On superintegrable Stäckel systems: elliptic and parabolic coordinates.

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Abstract

Recently we proposed a generic construction of the additional integrals of motion for the Stäckel systems applying addition theorems to the angle variables. In this note we show some trivial examples associated with angle variables for elliptic and parabolic coordinate systems on the plane.

1 Introduction

In classical mechanics Hamiltonian system on a $2n$-dimensional phase space M is called completely integrable in Liouville's sense if it possesses n functionally independent integrals of motion $H_1, \ldots, H_n$ in involution:

$$\frac{dH_i}{dt} = \{H, H_i\} = 0, \qquad \{H_i, H_j\} = 0, \qquad i, j = 1, \ldots, n,$$

where $H = H_1$ is the Hamilton function and $\{.,.\}$ is the Poisson bracket on M.

Superintegrable system is a system that is integrable in the Liouville sense and that possesses more functionally independent integrals of motion than degrees of freedom. The construction of superintegrable Stäckel systems using angle variables ω_k has been proposed in [12, 13, 14, 15].

In generic case the action variables ω_k are multi-valued functions on the whole phase space M. In fact, we can extract polynomial integrals of motion from angle variables only when we can apply addition theorems to the corresponding Abelian integrals. As there are only few addition theorems for the Abel equations [1, 3] we can easily classify the corresponding superintegrable systems, see [12, 13, 14, 15].

The goal of this brief note is to present some trivial examples of applying this generic theory associated with elliptic and parabolic coordinate systems on the plane. Superintegrable systems separable in spherical coordinates can be found in [2, 6]. The corresponding addition integrals of motion are related with an addition theorem for the logarithmic angle variables. Of course, there is a trivial generalization of the proposed method for all the orthogonal coordinate systems in $\mathbb{R}^3$ (ellipsoidal, paraboloidal, cylindrical, prolate and oblate spheroidal coordinates etc).

The non-Stäckel superintegrable systems in classical and quantum mechanics have been considered in [7, 8]. In contrast with the Stäckel case we do not have a generic theory for constructing such superintegrable systems.

2 The Stäckel systems

The system associated with the name of Stäckel [11] is a holonomic system on the phase space $\mathbb{R}^{2n}$, with the canonical variables $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$:

$$\Omega = \sum_{j=1}^{n} dp_j \wedge dq_j\,, \qquad \{p_j, q_k\} = \delta_{jk}\,. \tag{2.1}$$

The nondegenerate $n\times n$ Stäckel matrix S, whose j column depends only on coordinate q_j, defines n functionally independent integrals of motion

$$H_k=\sum_{j=1}^{n}(S^{-1})_{jk}\left(p_j^2+V_j(q_j)\right), \tag{2.2}$$

the separated relations

$$p_j^2=\sum_{k=1}^{n}H_kS_{kj}(q_j)-V_j(q_j)\,, \tag{2.3}$$

and the action variables w_k

$$\omega_i=\frac{1}{n}\sum_{j=1}^{n}\int^{q_j}\frac{S_{ij}(\lambda)}{\sqrt{\sum_{k=1}^{n}H_kS_{kj}(\lambda)-V_j(\lambda)}}\,\mathrm{d}\lambda, \tag{2.4}$$

so that

$$\{H_j,H_k\}=\{\omega_i,\omega_k\}=0,\qquad \{H_j,\omega_k\}=\delta_{ij}. \tag{2.5}$$

In generic case the action variables (2.4) are sums of the multi-valued Abelian integrals. However, if we are able to apply the known addition theorems for these Abelian integrals then we can get additional integrals of motion [12, 13, 14].

Let us discuss addition theorems for the logarithmic (exponential) and elliptic functions [1, 3]. In the first case polynomials

$$P_j=\sum_{k=1}^{n}H_kS_{kj}(\lambda)-V_j(\lambda)$$

are the second order polynomials and

$$\omega=\frac{1}{n}\sum_{j=1}^{m}\int^{q_j}\frac{d\lambda}{\sqrt{k_j^2\lambda^2+b_j\lambda+c_j}}=\sum_{j=1}^{m}\int^{q_j}\ln\left(p_j+\frac{k_j^2q_j+b_j}{k_j}\right)^{\frac{1}{nk_j}}.$$

