

Oriented Associativity Equations and Symmetry Consistent Conjugate Curvilinear Coordinate Nets

Maxim V. Pavlov¹, Artur Sergyeyev²

¹Department of Mathematical Physics,
Lebedev Physical Institute of Russian Academy of Sciences,
Leninskij Prospekt, 53, Moscow, Russia;

²Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 74601 Opava, Czech Republic

e-mail: maxim@math.sinica.edu.tw, Artur.Sergyeyev@math.slu.cz

Abstract

This paper is devoted to description of the relationship among oriented associativity equations, symmetry consistent conjugate curvilinear coordinate nets, and the widest associated class of semi-Hamiltonian hydrodynamic-type systems.

In honour of Franco Magri

Contents

1	Introduction	2
2	Linear Spectral Problems. Basic Sets of Solutions	3
3	Reconstruction of Oriented Associativity Equations	4
4	Linear Spectral Problems for Oriented Associativity Equations	7
5	Reduction to Canonical Form	8
6	Hamiltonian Structure of Oriented Associativity Equations	11
7	Centroaffine Geometry	12
8	Inverse Construction	14
9	The Widest Class of Semi-Hamiltonian Hydrodynamic-Type Systems	16
10	Conclusion	17
11	Acknowledgements	18
	References	18

Keywords: Riemann invariants, Haantjes tensor, hydrodynamic-type systems, oriented associativity equations, centroaffine geometry

1 Introduction

The nonlinear partial differential system¹

$$\frac{\partial^2 c^i}{\partial a^j \partial a^m} \frac{\partial^2 c^m}{\partial a^k \partial a^n} = \frac{\partial^2 c^i}{\partial a^k \partial a^m} \frac{\partial^2 c^m}{\partial a^j \partial a^n} \quad (1)$$

describing a displacement vector appears in [4]. Following [14] we call this system the *oriented associativity equations*. This system admits the scalar linear spectral problem (cf. [4])

$$\frac{\partial^2 h}{\partial a^i \partial a^j} = \lambda \frac{\partial^2 c^m}{\partial a^i \partial a^j} \frac{\partial h}{\partial a^m} \quad (2)$$

or, alternatively, the vector linear spectral problem (cf. e.g. [15] and references therein)

$$\frac{\partial b^i}{\partial a^k} = \lambda \frac{\partial^2 c^i}{\partial a^k \partial a^m} b^m. \quad (3)$$

Integrable system (1) was extensively investigated in a number of papers (see, for instance, [9]) dedicated to the so-called coisotropic deformations. Some other aspects were considered in [8], [11], [12], [13], [15].

This paper is devoted to integrability of system (1). We consider a connection of (1) with a *widest* class of semi-Hamiltonian hydrodynamic-type systems

$$a_{tk}^i = \partial_x \frac{\partial c^i}{\partial a^k}; \quad (4)$$

we present a geometrical interpretation for (1) and linear spectral problems (2) and (3), and describe some transformations preserving (1), (2) and (3).

The celebrated WDVV equation (see, for instance, [4], [5], [6], [10])

$$\frac{\partial^3 F}{\partial a^i \partial a^j \partial a^m} \eta^{mn} \frac{\partial^3 F}{\partial a^n \partial a^k \partial a^s} = \frac{\partial^3 F}{\partial a^i \partial a^k \partial a^m} \eta^{mn} \frac{\partial^3 F}{\partial a^n \partial a^j \partial a^s}, \quad j \neq k \quad (5)$$

can be obtained from (1) by the potential reduction (where η^{ks} is a constant nondegenerate symmetric metric)

$$c^i = \eta^{im} \frac{\partial F}{\partial a^m}. \quad (6)$$

In this paper, we follow [4], and step by step unravel the relationship among (1) and the widest class of the so-called conjugate curvilinear coordinate nets determined by (see, for instance, [1])

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k, \quad (7)$$

$$\delta \beta_{ik} = 0, \quad i \neq k, \quad (8)$$

where $\partial_i = \partial / \partial r^i$, $\delta = \Sigma \partial_m$ is the so-called shift symmetry operator, and the rotation coefficients β_{ik} depend on N Riemann invariants r^k . The first subset of the above equations is the famous Darboux system, while the second condition means that the rotation coefficients depend only on differences of the Riemann invariants. For the sake of simplicity we shall call these conjugate curvilinear coordinate nets *symmetry consistent*.

¹Here and below the sum over *any* of repeated indices in the opposite locations (that is, one subscript and one superscript) is understood; otherwise the sum is indicated explicitly.

The paper is organized as follows. In Section 2 we introduce the metric and the basic set of solutions of linear systems (determining the symmetry consistent conjugate curvilinear coordinate nets) whose compatibility conditions imply the oriented associativity equations. In Section 3 we construct the transformation from the symmetry consistent conjugate curvilinear coordinate nets to the oriented associativity equations. In Section 4 we construct the relationship between linear spectral problems for the oriented associativity equations and the symmetry consistent conjugate curvilinear coordinate nets. In Section 5 we equip the oriented associativity equations by the unity condition. We concentrate here on the three-component case. In Section 6 we prove that these oriented associativity equations are a Hamiltonian system. In Section 7 we prove that the oriented associativity equations can be interpreted as a system of equations describing N -component position vector of a hypersurface in centroaffine geometry. In Section 8 we construct the inverse transformation from the oriented associativity equations to the symmetry consistent conjugate curvilinear coordinate nets. In Section 9 we construct an infinite set of particular solutions for semi-Hamiltonian hydrodynamic type systems whose rotation coefficients depend on differences of the Riemann invariants only. Finally, in Section 10 we just emphasize the relations among these very important systems.

2 Linear Spectral Problems. Basic Sets of Solutions

Consider two linear systems

$$\partial_i H_k = \beta_{ik} H_i, \quad \partial_i \psi_k = \beta_{ki} \psi_i, \quad i \neq k, \quad (9)$$

whose rotation coefficients β_{ik} depend on differences of the Riemann invariants r^k only (see (8)). This means that a particular set of solutions H_k, ψ_i satisfies two extra equations

$$\delta H_i = \lambda H_i, \quad \delta \psi_i = \lambda \psi_i. \quad (10)$$

The first set of compatibility conditions $\partial_j(\partial_i H_k) = \partial_i(\partial_j H_k), \partial_j(\partial_i \psi_k) = \partial_i(\partial_j \psi_k)$ leads to a full set of equations (7) describing conjugate curvilinear coordinate nets, while the second compatibility conditions $\partial_j(\delta H_k) = \delta(\partial_j H_k), \partial_j(\delta \psi_k) = \delta(\partial_j \psi_k)$ yield (8).

Remark: The symmetry consistent conjugate curvilinear coordinate nets are well known in classical differential geometry (see e.g. [1] and [17]). Moreover, system (7), (8) was derived more recently in the context of algebro-geometric solutions for multidimensional integrable systems (see [3]). It is interesting to note that this system also arises in quantum statistical physics, see Slavnov [16], and in theory of the discrete analogue of conjugate curvilinear coordinate nets known as D-invariant lattices (see [2]).

In this case N infinite series of solutions of systems (9) can be recursively found by quadratures (see [17])

$$\delta H_k^{(n+1,s)} = H_k^{(n,s)}, \quad \delta \psi_i^{(n+1,s)} = \psi_i^{(n,s)}, \quad s = 1, 2, \dots, N, \quad n = 0, 1, \dots, \quad (11)$$

where $\delta \psi_i^{(0,s)} = 0$ and $\delta H_i^{(0,s)} = 0$.

Choose N particular solutions $H_{(k)i} \equiv H_i^{(0,k)}$ as the *basic* set of solutions. Then introduce a non-degenerate (and non-constant in the generic case) symmetric metric

$$\bar{g}_{ik} = \sum_{m=1}^N H_{(i)m} H_{(k)m}. \quad (12)$$

Lemma: N particular solutions $\psi_i^{(n)} = \psi_i^{(0,n)}$ can be chosen in the form

$$\psi_i^{(s)} = \bar{g}^{sn} H_{(n)i}, \quad (13)$$

where \bar{g}^{sn} is an inverse metric for \bar{g}_{ik} .

Proof: Substituting (13) into the second equation of (9) yields

$$\partial_i(\bar{g}^{sn}H_{(n)k}) = \beta_{ki}\bar{g}^{sn}H_{(n)i}, \quad i \neq k.$$

Upon removing the parentheses on the l.h.s. and multiplying both sides by \bar{g}_{js} the above relations boil down to

$$H_{(n)k}\bar{g}^{ns}\partial_i\bar{g}_{sj} = (\beta_{ik} - \beta_{ki})H_{(j)i}, \quad i \neq k.$$

Taking into account (recall that $\delta H_{(s)i} = 0$) the equations

$$\partial_i H_{(s)i} = -\sum_{m \neq i} \beta_{mi} H_{(s)m}, \quad \partial_i \bar{g}_{jk} = \sum_{m \neq i} (\beta_{im} - \beta_{mi})(H_{(j)i}H_{(k)m} + H_{(k)i}H_{(j)m}), \quad (14)$$

one obtains an identity. The theorem is proved.

