The Algebro-Geometric Initial Value Problem for the Ruijsenaars-Toda Hierarchy and Quasi-Periodic Solutions

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Abstract

We provide a detailed treatment of Ruijsenaars-Toda (RT) hierarchy with special emphasis on its the theta function representation of all algebro-geometric solutions. The basic tools involve hyperelliptic curve $\mathcal{K}_p$ associated with the Burchnall-Chaundy polynomial, Dubrovin-type equations for auxiliary divisors and associated trace formulas. With the help of a foundamental meromorphic function ϕ, Baker-Akhiezer vector Ψ on $\mathcal{K}_p$, the complex-valued algebro-geometric solutions of RT hierarchy are derived.

Keywords: Ruijsenaars-Toda hierarchy, complex-valued algebro-geometric solutions, hyperelliptic curve, Dubrovin-type equation, Baker-Akhiezer vector.

1 Introduction

Nonlinear integrable lattice systems have been studied extensively in relation with various aspects and they usually possess rich mathematical structure such as Lax pairs,

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Hamilton structure, conservation law, etc. The Toda lattice is one of the most important integrable systems [8, 11]. It is well-known that soliton equations such as the KdV, modified KdV, and nonlinear Schrödinger equations are closely related to/or derived from the Toda equation by suitable limiting procedures [11, 13]. Various kinds of Toda lattice have been discussed since it was proposed [9, 10, 11, 12, 20]. Among them, a remarkable discovery was made by Ruijsenaars in the area of integrable lattice systems [1]. He found a relativistic integrable generalization of non-relativistic Toda lattice through solving a relativistic version of the Calogero-Moser system. The Lax representation, inverse scattering problem of the Ruijsenaars-Toda lattice and its connection with soliton dynamics were investigated. A general approach to constructing relativistic generalizations of integrable lattice systems, applicable to the whole lattice KP hierarchy, was proposed by Gibbons and Kupershmidt [7]. After that, a series of techniques to construct relativistic lattice equations were developed by systematic procedure and Hirota's bilinear method [4, 18].

The Ruijsenaars-Toda lattice, sometimes also called relativistic Toda (RT) lattice, takes the form [4]

$$\begin{aligned} &\beta_{k,t} = (1 + h\beta_k)(\alpha_k - \alpha_{k-1}), \\ &\alpha_{k,t} = \alpha_k \left(\beta_{k+1} - \beta_k + h\alpha_{k+1} - h\alpha_{k-1}\right) \end{aligned} \tag{1.1}$$

in Flaschka variables or

$$\begin{aligned} x_{k,tt} = &\left(1 + hx_{k,t}\right)\left(1 + hx_{k+1,t}\right) \frac{e^{x_{k+1}-x_k}}{1 + h^2 e^{x_{k+1}-x_k}} \\ &- (1 + hx_{k-1,t})(1 + hx_{k,t}) \frac{e^{x_k - x_{k-1}}}{1 + h^2 e^{x_k - x_{k-1}}} \end{aligned} \tag{1.2}$$

in Newtonian form, where the small time step $h = c^{-1}$ and c is light speed. In the non-relativistic limit $c \to \infty$, the RT equation (1.1) reduced to the well-known Toda lattice equation [8],

$$\beta_{k,t} = \alpha_k - \alpha_{k-1}, \qquad \alpha_{k,t} = \alpha_k(\beta_{k+1} - \beta_k) \tag{1.3}$$

in Flaschka variables, or

$$x_{k,tt} = e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}} \tag{1.4}$$

in Newtonian form. Eq (1.1) is the Poincare-invariant generalizations of the Galilei-invariant Toda systems (1.3).

Mathematical frame work such as Lax representation, Bäcklund transformation, Hamiltonian structure of RT lattice Eq (1.1), etc, were investigated by some authors ([14]-[17]). Cosentino obtained a soliton solution by using the IST method. Hietarinta and Junkichi Satsuma transformed the RT eq (1.1) into trilinear form through a suitable dependent variable transform. Later Yasuhiro Ohta, etc. decomposed the RT lattice eq (1.1) into three Toda systems, the Toda lattice itself, Bäcklund transformation of Toda lattice, and discrete time Toda lattice and explicitly derived the solutions in terms of the Casorati determinant [18]. The solution they obtained converges to that of TL eq (1.3) in the limit of $c \to \infty$.

Algebro-geometric solutions (finite-gap solutions or quasi-period solutions), an important character of integrable system, is a kind of explicit solutions closely related to the inverse spectral theory [28, 30]. Around 1975, several independent groups in UUSR and USA, namely, Novikov, Dubrovin and Krichever in Moscow, Matveev and Its in Leningrad, Lax, McKean, van Moerbeke and M. Kac in New York, and Marchenko, Kotlyarov and Kozel in Kharkov, developed the so-called finite finite-gap theory of nonlinear KdV equation based on the works of Drach, Burchnall and Chaunchy, and Baker [26, 29, 31]. The algebro-geometric method they established allowed us to find an important class of exact solutions to the soliton equations. As a degenerated case of this solutions, the multisoliton solutions and elliptic functions may be obtained [28, 36]. Its and Matveev first derived explicit expression of the quasi-period solution of KdV equation in 1975 [29], which is closely related to the finite-gap spectrum of the associated differential operator. Further exciting results appeared later, including the finite-gap solutions of Toda lattice, the Kadomtsev-Petviashvili equation and others [8, 31, 36], which could be found in the wonderful work of Belokolos, et al [28]. In recent years, a systematic approach based on the nonlinearization technique of Lax pairs or the restricted flow technique to derive the algebro-geometric solutions of (1+1)- and (2+1)-dimensional soliton equations has been obtained [32]-[35]. An alternate systematic approach proposed by Gesztesy and Holden can be used to construct algebro-geometric solutions has been extended to the whole (1+1) dimensional continuous and discrete hierarchy models [37]-[39],[41, 42].

In this paper, we mainly discussed the algebro-geometric quasi-period solutions of RT hierarchy for fixed constant $h = c^{-1} \neq 0$. In following section 2, the RT equation (1.1) is extended to a whole RT hierarchy through the polynomial recursive relation. The asymptotic spectral expansions of eigenfunction functions associated associated with spectral problem are further investigated in section 3. In section 4, we give a detailed study of algebro-geometric solutions for the stationary RT hierarchy. We first derive the hyperelliptic curve $\mathcal{K}_p$ in connection with the stationary KdV hierarchy. Then a fundamental meromorphic function ϕ on $\mathcal{K}_p$, the Baker-Akhiezer vector Ψ and the common eigenfunction of zero-curvature pair U, V_p, are introduced to study the trace formula and asymptotic properties of ϕ and Ψ, respectively. With the help of Riemann theta function associated with $\mathcal{K}_p$, one finds the theta function representations for ϕ and Ψ by alluding to Riemann's vanishing theorem and the Riemann-Roch theorem. In section 5, we derive the complex-valued algebro-geometric solutions of RT hierarchy with a given inital value problem by using the results in sections 3 and 4. Finally, in section 6 we give Lagrange interpolation representation that will be used in this paper.

2 Discrete Ruijsenaars-Toda hierarchy

In section, we derive RT hierarchy.

Definition 1. *We denote by $l(\mathbb{Z})$ the set of all the complex-valued sequences $\{f(n)\}_{n=-\infty}^{+\infty}$. This is a vector space with respect to the naturally defined operation. A subspace $l^2(\mathbb{Z}) \subset l(\mathbb{Z})$ is defined by the set of $\{f \in l(\mathbb{Z}) | \sum_{n=-\infty}^{+\infty} |f(n)|^2 < +\infty, n \in \mathbb{Z}\}$.*

Definition 2. *We denote by $S^{\pm}$ the shift operators acting on $\psi = \{\psi(n)\}_{n=-\infty}^{+\infty} \in l(\mathbb{Z})$ according to $(S^{\pm}\psi)(n) = \psi(n \pm 1)$. We also define $\psi^{\pm} = S^{\pm}\psi, \psi \in l(\mathbb{Z})$.*

Introduce the following spectral problem [15][27]

$$\begin{aligned} & S^{+}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = U(\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ & U(\lambda) = \begin{pmatrix} 0 & 1 \\ (h\lambda-1)\alpha & \lambda+\beta \end{pmatrix}, \quad h \in \mathbb{C}\backslash\{0\}. \end{aligned} \tag{2.1}$$

Here $\alpha = \alpha(n,t), \beta = \beta(n,t) \in l(\mathbb{Z}), (n,t) \in \mathbb{Z}\times\mathbb{R}$ in time-dependent case and $\alpha = \alpha(n), \beta = \beta(n) \in l(\mathbb{Z}), n \in \mathbb{Z}$ in stationary case.

We introduce the following recursion relation

$$\begin{aligned} & b_{l+1} + \beta b_l + a_l + a_l^{-} = 0, \\ & h\alpha b_{l+1}^{-} - h\alpha^{+} b_{l+1}^{+} - \alpha b_l^{-} + \alpha^{+} b_l^{+} - \beta a_l^{-} + \beta a_l - a_{l+1}^{-} + a_{l+1} = 0. \end{aligned} \tag{2.2}$$

If we take the initial term $a_0 = -\frac{1}{2}, b_0 = 0$, then $\{a_l, b_l\}_{l\in\mathbb{N}_0}$ can be derived from (2.2) recursively. For example,

$$\begin{aligned} & b_1 = 1, \\ & a_1 = h\alpha^{+} - \frac{\delta_1}{2}, \\ & b_2 = -h(\alpha^{+}+\alpha) - \beta + \delta_1, \\ & a_2 = -h^2\alpha^{++}\alpha^{+} - h^2(\alpha^{+})^2 + h^2\alpha^{+}\alpha - h\alpha^{+}\beta^{+} - h\alpha^{+}\beta - \alpha^{+} + h\alpha^{+}\delta_1 - \frac{1}{2}\delta_2, \\ & b_3 = h^2\alpha^{++}\alpha^{+} + h^2(\alpha^{+})^2 - h^2\alpha^{+}\alpha + h\alpha^{+}\beta^{+} + 2h\alpha^{+}\beta + \alpha^{+} + h^2\alpha^{+}\alpha + h^2\alpha^2, \\ & - h^2\alpha\alpha^{-} + 2h\alpha\beta + h\alpha\beta^{-} + \alpha + \left(-h\alpha^{+} - h\alpha - \beta\right)\delta_1 + \delta_2, \\ & \ldots \quad \ldots \end{aligned} \tag{2.3}$$

where the $\delta_l \in \mathbb{C}$ ($l \in \mathbb{N}_0$) are constants.

Remark 3. (i) *If we denote by* $\bar{a}_l = a|_{\delta_j=0,j=1,\cdots,l}, \bar{b}_l = b|_{\delta_j=0,j=1,\cdots,l}, l \in \mathbb{N}$*, that is,*

$$\begin{aligned}
&\bar{a}_0 = -\frac{1}{2}, \\
&\bar{b}_0 = 0, \\
&\bar{a}_1 = h\alpha^+, \\
&\bar{b}_1 = 1, \\
&\bar{a}_2 = -h^2\alpha^{++}\alpha^+ - h^2(\alpha^+)^2 - h^2\alpha^+\alpha - h\alpha^+\beta^+ - h\alpha^+\beta - \alpha^+, \\
&\bar{b}_3 = h^2\alpha^{++}\alpha^+ + h^2(\alpha^+)^2 - h^2\alpha^+\alpha + h\alpha^+\beta^+ + 2h\alpha^+\beta + \alpha^+ + h^2\alpha^+\alpha + h^2\alpha^2, \\
&- h^2\alpha\alpha^- + 2h\alpha\beta + h\alpha\beta^- + \alpha, \\
&\cdots \quad \cdots,
\end{aligned} \tag{2.4}$$

then we have

$$a_l = \sum_{s=0}^{l} \delta_{l-s}\bar{a}_s, \quad b_l = \sum_{s=0}^{l} \delta_{l-s}\bar{b}_s, \quad \delta_0 = 1, \quad l \in \mathbb{N}_0, \tag{2.5}$$

(ii) *The constants* $\delta_j \in \mathbb{C}, \quad j \in \mathbb{N}_0$ *can be expressed in terms of the zeros of* $E_i, \quad i = 0,\cdots,p+1$ *of the associated polynomial* $R_{2p+2}(\lambda)$ *defined in (4.6). (Theorem 6)*

The zero-curvature equation is

$$U_t + UV_p - V_p^+U = 0 \tag{2.6}$$

in time-dependent case and

$$UV_p - V_p^+U = 0 \tag{2.7}$$

in stationary case. We make the ansatz

$$V_p(\lambda) = \begin{pmatrix} A_{p+1}^- & B_{p+1}^- \\ C_{p+1}^- & D_{p+1}^- \end{pmatrix}, \tag{2.8}$$

where

$$
\begin{aligned}
A_{p+1} &= a_0\lambda^{p+1} + a_1\lambda^p + \cdots + a_{p+1} + b_{p+2} = \sum_{j=0}^{p+1} a_{p+1-j}\lambda^j + b_{p+2}, \\
B_{p+1} &= b_0\lambda^{p+1} + \cdots + b_{p+1} = \sum_{j=0}^{p+1} b_j \lambda^{p+1-j}, \\
C_{p+1} &= (h\lambda - 1)\alpha^+ B_{p+1}^+ = (h\lambda - 1)\alpha^+ \left(b_0^+\lambda^{p+1} + \cdots + b_{p+1}^+\right), \\
D_{p+1} &= -a_0\lambda^{p+1} - \cdots - a_{p+1} = -\sum_{j=0}^{p+1} a_{p+1-j}\lambda^j
\end{aligned} \tag{2.9}
$$

and $\{a_l\}_{l=0}^{p+1}, \{b_l\}_{l=0}^{p+2} \in l(\mathbb{Z})$ are undetermined sequences.