So, we can easily get polynomial or rational function

$$Z=e^{z\omega}=\prod_{j=1}^{m}\left(p_j+\frac{k_j^2q_j+b_j}{k_j}\right)^{\frac{z}{nk_j}} \tag{2.6}$$

at the special choice of k_j and z. In the second case $m-1$ angle variables $\omega_k,\ldots,\omega_{k+m-1}$ with $1<m\leq n$ have to satisfy to the Abel equations

$$\begin{aligned}
&\frac{d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}}+\frac{d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}}+\cdots+\frac{d\mathrm{x}_m}{\sqrt{f(\mathrm{x}_m)}}=d\omega_k,\\
&\frac{\mathrm{x}_1d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}}+\frac{\mathrm{x}_2d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}}+\cdots+\frac{\mathrm{x}_md\mathrm{x}_m}{\sqrt{f(\mathrm{x}_m)}}=d\omega_{k+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\mathrm{x}_1^{m-2}d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}}+\frac{\mathrm{x}_2^{m-2}d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}}+\cdots+\frac{\mathrm{x}_m^{m-2}d\mathrm{x}_m}{\sqrt{f(\mathrm{x}_m)}}=d\omega_{k+m-1}\,,
\end{aligned} \tag{2.7}$$

with a common polynomial of fixed degree $2m$

$$f(\mathrm{x})\equiv A_{2m}\mathrm{x}^{2m}+A_{2m-1}\mathrm{x}^{2m-1}+\cdots+A_1\mathrm{x}+A_0 \tag{2.8}$$

If this Richelot conditions holds, then there are some additional integrals of motion

$$C_k = \frac{\left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}\cdot\frac{1}{a_k-\mathrm{x}_1}+\cdots+\frac{\sqrt{f(\mathrm{x}_m)}}{F'(\mathrm{x}_m)}\cdot\frac{1}{a_k-\mathrm{x}_m}\right]^2}{\left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}+\cdots+\frac{\sqrt{f(\mathrm{x}_m)}}{F'(\mathrm{x}_m)}\right]^2 - A_{2m}}\,F(a_k)\,. \tag{2.9}$$

Here a_k are values of x at the branch points of the corresponding hyperelliptic curve and $F(\mathrm{x}) = (\mathrm{x}-\mathrm{x}_1)(\mathrm{x}-\mathrm{x}_2)\cdots(\mathrm{x}-\mathrm{x}_m)$ [10, 15]. At $m = 2$ we have famous Euler algebraic integral [3].

Below we show how these addition theorems could help us to classify superintegrable systems.

3 Elliptic coordinate system

Let us consider elliptic coordinates on the plane $q_{1,2}$ defined by

$$1-\frac{x^2}{\lambda-\kappa}-\frac{y^2}{\lambda+\kappa}=\frac{(\lambda-q_1)(\lambda-q_2)}{\lambda^2-\kappa^2}\,,\qquad \kappa\in\mathbb{R}\,.$$

The corresponding momenta reads as

$$p_1=\frac{2p_x x}{q_1-\kappa}+\frac{2p_y y}{q_1+\kappa}\,,\qquad p_2=\frac{2p_x x}{q_2-\kappa}+\frac{2p_y y}{q_2+\kappa}\,.$$

The Stäckel matrix and the separated relations

$$S\begin{pmatrix}\frac{q_1}{q_1^2-\kappa^2} & \frac{q_2}{q_2^2-\kappa^2}\\[2mm] \frac{1}{q_1^2-\kappa^2} & \frac{1}{q_2^2-\kappa^2}\end{pmatrix},\qquad \begin{array}{l} p_1^2+V_1-\frac{q_1H_1}{q_1^2-\kappa^2}-\frac{H_2}{q_1^2-\kappa^2}=0\,,\\[2mm] p_2^2+V_2-\frac{q_2H_1}{q_2^2-\kappa^2}-\frac{H_2}{q_2^2-\kappa^2}=0\,,\end{array} \tag{3.1}$$

give rise to the following Hamiltonians in the involution

$$\begin{aligned} H_1 &= \frac{(q_1^2-\kappa^2)(p_1^2+V_1)}{q_2-q_1}+\frac{(q_2^2-\kappa^2)(p_2^2+V_2)}{q_2-q_1}\,,\\ H_2 &= \frac{q_2(q_1^2-\kappa^2)(p_1^2+V_1)}{q_1-q_2}-\frac{q_1(q_2^2-\kappa^2)(p_2^2+V_2)}{q_1-q_2}\,.\end{aligned}$$

The Hamiltonian H_1 commutes with the second angle variable w_2, which is equal to

$$\begin{aligned} w_2 &= \frac12\int^{q_1}\frac{d\lambda}{\sqrt{(\lambda^2-\kappa^2)(\lambda H_1+H_2-V_1\lambda^2+V_1\kappa^2)}}\\ &+ \frac12\int^{q_2}\frac{d\lambda}{\sqrt{(\lambda^2-\kappa^2)(\lambda H_1+H_2-V_2\lambda^2+V_2\kappa^2)}}\,.\end{aligned}$$

Polynomials

$$P_{1,2}=(\lambda^2-\kappa^2)(\lambda H_1+H_2-V_{1,2}\lambda^2+V_{1,2}\kappa^2) \tag{3.2}$$

standing under square root in these integrals are at least third order polynomials on λ. So, in this case we can not apply addition theorem for the logarithms.

