# A finite-dimensional quantum model for the stock market

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**Abstract**

We present a finite-dimensional version of the quantum model for the stock market proposed in [C. Zhang and L. Huang, A quantum model for the stock market, *Physica A* **389** (2010) 5769]. Our approach is an attempt to make this model consistent with the discrete nature of the stock price and is based on the mathematical formalism used in the case of the quantum systems with finite-dimensional Hilbert space. The rate of return is a discrete variable corresponding to the coordinate in the case of quantum systems, and the operator of the conjugate variable describing the trend of the stock return is defined in terms of the finite Fourier transform. The stock return in equilibrium is described by a finite Gaussian function, and the time evolution of the stock price, directly related to the rate of return, is obtained by numerically solving a Schrödinger type equation.

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## 1. Introduction

The methods of quantum mechanics are among the tools used in order to get a deeper insight in the complexity of the financial markets [1-9]. The time evolution of the price and the trend of the stock in the financial markets depends on many factors including the political environment, market information, economic policies of the government, psychology of traders, etc. Prices at which traders are willing to buy or sell a financial asset are not

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more determined by the development of industry, trade, services, situation at the market of natural resources and so on [7]. The information exchange and market psychology play important role in price dynamics.

The mathematical modeling of price dynamics of the financial market is a very complex problem. We could never take into account all economic and non-economic conditions that have influences to the market. Therefore, we usually consider some very simplified and idealized models, a kind of toy models which mimic certain features of a real stock market. For a better understanding of the complexity of the financial markets one can use several complementary models. Certain features are better described by the quantum models. The trade of a stock can be regarded as the basic process that measures its momentary price. The stock price can only be determined at the time of sale when it is between traders. We can never simultaneously know both the ownership of a stock and its price [8, 9].

The stock price $\wp$ is a discrete variable, not a continuous one. It is an integer multiple of a certain minimal quantity, a sort of quantum of cash [5]. We shall take into consideration only the case of stock markets with a price limit rule: the rate of return

$$\mathcal{R} = \frac{\wp - \wp_0}{\wp_0} = \frac{\wp}{\wp_0} - 1$$

in a trading day can not be more than $\pm q\%$ comparing with the previous day's closing price $\wp_0$, that is, we have

$$-\frac{q}{100} \leq \mathcal{R} \leq \frac{q}{100}.$$

For example [8], in most Chinese stock markets $q = 10$.

In their activity, the traders do not take into consideration an arbitrary small rate of return. They approximate the rate of return by integer multiples of a minimal one. We shall consider $\mathcal{R}$ as a discrete variable with the only possible values

$$-\frac{q}{100}, \quad -\frac{q-1}{100}, \quad ... \quad , -\frac{1}{100}, \quad 0, \quad \frac{1}{100}, \quad ... \quad , \frac{q-1}{100}, \quad \frac{q}{100}$$

and describe the rate of return at a fixed moment of time by a wave function

$$\psi : \{-q, -q+1, ..., q-1, q\} \longrightarrow \mathbb{C}$$

chosen such that $|\psi(n)|^2$ is the probability to have a return rate equal to $\frac{n}{100}$.

## 2. Finite-dimensional quantum systems

Let $d=2q+1\in\{1,3,5,7,...\}$, $\mathcal{H}$ be a $d$-dimensional complex Hilbert space, and let $\mathbb{Z}_d$ be the ring of integers modulo $d$ which can be written as

$$\mathbb{Z}_d \equiv \{-q,-q+1,...,q-1,q\}$$

by choosing the $d$ integers $-q$, $-q+1$,...,$q-1$, $q$ as a set of representatives. We choose an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}_d}$ and define the 'position' operator

$$\hat{Q}:\mathcal{H}\longrightarrow\mathcal{H},\qquad \hat{Q}=\sqrt{\tfrac{2\pi}{d}}\sum_{n=-q}^{q} n\,|n\rangle\langle n|. \tag{1}$$

