data [6–12], OPERA was not expected to find a deviation $|v_\nu - c|$ in the neutrino propagation velocity of the order-of-magnitude given in Eq. (15). In light of Eqs. (12) and (13), the OPERA signal would otherwise correspond to a particle with a rest mass in the range of a hundred MeV, or, with an effective mass of the neutrino that grows linearly with the energy. Unfortunately, neither the Higgs mechanism nor the Gross-Neveu model, induce a mass that depends on the energy. Once the vacuum expectation value of the background field that generates the mass is fixed, the mass of the constituent particle is also fixed. We find that it is indicated to investigate genuine neutrino mass running due to interactions which have hitherto not been introduced into the standard model. In Sec. IV of this paper, we write down a chiral Yukawa interaction which might induce a neutrino mass running with the experimentally observed parameters.

Acknowledgments

Helpful conversations with B. J. Wundt are gratefully acknowledged. The author acknowledges support from the National Science Foundation and by a Precision Measurement Grant from the National Institute of Standards and Technology.

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