Then, obviously, three additional identities (where δ_k^i and δ_{ik} are the Kronecker symbols)

$$\bar{g}^{ik} = \sum_{m=1}^N \psi_m^{(i)} \psi_m^{(k)}, \quad \delta_k^i = \sum_{m=1}^N \psi_m^{(i)} H_{(k)m}, \quad \delta_{ik} = \sum_{m=1}^N \psi_i^{(m)} H_{(m)k} \quad (15)$$

hold. In the Egorov case $\beta_{ik} = \beta_{ki}$ and the metric $\bar{g}^{ik} = \text{const}$ (see (14)).

3 Reconstruction of Oriented Associativity Equations

In contrast with the previous section, we consider N Riemann invariants r^k as functions of N independent variables t^n , i.e. we introduce N commuting hydrodynamic-type systems

$$r_{t^k}^i = \frac{H_{(k)i}}{\bar{H}_i} r_x^i, \quad (16)$$

where \bar{H}_i is an arbitrary solution of the first linear system in (9). These hydrodynamic-type systems are semi-Hamiltonian (i.e. possess infinite set of conservation laws parameterized by N arbitrary functions of a single variable, see [17]). In such a case, they can be written in the conservative form

$$\partial_{t^k} h = \partial_x g_k, \quad (17)$$

where $\partial_i h = \psi_i \bar{H}_i$, $\partial_i g_k = \psi_i H_{(k)i}$ and ψ_i is an arbitrary solution (parameterized by N arbitrary functions of a single variable, see [17]) of the second system in (9). Introduce N conservation law densities a^k such that $\partial_i a^k = \psi_i^{(k)} \bar{H}_i$ and N conservation law densities c^k such that $\partial_i c^k = \bar{\psi}_i^{(k)} \bar{H}_i$, where $\delta \bar{\psi}_i^{(k)} = \psi_i^{(k)}$ (i.e. $\bar{\psi}_i^{(k)} \equiv \psi_i^{(1,k)}$, see (11) and (13)).

Theorem: N commuting hydrodynamic-type systems (16) can be written in the conservative form (4), whose compatibility conditions are oriented associativity equations (1).

Proof: The relation $\bar{\psi}_i^{(k)} = c_s^k \psi_i^{(s)}$ follows from

$$dc^k = \sum_{m=1}^N \bar{\psi}_m^{(k)} \bar{H}_m dr^m = c_s^k da^s = c_s^k \sum_{m=1}^N \psi_m^{(s)} \bar{H}_m dr^m,$$

where $c_k^i \equiv \partial c^i / \partial a^k$ (see (4)). Taking into account the second identity from (15), one can obtain

$$c_k^i = \sum_{m=1}^N \bar{\psi}_m^{(i)} H_{(k)m}. \quad (18)$$

Then (see (9))

$$\begin{aligned}
\partial_i c_j^k &= \partial_i \left(\bar{\psi}_i^{(k)} H_{(j)i} + \sum_{m \neq i} \bar{\psi}_m^{(k)} H_{(j)m} \right) \\
&= \bar{\psi}_i^{(k)} \partial_i H_{(j)i} + H_{(j)i} \partial_i \bar{\psi}_i^{(k)} + H_{(j)i} \sum_{m \neq i} \beta_{im} \bar{\psi}_m^{(k)} + \bar{\psi}_i^{(k)} \sum_{m \neq i} \beta_{mi} H_{(j)m} \\
&= \bar{\psi}_i^{(k)} \left(\partial_i H_{(j)i} + \sum_{m \neq i} \beta_{mi} H_{(j)m} \right) + H_{(j)i} \left(\partial_i \bar{\psi}_i^{(k)} + \sum_{m \neq i} \beta_{im} \bar{\psi}_m^{(k)} \right).
\end{aligned}$$

If we take into account that the expression in the first brackets vanishes because $\delta H_{(j)i} = 0$, and the expression in the second brackets is nothing but $\psi_i^{(k)}$ (since $\delta \bar{\psi}_i^{(k)} = \psi_i^{(k)}$), then one can conclude that

$$\partial_i c_j^k = \psi_i^{(k)} H_{(j)i}. \quad (19)$$

On the other hand, if the hydrodynamic-type systems (16) possess N conservation laws (4), then

$$\partial_m a^i \cdot r_{t^k}^m = \partial_m c_k^i \cdot r_x^m.$$

Substituting the r.h.s. of (16) for $r_{t^k}^i$ leads to (recall that $\partial_i a^k = \psi_i^{(k)} \bar{H}_i$)

$$\partial_i c_j^k = \frac{H_{(j)i}}{\bar{H}_i} \psi_i^{(k)} \bar{H}_i = \psi_i^{(k)} H_{(j)i},$$

which coincide with (19). The theorem is proved.

Remark: The second derivatives of the functions c^i with respect to the field variables a^j, a^k can be easily derived from (19). Indeed, the relations

$$\partial_i c_j^k = c_{j_s}^k \partial_i a^s = c_{j_s}^k \psi_i^{(s)} \bar{H}_i = \psi_i^{(k)} H_{(j)i}$$

lead (upon multiplying the third and fourth blocks of the above expression by the ratio $H_{(p)i}/\bar{H}_i$, and summing according to the second formula from (15)) to

$$c_{j_k}^i = \sum_{m=1}^N \frac{\psi_m^{(i)} H_{(j)m} H_{(k)m}}{\bar{H}_m}, \quad (20)$$

which, obviously, satisfy (1) by virtue of (15). Its symmetric form

$$c_{ijk} = \bar{g}_{is} c_{jk}^s = \sum_{m=1}^N \frac{H_{(i)m} H_{(j)m} H_{(k)m}}{\bar{H}_m}$$

is well known in the theory of WDVV associativity equations and Frobenius manifolds, where such expressions for c_{ijk} can be found using the theory of meromorphic differentials on algebraic Riemann surfaces (see, for instance, [4] and [10]).

Remark: Since

$$\sum_{m=1}^N \partial_m a^k \cdot \frac{\partial r^m}{\partial a^s} = \delta_s^k, \quad \sum_{m=1}^N \partial_k a^m \cdot \frac{\partial r^s}{\partial a^m} = \delta_k^s,$$

one can easily derive (recall (15) and $\partial_i a^k = \psi_i^{(k)} \bar{H}_i$)

$$\frac{\partial r^i}{\partial a^k} = \frac{H_{(k)i}}{\bar{H}_i}. \quad (21)$$

Thus, the characteristic velocities $v_{(k)}^i(\mathbf{a}) \equiv H_{(k)i}/\bar{H}_i$ (see (16)) of N commuting hydrodynamic-type systems (4) are nothing but $\partial r^i/\partial a^k$. Thus, the compatibility conditions $\partial v_{(k)}^i/\partial a^j = \partial v_{(j)}^i/\partial a^k$ should be satisfied. Indeed, multiplying both sides by a_i^p and summing over i , one obtains an identity, which follows from

$$\sum_{m=1}^N v_{(k)}^m \frac{\partial a_m^p}{\partial a^j} = \sum_{m=1}^N v_{(j)}^m \frac{\partial a_m^p}{\partial a^k},$$

where $\partial a_m^i/\partial a^k = a_{ms}^i v_{(k)}^s$. Thus, we conclude that

$$\frac{\partial r^i}{\partial a^k} = v_{(k)}^i(\mathbf{a}). \quad (22)$$

Any semi-Hamiltonian hydrodynamic-type system (see [17])

$$a_t^i = v_k^i a_x^k, \quad (23)$$

is associated with the non-degenerate metric tensor \bar{g}^{ik} . The necessary and sufficient conditions for existence of this tensor are given by (here ∇_k is the covariant derivative) the Tsarev lemma (see [17])

$$\bar{g}^{ik} v_k^j = \bar{g}^{jk} v_k^i, \quad \nabla_i v_j^k = \nabla_j v_i^k. \quad (24)$$

However, for hydrodynamic-type systems (4) we should not solve this system, because we already know that the metric tensor \bar{g}_{ij} in the field variables a^k is given by (12). Indeed, we have (see (12), (15))

$$\begin{aligned} ds^2 &= \sum_{i=1}^N \sum_{k=1}^N \bar{g}_{ik} da^i da^k = \sum_{i=1}^N \sum_{k=1}^N \sum_{s=1}^N \sum_{n=1}^N \left(\sum_{m=1}^N H_{(i)m} H_{(k)m} \right) (\psi_s^{(i)} \bar{H}_s dr^s) (\psi_n^{(k)} \bar{H}_n dr^n) \\ &= \sum_{s=1}^N \sum_{n=1}^N \sum_{m=1}^N \left(\sum_{i=1}^N \psi_s^{(i)} H_{(i)m} \right) \left(\sum_{k=1}^N \psi_n^{(k)} H_{(k)m} \right) \bar{H}_n \bar{H}_s dr^s dr^n = \sum_{m=1}^N \bar{H}_m^2 (dr^m)^2. \end{aligned}$$

So, we arrive at the conclusion that the diagonal metric $g_{kk} = \bar{H}_k^2$ in the Riemann invariants, in perfect agreement with Tsarev's definition of semi-Hamiltonian metric (see [17]).