Plunging (2.8) (2.9) into (2.6) and (2.7), we have the following result:

Theorem 4. *Suppose $U(\lambda), V_p(\lambda)$ defined in (2.1) (2.8) (2.9) satisfy zero-curvature equation (2.6) (2.7). Then the coefficients $\{a_l\}_{l=0}^{p+1}, \{b_l\}_{l=0}^{p+1}$ of $A_{p+1}, B_{p+1}, C_{p+1}, D_{p+1}$ in (2.8) satisfy*

$$
\begin{aligned}
& b_{l+1} + \beta b_l + a_l + a_l^- = 0, \\
& h\alpha b_{l+1}^- - h\alpha^+ b_{l+1}^+ - \alpha b_l^- + \alpha^+ b_l^+ - \beta a_l^- + \beta a_l - a_{l+1}^- + a_{l+1} = 0 \quad l = 0, 1, \cdots, p.
\end{aligned} \tag{2.10}
$$

Moreover, (2.6) (2.7) is equivalent to

$$\alpha_{t_p} = \alpha(b_{p+2} - b_{p+2}^-), \tag{2.11}$$

$$\beta_{t_p} = \beta(a_{p+1}^- - a_{p+1}) + \alpha b_{p+1}^- - \alpha^+ b_{p+1}^+ \tag{2.12}$$

in time-dependent case and

$$\alpha(b_{p+2}^- - b_{p+2}) = 0, \tag{2.13}$$

$$\beta(a_{p+1}^- - a_{p+1}) + \alpha b_{p+1}^- - \alpha^+ b_{p+1}^+ = 0, \tag{2.14}$$

or

$$\alpha A_{p+1}^- + \alpha A_{p+1} + (\lambda + \beta)\alpha B_p = 0, \tag{2.15}$$

$$(h\lambda - 1)\alpha B_p^- - (\lambda + \beta)A_{p+1}^- - (h\lambda - 1)\alpha^+ B_p^+ - (\lambda + \beta)A_{p+1} = 0 \tag{2.16}$$

in stationary case. Especially, if $\{a_l\}_{l=0}^{p+1}, \{b_l\}_{l=0}^{p+2}$ are defined by (2.2) or (2.3), (2.11)-(2.14) give rise to RT hierarchy.

Proof. Equation (2.7) is equivalent to

$$0 = UV_p - V_p^+ U = \begin{pmatrix} C_{p+1}^- - (h\lambda - 1)\alpha B_{p+1} & D_{p+1}^- - A_{p+1} - (\lambda+\beta)B_{p+1} \\ \\ (h\lambda-1)\alpha A_{p+1}^- + (\lambda+\beta)C_{p+1}^- & (h\lambda-1)\alpha B_p^- + (\lambda+\beta)D_{p+1}^- \\ -(h\lambda-1)\alpha D_{p+1} & -C_{p+1} - (\lambda+\beta)D_{p+1} \end{pmatrix} \tag{2.17}$$

that is,

$$C_{p+1}^- - (h\lambda - 1)\alpha B_{p+1} = 0, \tag{2.18}$$

$$D_{p+1}^- - A_{p+1} - (\lambda+\beta)B_{p+1} = 0, \tag{2.19}$$

$$(h\lambda-1)\alpha A_{p+1}^- + (\lambda+\beta)C_{p+1}^- - (h\lambda-1)\alpha D_{p+1} = 0, \tag{2.20}$$

$$(h\lambda-1)\alpha B_p^- + (\lambda+\beta)D_{p+1}^- - C_{p+1} - (\lambda+\beta)D_{p+1} = 0. \tag{2.21}$$

It's easy to derive the relation of the coefficients $\{a_l\}_{l=0}^{p+1}, \{b_l\}_{l=0}^{p+1}$ of $A_{p+1}, B_{p+1}, C_{p+1}, D_{p+1}$ in (2.9) from (2.18)-(2.21). Recursive relation (2.10) is the same with (2.2) for $l = 1, 2, \cdots, p$. Therefore if we define the coefficients of $A_{p+1}, B_{p+1}, C_{p+1}, D_{p+1}$ according to (2.2) or (2.3), the zero-curvature equation (2.6) is equivalent to

$$\begin{aligned} 0 = U_{t_p} + UV_p - V_p^+ U = & \begin{pmatrix} 0 & 0 \\ (h\lambda-1)\alpha_{t_p} & \beta_{t_p} \end{pmatrix} \\ + & \begin{pmatrix} 0 & 0 \\ (h\lambda-1)(\alpha b_{p+2}^- - \alpha b_{p+2}) & -\alpha b_{p+1}^- + \alpha^+ b_{p+1}^+ - \beta(a_{p+1}^- - a_{p+1}) \end{pmatrix}. \end{aligned} \tag{2.22}$$

We have (2.11) (2.12), which gives rise to RT hierarchy varying $p \in \mathbb{N}_0$ (Remark 5). Next we illustrate the equation (2.15) and (2.16). In the stationary case, from equation (2.7), we note that V_p is similar to V_p^+. Hence the trace of V_p and V_p^+ have the following relation

$$A_{p+1}^- + D_{p+1}^- = A_{p+1} + D_{p+1}. \tag{2.23}$$

In other words, $A_{p+1} + D_{p+1}$ is a lattice constant only depends on λ (not rely on $n \in \mathbb{Z}$). If we add a constant coefficient polynomial with λ times unit matrix on $V_p(\lambda)$. The stationary zero-curvature equation (2.7) is invariant. Without loss of generality, we can assume $A_{p+1} + D_{p+1} = 0$. Equation (2.15) (2.16) holds. □

Remark 5. *In the case $p = 0$, equation (2.11) (2.12) is*

$$\begin{aligned} \alpha_{t_0} &= h\alpha(\alpha^- - \alpha^+) + \alpha(\beta^- - \beta), \\ \beta_{t_0} &= h\beta(\alpha - \alpha^+) - \alpha + \alpha^+, \end{aligned} \tag{2.24}$$

which is the relativistic Toda lattice [1] [6] [15] [19][21] [27].
In the cases $p = 1$,

$$\begin{aligned} &\alpha_{t_1} = \alpha(h^2\alpha^{++}\alpha^+ + h^2(\alpha^+)^2 - h^2\alpha^+\alpha + h\alpha^+\beta^+ + \alpha^+ + h\alpha\beta - h\alpha\beta^- - h^2\alpha\alpha^- \\ &- h^2(\alpha^-)^2 + h^2\alpha^-\alpha^{--} - 2h\alpha^-\beta^- - h\alpha^-\beta^{--} - \alpha^-) + \alpha(-h\alpha^+ - \beta + h\alpha^- + \\ &\beta^-)\delta_1, \\ &\beta_{t_1} = -h\alpha^2 - h\alpha\alpha^- - \alpha\beta^- + h\alpha^+\alpha^{++} + h(\alpha^+)^2 + \alpha^+\beta^+ + \beta(-h^2\alpha^+\alpha - h^2\alpha^2 \\ &+ h^2\alpha\alpha^- - h\alpha\beta - h\alpha\beta^- - \alpha + h^2\alpha^{++}\alpha^+ + h^2(\alpha^+)^2 - h^2\alpha^+\alpha + h\alpha^+\beta^+ + h\alpha^+\beta \\ &+ \alpha^+) + (\alpha - \alpha^+ + h\alpha\beta - h\alpha^+\beta)\delta_1. \end{aligned} \tag{2.25}$$

3 Asymptotic Spectral Parameter Expansions

Next we study the asymptotic behavior of B_{p+1}/y near the point $P_{\infty\pm}$. Assume $\Psi_\pm = (\Psi_{1,\pm}, \Psi_{2,\pm})^T$ is a fundamental system of solutions of spectral problem

$$\begin{pmatrix} 0 & 1 \\ (h\lambda - 1)\alpha & \lambda + \beta \end{pmatrix} \begin{pmatrix} \Psi_{1\pm}^- \\ \Psi_{2\pm}^- \end{pmatrix} = \begin{pmatrix} \Psi_{1\pm} \\ \Psi_{2\pm} \end{pmatrix}. \tag{3.1}$$

Introducing $\phi_1 = \Psi_{2,+}/\Psi_{1,+}, \phi_2 = \Psi_{2,-}/\Psi_{1,-}$, Then ϕ_1, ϕ_2 satisfy the following Riccati-type equation

$$\phi_1\phi_1^- - (\lambda + \beta)\phi_1^- - (h\lambda - 1)\alpha = 0, \tag{3.2}$$

$$\phi_2\phi_2^- - (\lambda + \beta)\phi_2^- - (h\lambda - 1)\alpha = 0. \tag{3.3}$$

Lemma 1. *Let $\mathcal{A} = -(\phi_1 + \phi_2)/2(\phi_1 - \phi_2)$, $\mathcal{B} = 1/(\phi_1 - \phi_2)$, then $\mathcal{A}$ and $\mathcal{B}$ have the following relationship:*

$$\mathcal{A} + \mathcal{A}^- + (\lambda + \beta)\mathcal{B} = 0, \tag{3.4}$$

$$(h\lambda - 1)\alpha\mathcal{B}^- - (\lambda+\beta)\mathcal{A}^- - (h\lambda-1)\alpha^+\mathcal{B}^+ + (\lambda+\beta)A = 0, \tag{3.5}$$

$$\mathcal{A}^2 - \alpha^+(h\lambda-1)\mathcal{B}\mathcal{B}^+\lambda = \frac{1}{4}, \tag{3.6}$$

$$\mathcal{B} = \frac{1}{\phi_1 - \phi_2} = \frac{\phi_1^-\phi_2^-}{\alpha(h\lambda-1)(\phi_2^- - \phi_1^-)}. \tag{3.7}$$

Proof. Equation(3.2) minus equation (3.3), one derives

$$\phi_1\phi_1^- - \phi_2\phi_2^- - (\lambda+\beta)(\phi_1^- - \phi_2^-) = 0 \tag{3.8}$$

that is,

$$(\phi_1^- - \phi_2^-)(\phi_1+\phi_2) + (\phi_1-\phi_2)(\phi_1^- + \phi_2^-) - 2(\lambda+\beta)(\phi_1^- - \phi_2^-) = 0. \tag{3.9}$$

Then we have (3.4). We calculate

$$\phi_2^-\phi_1\phi_1^- - \phi_2\phi_2^-\phi_1^- = \alpha(h\lambda-1)(\phi_2^- - \phi_1^-). \tag{3.10}$$

Hence equality (3.7) holds. Using (3.7), it's easy to show

$$(h\lambda-1)\alpha\frac{1}{\phi_1^- - \phi_2^-} - (h\lambda-1)\alpha^+\frac{1}{\phi_1^+ - \phi_2^+} + \frac{1}{2}(\lambda+\beta)\frac{\phi_1^- + \phi_2^-}{\phi_1^- - \phi_2^-} - \frac{1}{2}(\lambda+\beta)\frac{\phi_1+\phi_2}{\phi_1-\phi_2}. \tag{3.11}$$

Equality (3.6) is the direct result of (3.7) and the definition of $\mathcal{A}, \mathcal{B}$. □

From lemma 4 we have

Lemma 2. *Assume the definition of* $\mathcal{A}, \mathcal{B}$ *hold in lemma 10, then* $\mathcal{A}, \mathcal{B}$ *satisfy*

$$(\lambda+\beta)^2 - (\mathcal{A}+\mathcal{A}^+)(\mathcal{A}+\mathcal{A}^-)(h\lambda-1)\alpha^+ = \frac{1}{4}(\lambda+\beta)^2 \tag{3.12}$$

and

$$\big((\lambda+\beta)^2\mathcal{B} + (h\lambda-1)\alpha\mathcal{B}^- - (h\lambda-1)\alpha^+\mathcal{B}^+\big)^2 - 4\mathcal{B}\mathcal{B}^+\alpha^+(h\lambda-1)(\lambda+\beta)^2 = (\lambda+\beta)^2 \tag{3.13}$$

respectively.

Proof. This is the direct result of Lemma 4. □

Theorem 6. *Assume α, β satisfy the p-th stationary RT hierarchy, and suppose $P = (\lambda, y) \in \mathcal{K}_p \backslash \{P_{\infty\pm}\}$. Then $\mathcal{A}, \mathcal{B}$ has the following convergent expansion ,*

$$\mathcal{A} \xlongequal[\zeta=\lambda^{-1}\in C_r]{|\zeta|\to 0} \pm \sum_{l=0}^{\infty} \bar{a}_l \zeta^l, \tag{3.14}$$

$$\mathcal{B} \xlongequal[\zeta=\lambda^{-1}\in C_r]{|\zeta|\to 0} \pm \sum_{l=0}^{\infty} \bar{b}_l \zeta^l, \tag{3.15}$$

where $\bar{a}_l, \bar{b}_l \quad (l = 0, 1, \cdots)$ are the homogeneous coefficients of a_l, b_l and simultaneously as $P \to P_{\infty\pm}$

$$A_{p+1}/(iy) = \mp \sum_{l=0}^{\infty} \bar{a}_l \zeta^l \qquad \zeta \to 0, \tag{3.16}$$

$$B_p/(iy) = \mp \sum_{l=0}^{\infty} \bar{b}_l \zeta^l \qquad \zeta \to 0, \tag{3.17}$$

where $\zeta = \lambda^{-1}$ is the local coordinate near $P_{\infty\pm}$. Moreover, one infers for the constant $\delta_l, \quad l = 0, \cdots, p$ in B_{p+1} that

$$\delta_l = c_l(\underline{E}) \quad l = 0, \cdots, p \tag{3.18}$$

and

$$b_l = \sum_{k=0}^{l} c_{l-k}(\underline{E}) \bar{b}_k \quad l = 0, \cdots, p+1, \tag{3.19}$$

$$\begin{aligned} \bar{b}_l &= \sum_{k=0}^{min\{l,p+1\}} \hat{c}_{l-k}(\underline{E}) b_k \\ &= \sum_{k=0}^{min\{l,p\}} \hat{c}_{l-k}(\underline{E}) b_k \quad l \in \mathbb{N}_0. \end{aligned} \tag{3.20}$$

Proof. Identifying

$$\Psi_+(\lambda, \cdot) \quad \text{with} \quad \Psi(P, \cdot, 0) \quad \text{and} \quad \Psi_-(\lambda, \cdot) \quad \text{with} \quad \Psi_-(P^*, \cdot, 0) \tag{3.21}$$

and similarly, identifying

$$\phi_1(\lambda, \cdot) \quad \text{with} \quad \phi(P, \cdot) \quad \text{and} \quad \phi_2(\lambda, \cdot) \quad \text{with} \quad \phi_1(P^*, \cdot), \tag{3.22}$$

a comparison of (3.4)-(5.73) and lemma the result of lemma 4 shows that we may also identify

$$\mathcal{A} \quad \text{with} \quad \pm\frac{A_{p+1}}{iy}, \quad \mathcal{B} \quad \text{with} \quad \frac{B_{p+1}}{iy}, \tag{3.23}$$