It is easy to see that we can apply addition theorem for the elliptic functions at

$$V_1=V_1=\alpha\,.$$

Namely, if we put $\lambda = \mathrm{x}$ and $\lambda = \mathrm{y}$ in the first and second integrals (3.2), we could apply the Euler addition theorem

$$\frac{d\mathrm{x}}{\sqrt{\mathrm{X}}} + \frac{d\mathrm{y}}{\sqrt{\mathrm{Y}}} = \frac{d\mathrm{s}}{\sqrt{\mathrm{S}}} \tag{3.3}$$

to angle variable ω_2. Here X is an arbitrary quartic

$$\mathrm{X} = a\mathrm{x}^4 + 4b\mathrm{x}^3 + 6c\mathrm{x}^2 + 4d\mathrm{x} + e \tag{3.4}$$

and Y, S are the same functions of another variables y and s. In this case, symmetrical bi-quadratic form of x and y

$$F(\mathrm{x},\mathrm{y}) = a\mathrm{x}^2\mathrm{y}^2 + 2b\mathrm{xy}(\mathrm{x}+\mathrm{y}) + c(\mathrm{x}^2 + 4\mathrm{xy} + \mathrm{y}^2) + 2d(\mathrm{x}+\mathrm{y}) + e = 0 \tag{3.5}$$

defines the conic section on the plane (x,y). According to [1, 3, 10], there is a famous Euler integral

$$C = \frac{F(\mathrm{x},\mathrm{y}) - \sqrt{\mathrm{X}}\sqrt{\mathrm{Y}}}{2(\mathrm{x}-\mathrm{y})^2} = \frac{1}{4}\left(\frac{\sqrt{\mathrm{X}} - \sqrt{\mathrm{Y}}}{\mathrm{x}-\mathrm{y}}\right)^2 - \frac{a(\mathrm{x}+\mathrm{y})^2}{4} - b(\mathrm{x}+\mathrm{y}) - c. \tag{3.6}$$

For the quartic (3.2) associated with the angle variable (3.2) this Euler integral looks like

$$H_3 = \frac{(p_1-p_2)(q_1^2-\kappa^2)(q_2^2-\kappa^2)}{(q_1-q_2)^3}\Big(\alpha(q_1-q_2)^2 - (p_1-p_2)^2\kappa^2 + (p_1q_1 - p_2q_2)^2\Big).$$

Here $q_{1,2}$ and $p_{1,2}$ are elliptic coordinates and momenta.

It is a third order polynomial in momenta which commutes with the Hamiltonian

$$\{H_1, H_2\} = 0\,, \qquad \{H_2, H_3\} = H_4 \neq 0\,.$$

The algebra of the polynomial integrals of motion H_1, H_2, H_3 can be closed only after some other polynomial generators are added.

Thus, we easily find the additional integrals of motion for the Hamilton function of the oscillator

$$4H_1 = p_x^2 + p_y^2 + a(x^2+y^2)$$

using the separation of variables in elliptic coordinate system and the corresponding angle variables. Another result is that there is only one superintegrable system separable in elliptic coordinates and associated with the known addition theorems for Abelian integrals.

4 Parabolic coordinate system

Let us consider parabolic coordinates on the plane $q_{1,2}$ defined by

$$x = q_1q_2, \qquad y = \frac{q_1^2 - q_2^2}{2}$$

and the corresponding momenta

$$p_x = \frac{p_1q_2 + p_2q_1}{q_1^2+q_2^2}\,, \qquad p_y = \frac{q_1p_1 - q_2p_2}{q_1^2+q_2^2}\,,$$

The Stäckel matrix and the separated relations

$$S\begin{pmatrix} q_1^2 & q_2^2 \\ 1 & -1 \end{pmatrix}, \qquad \begin{array}{l} p_1^2 + V_1(q_1) - q_1^2H_1 - H_2 = 0\,, \\ p_2^2 + V_2(q_2) - q_2^2H_1 + H_2 = 0\,, \end{array} \tag{4.1}$$

give rise to the Hamiltonians

$$H_1 = \frac{p_1^2 + p_2^+ V_1(q_1) + V_2(q_2)}{q_1^2 + q_2^2}\,, \qquad H_2 = \frac{p_1^2 q_2^2 - p_2^2 q_1^2 + q_2^2 V_1(q_1) - q_1^2 V_2(q_2)}{q_1^2 + q_2^2}\,.$$

The Hamiltonian H_1 commutes with the second angle variable w_2, which is equal to

$$w_2 = \frac{1}{2}\int^{q_1} \frac{d\lambda}{\sqrt{\lambda^2 H_1 + H_2 - V_1(\lambda)}} + \frac{1}{2}\int^{q_2} \frac{d\lambda}{\sqrt{\lambda^2 H_1 - H_2 - V_2(\lambda)}}\,.$$

In contrast with the elliptic coordinates, polynomials

$$P_{1,2} = \lambda^2 H_1 + H_2 - V_{1,2}(\lambda)$$

standing under square root in these integrals are at least second order polynomials on λ. So, we can apply both known addition theorems to these Abelian integrals.