Anyway, taking into account the expression for the covariant derivative $\nabla_l v_k^i = \partial_l v_k^i - \Gamma_{lk}^m v_m^i + \Gamma_{lm}^i v_k^m$, conditions (24) for hydrodynamic-type systems (4) reduce to a more compact form

$$\bar{g}^{ik} c_{ks}^j = \bar{g}^{jk} c_{ks}^i, \quad \Gamma_{im}^k c_{js}^m = \Gamma_{jm}^k c_{is}^m. \quad (25)$$

By virtue of (15) and (20), the first group of equations yields an identity. To consider the second group of equations, at first we should compute

$$\Gamma_{jk}^i = \frac{1}{2} \bar{g}^{im} \left(\frac{\partial \bar{g}_{mk}}{\partial a^j} + \frac{\partial \bar{g}_{mj}}{\partial a^k} - \frac{\partial \bar{g}_{jk}}{\partial a^m} \right).$$

Taking into account (14), (15) and (21) yields

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{m=1}^N \sum_{s=1}^N \bar{g}^{im} \left(\partial_s \bar{g}_{mk} \frac{\partial r^s}{\partial a^j} + \partial_s \bar{g}_{mj} \frac{\partial r^s}{\partial a^k} - \partial_s \bar{g}_{jk} \frac{\partial r^s}{\partial a^m} \right) = \sum_{s=1}^N \frac{H_{(j)s} H_{(k)s}}{\bar{H}_s} \sum_{p=1}^N (\beta_{sp} - \beta_{ps}) \psi_p^{(i)}.$$

Then, upon taking into account (15) and (20) once again, the second group of equations in (25) becomes an identity.

4 Linear Spectral Problems for Oriented Associativity Equations

As it was already mentioned in Introduction, the oriented associativity equations (1) can be obtained as compatibility conditions of the scalar linear spectral problem (2)

$$h_{ik} = \lambda c_{ik}^s h_s, \quad (26)$$

where we denote $c_{jk}^i \equiv \partial^2 c^i / \partial a^j \partial a^k$, $h_i \equiv \partial h / \partial a^i$, $h_{ik} \equiv \partial^2 h / \partial a^i \partial a^k$.

Lemma: *The function $h(\mathbf{a})$ is a generating function of conservation law densities (see (17)).*

Proof: If $h(\mathbf{a})$ is a conservation law density, then (see (21) and cf. (18))

$$h_k = \sum_{m=1}^N \partial_m h \cdot \frac{\partial r^m}{\partial a^k} = \sum_{m=1}^N \psi_m \bar{H}_m \cdot \frac{H_{(k)m}}{\bar{H}_m} = \sum_{m=1}^N \psi_m H_{(k)m}.$$

Also, taking into account (26) and (19), one can compute

$$\begin{aligned} dh_k &= h_{km} da^m = \lambda c_{km}^s h_s da^m = \lambda h_s dc_k^s \\ &= \lambda \sum_{s=1}^N \sum_{n=1}^N \sum_{m=1}^N (\psi_n H_{(s)n}) (\psi_m^{(s)} H_{(k)m} dr^m) = \lambda \sum_{n=1}^N \sum_{m=1}^N \left(\sum_{s=1}^N H_{(s)n} \psi_m^{(s)} \right) \psi_n H_{(k)m} dr^m, \end{aligned}$$

i.e. we have

$$dh_k = \lambda \sum_{m=1}^N \psi_m H_{(k)m} dr^m.$$

So, we arrive at the relation

$$\partial_i \left(\sum_{m=1}^N \psi_m H_{(k)m} \right) = \lambda \psi_i H_{(k)i},$$

which immediately reduces to the second equation in (10). The lemma is proved.

Now, consider the commuting hydrodynamic-type system (τ is a group parameter)

$$r_\tau^i = \frac{H_i}{H_i} r_x^i, \quad \Leftrightarrow \quad a_\tau^k = b_x^k, \quad (27)$$

where H_i is an arbitrary solution of first linear system (9) and $\partial_i b^k = \psi_i^{(k)} H_i$. As it was mentioned in Introduction, oriented associativity equations (1) can be obtained as compatibility conditions of the vector linear spectral problem (3)

$$b_k^i = \lambda c_{ks}^i b^s, \quad (28)$$

where we denote $b_k^i \equiv \partial b^i / \partial a^k$.

Lemma: *The functions*

$$b^k(\mathbf{a}) = \lambda^{-1} \sum_{m=1}^N \psi_m^{(k)} H_m$$

are conservation law fluxes for generating functions of commuting flows (27).

Proof: Taking into account (28), (19) and (15), we find that

$$db^k = \lambda c_{ns}^k b^s da^n = \lambda b^s dc_s^k = \sum_{s=1}^N \left(\sum_{m=1}^N \psi_m^{(s)} H_m \right) \left(\sum_{n=1}^N \psi_n^{(k)} H_{(s)n} dr^n \right)$$

$$= \sum_{m=1}^N \sum_{n=1}^N \left(\sum_{s=1}^N \psi_m^{(s)} H_{(s)n} \right) \psi_n^{(k)} H_m dr^n = \sum_{m=1}^N \psi_m^{(k)} H_m dr^m.$$

Thus, indeed, $\partial_i b^k = \psi_i^{(k)} H_i$. On the other hand, we have

$$\lambda \partial_i b^k = \sum_{m \neq i} \psi_m^{(k)} \partial_i H_m + \psi_i^{(k)} \partial_i H_i + \sum_{m \neq i} H_m \partial_i \psi_m^{(k)} + H_i \partial_i \psi_i^{(k)},$$

i.e.

$$\lambda \partial_i b^k = H_i \sum_{m \neq i} \beta_{im} \psi_m^{(k)} + \psi_i^{(k)} \left(\delta H_i - \sum_{m \neq i} \beta_{mi} H_m \right) + \psi_i^{(k)} \sum_{m \neq i} \beta_{mi} H_m + H_i \left(\delta \psi_i^{(k)} - \sum_{m \neq i} \beta_{im} \psi_m^{(k)} \right).$$

Since the first sum equals to the fourth sum, and the second sum equals to the third sum, we arrive at the conclusion that $\lambda \partial_i b^k = \psi_i^{(k)} \delta H_i$ (recall that $\delta \psi_i^{(k)} = 0$), which agrees with the previously computed $\partial_i b^k = \psi_i^{(k)} H_i$, if and only if H_i satisfies the first equation in the linear spectral problem (10). The lemma is proved.

Remark: One commuting flow (27) can be found without integration, i.e. (y is a group parameter)

$$a_y^i = (a^s c_s^i - c^i)_x. \quad (29)$$

Indeed (see (20)), we have

$$\begin{aligned} \partial_i (a^s c_s^k - c^k) &= a^s c_{sn}^k \partial_i a^n = a^s c_{sn}^k \psi_i^{(n)} \tilde{H}_i = a^s \sum_{n=1}^N \sum_{m=1}^N \frac{\psi_m^{(k)} H_{(s)m} H_{(n)m}}{\tilde{H}_m} \psi_i^{(n)} \tilde{H}_i \\ &= a^s \sum_{m=1}^N \frac{\psi_m^{(k)} H_{(s)m}}{\tilde{H}_m} \left(\sum_{n=1}^N \psi_i^{(n)} H_{(n)m} \right) \tilde{H}_i = a^s \psi_i^{(k)} H_{(s)i}. \end{aligned}$$

Since $\partial_i (a^s c_s^k - c^k) = \psi_i^{(k)} \tilde{H}_i$, where \tilde{H}_i is some solution of the first system in (9), we conclude that $\tilde{H}_i = a^s H_{(s)i}$. Substituting this expression into the first system in (9) leads to an identity, and the substitution into the first equation in (10) yields

$$\delta \tilde{H}_i = H_{(s)i} \delta a^s = \sum_{s=1}^N H_{(s)i} \sum_{m=1}^N \psi_m^{(s)} \tilde{H}_m = \sum_{m=1}^N \left(\sum_{s=1}^N \psi_m^{(s)} H_{(s)i} \right) \tilde{H}_m = \tilde{H}_i.$$

Thus, the family of hydrodynamic-type systems (4) has a simple commuting flow (29), which in the diagonal form (see the first equation in (27)) has characteristic velocities \tilde{H}_i / \bar{H}_i such that $\delta \tilde{H}_i = \tilde{H}_i$.