The sign depending on whether P tends to $P_{\infty\pm}$. Hence we only investigate $\frac{A_{p+1}}{iy}$ and $\frac{B_{p+1}}{iy}$. Dividing B_{p+1} by iy, one obtains

$$\frac{A_{p+1}(\lambda)}{iy} = \left(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\lambda^{-k}\right)\left(\sum_{l=0}^{p+1}a_l\lambda^{-l}\right) = \sum_{l=0}^{\infty}\check{a}_l\lambda^{-l}, \tag{3.24}$$

$$\frac{B_{p+1}(\lambda)}{iy} = \left(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\lambda^{-k}\right)\left(\sum_{l=0}^{p+1}b_l\lambda^{-l}\right) = \sum_{l=0}^{\infty}\check{b}_l\lambda^{-l}, \tag{3.25}$$

where $\check{a}_l, \check{b}_l$ is undermined. Obviously $B_p(\lambda)/(iy)$ remain satisfies (2.15) (2.16), hence the coefficient of $\check{b}_l$ still have the recursion relation (2.2). So they have the form of a_l, b_l, but for the constant δ_l. Plunging (3.24)(3.25) into (3.12)-(3.13), we arrive at $\check{a}_0 = \frac{1}{2} = -\bar{a}_0, \check{a}_1 = -h\alpha^+ = -\bar{a}_1, \check{b}_0 = 0 = \bar{b}_0, \check{b}_1 = 1 = \bar{b}_1$ and we can inductively to show that $\check{a}_l = F_l(\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{l-1}) = \bar{a}_l, \check{b}_l = G_l(\bar{b}_0, \bar{b}_1, \cdots, \bar{b}_{l-1}) = \bar{b}_l$, where $F((x_0, x_1, \cdots, x_{l-1})), G((x_0, x_1, \cdots, x_{l-1}))$ is the polynomial of $(x_0, x_1, \cdots, x_{l-1})$. Then we have (3.14)-(3.17). From (3.25), comparing the coefficients of λ^{-k} of both sides, we have (3.20). Obviously

$$\sum_{l=0}^{k}\hat{c}_{k-l}(\underline{E})c_l(\underline{E}) = \delta_{k,0} = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}. \tag{3.26}$$

If we label $c_{-k}(\underline{E}) = \hat{c}_{-k}(\underline{E}) = 0, \quad k \in \mathbb{N}$, then one computes

$$\begin{aligned}
\sum_{k=0}^{l} c_{l-k}(\underline{E})\bar{b}_k &= \sum_{m=0}^{l} c_{l-k}(\underline{E}) \left(\sum_{s=0}^{\min\{k,p+1\}} \hat{c}_{k-s}(\underline{E})b_s\right) = \sum_{k=0}^{l} c_{l-k}(\underline{E}) \left(\sum_{s=0}^{k}\hat{c}_{k-s}(\underline{E})b_s\right) \\
&= \sum_{k=0}^{l}\sum_{s=0}^{k} c_{l-k}(\underline{E})\hat{c}_{k-s}(\underline{E})b_s = \sum_{k=0}^{l}\sum_{s=0}^{l} c_{l-k}(\underline{E})\hat{c}_{k-s}(\underline{E})b_s \\
&= \sum_{s=0}^{k}\sum_{k=0}^{l} c_{l-k}(\underline{E})\hat{c}_{k-s}(\underline{E})b_s = \sum_{s=0}^{l}\left(\sum_{k=s}^{l} c_{l-k}(\underline{E})\hat{c}_{k-s}(\underline{E})\right)b_s \\
&= b_l, \qquad l = 0, \cdots, p+1.
\end{aligned} \tag{3.27}$$

Equalities (3.18) and (3.20) hold. □

4 Algebro-geometric solutions of stationary RT hierarchy

Throughout this paper we suppose the following hypothesis.

Hypothesis 1. *In stationary case we assume*

$$\alpha(n), \beta(n) \in l(\mathbb{Z}), \quad \alpha(n) \neq 0, \quad n \in \mathbb{Z} \tag{4.1}$$

In time dependent case we assume

$$\alpha(n,t), \beta(n,t) \in l(\mathbb{Z}), \quad \alpha(n,t) \neq 0, \quad (n,t) \in \mathbb{Z} \times \mathbb{R} \tag{4.2}$$

The stationary zero-curvature equation is

$$UV_p - V_p^+ U = 0, \tag{4.3}$$

where U, V_p are defined by (2.1) (2.8) and (2.9). Taking determinants in (4.3), one derives

$$R_{2p+2}(\lambda) = -A_{p+1}^2(\lambda, n) - B_{p+1}(\lambda, n) C_{p+1}(\lambda, n) \tag{4.4}$$

is lattice constant, that is, $R_{2p+2}(\lambda)$ only depends on λ (n-independent).
Let

$$R_{2p+2}(\lambda) = -(A_{p+1}^-)^2 - B_{p+1}^- C_{p+1}^- = -A_{p+1}^2 - B_{p+1} C_{p+1} = -A_{p+1}^2 - (h\lambda - 1) m^+ B_{p+1} B_{p+1}^+. \tag{4.5}$$

Assume

$$R_{2p+2}(\lambda) = -\frac{1}{4} y^2 = -\frac{1}{4} \prod_{m=0}^{2p+1} (\lambda - E_m). \tag{4.6}$$

Here we introduce the algebra curve associated with stationary RT system (2.13) (2.14) as

$$\mathcal{K}_p : \qquad \mathcal{F}(\lambda, y) = y^2 - \prod_{m=0}^{2p+1} (\lambda - E_m) = 0. \tag{4.7}$$

Hypothesis 2. *Throughout this section we assume the affine part of $\mathcal{K}_p$ to be nonsingular, that is*

$$E_m \neq E_{m'} \qquad m \neq m' \qquad m = 0, \cdots, 2p+1. \tag{4.8}$$

For notational convenience we still denote its compactification by $\mathcal{K}_p$.

Obviously, $\mathcal{K}_p$ is an nonsingular affine algebra curve[1] of 2-order. One can introduce the complex structure on $\mathcal{K}_p$ to yield a compact Riemann surface of genus p [22]. This is a hyperelliptic Riemann surface. It can be regarded as a double covering of sphere surface $\mathbb{C}_\infty = \mathbb{C}\bigcup\{\infty\}$. Hypothesis 2 has ensured that there just exists p linear independent holomorphic differential forms on $\mathcal{K}_p$. We write

$$
\begin{aligned}
&B_p(\lambda) = \prod_{j=1}^{p}(\lambda-\mu_j), \quad C_{p+1}(\lambda) = h(\lambda - \frac{1}{h})\prod_{j=1}^{p}(\lambda-\mu_j^+), \\
&\mu_j^+, \mu_j \in l(\mathbb{Z}), \quad j = 1,\cdots,p, \quad \mu_j^+(n) = \mu_j(n+1), \quad n \in \mathbb{Z}.
\end{aligned}
\tag{4.9}
$$

Next we 'lift' the point μ_j^+, μ_j from the $\mathbb{C}_\infty$ to the compact Riemann surface $\mathcal{K}_p$

$$
\begin{aligned}
&\hat{\mu}_j(n) = (\mu_j(n), 2iA_{p+1}(\mu_j(n), n)), \\
&\hat{\mu}_j^+(n) = (\mu_j^+(n), -2iA_{p+1}(\mu_j^+(n), n)), \\
&j = 1,\cdots,p \quad n \in \mathbb{Z}
\end{aligned}
\tag{4.10}
$$

and

$$
P_0 = (h^{-1}, A(h^{-1}, n)), \quad P_{\infty+}, \quad P_{\infty-},
$$

where $P_{\infty\pm}$ are two points at infinity on $\mathcal{K}_p$. We define a fundamental meromorphic function on $\mathcal{K}_p$

$$
\begin{aligned}
\phi(P,n) &= \frac{\frac{1}{2}iy - A_{p+1}(\lambda,n)}{B_{p+1}(\lambda,n)} \\
&= \frac{(h\lambda-1)\alpha^+ B_{p+1}^+(\lambda,n)}{\frac{1}{2}iy + A_{p+1}(\lambda,n)}, \\
P &= (\lambda,y) \in \mathcal{K}_p, \quad n \in \mathbb{Z}.
\end{aligned}
\tag{4.11}
$$

The divisor $(\phi(\cdot,n))$ of $\phi(\cdot,n)$ is

$$
\begin{aligned}
&(\phi(\cdot,n)) = \mathcal{D}_{P_0\underline{\hat{\mu}}^+(n)} - \mathcal{D}_{P_{\infty+}\underline{\hat{\mu}}(n)}, \\
&\underline{\hat{\mu}}(n) = (\hat{\mu}_1(n),\cdots,\mu_p(n)), \qquad \underline{\hat{\mu}}^+(n) = \left(\hat{\mu}_1^+(n),\cdots,\mu_p(n)\right) \in Sym^p(\mathcal{K}_p).
\end{aligned}
\tag{4.12}
$$

Here $Sym^p(\mathcal{K}_p)$ denotes the space of all the nonnegative divisors of degree p. With a differential structure on it we can prove $Sym^p(\mathcal{K}_p)$ is isomorphism to $\mathcal{K}_p \times \mathcal{K}_p \cdots \times \mathcal{K}_p$

[1] $\mathcal{K}_p$ may have one singular point in projective space.

(p times) [23]. Given the function $\phi(P,n)$, we define the stationary Baker-Akhiezer vector $\Psi(P,n,n_0)$ by

$$\Psi(P,n,n_0)=\begin{pmatrix}\Psi_1(P,n,n_0)\\ \Psi_2(P,n,n_0)\end{pmatrix}, \tag{4.13}$$

$$\Psi_1(P,n,n_0)=\begin{cases}\prod_{n'=n_0}^{n-1}\phi(P,n') & n'>n_0\\ 1 & n'=n_0\\ \prod_{n'=n}^{n_0-1}\phi(P,n')^{-1} & n'<n_0,\end{cases} \tag{4.14}$$

$$\Psi_2(P,n,n_0)=\phi(P,n_0)\times\begin{cases}\prod_{n'=n_0+1}^{n-1}\left(\frac{\alpha(n')(h\lambda-1)}{\phi^-(P,n')}+\lambda+\beta(n')\right) & n'>n_0\\ 1 & n'=n_0\\ \prod_{n'=n+1}^{n_0}\left(\frac{\alpha(n')(h\lambda-1)}{\phi^-(P,n')}+\lambda+\beta(n')\right)^{-1} & n'<n_0.\end{cases} \tag{4.15}$$

The divisor $(\Psi_1(\cdot,n,n_0))$ of $\Psi_1(\cdot,n,n_0)$ is

$$(\Psi_1(\cdot,n,n_0))=\mathcal{D}_{\underline{\hat{\mu}}(n)}-\mathcal{D}_{\underline{\hat{\mu}}(n_0)}+(n-n_0)(\mathcal{D}_{P_0}-\mathcal{D}_{P_{\infty+}}). \tag{4.16}$$

We introduce the holomorphic sheet exchange map on $\mathcal{K}_p$

$$*:\quad \mathcal{K}_p\to\mathcal{K}_p\qquad P=(\lambda,y)\mapsto P^*=(\lambda,-y),\quad P_{\infty\pm}\mapsto P^*_{\infty\pm}=P_{\infty\mp}. \tag{4.17}$$

There are some basic properties for ϕ,Ψ,Ψ_1,Ψ_2 in the following lemma:

Lemma 3. *Assume α,β satisfy the p-th stationary RT system (2.13) (2.14). hypothesis 1 and hypothesis 2 hold. Suppose $P=(z,y)\in\mathcal{K}_p\backslash\{P_{\infty\pm}\}$. Then ϕ satisfy Riccati-type equation*

$$\phi(P)\phi^-(P)-(\lambda+\beta)\phi^-(P)-(h\lambda-1)\alpha=0. \tag{4.18}$$

Moreover,

$$\begin{aligned}\phi(P)\phi(P^*)&=-\frac{(h\lambda-1)\alpha^+B^+_{p+1}}{B_{p+1}},\\ \phi(P)+\phi(P^*)&=-2\frac{A_{p+1}}{B_{p+1}},\\ \phi(P)-\phi(P^*)&=i\frac{y}{B_{p+1}},\end{aligned} \tag{4.19}$$

$$\Psi_2(P,n,n_0)=\Psi_1(P,n,n_0)\phi(P,n), \tag{4.20}$$

$$U(\lambda)\Psi^-(P)=\Psi(P), \tag{4.21}$$

$$V_p(\lambda)\Psi^-(P)=\frac{1}{2}iy\Psi(P). \tag{4.22}$$

Proof. Directly calculate

$$\begin{aligned}
&\phi(P)\phi^-(P)-(\lambda+\beta)\phi^-(P)-(h\lambda-1)\alpha\\
&=\frac{\frac{1}{2}iy-A_{p+1}}{B_{p+1}}\frac{\frac{1}{2}iy-A^-_{p+1}}{B^-_{p+1}}+(\lambda+\beta)\frac{\frac{1}{2}iy-A^-_{p+1}}{B^-_{p+1}}-(h\lambda-1)\alpha\\
&=\frac{1}{B_{p+1}B^-_{p+1}}[(\frac{1}{2}iy-A_{p+1})(\frac{1}{2}iy-\lambda A^-_{p+1})-(\lambda+\beta)B_{p+1}(\frac{1}{2}iy-A^-_{p+1})-(h\lambda-1)B_{p+1}B^-_{p+1}]\\
&=0.
\end{aligned}$$

Here we used the definition of ϕ in (4.11) and (2.15) (2.16). (4.19) is the immediate consequence of (4.11). Next we will prove (4.20)-(4.22). From the definition of Ψ_1, Ψ_2, they have the following relation

$$\frac{\Psi_2(P,n,n_0)}{\Psi_1(P,n,n_0)}=\frac{(h\lambda-1)\alpha\phi^-(P,n)^{-1}+\lambda+\beta}{\phi^-(P,n)}\frac{\Psi_2^-}{\Psi_1^-}. \tag{4.23}$$

Next we use induction to prove

$$\Psi_2(P,n,n_0)=\Psi_1(P,n,n_0)\phi(P,n). \tag{4.24}$$

For $n=n_0$, (4.24) holds.
For $n>n_0$, assume (4.24) holds for $n-1$. One finds (4.23) is

$$\frac{\Psi_2(P,n,n_0)}{\Psi_1(P,n,n_0)}=\frac{(h\lambda-1)\alpha\phi^-(P,n)^{-1}+\lambda+\beta}{\phi^-(P,n)}\phi^-(P,n)=(h\lambda-1)\alpha\phi^-(P,n)^{-1}+\lambda+\beta.$$

Hence $\frac{\Psi_2(P,n,n_0)}{\Psi_1(P,n,n_0)}$ satisfy Riccati equation

$$\frac{\Psi_2(P,n,n_0)}{\Psi_1(P,n,n_0)}\phi^-(P,n)-(\lambda+\beta)\phi^-(P,n)-(h\lambda-1)\alpha=0.$$

Noticed (4.18), one can derive (4.20).
For $n<n_0$, this case is similar with $n>n_0$. From the definition (4.14), we finds

$$\Psi_1(P,n,n_0)=\Psi_1^-(P,n,n_0)\phi^-(P,n).$$

Then we have

$$\Psi_1(P,n,n_0) = \Psi_2^-(P,n,n_0) \tag{4.25}$$

Similarly, from the definition of Ψ_2 in (4.15) and (4.20), one shows

$$\begin{aligned}\Psi_2(P,n,n_0) &= \left(\frac{(h\lambda-1)\alpha}{\phi^-(P,n)} + \lambda + \beta\right)\Psi_2^-(P,n,n_0) \\ &= (h\lambda-1)\alpha\Psi_1^-(P,n,n_0) + (\lambda+\beta)\Psi_2^-(P,n,n_0).\end{aligned} \tag{4.26}$$

We have (4.21). Equation (4.22) is the direct result of (4.11) and (4.20). □

Combining the polynomial recursion approach in the section 1 with (4.9) yields the following trace formula, which means a_l, b_l can be expressed by the symmetric functions of the zeros μ_j of B_{p+1}. For simplicity, we only show one of them.