The finite Fourier transform

$$F:\mathcal{H}\longrightarrow\mathcal{H},\qquad F=\frac{1}{\sqrt{d}}\sum_{n,n'=-q}^{q}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nn'}|n\rangle\langle n'| \tag{2}$$

allows us to consider a second orthonormal basis $\{|\tilde{k}\rangle\}_{k\in\mathbb{Z}_d}$, where

$$|\tilde{k}\rangle=F|k\rangle=\frac{1}{\sqrt{d}}\sum_{n=-q}^{q}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}|n\rangle \tag{3}$$

and to define the 'momentum' operator [10]

$$\hat{P}:\mathcal{H}\longrightarrow\mathcal{H},\qquad \hat{P}=\sqrt{\tfrac{2\pi}{d}}\sum_{k=-q}^{q} k\,|\tilde{k}\rangle\langle\tilde{k}|. \tag{4}$$

Each state $|\psi\rangle\in\mathcal{H}$ can be expanded as

$$|\psi\rangle=\sum_{n=-q}^{q}\psi(n)\,|n\rangle=\sum_{k=-q}^{q}\tilde{\psi}(k)\,|\tilde{k}\rangle \tag{5}$$

where the functions

$$\psi:\mathbb{Z}_d\longrightarrow\mathbb{C},\ \psi(n)=\langle n|\psi\rangle\qquad\text{and}\qquad\tilde{\psi}:\mathbb{Z}_d\longrightarrow\mathbb{C},\ \tilde{\psi}(k)=\langle\tilde{k}|\psi\rangle$$

satisfying the relations

$$\psi(n)=\frac{1}{\sqrt{d}}\sum_{k=-q}^{q}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\tilde{\psi}(k),\qquad \tilde{\psi}(k)=\frac{1}{\sqrt{d}}\sum_{n=-q}^{q}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\psi(n) \tag{6}$$

are the corresponding 'wave-functions' in the position and momentum representations [10]. The operators $\hat{Q}$ and $\hat{P}$ satisfy the relations

$$F\hat{Q}F^{-1}=\hat{P} \qquad\qquad F\hat{P}F^{-1}=-\hat{Q}. \tag{7}$$

The other physical observables are also described by Hermitian operators . The operator $\hat{H}:\mathcal{H}\longrightarrow\mathcal{H}$ corresponding to the energy, called the Hamiltonian, plays an important role in the description of the time evolution of the wave function. The wave function describing the time evolution of the system

$$\mathbb{Z}_d\times\mathbb{R}\longrightarrow\mathbb{C}:(n,t)\mapsto\Psi(n,t)$$

satisfies the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(n,t)=\hat{H}\Psi(n,t).$$

Generally, $\hat{H}$ is the sum of two terms

$$\hat{H}=\frac{1}{2\mu}\hat{P}^2+\mathcal{V}(\hat{Q},t)$$

representing the kinetic and potential energy, respectively.

In case of the free-evolution, that is,

$$\hat{H}=\frac{1}{2\mu}\hat{P}^2$$

the time dependent state corresponding to the initial state $|\psi\rangle$ is

$$\Psi:\mathbb{Z}_d\times\mathbb{R}\longrightarrow\mathbb{C},\qquad \Psi(n,t)=\langle n|\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{P}^2}|\psi\rangle.$$

From the relation

$$\begin{array}{rl}
\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{P}^2}|\psi\rangle & =\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{P}^2}\sum_{n=-q}^{q}\psi(n)|n\rangle=\sum_{n=-q}^{q}\psi(n)\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{P}^2}|n\rangle\\
& =\frac{1}{\sqrt{d}}\sum_{n=-q}^{q}\psi(n)\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{P}^2}\sum_{k=-q}^{q}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk}\,|\tilde{k}\rangle\\
& =\frac{1}{\sqrt{d}}\sum_{n=-q}^{q}\psi(n)\sum_{k=-q}^{q}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\frac{2\pi}{d}k^2}|\tilde{k}\rangle
\end{array}$$

it follows

$$\mathrm{e}^{-\mathrm{i}\frac{t+2\mu d}{2\mu}P^2}|\psi\rangle=\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}P^2}|\psi\rangle$$

whence

$$\Psi(n,t+2\mu d) = \Psi(n,t).$$

This means that the time evolution of the state of the system is periodic with period $2\mu d$, independently of the considered initial state.