5 Reduction to Canonical Form

Hydrodynamic-type systems (4) have N additional natural conservation laws

$$c_{t^k}^i = \partial_x \frac{\partial Q^i}{\partial a^k}. \quad (30)$$

Indeed, suppose that the hydrodynamic-type systems (4) possess N additional conservation laws $c_{t^k}^i = \partial_x Q_k^i$. Then $dQ_k^i = c_m^i dc_m^k$, i.e. $Q_{ks}^i = c_m^i c_{ks}^m$. Thus, we conclude that $Q_{ks}^i = Q_{sk}^i$ and N functions Q^i determine the fluxes of these conservation laws. The compatibility conditions $\partial Q_{ks}^i / \partial a^j = \partial Q_{js}^i / \partial a^k$ hold by virtue of (1).

Rewrite N commuting hydrodynamic-type systems (4) in the differential form, i.e. (see (18)),

$$dz^i = a^i dx + \sum_{k=1}^N c_k^i(\mathbf{a}) dt^k \equiv a^i dx + \sum_{m=1}^N \sum_{k=1}^N \bar{\psi}_m^{(i)} H_{(k)m} dt^k.$$

Thus, we see that functions $c_k^i(\mathbf{a})$ do not depend on the choice of Lamé coefficients \bar{H}_k , while (recall again) $\partial_i a^k = \psi_i^{(k)} \bar{H}_i$. Below we shall consider the reduced version, i.e.

$$dz^i = \sum_{m=1}^N \sum_{k=1}^N \bar{\psi}_m^{(i)} H_{(k)m} dt^k.$$

In such a case, we pick the first “time” variable t^1 and choose the new field variables $\tilde{a}^i \equiv c_1^i(\mathbf{a})$, and then instead of (4) we shall consider just $N - 1$ commuting flows

$$\tilde{a}_{t^k}^i = \partial_{t^1} \frac{\partial \tilde{c}^i}{\partial \tilde{a}^k}, \quad (31)$$

where $\partial \tilde{c}^i / \partial \tilde{a}^k = \partial c^i / \partial a^k$, $k = 2, 3, \dots, N$. Thus, all functions \tilde{c}^i can be found in quadratures, i.e.

$$d\tilde{c}^i = \sum_{m=1}^N c_m^i dc_1^m.$$

Indeed, rewrite the conservation laws (30) in the differential form

$$dy^i = c^i(\mathbf{a}) dx + \sum_{k=1}^N Q_k^i(\mathbf{a}) dt^k = c^i(\mathbf{a}) dx + Q_1^i(\mathbf{a}) dt^1 + \sum_{k=2}^N Q_k^i(\mathbf{a}) dt^k.$$

Choose $\tilde{c}^i = Q_1^i(\mathbf{a})$, then $d\tilde{c}^i = Q_{1s}^i da^s = c_m^i c_{1s}^m da^s = c_m^i dc_1^m = c_m^i d\tilde{a}^m$. So, $\partial \tilde{c}^i / \partial \tilde{a}^k = \partial c^i / \partial a^k$.

Obviously, the hydrodynamic-type systems (31) can be written in the diagonal form

$$r_{t^k}^i = \frac{H_{(k)i}}{H_{(1)i}} r_{t^1}^i. \quad (32)$$

Indeed, the first of these commuting flows (16) can be written in the form

$$r_x^i = \frac{\bar{H}_i}{H_{(1)i}} r_{t^1}^i.$$

Then all other commuting flows (16) reduce to (32) upon substituting for r_x^i from the above formula.

Below we shall omit tildes over field variables a^k and c^n , because we are going to consider just the oriented associativity equations supplemented with the so-called “unity” condition (cf. [4])

$$c_{1k}^i = \delta_k^i, \quad (33)$$

which is equivalent to considering a canonical set of $N - 1$ commuting flows (31). We shall call such oriented associativity equations *normalized*.

Thus, the conservative representations (31) reduce to the form

$$a_{t^1}^1 = \partial_{t^1} u_k^1, \quad a_{t^k}^i = \partial_{t^1} (a^1 \delta_k^i + u_k^i), \quad i, k = 2, 3, \dots, N,$$

where the new unknown functions $u^n(a^2, a^3, \dots, a^N)$ appear from integration of (33), i.e.

$$c^1 = \frac{1}{2}(a^1)^2 + u^1, \quad c^k = a^1 a^k + u^k, \quad k = 2, 3, \dots, N. \quad (34)$$

In this case, the shift symmetry operator δ can be easily expressed via the field variables a^k instead of the Riemann invariants r^n , i.e.

$$\delta = \sum_{m=1}^N \frac{\partial}{\partial r^m} = \sum_{k=1}^N \left(\sum_{m=1}^N \frac{\partial a^k}{\partial r^m} \right) \frac{\partial}{\partial a^k} = \sum_{k=1}^N \left(\sum_{m=1}^N \psi_m^{(k)} H_{(1)m} \right) \frac{\partial}{\partial a^k} = \frac{\partial}{\partial a^1}. \quad (35)$$

Thus, two equations of scalar and vector linear spectral problems (26) and (28), namely

$$h_1 = \lambda h, \quad b_1^k = \lambda b^k \quad (36)$$

coincide with the eigenvalue problem for the above shift symmetry operator, while the remaining equations in (26) and (28) become, respectively,

$$h_{ik} = \lambda \sum_{s=1}^N u_{ik}^s h_s, \quad i, k = 2, 3, \dots, N, \quad (37)$$

$$b_k^i = \lambda \left((\delta_k^i - \delta_1^i \delta_k^1) b^1 + \sum_{m=2}^N u_{km}^i b^m \right), \quad i = 1, 2, \dots, N, \quad k = 2, 3, \dots, N.$$

Example: Consider the simplest nontrivial case $N = 3$. Then two commuting flows are given by (here $x = t^1, t = t^2, y = t^3$)

$$\begin{aligned} a_t &= \partial_x u_b, & b_t &= \partial_x(a + v_b), & c_t &= \partial_x w_b; \\ a_y &= \partial_x u_c, & b_y &= \partial_x v_c, & c_y &= \partial_x(a + w_c), \end{aligned} \quad (38)$$

where $a = a^1, b = a^2, c = a^3, u = u^1(b, c), v = u^2(b, c), w = u^3(b, c)$ and the subscripts indicate the corresponding partial derivatives. The compatibility conditions for (38), which read $(a_y)_t = (a_t)_y, (b_y)_t = (b_t)_y, (c_y)_t = (c_t)_y$, lead to three algebraic equations relating the second-order derivatives

$$u_{bb} = v_{bc} w_{bb} - v_{bb} w_{bc} + w_{bc}^2 - w_{bb} w_{cc}, \quad u_{bc} = v_{cc} w_{bb} - v_{bc} w_{bc}, \quad u_{cc} = v_{bc}^2 - v_{bb} v_{cc} + v_{cc} w_{bc} - v_{bc} w_{cc},$$

which are integrable by the inverse spectral transform. The scalar Lax pair for the latter (see (37)) is given by

$$\begin{pmatrix} h_a \\ h_b \\ h_c \end{pmatrix}_b = \lambda \begin{pmatrix} 0 & 1 & 0 \\ u_{bb} & v_{bb} & w_{bb} \\ u_{bc} & v_{bc} & w_{bc} \end{pmatrix} \begin{pmatrix} h_a \\ h_b \\ h_c \end{pmatrix}, \quad \begin{pmatrix} h_a \\ h_b \\ h_c \end{pmatrix}_c = \lambda \begin{pmatrix} 0 & 0 & 1 \\ u_{bc} & v_{bc} & w_{bc} \\ u_{cc} & v_{cc} & w_{cc} \end{pmatrix} \begin{pmatrix} h_a \\ h_b \\ h_c \end{pmatrix},$$

while the adjoint Lax pair has transposed matrices, i.e.

$$\begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}_b = \lambda \begin{pmatrix} 0 & u_{bb} & u_{bc} \\ 1 & v_{bb} & v_{bc} \\ 0 & w_{bb} & w_{bc} \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}, \quad \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}_c = \lambda \begin{pmatrix} 0 & u_{bc} & u_{cc} \\ 0 & v_{bc} & v_{cc} \\ 1 & w_{bc} & w_{cc} \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}.$$