Lemma 4 (trace formula). *Suppose $g \in l(\mathbb{Z})$ satisfy the pth stationary RT hierarchy. Then*

$$-h(\alpha+\alpha^+) - \beta + \delta_1 = -\sum_{j=1}^{p} \mu_j. \tag{4.27}$$

Proof. Comparing the powers of λ^{p-1} in (2.10) and (3.6) for B_{p+1}, that is,

$$b_0\lambda^{p+1} + b_1\lambda^p + \cdots + b_{p+1} = \prod_{j=1}^{p}(\lambda-\mu_j). \tag{4.28}$$

Then equation (4.27) holds.

□

Remark 7. *It's a bit pity that we can not derive the algebro-geometric solutions of RT hierarchy directly from trace formula because α and β are mutually determined in one equality. However, it can be useful once one has derived the theta function representation of one of them (Remark 8).*

Next we turn to study the asymptotic properties of ϕ, Ψ_1 to prepare for later calculation. The asymptotic behavior of Ψ_2 is derived naturally from (4.20). It is a crucial step to construct the stationary algebro-geometric solutions of RT hierarchy.

Lemma 5. *Suppose that* α, β *satisfy the* p*-th stationary RT hierarchy. Moreover, let* $P = (\lambda, y) \in \mathcal{K}_p \backslash \{P_{\infty\pm}, P_0\}$, $(n, n_0) \in \mathbb{Z} \times \mathbb{Z}$*. Then,*

$$\phi(P) = \begin{cases} \zeta^{-1} + (\beta - h\alpha) + O(\zeta) & P \to P_{\infty+} \quad \lambda = \zeta^{-1} \\ h\alpha^+ + (h^2\alpha^+\alpha + \alpha - h\alpha\beta)\zeta + O(\zeta^2) & P \to P_{\infty-} \quad \lambda = \zeta^{-1} \\ \frac{h\alpha^+}{h+\beta^+}\zeta + O(\zeta^2) & P \to P_0 \quad \lambda = \zeta^2, \end{cases} \tag{4.29}$$

$$\Psi_1(P, n, n_0) = \begin{cases} \zeta^{n_0 - n}\left(1 + O(\zeta)\right) & \textit{as} \quad P \to P_{\infty+} \quad \lambda = \zeta^{-1} \\ \Lambda\left(h\alpha^+\right)\left(1 + O(\zeta)\right) & \textit{as} \quad P \to P_{\infty-} \quad \lambda = \zeta^{-1} \\ \Lambda\left(\frac{h\alpha^+}{h+\beta^+}\right)\zeta^{n-n_0}\left(1 + O(\zeta)\right) & \textit{as} \quad P \to P_0 \quad \lambda = \zeta^{-1}, \end{cases} \tag{4.30}$$

Here

$$\Lambda\left(\varphi\right) = \begin{cases} \prod_{n'=n_0}^{n-1} \varphi(n') & n' > n_0 \\ 1 & n' = n_0 \\ \prod_{n'=n}^{n_0-1} \varphi^{-1}(n') & n' > n_0. \end{cases} \tag{4.31}$$

The divisor $(\Psi_1(\cdot, n, n_0))$ *of* $\Psi_1(P, n, n_0)$ *is*

$$(\Psi_1(\cdot, n, n_0)) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_0} - \mathcal{D}_{P_{\infty+}}). \tag{4.32}$$

Proof. Assume ϕ has the following asymptotic expansion

$$\phi = \begin{cases} \phi_{-1}\zeta^{-1} + \phi_0 + \phi_1\zeta + \cdots, & \text{as} \quad P \to P_{\infty+} \\ \phi_0 + \phi_1\zeta + \phi_2\zeta^2 + \cdots, & \text{as} \quad P \to P_{\infty-} \\ \phi_1\zeta + \phi_2\zeta^2 + \cdots, & \text{as} \quad P \to P_0. \end{cases} \tag{4.33}$$

Here we used local coordinate $\zeta = \lambda^{-1}$ near $P_{\infty\pm}$ and $\zeta = \lambda - h^{-1}$ near P_0. Then plunging it into (4.18) and comparing the powers in both sides we prove (4.29). (4.32) is from (4.16) and Abel Theorem.

□

We choose a fixed base point Q_0 on $\mathcal{K}_p\backslash\{P_0, P_{\infty\pm}\}$. Let $\omega^{(3)}_{P_0P_{\infty+}}$ be a normal differential of the third kind holomorphic on $\mathcal{K}_p\backslash\{P_{\infty+}, P_0\}$ with simple poles at $P_{\infty+}$ and P_0 and residues -1 and 1, respectively, that is,

$$\omega^{(3)}_{P_0P_{\infty+}} = \frac{y + A(h^{-1}, n)}{\lambda - h^{-1}}\frac{d\lambda}{2y} - \frac{1}{2y}\prod_{j=1}^{p}(\lambda - \lambda_j)d\lambda \xlongequal{\zeta\to 0} \begin{cases}(-\zeta^{-1} + O(1))d\zeta & P \to P_{\infty+} \\ (\zeta^{-1} + O(1))d\zeta & P \to P_0\end{cases} \tag{4.34}$$

and

$$\omega^{(3)}_{P_{\infty+}P_{\infty-}} = \frac{1}{y}\prod_{j=1}^{p}\left(\lambda - \lambda_j'\right)d\lambda \tag{4.35}$$

be a normal differential of the third kind holomorphic on $\mathcal{K}_p\backslash\{P_{\infty+}, P_0\}$ with simple poles at $P_{\infty+}$ and P_0 and residues 1 and -1, where $\zeta = \lambda^{-1}$ for P near $P_{\infty\pm}$, $\zeta = \lambda - h^{-1}$ for P near P_0, $\sigma \in \{\pm 1\}$. and $\{\lambda_j\}_{j=1,\cdots,p}$, $\{\lambda_j'\}_{j=1,\cdots,p}$ are constants uniquely determined by normalized process.

Moreover,

$$\int_{a_j}\omega^{(3)}_{P_0P_{\infty+}} = 0, \quad j = 1,\cdots,p, \tag{4.36}$$

$$\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}} \xlongequal{\zeta\to 0} \begin{pmatrix} 0 \\ \ln\zeta\end{pmatrix} + \begin{pmatrix} e_{0.-} \\ e_{0,+}\end{pmatrix} + O(\zeta) \quad \begin{matrix} P \to P_{\infty-} \\ P \to P_{\infty+}\end{matrix}, \tag{4.37}$$

$$\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}} \xlongequal{\zeta\to 0} -\ln\zeta + d_0 + O(\zeta) \quad P \to P_0, \tag{4.38}$$

where we choose a homology basis $\{a_j, b_j\}_{j=1}^{p}$ on $\mathcal{K}_p$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1,\cdots,p \tag{4.39}$$

and $e_{0,\pm}, d_0 \in \mathbb{C}$. One easily verifies that $d\lambda/y$ is a differential on $\mathcal{K}_p$ with zeros of order $p-1$ at $P_{\infty\pm}$ and hence

$$\eta_j = \frac{\lambda^{j-1}}{y}d\lambda, \quad j = 1,\cdots,p$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_p$. Introducing the following invertible matrix $C_{j,k} \in \mathbb{C}$

$$C = (C_{j,k})_{j,k=1,\cdots,p}, \quad C_{j,k} = \int_{a_k}\eta_j, \tag{4.40}$$

$$\underline{c}(k) = (c_1(k),\cdots, c_p(k)), \quad c_j(k) = (C^{-1})_{j,k} \quad k = 1,\cdots, p. \tag{4.41}$$

It's easy to show that the normalized holomorphic differentials $\{\omega_j\}_{j=1,\cdots,p}$ can be written into

$$\omega_j = \sum_{l=1}^{p} c_j(l)\eta_l, \quad \int_{a_k} \eta_j = \delta_{j,k}, \quad j,k = 1,\cdots, p, \tag{4.42}$$

$$\underline{\omega} = (\omega_1,\cdots, \omega_p)\,. \tag{4.43}$$

Assume $\eta \in \mathbb{C}$ and $|\eta| <$min$\{|E_0|^{-1}, |E_1|^{-1}, |E_2|^{-1},\cdots, |E_{2p+1}|^{-1}\}$ and abbreviate

$$\underline{E} = (E_0, E_1,\cdots, E_{2p+1}).$$

Then

$$\left(\prod_{m=0}^{2p+1}(1 - E_m\eta)\right)^{-1/2} = \sum_{k=0}^{+\infty} \hat{c}_k(\underline{E})\eta^k,$$

where

$$\hat{c}_0(\underline{E}) = 1, \quad \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m,$$

$$\hat{c}_k(\underline{E}) = \sum_{j_0,\cdots,j_{2p+1}=0, j_0+\cdots+j_{2p+1}=k}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})!}{2^{2k}(j_0!)^2(j_{2p+1}!)^2} E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}, \quad k \in \mathbb{N}, \quad etc.$$

Similarly,

$$\left(\prod_{m=0}^{2p+1}(1 - E_m\eta)\right)^{1/2} = \sum_{k=0}^{+\infty} c_k(\underline{E})\eta^k,$$

where

$$c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{2p+1} E_m,$$

$$c_k(\underline{E}) = \sum_{j_0,\cdots,j_{2p+1}=0, j_0+\cdots+j_{2p+1}=k}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})! E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2(2j_0-1)\cdots(2j_{2p+1}-1)} \quad k \in \mathbb{N}, \quad etc.$$

Obviously,

$$y(P) = \mp\zeta^{-p-1}\sum_{k=0}^{+\infty} c_k(\underline{E})\zeta^k = \mp\left(1 - \frac{1}{2}\left(\sum_{m=0}^{2p+1}\zeta + O(\zeta^2)\right)\right) \quad \text{as} \quad P \to P_{\infty\pm}, \quad \zeta = \lambda^{-1}. \tag{4.44}$$

It's easy to calculate the expansion of $\omega^{(3)}_{P_0P_{\infty+}}$ near $P_{\infty+}$ concretely. That is,

$$\omega^{(3)}_{P_0P_{\infty+}} = -\zeta^{-1}\left(1+(\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\lambda_j)\zeta+O(\zeta^2)\right), \tag{4.45}$$

$$\int_{Q_0}^{P}\omega_{P_0P_{\infty+}} = \tilde{a}\zeta^{-1}\left(1+(\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\sum_{j=1}^{p}\lambda_j)\zeta+O(\zeta^2)\right), \quad \text{as} \quad P\to P_{\infty+}, \tag{4.46}$$

where $\tilde{a}$ is a integration constant.

In the following it will be convenient to introduce the abbreviations

$$\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad P\in\mathcal{K}_p, \quad \underline{Q}=\{Q_1,\cdots,Q_p\}\in Sym^p(\mathcal{K}_p), \tag{4.47}$$

where $\underline{\Xi}_{Q_0}$ is the vector of Riemann constants and the Abel maps $\underline{A}_{Q_0}(\cdot),\underline{\alpha}_{Q_0}(\cdot)$ are defined by (period lattice $L_p=\{\underline{z}\in\mathbb{Z}^g|\underline{z}=\underline{n}+\underline{m}\tau,\underline{n},\underline{m}\in\mathbb{Z}^g\}$)

$$\underline{A}_{Q_0}:\mathcal{K}_p\to\mathcal{J}(\mathcal{K}_p)=\mathbb{Z}^p/L_p \quad P\mapsto\underline{A}_{Q_0}(P)=(A_{Q_0,1}(P),\cdots,A_{Q_0,p}(P))=\left(\int_{Q_0}^{p}\omega_1,\cdots,\int_{Q_0}^{P}\omega_p\right)$$

and

$$\underline{\alpha}_{Q_0}:Div(\mathcal{K}_p)\to\mathcal{J}(\mathcal{K}_p),\mathcal{D}\mapsto\underline{\alpha}_{Q_0}(\mathcal{D})=\sum_{P\in\mathcal{K}_g}\mathcal{D}(P)\underline{A}_{Q_0}(P).$$

Theorem 8. *Suppose that g satisfy the p-th stationary RT hierarchy,let $P\in\mathcal{K}_p\backslash\{P_{\infty\pm},P_0\}$ and $(n,n_0)\in\mathbb{Z}^2$. Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is non-special.Moreover,*

$$\phi(P,n)=C(n)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}}\right) \tag{4.48}$$

and

$$\Psi_1(P,n,n_0)=C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\left((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}}\right) \tag{4.49}$$

where

$$C(n)=\tilde{a}^{-1}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))} \tag{4.50}$$

and

$$C(n,n_0)=\begin{cases}\prod_{n'=n_0}^{n-1}C(n') & n>n_0\\ 1 & n=n_0\\ \prod_{n'=n}^{n_0-1}C(n')^{-1} & n<n_0.\end{cases} \tag{4.51}$$

The Abel map linearizes the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)})=\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)})-\underline{A}_{P_{\infty+}}(P_0)(n-n_0) \tag{4.52}$$

and α,β are the form of

$$\alpha^+=\frac{1}{h}(\tilde{a})^{-2}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))} \tag{4.53}$$

and

$$\begin{aligned}\beta^+=&(\tilde{a})^{-2}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))}\\ &+\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\sum_{j=1}^{p}\lambda_j-\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)+\underline{\omega}))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))+\underline{\omega})}\right)|_{\underline{\omega}=0}.\end{aligned} \tag{4.54}$$