In the limit $d \to \infty$, the relations (1), (3), (5), (6) correspond to

$$\hat{Q} = \int dx\, x\, |x\rangle\langle x|, \qquad |\psi\rangle = \int dx\, \psi(x)\, |x\rangle = \int dp\, \tilde{\psi}(p)\, |\tilde{p}\rangle$$
$$|\tilde{p}\rangle = \frac{1}{\sqrt{2\pi}} \int dx\, \mathrm{e}^{\mathrm{i}px} |x\rangle, \qquad \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx\, \mathrm{e}^{-\mathrm{i}px} \psi(x).$$

## 3. A quantum model for the stock market

We consider a stock market with a price limit rule [8], for which the rate of return $\mathcal{R}$ in a trading day can not be more than $\pm q\%$. The space $\mathcal{H}$ of all the functions

$$\psi : \{-q, -q+1, ..., q-1, q\} \longrightarrow \mathbb{C}$$

considered with the scalar product

$$\langle\psi_1|\psi_2\rangle = \sum_{n=-q}^{q} \overline{\psi_1(n)}\, \psi_2(n)$$

is a Hilbert space of dimension $d=2q+1$, isomorphic to $\mathbb{C}^d$. The function $\psi$ has the norm $||\psi|| = \sqrt{\langle\psi|\psi\rangle}$, and the corresponding normalized function is

$$\Psi : \{-q, -q+1, ..., q-1, q\} \longrightarrow \mathbb{C}, \qquad \Psi(n) = \frac{1}{||\psi||}\psi(n).$$

The set $\{|n\rangle\}_{n=-q}^{q}$, where $|n\rangle$ represents the function

$$\delta_n : \{-q, -q+1, ..., q-1, q\} \longrightarrow \mathbb{C}, \qquad \delta_n(m) = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

in Dirac's notation, is an orthonormal basis in $\mathcal{H}$.

By following the analogy with the quantum mechanics, we describe the rate of return at a fixed moment of time by a normalized function $\Psi$ and interpret $|\Psi(n)|^2$ as the probability to have a return rate equal to $\frac{n}{100}$. The operator

$$\hat{R} : \mathcal{H} \longrightarrow \mathcal{H}, \qquad \hat{R} = \frac{1}{100} \sum_{n=-q}^{q} n\, |n\rangle\langle n|$$

similar to $\hat{Q}$, is the operator corresponding to the rate of return $\mathcal{R}$. The function $\delta_n$ for which we use the alternative notation $|n\rangle$ is an eigenfunction of $\hat{R}$ corresponding to the eigenvalue $\frac{n}{100}$

$$\hat{R}\,\delta_n = \tfrac{n}{100}\,\delta_n.$$

The mean value of $\hat{R}$ in the case of a normalized function $\Psi$ is

$$\langle\hat{R}\rangle = \langle\Psi|\hat{R}|\Psi\rangle = \tfrac{1}{100}\sum_{n=-q}^{q} n\,|\Psi(n)|^2.$$

The operator $\hat{R}$ of the rate of return is directly related to the price operator

$$\hat{\wp} = \wp_0 + \wp_0\,\hat{R}$$

where the constant $\wp_0$ is the previous day's closing price.

In our discrete version, we can not define the operator $\hat{T}$ corresponding to the trend of return rate by using the method from [8] as a differential operator. We define it directly by using the finite Fourier transform as

$$\hat{T}:\mathcal{H}\longrightarrow\mathcal{H},\qquad \hat{T} = F\hat{R}F^{-1}.$$

The eigenfunctions of $\hat{T}$

$$|\tilde{k}\rangle = F|k\rangle = \frac{1}{\sqrt{d}}\sum_{n=-q}^{q} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}|n\rangle$$

that is, the functions

$$\psi_k:\{-q,-q+1,...,q-1,q\}\longrightarrow\mathbb{C},\qquad \psi_k(n) = \langle n|\tilde{k}\rangle = \frac{1}{\sqrt{d}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}$$

form an orthonormal basis in $\mathcal{H}$ and

$$\hat{T}\,\psi_k = \tfrac{k}{100}\,\psi_k,\qquad \hat{T} = \tfrac{1}{100}\textstyle\sum_{k=-q}^{q} k\,|\tilde{k}\rangle\langle\tilde{k}|.$$