Remark: Both commuting hydrodynamic-type systems (37) also have the same Lax matrices, i.e.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = \begin{pmatrix} 0 & u_{bb} & u_{bc} \\ 1 & v_{bb} & v_{bc} \\ 0 & w_{bb} & w_{bc} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}_y = \begin{pmatrix} 0 & u_{bc} & u_{cc} \\ 0 & v_{bc} & v_{cc} \\ 1 & w_{bc} & w_{cc} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x.$$

6 Hamiltonian Structure of Oriented Associativity Equations

The system of quadratic equations

$$u_{bb} = v_{bc}w_{bb} - v_{bb}w_{bc} + w_{bc}^2 - w_{bb}w_{cc}, \quad u_{bc} = v_{cc}w_{bb} - v_{bc}w_{bc}, \quad u_{cc} = v_{bc}^2 - v_{bb}v_{cc} + v_{cc}w_{bc} - v_{bc}w_{cc},$$

is nothing but the oriented associativity equations in three-dimensional case with the unity condition (33) (see also (34)). Introduce a new set of field variables $q^1 = u_{bb}$, $q^2 = u_{bc}$, $q^3 = v_{bb}$, $q^4 = v_{bc}$, $q^5 = w_{bb}$, $q^6 = w_{bc}$. Then this quadratic system becomes a six-component hydrodynamic-type system

$$\begin{aligned} q_c^1 &= q_b^2, & q_c^2 &= \partial_b \frac{q^2 q^6 + q^1 q^4 - q^2 q^3}{q^5}, \\ q_c^3 &= q_b^4, & q_c^4 &= \partial_b \frac{q^2 + q^4 q^6}{q^5}, \\ q_c^5 &= q_b^6, & q_c^6 &= \partial_b \frac{(q^6)^2 - q^3 q^6 + q^4 q^5 - q^1}{q^5}. \end{aligned} \quad (39)$$

As it was already mentioned in Section 5, in the present paper we restrict ourselves to the generic case when all characteristic velocities $v_{(k)}^i(\mathbf{a})$ are pairwise distinct. We seek the Riemann invariants $r(a, b, c)$ for both commuting hydrodynamic-type systems (38) written in the diagonal form

$$r_t = v_{(2)} r_x, \quad r_y = v_{(3)} r_x. \quad (40)$$

Taking into account that $r_a = 1$ (thanks to the shift symmetry operator $\delta = \partial_a$, see (35)), one can obtain

$$r_b = v_{(2)}, \quad r_c = v_{(3)}, \quad (41)$$

where the characteristic velocities $v_{(2)}$ and $v_{(3)}$ of (38) are related polynomially,

$$v_{(3)} = \frac{(v_{(2)})^2 - q^3 v_{(2)} - q^1}{q^5},$$

while $v_{(2)}$ satisfies the characteristic equation (45) for the first hydrodynamic-type system from (38), i.e.

$$(v_{(2)})^3 - (q^3 + q^6)(v_{(2)})^2 + (q^3 q^6 - q^4 q^5 - q^1)v_{(2)} + q^1 q^6 - q^2 q^5 = 0.$$

Since the three characteristic velocities are distinct, the commuting hydrodynamic-type systems (38) can be written in diagonal form (40), where the Riemann invariants can be found by quadratures (see (41)), i.e. (cf. (46))

$$r^k = a + \int (v_{(2)}^k db + v_{(3)}^k dc), \quad k = 1, 2, 3.$$

Remark: Hydrodynamic-type system (39) possesses at least two additional local conservation laws. Indeed, the existence of three Riemann invariants (see (41)) implies three additional conservation laws

$$\partial_c v_{(2)}^k = \partial_b \frac{(v_{(2)}^k)^2 - q^3 v_{(2)}^k - q^1}{q^5}, \quad k = 1, 2, 3.$$

Hence, the roots $v_{(2)}^k$ are conservation law densities. However, five conservation law densities $q^3, q^6, v_{(2)}^1, v_{(2)}^2, v_{(2)}^3$ are related by the linear equation

$$q^3 + q^6 = v_{(2)}^1 + v_{(2)}^2 + v_{(2)}^3 \quad (42)$$

by virtue of the Viète theorem. This means that just two of the above three conservation laws are new.

Main Result of this Section: Upon expressing q^1, q^2 and q^6 using the Viète theorem via q^3, q^4, q^5 and $v_{(2)}^1, v_{(2)}^2, v_{(2)}^3$, the hydrodynamic-type system (39) can be written in the local Hamiltonian form (cf. [6])

$$s_c^i = \tilde{g}^{ik} \partial_b \frac{\partial H}{\partial s^k},$$

where the flat coordinates are

$$s^1 = v_{(2)}^1, \quad s^2 = v_{(2)}^2, \quad s^3 = v_{(2)}^3, \quad s^4 = q^4, \quad s^5 = q^5, \quad s^6 = 2q^3 - (v_{(2)}^1 + v_{(2)}^2 + v_{(2)}^3),$$

the metric is

$$\tilde{g}_{ik} = -\frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{g}^{ik} = - \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

the momentum density P is

$$q^1 = \frac{1}{2} \tilde{g}_{ik} s^i s^k = \frac{1}{4} (s^1 + s^2 + s^3)^2 - (s^1 s^2 + s^1 s^3 + s^2 s^3) - \frac{1}{4} (s^6)^2 - s^4 s^5,$$

and the Hamiltonian density H is

$$q^2 = \frac{2s^1 s^2 s^3 + (s^1 + s^2 + s^3 - s^6) q^1}{2s^5},$$

while the other conservation law densities q^3, q^4, q^5, q^6 (recall (42)) are just linear combinations of the flat coordinates s^k , i.e.

$$q^3 = \frac{1}{2} (s^1 + s^2 + s^3 + s^6), \quad q^4 = s^4, \quad q^5 = s^5, \quad q^6 = \frac{1}{2} (s^1 + s^2 + s^3 - s^6).$$

7 Centraffine Geometry

Thanks to the presence of the shift symmetry operator (i.e. $h_1 = \lambda h$, see the first equation in (36)) linear spectral problem (37) for normalized oriented associativity equations (see (33)) becomes

$$h_{ik} = \lambda \sum_{s=2}^N u_{ik}^s h_s + \lambda^2 u_{ik}^1 h, \quad i, k = 2, \dots, N.$$

This linear system arises in centraffine geometry. Following [5], consider a linear overdetermined system (which is required to be compatible of rank N)

$$\frac{\partial^2 \mathbf{r}}{\partial a^i \partial a^j} = \lambda \Gamma_{ij}^k \frac{\partial \mathbf{r}}{\partial a^k} + \lambda^2 g_{ij} \mathbf{r}, \quad i, j = 2, \dots, N, \quad (43)$$

for the N -component position vector $\mathbf{r} = \mathbf{r}(a^2, \dots, a^N)$ of a hypersurface M^{N-1} in centraffine geometry, where λ is a spectral parameter, $g_{ij}(a^2, \dots, a^N)$ is a pseudo-Riemannian metric, and $\Gamma_{ij}^k(a^2, \dots, a^N)$ are components of a torsionless affine connection (which in general is not the Levi-Civita connection of flat metric g_{ij}). The conformal class of g_{ij} is nothing but the second fundamental form of M^{N-1} .

The compatibility conditions for this system have the form (see [5])

$$\begin{aligned}\Gamma_{ij}^s g_{sk} &= \Gamma_{jk}^s g_{si} = \Gamma_{ik}^s g_{sj}, & \partial_k g_{ij} &= \partial_i g_{jk} = \partial_j g_{ik}, & \partial_k \Gamma_{ij}^s &= \partial_i \Gamma_{jk}^s = \partial_j \Gamma_{ik}^s, \\ \Gamma_{ij}^s \Gamma_{sk}^m + g_{ij} \delta_k^m &= \Gamma_{jk}^s \Gamma_{si}^m + g_{jk} \delta_i^m = \Gamma_{ik}^s \Gamma_{sj}^m + g_{ik} \delta_j^m,\end{aligned}\tag{44}$$

The equations $\partial_k g_{ij} = \partial_i g_{jk} = \partial_j g_{ik}$, $\partial_k \Gamma_{ij}^s = \partial_i \Gamma_{jk}^s = \partial_j \Gamma_{ik}^s$ mean that the components of the metric tensor (in these coordinates a^k) are second derivatives of a single function $g(\mathbf{a})$ with respect to the corresponding coordinates a^k (i.e. $g_{ik} \equiv \partial^2 g / \partial a^i \partial a^k$), while the components of the affine connection Γ_{jk}^i are also second derivatives of a single vector function $\vec{\Gamma}(\mathbf{a})$ (i.e. $\Gamma_{ik}^s \equiv \partial^2 \Gamma^s / \partial a^i \partial a^k$, where Γ^s are components of the vector function $\vec{\Gamma}(\mathbf{a})$). Following [5], one can split the affine connection on two parts, i.e.

$$\Gamma_{ik}^s = \frac{1}{2} g^{sm} g_{mik} + f_{ik}^s,$$

where the first block $\frac{1}{2} g^{sm} g_{mik}$ is a Levi-Civita connection, while the difference of connections f_{jk}^i is a (1,2)-tensor. Then the equations $\Gamma_{ij}^s g_{sk} = \Gamma_{jk}^s g_{si} = \Gamma_{ik}^s g_{sj}$ (see (44)) imply that the tensor $f_{ijk} = f_{ij}^s g_{sk}$ is totally symmetric, defining the centroaffine cubic form $C = f_{ijk} dx^i dx^j dx^k$ of the hypersurface M^{N-1} which together with the centroaffine metric $M = g_{ij} dx^i dx^j$ (satisfying compatibility conditions (44)) uniquely characterize a hypersurface. The rest of system (44) are precisely the normalized oriented associativity equations, where $u^1 \equiv g$ and all other $u^k \equiv \Gamma^k$.