Proof. The proof that the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is non-special see Lemma 11, where t_r is regarded as a parameter. Hence the theta functions defined in this lemma are meaningful and not identical to zero. Obviously, $\phi(P,n)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n)))}\exp(\int_{Q_0}^{P}\omega_{P_{\infty+}P_0}^{(3)})$ is holomorphic function on compact Riemann surface $\mathcal{K}_p$(Riemann-Roch Theorem [14]). So it is a constant $C(n)$ related to n and $\phi(P,n)$ has the form (4.48). One have the following expansion as $P\to P_{\infty+}$, $\quad(\zeta=\lambda^{-1})$

$$\begin{aligned}&\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}\\ &=\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}\left(1-\sum_{c=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)+\underline{\omega}))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n))+\underline{\omega})}\right)|_{\underline{\omega}=0}\zeta+O(\zeta^2)\right).\end{aligned} \tag{4.55}$$

Then as $P \to P_{\infty+}$,

$$\phi(P,n)$$
$$= \tilde{a}C(n)\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}\left(1-\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)+\underline{\omega}))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))+\underline{\omega})}\right)|_{\underline{\omega}=0}\zeta+O(\zeta^2)\right)$$
$$\times\zeta^{-1}\left(1+(\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\sum_{j=1}^{p}\lambda_j)\zeta+O(\zeta^2)\right). \tag{4.56}$$

Comparing with (4.29) we have

$$C(n)=\tilde{a}^{-1}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}, \tag{4.57}$$

$$\beta-h\alpha=\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\sum_{j=1}^{p}\lambda_j-\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)+\underline{\omega}))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))+\underline{\omega})}\right)|_{\underline{\omega}=0}. \tag{4.58}$$

Similarly, as $P \to P_{\infty-}$,

$$\phi(P,n)$$
$$= \tilde{a}^{-1}C(n)\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))}\left(1+\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)+\underline{\omega}))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))+\underline{\omega})}\right)|_{\underline{\omega}=0}\zeta+O(\zeta^2)\right)$$
$$\times\left(1+(-\frac{h^{-1}}{2}+\frac{1}{4}\sum_{m=0}^{2p+1}E_m-\frac{1}{2}\sum_{j=1}^{p}\lambda_j)\zeta+O(\zeta^2)\right), \tag{4.59}$$

which gives rise to

$$\begin{aligned} h\alpha^+ &= \tilde{a}^{-1}C(n)\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))} \\ &= (\tilde{a})^{-2}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))}. \end{aligned} \tag{4.60}$$

Then we have (4.53) (4.54). (4.53)-(4.51) arise from the definition of Ψ_1. (4.52) is the direct result of (4.16) and Abel Theorem ([22]).

□

Remark 9. *From trace formula (4.27), (3.18), (4.35) (4.53) and residue formula about the theta function representation of symmetric function $\sum_{j=1}^{p}\mu_j$ [38], β admits another representation*

$$\begin{aligned}
\beta &= -h(\alpha+\alpha^+)+\delta_1+\sum_{j=1}^{p}\mu_j \\
&= (\tilde{a})^{-2}\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))}+\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^-(n)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^-(n)))}\right) \\
&-\sum_{j=0}^{2p+1}E_i+\sum_{j=1}^{p}\lambda_j^{'}-\sum_{j=1}^{p}c_j(k)\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n))+\underline{\omega}))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n)))+\underline{\omega})}\right)|_{\underline{\omega}=0}.
\end{aligned} \tag{4.61}$$

We consider the trivial case $p=0$ which excluded in Theorem 7.

Remark 10. *Assume $p=0$, $P=(\lambda,y)\in\mathcal{K}_0\backslash\{P_{\infty\pm}\}$ and $(n,n_0)\in\mathbb{Z}^2$. Then*

$$V_0^+(\lambda,n)=\begin{pmatrix} -\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2} & 1 \\ (h\lambda-1)\alpha^+ & \frac{1}{2}\lambda-h\alpha^++\frac{\delta_1}{2}\end{pmatrix}, \tag{4.62}$$

$$\mathcal{K}_0: \quad \mathcal{F}_0(\lambda,y)=y^2-(\lambda-E_0)(\lambda-E_1)=0 \qquad E_0,E_1\in\mathbb{C}. \tag{4.63}$$

α,β *satisfy*

$$\begin{cases} h\alpha^++h\alpha+\beta=-\frac{E_0+E_1}{2} \\ -4\left(\left(h\alpha^++\frac{E_0+E_1}{4}\right)\left(-h\alpha^++\frac{\delta_1}{2}\right)+\alpha^+\right)=E_0E_1 \end{cases}. \tag{4.64}$$

$$\alpha=-\frac{E_0+E_1}{2h}\pm\frac{1}{2}\sqrt{\left(\frac{E_0+E_1}{2h}\right)^2+\left(\frac{E_0-E_1}{2}\right)^2}, \qquad \beta=\frac{E_0+E_1}{2}. \tag{4.65}$$

$$B_1=1, \quad C_1=(h\lambda-1)\alpha^+, \quad A_1=-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2}, \quad D_1=\frac{1}{2}\lambda-h\alpha^++\frac{\delta_1}{2}. \tag{4.66}$$

$$\phi(P,n_0)=\frac{1}{2}iy-\left(-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2}\right), \tag{4.67}$$

$$\Psi_1(P,n,n_0)=\left(\frac{1}{2}iy-\left(-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2}\right)\right)^{n-n_0}. \tag{4.68}$$

5 Algebro-geometric solutions of time-dependent cases

In this section we analysis the quasi-period solution of the time-dependent RT hierarchy.

Hypothesis 3. *Assume*

$$\begin{aligned} &\alpha(n,t),\beta(n,t)\in l(\mathbb{Z}),\quad \alpha(n,t)\neq 0,\quad (n,t)\in\mathbb{Z}\times\mathbb{R}\\ &\alpha(n,\cdot),\beta(n,\cdot)\in C^1(\mathbb{R}). \end{aligned} \tag{5.1}$$

Assume Hypothesis 1 ,Hypothesis 2 and 3 hold in this section. The basic problem of analysis of algebro-geometric solutions of the RT hierarchy consists of solving the time-dependent rth RT flow with initial data a stationary solution of the pth system in the hierarchy. That is

$$U_{t_r}(\lambda,t_r)+U(\lambda,t_r)\tilde{V}_r(\lambda,t_r)-\tilde{V}_r^+(\lambda,t_r)U(\lambda,t_r)=0, \tag{5.2}$$

$$U(\lambda,t_r)V_p(\lambda,t_r)-V_p^+(\lambda,t_r)U(\lambda,t_r)=0, \tag{5.3}$$

$$U(\lambda)=\begin{pmatrix} 0 & 1\\ (h\lambda-1)\alpha & \lambda+\beta \end{pmatrix},\quad \tilde{V}_r(\lambda)=\begin{pmatrix} \tilde{A}_{r+1} & \tilde{B}_{r+1}\\ \tilde{C}_{r+1} & \tilde{A}_{r+1} \end{pmatrix},\quad V_p=\begin{pmatrix} A_{p+1} & B_{p+1}\\ C_{p+1} & D_{p+1} \end{pmatrix}. \tag{5.4}$$

We make the anatz

$$
\begin{aligned}
A_{p+1} &= \sum_{j=0}^{p+1} a_{p+1-j}\lambda^j + b_{p+2}, \\
B_{p+1} &= \sum_{j=1}^{p+1} b_j \lambda^{p+1-j}, \\
C_{p+1} &= (h\lambda - 1)\alpha^+ B^+ = (h\lambda-1)\alpha^+ \left(b_0^+\lambda^{p+1} + \cdots + b_{p+1}^+\right), \\
D_{p+1} &= -\sum_{j=0}^{p+1} a_{p+1-j}\lambda^j, \\
\tilde{A}_{r+1} &= \sum_{j=0}^{r+1} \tilde{a}_{r+1-j}\lambda^j + \tilde{b}_{r+2}, \\
\tilde{B}_{r+1} &= \sum_{j=1}^{r+1} \tilde{b}_j \lambda^{r+1-j}, \\
\tilde{C}_{r+1} &= (h\lambda-1)\alpha^+\tilde{B}_{p+1}^+ = (h\lambda-1)\alpha^+\left(\tilde{b}_0^+\lambda^{r+1} + \cdots + \tilde{b}_{r+1}^+\right), \\
\tilde{D}_{r+1} &= -\sum_{j=0}^{r+1} \tilde{a}_{r+1-j}\lambda^j.
\end{aligned}
\tag{5.5}
$$

Here $\{a_l, b_l\}$,and$\{\tilde{a}_l, \tilde{b}_l\}$ are defined by (2.3), but correspond to different constant $\{\delta_l\}$ and $\{\tilde{\delta}_l\}, \quad l \in \mathbb{N}_0$.

Explicitly, equation (5.2) and (5.3) are equivalent to

$$\alpha A_{p+1}^- + (\lambda+\beta)\alpha B_{p+1} - \alpha A_{p+1} = 0 \tag{5.6}$$

$$(h\lambda-1)\alpha B_{p+1}^- + (\lambda+\beta)A_{p+1}^- - (h\lambda-1)\alpha^+ B_{p+1}^+ - (\lambda+\beta)A_{p+1} = 0 \tag{5.7}$$

$$\tilde{D}_{r+1} = \tilde{A}_{r+1} + (\lambda+\beta)\tilde{B}_{r+1} \tag{5.8}$$

$$\alpha_{t_r} + \alpha\tilde{A}_{r+1}^- + (\lambda+\beta)\alpha\tilde{B}_{r+1} - \alpha\tilde{D}_{r+1} = 0 \tag{5.9}$$

$$\beta_{t_r} + (h\lambda-1)\alpha\tilde{B}_{r+1}^- + (\lambda+\beta)\tilde{D}_{r+1}^- - (h\lambda-1)\alpha^+\tilde{B}_{r+1}^+ - (\lambda+\beta)\tilde{A}_{r+1} = 0 \tag{5.10}$$

In particular, (4.4) holds in the present t_r-dependent setting, that is,

$$R_{2p+2}(\lambda, t_r) = -A_{p+1}^2(\lambda, t_r) - (h\lambda-1)\alpha^+(\lambda, t_r)B_p(\lambda, t_r)B_p^+(\lambda, t_r). \tag{5.11}$$

As in the stationary context, we introduce

$$\hat{\mu}_j(n,t_r) = (\mu_j(n,t_r), 2iA_{p+1}(\mu_j(n,t_r),n)),$$
$$\hat{\mu}_j^+(n,t_r) = (\mu_j^+(n,t_r), -2iA_{p+1}(\mu_j^+(n,t_r),n,t_r)), \tag{5.12}$$
$$j = 1,\cdots,p \quad (n,t_r) \in \mathbb{Z}\times\mathbb{R}$$

and note the regularity assumptions (5.1) on α,β imply the continuity of μ_j with respect to $t_r \in \mathbb{R}$.

In analogy to the stationary case, one may define the following meromorphic function $\phi(\cdot,n,t_r)$ on $\mathcal{K}_p$,

$$\phi(P,n,t_r) = \frac{\frac{1}{2}iy - A_{p+1}(\lambda,n,t_r)}{B_{p+1}(\lambda,n,t_r)} = \frac{(h\lambda-1)\alpha^+ B_{p+1}^+(\lambda,n,t_r)}{\frac{1}{2}iy + A_{p+1}(\lambda,n,t_r)}. \tag{5.13}$$

The divisor $(\phi(.,n,t_r))$ of ϕ is

$$(\phi(.,n,t_r)) = \mathcal{D}_{P_0\underline{\hat{\mu}}^+(n,t_r)} - \mathcal{D}_{P_{\infty+}\underline{\hat{\mu}}(n,t_r)}. \tag{5.14}$$

The time-dependent Baker-Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_r,t_{0,r}) = \begin{pmatrix} \Psi_1(P,n,n_0,t_r,t_{0,r}) \\ \Psi_2(P,n,n_0,t_r,t_{0,r}) \end{pmatrix}, \tag{5.15}$$

$$\Psi_1(P,n,n_0,t_r,t_{0,r}) = \exp\left(\int_{t_{0,r}}^{t_r} \left(\tilde{A}_{r+1}(\lambda,n_0,s) + \tilde{B}_{r+1}(\lambda,n_0,s))\phi(P,n_0,s)\right) ds\right) \tag{5.16}$$

$$\times \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P,n',t_r) & n' > n_0 \\ 1 & n' = n_0 \\ \prod_{n'=n}^{n_0-1} \phi(P,n',t_r)^{-1} & n' < n_0, \end{cases} \tag{5.17}$$

$$\Psi_2(P,n,n_0) = \exp\left(\int_{t_{0,r}}^{t_r} \left(\tilde{A}_{r+1}(\lambda,n_0,s) + \tilde{B}_{r+1}(\lambda,n_0,s)\phi(P,n_0,s)\right) ds\right) \tag{5.18}$$

$$\phi(P,n_0,t_r) \times \begin{cases} \prod_{n'=n_0+1}^{n-1} \left(\frac{\alpha(n',t_r)(h\lambda-1)}{\phi^-(P,n',t_r)} + \lambda + \beta(n',t_r)\right) & n' > n_0 \\ 1 & n' = n_0 \\ \prod_{n'=n+1}^{n_0} \left(\frac{\alpha(n',t_r)(h\lambda-1)}{\phi^-(P,n',t_r)} + \lambda + \beta(n',t_r)\right)^{-1} & n' < n_0, \end{cases} \tag{5.19}$$

$$P=(\lambda,y)\in\mathcal{K}_p\backslash\{P_{\infty\pm},P_0\},\quad (n,t_r)\in\mathbb{Z}\times\mathbb{R}. \tag{5.20}$$

One observes that

$$\begin{aligned}\Psi_1(P,n,n_0,t_r,t_{0,r})=\Psi_1(P,n_0,n_0,t_r,t_{0,r})\Psi_1(P,n,n_0,t_r,t_r)\\ P\in\mathcal{K}_p\backslash\{P_{\infty\pm},P_0\},(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2.\end{aligned} \tag{5.21}$$

The following lemma shows the properties of ϕ,Ψ as discussed in the stationary case.