We consider that the time evolution of the rate of return can be described by using a Schrödinger type equation

$$\mathrm{i}\,\frac{\partial\Psi}{\partial t} = \hat{H}\,\Psi$$

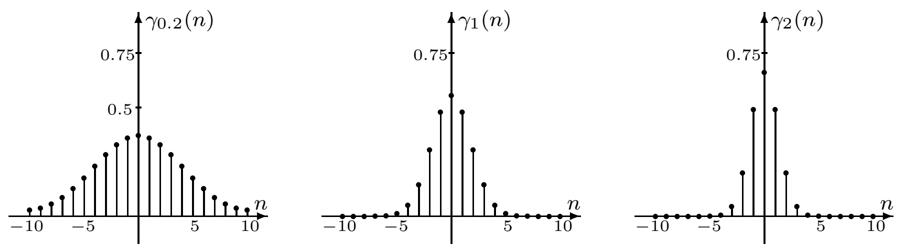


Figure 1: The functions $\gamma_{0.2}$ (left), $\gamma_1$ (center) and $\gamma_2$ (right) in the case $q = 10$.

where $\hat{H}$ is a suitable Hamiltonian of the form

$$\hat{H} = \frac{1}{2\mu}\hat{T}^2 + \mathcal{V}(\hat{R}, t)$$

with $\mu$ a positive parameter. In the particular case

$$\hat{H} = \frac{1}{2\mu}\hat{T}^2$$

we have

$$\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\hat{T}^2}|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{n=-q}^{q}\psi(n)\sum_{k=-q}^{q}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\mathrm{i}\frac{t}{2\mu}\frac{1}{10000}k^2}|\tilde{k}\rangle.$$

Therefore, the time evolution is periodic with a period of $40000\pi\mu$ seconds.

## 4. Market information and the price evolution

The rate of return of the stock market in equilibrium is usually described by a Gaussian function [8, 11]. In the case of our model we use a function

$$\gamma_\alpha : \{-q, -q+1, ..., q-1, q\} \longrightarrow \mathbb{R}, \qquad \gamma_\alpha(n) = \frac{1}{||g_\alpha||}\, g_\alpha(n)$$

(see Figure 1) depending on a parameter $\alpha \in (0, \infty)$, where

$$g_\alpha(n) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\frac{\alpha\pi}{d}(md+n)^2} \qquad \text{and} \qquad ||g_\alpha|| = \sqrt{\sum_{n=-q}^{q} g_\alpha^2(n)}.$$

It is well-known [12] that the function $g_1$ is an eigenfunction of the finite Fourier transform: $Fg_1 = g_1$, that is,

$$\frac{1}{\sqrt{d}} \sum_{n=-q}^{q} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} kn} g_1(n) = g_1(k).$$

In the case of a Hamiltonian

$$\hat{H} = \tfrac{1}{2\mu} \hat{T}^2 + \mathcal{V}(\hat{R}, t)$$

the kinetic part $\frac{1}{2}\hat{T}^2$ represents the efforts of the traders to change prices [7]. The intensive exchange of information in the world of finances is one of the main sources determining dynamics of prices. The potential part $\mathcal{V}(\hat{R}, t)$ of $\hat{H}$ describes the interactions between traders as well as external economic conditions and even meteorological conditions [7].

The total effect of market information affecting the stock price at a certain time determines either the stock price's rise or the stock price's decline. In order to illustrate our model, by following [8], we consider an idealized model in which we assume two type of information appear periodically. We choose a cosine function $\cos \omega t$ to simulate the fluctuation of the information and use the Hamiltonian

$$\hat{H} = \tfrac{1}{2\mu} \hat{T}^2 + \beta \, \hat{R} \cos \omega t.$$

where $\beta$ is a constant. The solution of the Schrödinger equation

$$\mathrm{i} \, \frac{\partial \Psi}{\partial t} = \left[ \frac{1}{2\mu} \hat{T}^2 + \beta \, \hat{R} \cos \omega t \right] \Psi$$

can be obtained by using a program in Mathematica (see the Appendix). We assume that at the opening time ($t = 0$) of the stock market, the wave function describing the rate of return is the function $\gamma_\alpha$ (corresponding to a certain $\alpha$), that is,

$$\Psi(n, 0) = \gamma_\alpha(n).$$

The role played by the kinetic part of the Hamiltonian and the role played by the market information depend on the values of the parameters $\mu$, $\beta$ and $\omega$. The distribution of the probabilities $|\Psi(n, t)|^2$ corresponding to the possible values $\frac{n}{100}$ of the rate of return at certain moments of time are presented in the figure 2.