The inverse construction: Now consider an additional variable a^1 and assume that the indices i, j, k run from 1 up to N . Since the vector function $\vec{c}(\mathbf{a})$ has the components determined from (34), we can define the quantities $c_{jk}^i(a^1, \dots, a^N)$

$$c_{1k}^i = \delta_k^i, \quad i, k = 1, \dots, N, \quad c_{jk}^i = \Gamma_{jk}^i, \quad c_{jk}^1 = g_{ik}, \quad i, j, k = 2, \dots, N$$

and $h = (\lambda \mathbf{r}, \partial \mathbf{r} / \partial a^1, \dots, \partial \mathbf{r} / \partial a^N)^T$. In such a case, linear system (43) is replaced again by more general linear system (26).

Note that these formulas bear considerable resemblance with the formulas for c_{jk}^i for the associativity equations (the famous WDVV equation) at p. 36 in [5]. In fact, our formulas reduce to those of Ferapontov if the metric g_{ij} is constant in the coordinates a^i : $g_{ij} = \tilde{\eta}_{ij} = \text{const}$.

Examples: The oriented associativity equations in three-dimensional case with the unity condition (33) are nothing but a system of quadratic equations (see (38) and below)

$$u_{bb} = v_{bc} w_{bb} - v_{bb} w_{bc} + w_{bc}^2 - w_{bb} w_{cc}, \quad u_{bc} = v_{cc} w_{bb} - v_{bc} w_{bc}, \quad u_{cc} = v_{bc}^2 - v_{bb} v_{cc} + v_{cc} w_{bc} - v_{bc} w_{cc}.$$

Three distinguished versions of the WDVV associativity equations can be singled out by the special choices of the metrics \bar{g}^{ik} (see (6)). If $\eta^{11} = \eta^{22} = \eta^{33} = 1$ (and all other entries of η^{ik} are equal to zero), then (see (5) and (34))

$$F = \frac{a^3}{6} + \frac{b^2 + c^2}{2} a + z(b, c),$$

where the three above quadratic equations reduce to a single one ($v = z_b, w = z_c$ and $u = (b^2 + c^2)/2$)

$$z_{bbc}^2 + z_{bcc}^2 = 1 + z_{bbb} z_{bcc} + z_{ccc} z_{bbc};$$

if $\eta^{11} = \eta^{23} = \eta^{32} = 1$, then

$$F = \frac{a^3}{6} + abc + z(b, c),$$

where the three aforementioned quadratic equations reduce to a single one ($u = bc, v = z_c, w = z_b$)

$$1 = z_{ccc} z_{bbb} - z_{bcc} z_{bbc};$$

if $\eta^{13} = \eta^{22} = \eta^{31} = 1$, then

$$F = \frac{1}{2}(a^2c + ab^2) + z(b, c),$$

where the three corresponding quadratic equations reduce to a single one ($u = z_c, v = z_b, w = b^2/2$)

$$z_{ccc} = z_{bbc}^2 - z_{bbb}z_{bcc}.$$

The first two examples are associated with the hypersurfaces endowed with flat centroaffine metrics (see [5] for details), while the third example is related to a non-flat centroaffine metric, because $g_{bb} = u_{bb}, g_{bc} = u_{bc}, g_{cc} = u_{cc}$ and all components of the Riemann curvature tensor do not vanish.

Thus, we established a link (in fact, an equivalence) between the oriented associativity equations with unity and centroaffine geometry in a general non-flat case (cf. [5]).

8 Inverse Construction

In the preceding sections we constructed the transformation from symmetry consistent conjugate curvilinear coordinate nets to the oriented associativity equations. In this section, we briefly discuss the inverse transformation.

- Any solution of oriented associativity equations (1) is associated with the corresponding hydrodynamic-type systems (4).
- Suppose that all characteristic velocities $v_{(k)}^i(\mathbf{a})$ of each hydrodynamic-type system (4) are distinct, i.e. the algebraic equations (for each fixed index k)

$$\det |c_{jk}^i - v_{(k)}^i \delta_j^i| = 0 \quad (45)$$

have just simple roots. In this paper we restrict our consideration to this semi-simple case only.

- If an N -component hydrodynamic-type system has pairwise distinct characteristic velocities (see (45)), N conservation laws (see (17)) and all components of the Haantjes tensor (see [7]) vanish, then this hydrodynamic-type system is semi-Hamiltonian (see [17]) and can be written in diagonal form (i.e., N Riemann invariants exist). The Nijenhuis tensor for the hydrodynamic-type system (23) reads (below in this Section $\partial_k \equiv \partial/\partial a^k$)

$$\mathbf{N}_{jk}^i = v_j^p \partial_p v_k^i - v_k^p \partial_p v_j^i - v_p^i (\partial_j v_k^p - \partial_k v_j^p),$$

and the Haantjes tensor has the form

$$\mathbf{H}_{jk}^i = N_{pn}^i v_j^p v_k^n - N_{jn}^p v_p^i v_k^n - N_{nk}^p v_p^i v_j^n + N_{jk}^p v_n^i v_p^n.$$

For each system (4) with the time variable t^s we readily find that the Nijenhuis tensor reduces to the form (recall that the summation is over the pairs of oppositely located repeated indices only)

$$\mathbf{N}_{(s)jk}^i = c_{js}^q c_{qks}^i - c_{ks}^q c_{qjs}^i,$$

while the Haantjes tensor becomes

$$\mathbf{H}_{(s)jk}^i = c_{js}^p c_{ks}^m (c_{ps}^q c_{qms}^i - c_{ms}^q c_{qps}^i) + c_{ps}^i c_{ks}^m (c_{ms}^q c_{qjs}^p - c_{js}^q c_{qms}^p) + c_{ps}^i c_{js}^m (c_{ks}^q c_{qms}^p - c_{ms}^q c_{qks}^p) + c_{ps}^i c_{ns}^p (c_{js}^q c_{qks}^n - c_{ks}^q c_{qjs}^n).$$

This expression formally contains eight blocks. However, the fourth and fifth blocks coincide. Thus, the Haantjes tensor reduces to the six-block form

$$\mathbf{H}_{(s)jk}^i = c_{js}^p c_{ks}^m (c_{ps}^q c_{qms}^i - c_{ms}^q c_{qps}^i) + c_{ps}^i c_{ms}^q (c_{ks}^m c_{qjs}^p - c_{js}^m c_{qks}^p) + c_{ps}^i c_{ns}^p (c_{js}^q c_{qks}^n - c_{ks}^q c_{qjs}^n).$$

Then one can rewrite the Haantjes tensor in the following form

$$\begin{aligned} \mathbf{H}_{(s)jk}^i &= c_{js}^p c_{ks}^m \partial_s (c_{qm}^i c_{ps}^q - c_{qp}^i c_{ms}^q) + c_{ps}^i c_{ks}^m \partial_s (c_{ms}^q c_{qj}^p - c_{mj}^q c_{qs}^p) + c_{ps}^i c_{js}^m \partial_s (c_{qs}^p c_{mk}^q - c_{qk}^p c_{ms}^q) \\ &\quad + (c_{qm}^i c_{js}^m - c_{ms}^i c_{qj}^m) c_{ks}^p c_{pss}^q + (c_{ms}^i c_{qk}^m - c_{qm}^i c_{ks}^m) c_{js}^p c_{pss}^q + c_{qs}^i (c_{ks}^m c_{mj}^p - c_{js}^m c_{mk}^p) c_{pss}^q, \end{aligned}$$

where each bracket vanishes by virtue of oriented associativity equations (1). So, indeed, the family of hydrodynamic-type systems (4) is semi-Hamiltonian.