Lemma 6. *Assume $\alpha(n,t_r),\beta(n,t_r)$ satisfy equation (5.2) (5.3),then $P=(\lambda,y)\in\mathcal{K}_p\backslash\{P_{\infty\pm}\}$ then ϕ satisfy Riccati-type equation*

$$\phi(P)\phi^-(P)-(\lambda+\beta)\phi^-(P)-(h\lambda-1)\alpha=0, \tag{5.22}$$

$$\phi_{t_r}(P)=(-\tilde{A}_{r+1}+\tilde{D}_{r+1})\phi-\tilde{B}_{r+1}\phi^2+(h\lambda-1)\alpha^+\tilde{B}^+_{r+1}. \tag{5.23}$$

Moreover,

$$\begin{aligned}\phi(P)\phi(P^*)&=-\frac{(h\lambda-1)\alpha^+B^+_{p+1}}{B_{p+1}},\\ \phi(P)+\phi(P^*)&=-2\frac{A_{p+1}}{B_{p+1}},\\ \phi(P)-\phi(P^*)&=i\frac{y}{B_{p+1}}\end{aligned} \tag{5.24}$$

and

$$\Psi_2(P,n,n_0,t_r,t_{0,r})=\phi(P,n,t_r)\Psi_1(P,n,n_0,t_r,t_{0,r}), \tag{5.25}$$

$$U(\lambda)\Psi^-(P)=\Psi(P), \tag{5.26}$$

$$V_p(\lambda)\Psi^-(P)=\frac{1}{2}iy\Psi(P), \tag{5.27}$$

$$\Psi_{t_r}(p)=\tilde{V}^+_r(\lambda)\Psi(P). \tag{5.28}$$

Moreover, as long as the zeros of $\mu_j(n_0,s)$ of $B_{p+1}(\cdot,n_0,s)$ are all simple for all $s\in\Omega$, $\Omega\subseteq\mathbb{R}$, is an open interval, Ψ_1 is meromorphic on $\mathcal{K}_p\backslash\{P_{\infty\pm}\}$ for $(n,n_0,t_r)\in\mathbb{Z}\times\Omega^2$.

Proof. From (5.22) we have

$$\phi_{t_r}\phi^-+\phi\phi^-_{t_r}=\beta_{t_r}\phi^-+(\lambda+\beta)\phi^-_{t_r}+\alpha_{t_r}(h\lambda-1). \tag{5.29}$$

Using (5.9) (5.10), one finds

$$
\begin{aligned}
&-\left(\phi^- + (\phi - (\lambda+\beta))\, S^-\right)\phi_{t_r} \\
&= \left((h\lambda-1)\alpha\tilde{B}^-_{r+1} + (\lambda+\beta)\tilde{D}^-_{r+1} - (h\lambda-1)\alpha^+\tilde{B}^+_{r+1} - (\lambda+\beta)\tilde{D}_{r+1}\right)\phi^- \\
&+ (h\lambda-1)\left(\alpha\tilde{A}^-_{r+1} + (\lambda+\beta)\alpha\tilde{B}_{r+1} - \alpha\tilde{D}_{r+1}\right).
\end{aligned}
\tag{5.30}
$$

In other sides, we calculate

$$
\begin{aligned}
&\left(\phi^- + (\phi - (\lambda+\beta))\, S^-)\right)\left(\tilde{A}_{r+1}\phi + \tilde{B}_{r+1}\phi^2 - \tilde{C}_{r+1} - \tilde{D}_{r+1}\phi\right) \\
&= \tilde{A}_{r+1}\phi\phi^- + \tilde{B}_{r+1}\phi^2\phi^- - \tilde{C}_{r+1}\phi^- - \tilde{D}_{r+1}\phi\phi^- + (h\lambda-1)\alpha\tilde{A}^-_{r+1} \\
&+ (h\lambda-1)\alpha\tilde{B}^-_{r+1}\phi^- - \tilde{C}^-_{r+1}\phi + (\lambda+\beta)\tilde{C}^-_{r+1} - \tilde{D}^-_{r+1}(h\lambda-1)\alpha \\
&= \tilde{A}_{r+1}\left((h\lambda-1)\alpha + (\lambda+\beta)\phi^-\right) + \tilde{B}_{r+1}\phi\left((h\lambda-1)\alpha + (\lambda+\beta)\phi^-\right) \\
&- \tilde{C}_{r+1}\phi^- - \tilde{D}_{r+1}\left((h\lambda-1)\alpha + (\lambda+\beta)\phi^-\right) + (h\lambda-1)\alpha\tilde{A}^-_{r+1} + (h\lambda-1)\alpha\tilde{B}^-_{r+1}\phi^- \\
&- C^-_{r+1}\phi + (\lambda+\beta)C^-_{r+1} - \tilde{D}^-_{r+1}(h\lambda-1)\alpha.
\end{aligned}
\tag{5.31}
$$

From (5.30) and (5.31), one finds

$$
\begin{aligned}
&\left(\phi^- + \left(\phi - (\lambda+\beta)S^-\right)\right)\left(-\phi_{t_r} + \tilde{A}_{r+1}\phi + \tilde{B}_{r+1}\phi^2 - \tilde{C}_{r+1} - \tilde{D}_{r+1}\phi\right) \\
&= -(\lambda+\beta)\tilde{D}^-_{r+1}\phi^- + \tilde{A}_{r+1}(\lambda+\beta)\phi^- + (\lambda+\beta)^2\tilde{B}_{r+1}\phi^- \\
&= 0.
\end{aligned}
\tag{5.32}
$$

In the last equality in (5.32), we used

$$\tilde{D}^-_{r+1} = \tilde{A}_{r+1} + (\lambda+\beta)\tilde{B}_{r+1}. \tag{5.33}$$

Eq (5.32) is

$$
-\phi_{t_r} + \tilde{A}_{r+1}\phi + \tilde{B}_{r+1}\phi^2 - \tilde{C}_{r+1} - \tilde{D}_{r+1}\phi = C(P,t_r)\times
\begin{cases}
\prod_{n'=1}^{n} B(P,n',t_r) & n\in\mathbb{N} \\
1 & n=0 \\
\prod_{n'=0}^{n+1} B(P,n',t_r)^{-1} & -n\in\mathbb{N},
\end{cases}
\tag{5.34}
$$

where

$$B(P,n',t_r) = \frac{\phi(P,n',t_r) - (\lambda + \beta(n',t_r))}{\phi^-(P,n',t_r)} \qquad (n',t_r)\in\mathbb{Z}\times\mathbb{R} \tag{5.35}$$

and $C(n,t_r)$ is some n-independent meromorphic function on $\mathcal{K}_p$. The asymptotic behavior of $\phi(P,n,t_r)$ in (5.53) then yields

$$B(P) \underset{}{\overset{P\to P_{\infty-}}{=\!=\!=\!=}} \frac{-1}{h\alpha}\zeta^{-1} + O(1). \tag{5.36}$$

Comparing the order of both sides of (5.34) and taking $n>0$ sufficiently large, one finds contradiction unless $C=0$. This proves (5.23). To prove (5.28), we rewrite (5.23) as

$$\begin{aligned}
\phi_{t_r} &= \left(-\tilde{A}_{r+1} + \tilde{A}^+_{r+1} + \left(\lambda+\beta^+\right)\tilde{B}^+_{r+1} - \tilde{B}_{r+1}\phi + \frac{\left(h\lambda-1\right)\alpha^+\tilde{B}^+_{r+1}}{\phi}\right)\phi \\
&= \left(-\tilde{A}_{r+1} + \tilde{A}^+_{r+1} + \left(\lambda+\beta^+\right)\tilde{B}^+_{r+1} - \tilde{B}_{r+1}\phi + \tilde{B}^+_{r+1}\phi^+ - \left(\lambda+\beta^+\right)\tilde{B}^+_{r+1}\right)\phi \\
&= \left(\tilde{A}^+_{r+1} + \tilde{B}^+_{r+1}\phi^+ - \tilde{A}_{r+1} - \tilde{B}_{r+1}\phi\right)\phi
\end{aligned} \tag{5.37}$$

Let

$$\Delta = \int_{t_{0,t}}^{t_r} \left(\tilde{A}_{r+1}(\lambda,n_0,s) + \tilde{B}_{r+1}(\lambda,n_0,s)\phi(P,n_0,s)\right) ds \tag{5.38}$$

for the case $n^{'} > n_0$

$$\begin{aligned}
\Psi_{1,t_r} &= \left(\exp(\Delta)\prod_{n^{'}=n_0}^{n-1}\phi(n^{'})\right)_{t_r} \\
&= \Delta_{t_r}\Psi_1 + \exp(\Delta)\sum_{n^{'}=n_0}^{n-1}\phi_{t_r}(n^{'})\prod_{n^{''}\neq n^{'}}\phi(n^{''}) \\
&= \left(\tilde{A}_{r+1}(\lambda,n_0,t_r) + \tilde{B}_{r+1}(\lambda,n_0,t_r)\phi(P,n_0,t_r)\right)\Psi_1 \\
&+ \exp(\Delta)\sum_{n^{'}=n_0}^{n-1}\left(\tilde{B}^+_{r+1}(n^{'})\phi^+(n^{'}) + \tilde{A}^+_{r+1}(n^{'}) - \tilde{B}_{r+1}(n^{'})\phi(n^{'}) - \tilde{A}_{r+1}(n^{'})\right)\phi(n^{'})\prod_{n^{''}\neq n^{'}}\phi(n^{''}) \\
&= (\tilde{A}_{r+1} + \tilde{B}_{r+1}\phi)\Psi_1
\end{aligned} \tag{5.39}$$

The case $n^{'<}n_0$ is similar to $n^{'} > n_0$.

$$\begin{aligned}\Psi_{2,t_r} &= \phi_{t_r}\Psi_1 + \phi\Psi_{1,t_r} \\ &= \left(\left(-\tilde{A}_{r+1} + \tilde{D}_r\right)\phi - \tilde{B}_{r+1}\phi^2 + (h\lambda - 1)\,\alpha^+\tilde{B}^+_{r+1}\right)\Psi_1 + \phi\left(\tilde{A}_{r+1} + \tilde{B}_{r+1}\phi\right)\Psi_1 \\ &= \left((h\lambda - 1)\,\alpha^+\tilde{B}^+_{r+1}\right)\Psi_1 + \tilde{D}_{r+1}\Psi_2\end{aligned} \tag{5.40}$$

Then (5.28) holds. Here we used ().

The fact that $\Psi_1(P, n, n_0, t_0, t_r)$ is meromorphic on $\mathcal{K}_p\backslash\{P_{\infty\pm}\}$ if $B_{p+1}(\cdot, n_0, t_r)$ has only simple zeros distinct from P_0 is a consequence of (5.13) (5.14) (5.16),(5.41) and of

$$\tilde{B}_{r+1}\phi \xlongequal{P\to\hat{\mu}_j(n_0,s)} \partial_s \ln\left(B_{p+1}(\lambda, n_0, s)\right) + O(1).$$

The proof of other equalities in this lemma is similar with lemma 1, which t_r is regarded as a parameter. □

Lemma 7. *Assume* α, β *satisfy (5.6)-(5.10). In addition, let* $(\lambda, n, t_r) \in \mathbb{C}\times\mathbb{Z}\times\mathbb{R}$.*Then,*

$$B_{p+1,t_r} = \tilde{A}_{r+1}B_{p+1} - \tilde{D}_{r+1}B_{p+1} - 2A_{p+1}\tilde{B}_{r+1}, \tag{5.41}$$

$$A_{p+1,t_r} = -(h\lambda - 1)\alpha^+\left(\tilde{B}_{r+1}B^+_{p+1} - \tilde{B}^+_{r+1}B_{p+1}\right). \tag{5.42}$$

Proof. From (5.24),

$$\phi(P) - \phi(P^*) = \frac{iy}{B_{p+1}} \tag{5.43}$$

then we have

$$\phi_{t_r}(P) - \phi_{t_r}(P^*) = -\frac{iy}{B_p^2}B_{p+1,t_r} \tag{5.44}$$

Plunging (5.23) into (5.44), one derive (5.41).

Similarly,

$$\phi(P) + \phi(P^*) = -2\frac{A_{p+1}}{B_{p+1}}$$

we have

$$\phi_{t_r}(P) + \phi_{t_r}(P^*) = -2\left(\frac{A_{p+1,t_r}}{B_{p+1}} - \frac{A_{p+1}B_{p+1,t_r}}{B^2_{p+1}}\right)$$

noted (5.41), (5.42) holds. □

Next we turn to the Dubrovin equation for the time variation of the zeros μ_j of B_{p+1} governed by the (5.8)-(5.10).

Lemma 8 (Dubrovin equation). *Assume (5.2) (5.3) hold on $\mathbb{Z} \times \Omega$ with $\Omega \subseteq \mathbb{R}$ is an open interval. In addition, assume that the zeros $\mu_j, \quad j = 1, \cdots, p$ of B_p remain distinct on $\mathbb{Z} \times \Omega$. Then $\{\hat{\mu}_j\}_{j=1,\cdots,p}$ defined in (5.12) satisfy the following first-order system of differential system on $\mathbb{Z} \times \Omega$,*

$$\mu_{j,t_r} = -\tilde{B}_{r+1}(\mu_j) y(\hat{\mu}_j) \prod_{k=1,k\neq j}^{p} (\mu_j - \mu_k)^{-1} \tag{5.45}$$

with

$$\hat{\mu}_j(n,\cdot) \in C^{\infty}(\Omega, \mathcal{K}_p), \qquad j = 1, \cdots, p, n \in \mathbb{Z}.$$

Proof. Using the product representation for B_{p+1}

$$B_{p+1} = \prod_{j=1}^{p} (\lambda - \mu_j) \tag{5.46}$$

and employing the (5.12) and (5.41), one computes

$$B_{p+1,t_r}(\mu_j) = -2A_{p+1}(\mu_j)\tilde{B}_{r+1}(\mu_j) = y(\hat{\mu}_j)\tilde{B}_{r+1}(\mu_j) = -\mu_{j,t_r} \prod_{k\neq j, j=1}^{p} (\mu_j - \mu_k)\,. \tag{5.47}$$

Then one finds (5.45) holds.