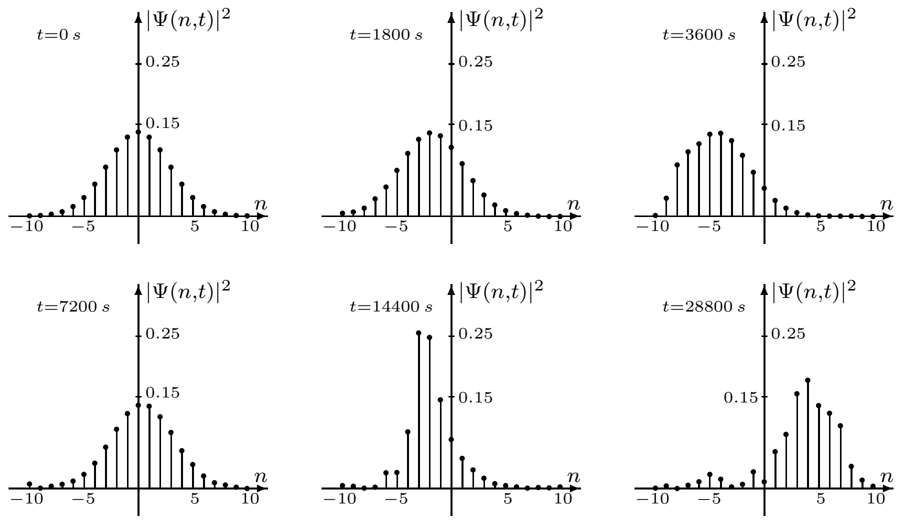


Figure 2: The probability $|\Psi(n,t)|^2$ to have a rate of return equal to $n\%$ for $t=0$, 1800, 3600, 7200, 14400 and 28800 in the particular case $\alpha=0.2$, $\mu=1$, $\beta=1/10$, $\omega=1/10000$.

## 5. Conclusion

Our finite-dimensional version, based on the mathematical formalism used in the case of finite quantum systems, keeps the essential characteristics of the model proposed by Zhang and Huang in [8]. In additions, it takes into consideration the discrete nature of the quantities used in the field of finance, and is much more accessible as concerns the numerical computations.

## Appendix

The time evolution of the probabilities $|\Psi(n,t)|^2$ corresponding to the possible values $\frac{n}{100}$ of the rate of return can be obtained by using the following program in Mathematica:

```
time = 3600;   alpha = 0.2; mu = 1; beta = 1/10; omega =  1/10000; d = 21; q := (d - 1)/2
g[n_] := N[  Sum[Exp[-Pi  alpha (m d + n)^2/d], {m, -Infinity, Infinity}]]
gamma := Normalize[Table[ g[n], {n, -q, q}]]
T[n_, m_] := N[(1/(100 d) ) Sum[k Exp[2 Pi I (n - m) k/d], {k, -q, q}]]
H[n_, m_, t_] := (1/(2 mu)) Sum[ T[n, k] T[k, m], {k, -q, q}]
                                       + beta (n/100) Cos[omega t] DiscreteDelta[n - m]
eqns = {Table[   psi[n]'[t] == Sum[- I H[n, m, t] psi[m][t], {m, 1, d}], {n, 1, d}],
                                        Table[psi[n][0] == gamma[[n]], {n, 1, d}]}
ndsolve = NDSolve[eqns, Table[psi[n], {n, d}], {t, 30000}]
Psi = Normalize[  Table[Evaluate[Abs[psi[n][time]] /. ndsolve][[1]], {n, 1, d}]]
ListPlot[Table[{n, Psi[[n + q + 1]]^2}, {n, -q, q}] , Filling -> Axis]
```

The reader can easily change the values of parameters `time`, `alpha` , `mu`, `beta` and the function `Cos[omega t]` in order to investigate other cases.

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