- In such a case (see (22)), the Riemann invariants can be found by quadratures

$$r^i = \sum_{m=1}^N \int v_{(m)}^i(\mathbf{a}) da^m, \quad (46)$$

where the characteristic velocities $v_{(k)}^i(\mathbf{a})$ also (see (45)) satisfy quadratic relations (cf. [4])

$$v_{(j)}^i v_{(k)}^i = \sum_{s=1}^N c_{jk}^s v_{(s)}^i.$$

This means that the functions a^n (upon inverting the point transformation (46)) can be expressed via the Riemann invariants r^k .

- Then following Tsarev's construction (see [17] for details), one can compute the so-called Lamè coefficients (the expressions on the r.h.s. are equal for all values of s)

$$\partial_k \ln \bar{H}_i = \frac{\partial_k v_{(s)}^i(\mathbf{a})}{v_{(s)}^k(\mathbf{a}) - v_{(s)}^i(\mathbf{a})}, \quad i \neq k. \quad (47)$$

- Then following Darboux (see [1] for details), one can find the rotation coefficients

$$\beta_{ik} = \frac{\partial_i \bar{H}_k}{\bar{H}_i}, \quad i \neq k, \quad (48)$$

which satisfy² system (7), (8) describing symmetry consistent conjugate curvilinear coordinate nets. Indeed, since the rotation coefficients are the same for all commuting flows (4), without loss of generality consider just $N - 1$ commuting hydrodynamic-type systems (31). Then (46) reduces to the form

$$r^i = \tilde{a}^1 + \sum_{m=2}^N \int \tilde{v}_{(m)}^i(\tilde{\mathbf{a}}) d\tilde{a}^m,$$

where $\tilde{v}_{(m)}^i(\tilde{\mathbf{a}})$ depend on the variables $\tilde{a}^2, \dots, \tilde{a}^N$ only. Thus, $\delta \tilde{v}_{(m)}^i(\tilde{\mathbf{a}}) = 0$, where the shift symmetry operator (see (35))

$$\delta = \frac{\partial}{\partial \tilde{a}^1} = \sum_{m=1}^N \frac{\partial r^m}{\partial \tilde{a}^1} \frac{\partial}{\partial r^m} = \sum_{m=1}^N \frac{\partial}{\partial r^m}.$$

²Since the Lamè coefficients \bar{H}_i are determined up to multiplication by arbitrary functions $\mu_i(r^i)$ (see (47)), they should be fixed by the restriction $\delta \beta_{ik} = 0$.

Since the Lamè coefficients $H_{(1)i}$ can be found from (cf. (47))

$$\partial_k \ln H_{(1)i} = \frac{\partial_k \tilde{v}_{(s)}^i(\tilde{\mathbf{a}})}{\tilde{v}_{(s)}^k(\tilde{\mathbf{a}}) - \tilde{v}_{(s)}^i(\tilde{\mathbf{a}})}, \quad i \neq k,$$

we have $\partial_k \delta \ln H_{(1)i} = 0$, i.e. $\delta \ln H_{(1)i} = \chi_i(r^i)$, where $\chi_i(r^i)$ are arbitrary functions. Thus, one can choose $H_{(1)i} = \tilde{\chi}_i(r^i) \tilde{H}_{(1)i}$ such that $\chi_i(r^i) = \partial_i \ln \tilde{\chi}_i(r^i)$. Then $\delta \tilde{H}_{(1)i} = 0$, i.e. the corresponding rotation coefficients β_{ik} depend on differences of the Riemann invariants, because $\delta \beta_{ik} = 0$, which follows from (48). So, we conclude that any solution of the oriented associativity equations (1) gives rise to some solution of system (7), (8) describing symmetry consistent conjugate curvilinear coordinate nets.

9 The Widest Class of Semi-Hamiltonian Hydrodynamic-Type Systems

This section is devoted to description and integrability of the widest class of semi-Hamiltonian hydrodynamic-type systems (27). A general solution of oriented associativity equations (1) leads (see Section 5) to a general solution of system (7), (8) describing symmetry consistent conjugate curvilinear coordinate nets as well as to the basic set of solutions $H_{(i)k}$ (see (11)). In this section we construct general solutions of linear spectral problems (9) and (10) which are important for various applications in the theory of semi-Hamiltonian hydrodynamic-type systems, for instance, in the generalized hodograph method (see [17]).

Recall a few major formulas from Sections 2, 3, 4.

Suppose that the basic set of solutions $H_{(i)k}$ (see (11)) of the linear spectral problem (see (9) and (10))

$$\delta H_i = \lambda H_i, \quad \partial_i H_k = \beta_{ik} H_i, \quad i \neq k \quad (49)$$

is found for a given set of rotation coefficients β_{ik} depending only on differences of the Riemann invariants r^n (see (8)) and satisfying (7). Then, the basic set of solutions $\psi_i^{(s)}$ of the adjoint linear problem (see (9) and (10)) is given by (13), where \bar{g}^{sn} is a non-degenerate symmetric metric which is inverse to (12).

Our goal is to find a general solution of the above linear spectral problem as well as a general solution of the adjoint linear spectral problem

$$\delta \psi_i = \lambda \psi_i, \quad \partial_i \psi_k = \beta_{ki} \psi_i, \quad i \neq k, \quad (50)$$

and general solutions of both other linear spectral problems (26) and (28).

Main result: *N infinite series of solutions $H_j^{(s,k)}, \psi_i^{(n,p)}$ (see (11)) can be found in quadratures.*

Consider the expansions in λ of solutions for both linear spectral problems (49) and (50), i.e. (see (11))

$$H_i = H_i^{(0,k)} + \lambda H_i^{(1,k)} + \lambda^2 H_i^{(2,k)} + \dots, \quad \psi_i = \psi_i^{(0,k)} + \lambda \psi_i^{(1,k)} + \lambda^2 \psi_i^{(2,k)} + \dots, \quad k = 1, \dots, N,$$

as well as the corresponding expansions of solutions of two other linear spectral problems (26) and (28), i.e.

$$h = \lambda^{-1} \delta_1^k + a^k + \lambda c^k + \lambda^2 h^{(2,k)} + \lambda^3 h^{(3,k)} + \dots, \quad b^s = \lambda^{-1} \delta_k^s + c_k^s + \lambda b_{(1,k)}^s + \lambda^2 b_{(2,k)}^s + \lambda^3 b_{(3,k)}^s + \dots,$$

where $\partial_i h^{(n,k)} = \psi_i^{(n,k)} H_{(1)i}$ and $\partial_i b_{(n,k)}^s = \psi_i^{(s)} H_i^{(n,k)}$, while (see Section 4)

$$h^{(n,k)} = \sum_{m=1}^N \psi_m^{(n+1,k)} H_{(1)m}, \quad b_{(n,k)}^s = \sum_{m=1}^N \psi_m^{(s)} H_m^{(n+1,k)}.$$

Instead of complicated integration of infinite sets of recursion relations (11) (see (49) and (50)), one can easily integrate the other infinite sets of recursion relations (see (26) and (28)),

$$dh_i^{(n+1,k)} = \sum_{m=1}^N \sum_{s=1}^N c_{im}^s h_s^{(n,k)} da^m, \quad db_{(n+1,k)}^i = \sum_{m=1}^N \sum_{s=1}^N c_{ms}^i b_{(n,k)}^s da^m, \quad i = 1, \dots, N, \quad n = 0, 1, \dots \quad (51)$$

The construction of all these four recursion relations includes the following steps:

- Since $\psi_i^{(1,k)} \equiv \bar{\psi}_i^{(k)} = c_s^k \psi_i^{(s)}$ (see the proof of the theorem in Section 3), where $\partial_i c_j^k = \psi_i^{(k)} H_{(j)i}$, the expressions c_s^k can be found in quadratures, i.e.

$$dc_s^k = \sum_{m=1}^N \psi_m^{(k)} H_{(s)m} dr^m.$$

Then (recall that $\partial_i a^s = \psi_i^{(s)} H_{(1)i}$, $\partial_i c^s = \psi_i^{(1,s)} H_{(1)i}$)

$$\psi_i^{(1,k)} = \sum_{s=1}^N \psi_i^{(s)} \sum_{m=1}^N \int \psi_m^{(k)} H_{(s)m} dr^m, \quad dc^k = \sum_{p=1}^N \sum_{s=1}^N \psi_p^{(s)} \left(\sum_{m=1}^N \int \psi_m^{(k)} H_{(s)m} dr^m \right) H_{(1)p} dr^p.$$

Thus, we had a basic set of solutions $H_{(i)k}$. Then we reconstructed the adjoint basic set of solutions $\psi_m^{(k)}$. Then we found N conservation law densities a^k and N conservation law densities c^k . So, we have found all structure constants c_{jk}^i with unity condition (33).