Remark 11. (i) *When attempting to solve the Dubrovin systems (5.45), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in Sym^p(\mathcal{K}_p) \quad t_{0,r} \in \Omega$ as initial conditions.*
(ii) *The differential-difference hierarchy differs from the soliton hierarchies for the continuous models such as KdV, AKNS, or Cassama-Holm hierarchy, etc. It does not seem to have simple Dubrovin equations that govern the n-dependence of $\underline{\hat{\mu}}(n, t_r)$. The Dubrovin equations for continuous interpolations of Dirichlet-type eigenvalues appeared in [40].*

□

As in the stationary case, we have

Lemma 9 (trace formula). *Suppose $\alpha, \beta \in l(\mathbb{Z})$ satisfy (5.2) (5.3). Then*

$$-h(\alpha+\alpha^{+})-\beta+\delta_1=-\sum_{j=1}^{p}\mu_j \tag{5.48}$$

Proof. Comparing the powers of λ^{p-1} in (5.5) and (5.46) for B_{p+1}, that is,

$$b_0\lambda^{p+1}+b_1\lambda^{p}+\cdots+b_{p+1}=\prod_{j=1}^{p}(\lambda-\mu_j) \tag{5.49}$$

. Hence equation (5.48) holds.

□

Lemma 10. *Suppose that α, β satisfy the pth time-dependent RT hierarchy. Moreover, let $P=(\lambda, y)\in\mathcal{K}_p\backslash\{\infty\pm, P_0\}$, $(n, n_0, t_r)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}$. Then,*

$$\phi(P)=\begin{cases}\zeta^{-1}+(\beta-h\alpha)+O(\zeta) & P\to P_{\infty+}\quad \lambda=\zeta^{-1}\\ h\alpha^{+}+(h^2\alpha^{+}\alpha+\alpha-h\alpha\beta)\zeta+O(\zeta^2) & P\to P_{\infty-}\quad \lambda=\zeta^{-1}\\ \frac{h\alpha^{+}}{h+\beta^{+}}\zeta+O(\zeta^2) & P\to P_0\quad \lambda=\zeta^{2}.\end{cases} \tag{5.50}$$

The component Ψ_1 of the Baker-Akhiezer vector Ψ has the asymptotic behavior

$$\Psi_1(P,n,n_0,t_r,t_{0,r}) \underset{}{\overset{\zeta\to 0}{=\!=}} \exp\left(\mp\frac{1}{2}(t_r-t_{0,r})\sum_{s=0}^{r+1}\tilde{\delta}_{r+1-s}\zeta^{-(s+1)}(1+O(\zeta))\right) \tag{5.51}$$

$$\times\begin{cases}\zeta^{n_0-n}\,(1+O(\zeta))\exp\left(\int_{t_{0,r}}^{t_r}\left(\sum_{s=0}^{r+1}\bar{b}_{s+2}(n_0,s)\right)ds\right) & P\to P_{\infty+}\\ \Lambda(h\alpha^{+})\,(1+O(\zeta))\exp\left(\int_{t_{0,r}}^{t_r}\left(\sum_{s=0}^{r+1}\bar{b}_{s+2}(n_0,s)\right)ds\right) & P\to P_{\infty-},\end{cases} \tag{5.52}$$

$$\Psi_1(P,n,n_0,t_r,t_{0,r}) \overset{\zeta\to 0}{=\!=} \Lambda(\frac{h\alpha^{+}}{h+\beta^{+}})\zeta^{n-n_0}(1+O(\zeta))\exp\left(\int_{t_0}^{t_r}\tilde{A}_{r+1}(n_0,s)ds\right) \tag{5.53}$$

Proof. The proof of (5.53) is analog with lemma 3. One observe that

$$\tilde{A}_{r+1}=\sum_{s=0}^{r+1}\tilde{\delta}_{r+1-s}\bar{A}_s, \tag{5.54}$$

$$\tilde{B}_{r+1}=\sum_{s=0}^{r+1}\tilde{\delta}_{r+1-s}\bar{B}_s, \tag{5.55}$$

where

$$\bar{A}_r = \tilde{A}_r|_{\tilde{\delta}_0=1,\tilde{\delta}_j=0,j=1,\cdots,r}, \quad \bar{B}_r = \tilde{B}_r|_{\tilde{\delta}_0=1,\tilde{\delta}_j=0,j=1,\cdots,r}$$

and

$$\bar{A}_0 = a_0 + b_1 = \frac{1}{2}, \qquad \bar{B}_0 = 0.$$

Then we have

$$\begin{aligned}
&\bar{A}_{s+1} + \bar{B}_{s+1}\phi \\
&= \bar{A}_{s+1} + \bar{B}_{s+1}\frac{\frac{1}{2}iy - A_{p+1}}{B_{p+1}} \\
&= \bar{A}_{s+1} + \bar{B}_{s+1}\frac{\frac{1}{2} - \frac{A_{p+1}}{iy}}{\frac{B_{p+1}}{iy}} \\
&= \left(\bar{a}_0\lambda^{s+1} + \cdots + \bar{a}_s\lambda + \bar{a}_{s+1} + \bar{b}_{s+2}\right) + \left(\bar{b}_0\lambda^{s+1} + \cdots + \bar{b}_{s+1}\right)\frac{\frac{1}{2} - \frac{A_{p+1}}{iy}}{\frac{B_{p+1}}{iy}} \\
&= \mp\frac{1}{2}\zeta^{-(s+1)} - \bar{b}_{s+2} + O(\zeta) \\
&= \mp\frac{1}{2}\zeta^{-(s+1)} + O(1). \qquad P \to P_{\infty\pm}.
\end{aligned}$$

we infer that as $P \to P_{\infty\pm}$

$$\tilde{A}_{r+1} + \tilde{B}_{r+1}\phi = \frac{1}{2}\sum_{s=0}^{r+1}\tilde{\delta}_{r+1-s}\zeta^{-s} + \sum_{s=0}^{r+1}\bar{b}_{s+2}$$

and as $P \to P_0$

$$\hat{A}_{r+1}(n_0,s) + \hat{B}_{r+1}(n_0,s)\phi(P,n_0,s) \overset{\zeta\to 0}{=\!=} \tilde{A}_{r+1}(n_0,s) + O(\zeta) \quad \lambda = \zeta - h^{-1} \tag{5.56}$$

From (5.53), we have (5.51)-(5.53). □

Lemma 11. *Suppose* α, β *satisfy the (5.6)-(5.10), Moreover, let* $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$*. Denote by* $\mathcal{D}_{\underline{\hat{\mu}}}$, $\quad \underline{\hat{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_p) \in Sym(\mathcal{K}_p)$*, then the Dirichlet divisor of degree* p *associated with* α, β *and* ϕ *defined according to (5.14), that is,*

$$\hat{\mu}(n,t_r) = (\mu_j(n,t_r), 2iA_{p+1}(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_p, \quad j = 1,\cdots,p.$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ *is non-special for all* $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$.

Proof. $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is non-special if and only if $\{\hat{\mu}_1(n),\cdots,\hat{\mu}_p(n)\}$ contains one pair of $\{\hat{\mu}_j,\hat{\mu}^*(n)\}$. Hence, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is non-special as long as the projection μ_j of $\hat{\mu}_j$ are mutually distinct, $\mu_j(n)\neq\mu_k(n)$ for $j\neq k$. If two or more projection coincide for some $n_0\in\mathbb{Z}$, for instance,

$$\mu_{j_1}(n_0)=\cdots=\mu_{j_k}(n_0)=\mu_0,\quad k>1.$$

There are two cases in the following associated with μ_0. If $\mu_0\in\{E_0,E_1,\cdots,E_{2p+1}\}$, then $A_{p+1}(\mu_0,n_0,t_r)\neq 0$. $\hat{\mu}_{j_1}(n_0,t_r),\cdots,\hat{\mu}_{j_k}(n_0,t_r)$ all meet in the same sheet and hence no special divisor can arise in this manner. If μ_0 equals to some E_{m_0} and $k>1$, one concludes $B_{p+1}(\lambda,n_0,t_r)\overset{\lambda\to E_{m_0}}{=\!=}O\left((\lambda-E_{m_0})^2\right)$ and $A_{p+1}(E_{m_0},n_0,t_r)=0$. But one observes $R_{2p+2}(\lambda,n_0,t_r)=-A_{p+1}^2-(h\lambda-1)B_{p+1}\alpha^+B_{p+1}^+=O\left((\lambda-E_{m_0})^2\right)$. This conclusion contradict with the hypothesis that the curve is nonsingular. We have $k=1$. Therefore no special divisor can arise in this manner. Then we have completed the proof. □

Let $\omega^{(2)}_{P_{\infty\pm},q}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty\pm}$ and principal parts

$$\omega^{(2)}_{P_{\infty\pm},q}\overset{\zeta\to 0}{=\!=}\left(\zeta^{-2-q}+O(1)\right)d\zeta,\quad P\to P_{\infty\pm},\quad \zeta=\lambda^{-1},q\in\mathbb{N}_0 \tag{5.57}$$

with vanishing a-periods,

$$\int_{a_j}\omega^{(2)}_{P_{\infty\pm},q}=0,\quad j=1,\cdots,p. \tag{5.58}$$

Moreover, we define

$$\tilde{\Omega}^{(2)}_r=\frac{1}{2}\left(\sum_{s=1}^{r+1}s\tilde{\delta}_{r+1-s}\left(\omega^{(2)}_{P_{\infty+},s-1}-\omega^{(2)}_{P_{\infty-},s-1}\right)\right). \tag{5.59}$$

The corresponding vector of b-periods of $\tilde{\Omega}^{(2)}_r/(2\pi i)$ is then denoted by

$$\underline{\tilde{U}}^{(2)}_r=\left(\tilde{U}^{(2)}_{r,1},\tilde{U}^{(2)}_{r,2},\cdots,\tilde{U}^{(2)}_{r,p}\right),\quad \tilde{U}^{(2)}_{r,j}=\frac{1}{2\pi i}\int_{bj}\tilde{\Omega}^{(2)}_r,\quad j=1,2,\cdots,p.$$

Theorem 12. *let $P\in\mathcal{K}_p\backslash\{P_{\infty\pm},P_0\}$ and $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$.Then for each $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, $\mathcal{D}_{\underline{\hat{\mu}}}(n,t_r)$ is non-special. Moreover,*

$$\phi(P,n,t_r)=C(n,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}}\right), \tag{5.60}$$

$$\Psi_1(P,n,n_0,t_r,t_{0,r}) = C(n,n_0,t_r,t_{0,r})\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,r})))}$$
$$\times \exp\left((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}} - (t_r - t_{0,r})\int_{Q_0}^{P}\tilde{\Omega}^{(2)}_r\right), \tag{5.61}$$

where

$$C(n,t_r) = \tilde{a}^{-1}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n,t_r)))} \tag{5.62}$$

and

$$C(n,n_0,t_r,t_{0,r}) = \begin{cases} \prod_{n'=n_0}^{n-1} C(n') & n > n_0 \\ 1 & n = n_0 \\ \prod_{n'=n}^{n_0-1} C(n')^{-1} & n < n_0. \end{cases} \tag{5.63}$$

The Abel map linearizes the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}) - \underline{A}_{P_{\infty+}}(P_0)(n-n_0) - \underline{\tilde{U}}^{(2)}_r(t_r - t_{0,r}). \tag{5.64}$$

Moreover, α, β are the form of

$$\alpha^+ = \frac{1}{h}(\tilde{a})^{-2}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n,t_r)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(n,t_r)))}, \tag{5.65}$$

$$\beta^+ = (\tilde{a})^{-2}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n,t_r)))}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}^(n,t_r)))}$$
$$+\frac{h^{-1}}{2} + \frac{1}{4}\sum_{m=0}^{2p+1}E_m - \frac{1}{2}\sum_{j=1}^{p}\lambda_j - \sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}^+(n,t_r))+\underline{\omega})}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(n,t_r))+\underline{\omega})}\right)|_{\underline{\omega}=0}. \tag{5.66}$$

Proof. The proof of (5.60) is analog with Theorem 6. To prove (5.61), one observes

$$\Psi_1(P,n,n_0,t_r,t_{0,r}) = \Psi_1(P,n_0,n_0,t_r,t_r)\Psi_1(P,n,n_0,t_r,t_r). \tag{5.67}$$

$\Psi_1(n,n_0,t_r,t_r)$ is of the form

$$\Psi_1(P,n,n_0,t_r,t_r) = C(n,n_0,t_r,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_r)))}$$
$$\times \exp\left((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_0P_{\infty+}}\right). \tag{5.68}$$

In the following, we study

$$\Psi_1(P, n_0, n_0, t_r, t_{0,r}) = \exp\left(\int_{t_{0,r}}^{t_r} \tilde{A}_{r+1}(\lambda, n_0, s) + \tilde{B}_{r+1}(\lambda, n_0, s)\phi(P, n_0, s)\right). \tag{5.69}$$

Introducing $\hat{\Psi}_1(P)$ on $\mathcal{K}_p\backslash\{P_{\infty\pm}\}$ by

$$\begin{aligned}\hat{\Psi}_1(P, n_0, t_r, t_{0,r}) = C(n_0, n_0, t_r, t_{0,r})\frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0, t_r)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0, t_{0,r})))} \\ \times \exp\left(-(t_r - t_{0,r})\int_{Q_0}^{P} \tilde{\Omega}_r^{(2)}\right),\end{aligned} \tag{5.70}$$

We intend to prove that

$$\begin{aligned}\Psi_1(P, n_0, n_0, t_r, t_{0,r}) = \hat{\Psi}_1(P, n_0, t_r, t_{0,r}), \\ p \in \mathcal{K}_p\backslash\{P_{\infty\pm}\}, n_0 \in \mathbb{Z}, t_r, t_{0,r} \in \mathbb{R}\end{aligned} \tag{5.71}$$

for an appropriate choice of the normalization constant $C(n, n_0, t_r, t_{0,r})$. (5.51)-(5.53) (5.70) shows that Ψ_1 and $\hat{\Psi}_1$ have the same essential singularities at $P_{\infty\pm}$. P_0 is also not zero point or singular point of Ψ_1 and $\hat{\Psi}_1$. (5.70) has zeros and poles at $\underline{\hat{\mu}}(n_0, t_r)$ and $\underline{\hat{\mu}}(n_0, t_{0,r})$. We remain to study the point $P \in \mathcal{K}_p\backslash\{P_{\infty\pm}, P_0\}$.

$$\begin{aligned}&\Psi_1(P, n_0, n_0, t_r, t_{0,r}) \\ &= \exp\left(\int_{t_0}^{t_r} ds\left(\tilde{A}_{r+1}(\lambda, n_0, s) + \tilde{B}_{r+1}(\lambda, n_0, s)\frac{\frac{1}{2}iy - A_{p+1}(\lambda, n_0, s)}{B_{p+1}(\lambda, n_0, s)}\right)\right) \\ &= \exp\left(\int_{t_0}^{t_{0,r}} ds\left(\frac{\tilde{B}_{r+1}\left(\mu_j(n_0, s)\right)y(\hat{\mu}_j(n_0, s))}{(\lambda - \mu_j(n_0, s))\prod_{k=1,k\neq j}^{p}\left(\mu_j(n_0, s) - \mu_k(n_0, s)\right)} + O\left(1\right)\right)\right) \\ &= \exp\left(\int_{t_0}^{t_r} ds\left(\frac{-\mu_{j,t_s}(n_0, s)}{\lambda - \mu_j(n_0, s)} + O\left(1\right)\right)\right) \\ &= \exp\left(\int_{t_0}^{t_r} ds\left(\frac{\partial}{\partial s}\ln\left(\mu_j(n_0, s) - \lambda\right) + O\left(1\right)\right)\right).\end{aligned} \tag{5.72}$$

Therefore

$$\Psi_1(P, n_0, n_0, t_r, t_{0,r}) = \begin{cases}(\mu_j(n_0, t_r) - \lambda)O\left(1\right) & \text{as} P \to \hat{\mu}_j(n_0, t_r) \neq \hat{\mu}_j(n_0, t_{0,r}) \\ O\left(1\right) & \text{as} \quad P \to \hat{\mu}_j(n_0, t_r) = \hat{\mu}_j(n_0, t_{0,r}) \\ \left(\mu_j(n_0, t_{0,r}) - \lambda\right)^{-1}O\left(1\right) & \text{as} \quad P \to \hat{\mu}_j(n_0, t_{0,r}) \neq \hat{\mu}_j(n_0, t_r).\end{cases}$$

Hence Ψ_1 and $\hat{\Psi}_1$ has the same local behavior and identical essential singularities at $P_{\infty\pm}$ and P_0. So $\frac{\Psi_1}{\hat{\Psi}_1}$ is a multiple constant which may depend on $n_0, t_r, t_{0,r}$ (Riemann-Roch type uniqueness theorem for the Baker-Akhiezer functions [37]). Then we have (5.61) for $t_{0,r}, t_r \in I$ and for n_0. By continuity with respect to divisors this extends to all $n_0 \in \mathbb{Z}$ since $\mathcal{D}_{\underline{\hat{\mu}}(n,s)}$ remain non-special for all $(n,s) \in \mathbb{Z} \times \mathbb{R}$. We calculate

$$\begin{aligned}
&\frac{\partial}{\partial_{t_r}} \underline{\alpha}_{Q_0,l}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \frac{\partial}{\partial_{t_r}} \sum_{j=1}^{p} \int_{Q_0}^{\hat{\mu}_j(n,t_r)} \omega_l \\
&= \sum_{j=1}^{p} \omega_l(\hat{\mu}_j)\mu_{j,t_r} \\
&= \sum_{j=1}^{p} \left(\sum_{k=1}^{p} c_l(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j(n,t_r))} \right) \left(-\tilde{B}_{r+1}(\mu_j(n,t_r)) y(\hat{\mu}_j(n,t_r)) \prod_{k=1,k\neq j}^{p} (\mu_j - \mu_k)^{-1} \right) \\
&= \sum_{j=1}^{p} \left(\sum_{k=1}^{p} c_l(k) \frac{\mu_j^{k-1}}{\prod_{k=1,k\neq j}^{p} (\mu_j - \mu_k)} \right) \left(-\tilde{B}_{r+1}(\mu_j(n,t_r)) \right) \\
&= -\sum_{k=1}^{p} c_l(k) \sum_{j=1}^{p} \frac{\mu_j^{k-1}}{\prod_{k=1,k\neq j}^{p} (\mu_j - \mu_k)} \left(\sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s} \left(\sum_{t=\max\{0,s-p\}}^{s} \hat{c}_t(\underline{E}) \Psi_{s-t}^{(j)}(\underline{\mu}) \right) \right) \\
&= -\sum_{k=1}^{p} \sum_{s=0}^{r} c_l(k) \tilde{\delta}_{r-s} \hat{c}_{k+s-p}(\underline{E}),
\end{aligned} \tag{5.73}$$

where we used the result of section 6 in the 5th equality. In other way, one finds as $P \to P_{\infty\pm}$ ($\lambda = \zeta^{-1}$)

$$\omega_j = \pm \sum_{j=1}^{p} c_j(k) \frac{\zeta^{p-j}}{\left(\prod_{m=0}^{2p+1} (1 - E_m \zeta) \right)^{\frac{1}{2}}} d\zeta = \pm \left(\sum_{q=0}^{\infty} \sum_{k=1}^{p} c_j(k) \hat{c}_{k-p+q}(\underline{E}) \zeta^q \right) d\zeta \tag{5.74}$$

then

$$\begin{aligned}
\tilde{U}_{r,j}^{(2)} &= \frac{1}{2\pi i} \int_{bj} \tilde{\Omega}_r^{(2)} = \frac{1}{2\pi i} \sum_{s=1}^{r+1} \frac{1}{2} s \tilde{\delta}_{r+1-s} \left(\int_{b_j} \omega_{P_{\infty+},s-1}^{(2)} - \int_{b_j} \omega_{P_{\infty-},s-1}^{(2)} \right) \\
&= \sum_{s=1}^{r+1} \tilde{\delta}_{r+1-s} \sum_{k=1}^{p} c_l(k) \hat{c}_{k-p+s}(\underline{E}).
\end{aligned} \tag{5.75}$$

Combining (5.73) with (5.75), we have

$$\frac{\partial}{\partial_{t_r}} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = -\underline{\tilde{U}}_r^{(2)}, \tag{5.76}$$

which proves (5.64). $\square$

Remark 13. *Assume* $p=0, r\in\mathbb{N}_0$, $P=(\lambda,y)\in\mathcal{K}_0\backslash\{P_{\infty\pm}\}$ *and* $(n,n_0)\in\mathbb{Z}^2$. *Then*

$$V_0^+(\lambda,n,t_r)=\begin{pmatrix} -\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2} & 1\\ (h\lambda-1)\alpha^+ & \frac{1}{2}\lambda-h\alpha^++\frac{\delta_1}{2}\end{pmatrix} \tag{5.77}$$

$$\mathcal{K}_0:\quad \mathcal{F}_0(\lambda,y)=y^2-(\lambda-E_0)(\lambda-E_1)=0 \qquad E_0,E_1\in\mathbb{C} \tag{5.78}$$

α,β *satisfy*

$$\begin{cases} h\alpha^++h\alpha+\beta=-\frac{E_0+E_1}{2}\\ -4\left(\left(h\alpha^++\frac{E_0+E_1}{4}\right)\left(-h\alpha^++\frac{\delta_1}{2}\right)+\alpha^+\right)=E_0E_1\end{cases} \tag{5.79}$$

$$\alpha=-\frac{E_0+E_1}{2h}\pm\frac{1}{2}\sqrt{\left(\frac{E_0+E_1}{2h}\right)^2+\left(\frac{E_0-E_1}{2}\right)^2},\qquad \beta=\frac{E_0+E_1}{2}. \tag{5.80}$$

$$B_1=1,\quad C_1=(h\lambda-1)\alpha^+,\quad A_1=-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2},\quad D_1=\frac{1}{2}\lambda-h\alpha^++\frac{\delta_1}{2}. \tag{5.81}$$

$$\phi(P,n_0)=\frac{1}{2}iy-\left(-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2}\right), \tag{5.82}$$

$$\Psi_1(P,n,n_0)=\left(\frac{1}{2}iy-\left(-\frac{1}{2}\lambda-h\alpha-\beta+\frac{\delta_1}{2}\right)\right)^{n-n_0}\exp\left(\int_{t_0}^{t_r}\tilde{A}_{r+1}+B_{r+1}\frac{\frac{1}{2}iy-A_1}{B_1}\right). \tag{5.83}$$

6 Appendix: The Lagrange Interpolation Representation of $\tilde{B}_{r+1}(\mu_j(n,t_r))$

We search for the interpolation representation of $\tilde{B}_r(\mu_j(n,t_r))$ as in the KdV, AKNS, Toda cases. Introducing the notation in [38][39],

$$\Psi_k(\underline{\mu})=(-1)^k\sum_{\underline{l}\in\mathcal{S}_k}\mu_{l_1}\cdots,\mu_{l_k}\quad \mathcal{S}_k=\{\underline{l}=(l_1,\cdots,l_k)\in\mathbb{N}^k|l_1<\cdots<l_k\le p\}\quad k=1,\cdots p, \tag{6.1}$$

$$\Phi_k^{(j)}(\underline{\mu})=(-1)^k\sum_{\underline{l}\in\tau_k^{(j)}}\mu_{l_1}\cdots,\mu_{l_k}\quad \tau_k^{(j)}=\{\underline{l}=(l_1,\cdots,l_k)\in\mathbb{N}^k|l_1<\cdots<l_k\le p\quad l_m\neq j\}$$

$$k=1,\cdots p-1,\quad j=1,\cdots,p. \tag{6.2}$$

and the formula

$$\sum_{l=0}^{k} \Psi_{k-l}(\underline{\mu})\mu_j^l = \Phi_k^{(j)}(\underline{\mu}), \quad k = 0,\cdots,n, \quad j = 1,\cdots,n, \tag{6.3}$$

one finds

$$B_{p+1}(\lambda) = \sum_{l=0}^{p+1} b_{p+1-s}\lambda^s = \prod_{j=1}^{p} (\lambda - \mu_j) = \sum_{l=0}^{p} \Psi_{p-l}(\underline{\mu})\lambda^l$$

and

$$b_l = \Psi_{l-1}\left(\underline{\mu}\right), \quad l = 0,\cdots,p+1. \quad \big(\text{we label} \quad \Psi_{-1}\left(\underline{\mu}\right) = 0\big)$$

In the case $r < p$,

$$\begin{aligned}\bar{B}_{r+1} &= \sum_{s=0}^{r+1} \bar{b}_{r+1-s}\lambda^s = \sum_{s=0}^{r+1} \left(\sum_{k=0}^{min\{r+1-s,p+1\}} \hat{c}_{r+1-s-k}(\underline{E})b_k \right) \lambda^s \\ &= \sum_{s=0}^{r+1} \left(\sum_{k=0}^{r+1-s} \hat{c}_{r+1-s-k}(\underline{E})b_k \right) \lambda^s \\ &= \sum_{s=0}^{r+1} \left(\sum_{k=0}^{r+1-s} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu}) \right) \lambda^s \\ &= \sum_{s=0}^{r+1} \hat{c}_s(\underline{E}) \sum_{t=0}^{r+1-s} \Psi_{r-s-t}(\underline{\mu})\lambda^t \\ &= \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{t=0}^{r-s} \Psi_{r-s-t}(\underline{\mu})\lambda^t.\end{aligned} \tag{6.4}$$

Using (6.3), we have

$$\begin{aligned}\bar{B}_{r+1}(\mu_j) &= \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{t=0}^{r-s} \Psi_{r-s-t}(\underline{\mu})\mu_j^t \\ &= \sum_{s=0}^{r} \hat{c}_s(\underline{E})\Psi_{r-s}^{(j)}(\underline{\mu}).\end{aligned} \tag{6.5}$$

In the case $r > p$,

$$
\begin{aligned}
\bar{B}_{r+1}(\lambda) &= \sum_{s=0}^{r+1} \bar{b}_{r+1-s}\lambda^s = \sum_{s=0}^{r+1} \left(\sum_{k=0}^{\min\{r+1-s,p+1\}} \hat{c}_{r+1-s-k}(\underline{E}) b_k \right) \lambda^s \\
&= \sum_{s=0}^{r-p}\sum_{k=0}^{p+1} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s + \sum_{s=r-p+1}^{r+1}\sum_{k=0}^{r+1-s} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s \\
&= \sum_{s=0}^{r-p}\sum_{k=0}^{p+1} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s + \sum_{s=r-p+1}^{r+1}\sum_{k=0}^{p+1} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s \\
&= \sum_{k=0}^{p+1}\sum_{s=0}^{r+1} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s \\
&= \sum_{s=0}^{r+1}\sum_{k=0}^{p+1} \hat{c}_{r+1-s-k}(\underline{E})\Psi_{k-1}(\underline{\mu})\lambda^s \\
&= \sum_{s=0}^{r+1}\sum_{k=0}^{p+1} \hat{c}_{s}(\underline{E})\Psi_{k-1}\lambda^{r+1-s-k} \\
&= \sum_{s=0}^{r-p} \hat{c}_s(\underline{E}) \left(\sum_{k=0}^{p+1} \Psi_{k-1}(\underline{\mu})\lambda^{p+1-k} \right) \lambda^{r-p-s} + \sum_{s=r-p+1}^{r+1} \hat{c}_s(\underline{E}) \left(\sum_{k=0}^{p+1} \Psi_{k-1}(\underline{\mu})\lambda^{r+1-s-k} \right) \\
&= \sum_{s=0}^{r-p} \hat{c}_s(\underline{E}) \left(B_p(\lambda)\right) \lambda^{r-p-s} + \sum_{s=r-p+1}^{r+1} \hat{c}_s(\underline{E}) \left(\sum_{k=0}^{r+1-s} \Psi_{k-1}(\underline{\mu})\lambda^{r+1-s-k} \right).
\end{aligned}
\tag{6.6}
$$

Then we have

$$
\bar{B}_{r+1}(\mu_j) = \sum_{s=r-p+1}^{r+1} \hat{c}_s(\underline{E}) \left(\sum_{k=0}^{r+1-s} \Psi_{k-1}(\underline{\mu})\mu_j^{r+1-s-k} \right) = \sum_{s=r-p+1}^{r+1} \hat{c}_s(\underline{E})\Psi^{(j)}_{r-s}(\underline{\mu}). \tag{6.7}
$$

Combining (6.5) with (6.7), one finds

$$
\bar{B}_{r+1}(\lambda) = \sum_{s=\max\{0,r-p+1\}}^{r+1} \hat{c}_s(\underline{E})\Psi^{(j)}_{r-s}(\underline{\mu}) = \sum_{s=\max\{0,r-p\}}^{r} \hat{c}_s(\underline{E})\Psi^{(j)}_{r-s}(\underline{\mu}). \tag{6.8}
$$

Hence

$$
\tilde{B}_{r+1}(\mu_j) = \sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s}\bar{B}_s(\mu_j) = \sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s} \left(\sum_{t=\max\{0,s-p\}}^{s} \hat{c}_t(\underline{E})\Psi^{(j)}_{s-t}(\underline{\mu}) \right). \tag{6.9}
$$

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