- Then step by step (see (51)) we can find N infinite series of higher conservation law densities $h^{(n,k)}$ as well as N infinite series of higher conservation law fluxes $b_{(n,k)}^s$.
- Since $\partial_i h^{(n,k)} = \psi_i^{(n,k)} H_{(1)i}$ and $\partial_i b_{(n,k)}^s = \psi_i^{(s)} H_i^{(n,k)}$, we can find higher solutions $H_i^{(n,k)}$, $\psi_j^{(n,k)}$.

Moreover, we want to find general solutions of linear systems (9) without extra conditions (10). This is essential for application of the generalized hodograph method (see [17]). However, just in some very special cases general solutions of linear systems (9) can be found explicitly. Nevertheless, using the approach suggested in [17], one can see that any initial data for the linear system (9) can be approximated by linear combinations infinitely many particular solutions $H_i^{(n,k)}$, $\psi_j^{(n,k)}$ with appropriately chosen coefficients $\xi_{n,k}$.

This means that the general solution of an arbitrary semi-Hamiltonian hydrodynamic-type system (27) whose rotation coefficients depend on differences of Riemann invariants only can be found using the generalized hodograph method (see [17]) and has the form

$$x\bar{H}_i + tH_i = \sum_{n=0}^{\infty} \sum_{k=1}^N \xi_{n,k} H_i^{(n,k)}.$$

10 Conclusion

In this paper, we considered three distinguished objects: oriented associativity equations, symmetry consistent conjugate curvilinear coordinate nets and semi-Hamiltonian hydrodynamic-type systems whose rotation coefficients depend on differences of the Riemann invariants only. We have shown that these objects are closely related, and thus the knowledge about one of them implies the knowledge about the others.

11 Acknowledgements

We thank Boris Dubrovin, Eugeni Ferapontov, Boris Konopelchenko, Franco Magri and Sergey Tsarev for their stimulating and clarifying discussions.

MVP is grateful to Professor Boris Dubrovin for hospitality in SISSA in Trieste (Italy) where part of this work has been done. MVP's work was partially supported by the RF Government grant #2010-220-01-077, ag. #11.G34.31.0005, by the grant of Presidium of RAS "Fundamental Problems of Nonlinear Dynamics" and by the RFBR grant 11-01-00197.

The research of AS was supported in part by the Grant Agency of the Czech Republic (GA ĀR) under the project P201/11/0356, and by the Ministry of Education, Youth and Sport of the Czech Republic (MŠMT ĀR) under RVO:47813059.

References

- [1] *G. Darboux*, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. Paris, Gauthier-Villars, 1910. *G. Darboux*, Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, t. 1–4, Paris (1887–1896).
- [2] *A. Doliwa, P.M. Santini*, The symmetric, d -invariant and Egorov reductions of the quadrilateral lattice. *J. Geom. Phys.* **36** No. 1-2 (2000) 60–102.
- [3] *B.A. Dubrovin*, Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties. *Funkcional. Anal. i Ego Prilož.* **11** No. 4 (1977) 28–41 (in Russian); English translation: *Funct. Anal. Appl.* **11** No. 4 (1977) 265–277.
- [4] *B.A. Dubrovin*, Integrable systems in topological field theory, *Nucl. Phys. B*, **379** (1992) 627–689. *B.A. Dubrovin*, Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models, *Comm. Math. Phys.* **145** (1992) 195–207. *B.A. Dubrovin*, Geometry of 2D topological field theories, *Lecture Notes in Math.* 1620, Springer-Verlag (1996) 120–348. *B.A. Dubrovin*, On the differential geometry of strongly integrable systems of hydrodynamic type, *Funkcional. Anal. i Priložen.* **24** No. 4 (1990) 25–30 (in Russian); English translation: *Func. Anal. Appl.*, **24** (1990), 280–285.
- [5] *E. V. Ferapontov*, Hypersurfaces with flat centroaffine metric and equations of associativity, *Geometriae Dedicata*, **103** (2004) 33–49.
- [6] *E.V. Ferapontov, C.A.P. Galvao, O.I. Mokhov, Y. Nutku*, Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory, *Comm. in Math. Phys.*, **186** (1997) 649-669.
- [7] *J. Haantjes*, On X_m -forming sets of eigenvectors, *Proc. Konink. Nederl. Akad. Wetensch. Proc. Ser. A.*, **58** No. 2 (1955); *Indagationes Mathematicae* **17** No. 2, 158–162. *A. Nijenhuis*, X_{n-1} -forming sets of eigenvectors. *Nederl. Akad. Wetensch. Proc. Ser. A.* **54**; *Indagationes Mathematicae* **13** (1951) 200–212.
- [8] *C. Hertling, Y. Manin*, Weak Frobenius manifolds, *Internat. Math. Res. Notices* **1999**, no. 6, 277–286.
- [9] *B. Konopelchenko, F. Magri*, Dispersionless integrable equations as coisotropic deformations: generalizations and reductions. (Russian) *Teoret. Mat. Fiz.* 151 (2007), no. 3, 439–457; translation in *Theoret. and Math. Phys.* **151** No. 3 (2007) 803–819. *B. Konopelchenko, F. Magri*, Coisotropic deformations of associative algebras and dispersionless integrable hierarchies. *Comm. Math. Phys.* **274** No. 3 (2007) 627–658. *B. Konopelchenko*, Discrete integrable systems and deformations of associative algebras. *J. Phys. A: Math. Theor.* **42** No. 45 (2009) 454003, 35 pp. *B. Konopelchenko*,

- G. Ortenzi*, Coisotropic deformations of algebraic varieties and integrable systems. *J. Phys. A: Math. Theor.* **42** No. 41 (2009) 415207, 18 pp. *B.G. Konopelchenko*, Continuous-discrete integrable equations and Darboux transformations as deformations of associative algebras. (Russian) *Teoret. Mat. Fiz.* **159** No. 3 (2009) 502–514; English translation: *Theoret. and Math. Phys.* **159** No. 3 (2009) 842–852. *B. Konopelchenko*, Quantum deformations of associative algebras and integrable systems. *J. Phys. A: Math. Theor.* **42** No. 9 (2009) 095201, 18 pp. *B. Konopelchenko*, On the deformation theory of structure constants for associative algebras. *Adv. Math. Phys.* (2010) Art. ID 389091, 21 pp.
- [10] *I.M. Krichever*, Algebraic-geometrical N orthogonal curvilinear coordinate systems and solutions of the associativity equations. *Funct. Anal. Appl.*, **31** No. 1 (1997) 25–39. *A.A. Akhmetshin, I.M. Krichever, Y.S. Volvovski*, A generating formula for the solutions of the associativity equations. *Russian Math. Surveys*, **54** No. 2 (1999) 427–429.
- [11] *P. Lorenzoni, M. Pedroni*, Natural connections for semi-Hamiltonian systems: The case of the ϵ -system, *Letters in Mathematical Physics*, **97** (2011), no. 1, 85–108.
- [12] *P. Lorenzoni, M. Pedroni, A. Raimondo*, F -manifolds and integrable systems of hydrodynamic type, *Archivum Mathematicum* **47** (2011), 163–180.
- [13] *Y. Manin*, F -manifolds with flat structure and Dubrovin’s duality, *Adv. Math.* **198** (2005), no. 1, 5–26.
- [14] *A. Losev, Y.I. Manin*, Extended modular operads, in: *Frobenius manifolds, quantum cohomology and singularities* (C. Hertling and M. Marcolli, Eds.), *Aspects of Math.*, **E36**, 181–211 (2004).
- [15] *A. Sergyeyev*, Infinite hierarchies of nonlocal symmetries of the Chen–Kontsevich–Schwarz type for the oriented associativity equations. *J. Phys. A: Math. Theor.* **42** No. 40 (2009) paper 404017, 15 pp., arXiv:0806.0177.
- [16] *N.A. Slavnov*, Differential equations for multipoint correlation functions in a one-dimensional impenetrable Bose-gas, *Theor. Math. Phys.*, **106** No. 1 (1996) 131–142.
- [17] *S.P. Tsarev*, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.*, **31** (1985) 488–491. *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, *Math. USSR Izvestiya*, **37** No. 2 (1991) 397–419. *S.P. Tsarev*, On the integrability of the averaged KdV and Benney equations, *Proc. NATO ARW “Singular Limits and Dispersive Waves”*, Lyon, France, 8–12 July, 1991, N. M. Ercolani (ed.) et al., New York, NY: Plenum, NATO ASI Ser., Ser. B, Phys. 320, 143–155 (1994). *S.P. Tsarev*, Differential-geometrical methods of integration of systems of hydrodynamic type. Doctor of Science Thesis, Moscow, 1993, p. 236.