In fact, these integrals are expressed via logarithmic functions iff:

$$\text{Case 1}: \qquad V_1 = b_1 q_1 + c_1\,, \qquad V_2 = b_2 q_2 + c_2\,, \tag{4.2}$$

$$\text{Case 2}: \qquad V_1 = a_1 q_1^{-2} + b_1\,, \qquad V_2 = a_2 q_2^{-2} + b_1\,. \tag{4.3}$$

The addition theorem for the elliptic function is applicable iff:

$$\text{Case 3}: \qquad V_1 = a_1 q_1^6 + b_1 q_1^4 + c_1 q_1^{-2}\,, \qquad V_2 = a_1 q_2^6 - b_1 q_2^4 + c_1 q_2^{-2}\,. \tag{4.4}$$

The corresponding Hamilton functions are deformations of the Kepler-Coulomb and oscillator Hamiltonians:

$$\begin{aligned}
\text{Case 1}: \qquad H_1 &= p_x^2 + p_y^2 + \frac{1}{2\sqrt{x^2+y^2}}\left(b_1\sqrt{x+\sqrt{x^2+y^2}} + b_2\sqrt{\sqrt{x^2+y^2}-x} + c_1 = c_2\right) \\
&= p_x^2 + p_y^2 + \frac{1}{2r}\left(b_1\sqrt{2}\cos\frac{\varphi}{2} + b_2\sqrt{2}\sin\frac{\varphi}{2} + c_1 + c_2\right); \\
\text{Case 2}: \qquad H_1 &= p_x^2 + p_y^2 + \frac{1}{2\sqrt{x^2+y^2}}\left(\frac{a_1}{x+\sqrt{x^2+y^2}} + \frac{a_2}{x-\sqrt{x^2+y^2}} + b_1 + b_2\right) \\
&= p_x^2 + p_y^2 + \frac{a_1}{2r^2(\cos\varphi+1)} - \frac{a_2}{2r^2(\cos\varphi-1)} + \frac{b_1+b_2}{2r}; \\
\text{Case 3}: \qquad H_1 &= p_x^2 + p_y^2 + \alpha\left(4\,x^2 + y^2\right) + 2\,\beta\,x + \frac{\gamma}{y^2}\,.
\end{aligned} \tag{4.5}$$

Here $r = \sqrt{x^2+y^2}$ and $\varphi = \arctan x/y$ are polar coordinates on the plane. According to [9] these systems remain superintegrable in the quantum case.

4.1 Case 1

In the first case the second angle variable equals

$$\begin{aligned}
\omega_2 &= \frac{1}{2}\int^{q_1} \frac{d\lambda}{\sqrt{q_1^2 H_1 - b_1 q_1 + H_2 - c_1}} - \frac{1}{2}\int^{q_2} \frac{d\lambda}{\sqrt{q_2^2 H_1 - b_2 q_2 - H_2 - c_2}} \\
&= \frac{\ln\left(p_1 - \frac{q_1 H_1 - b_1/2}{\sqrt{H_1}}\right)}{2\sqrt{H_1}} - \frac{\ln\left(p_2 - \frac{q_2 H_1 - b_2/2}{\sqrt{H_1}}\right)}{2\sqrt{H_1}}\,.
\end{aligned}$$

The application of the addition theorem (2.6) to ω_2 gives rise to the following rational integral of motion

$$Z = e^{2\sqrt{H_1}\omega_2} = \frac{2q_1 H_1 + 2p_1\sqrt{H_1} - b_1}{2q_2 H_1 + 2p_2\sqrt{H_1} - b_2}\,.$$

In order to calculate polynomial integral of motion let us consider a series expansion of the function

$$f=\frac{1}{\sqrt{H_1}}\,\left(\alpha Z+\beta Z^{-1}\right)=\frac{\alpha(4H_1H_2-4H_1c_1-b_1^2)-\beta(4H_1H_2+4H_1c_2+b_2^2)}{\sqrt{H_1(4H_1c_2+4H_1H_2+b_2^2)(b_1^2+4H_1c_1-4H_1H_2)}}+O(p_{1,2})\,,$$

by momenta $p_{1,2}$. Here we substitute the variables $q_{1,2}$ from the separated relations (4.1) into the rational integral Z and α,β are undefined polynomials in $H_{1,2}$.

Equating first coefficient of this expansion to zero one gets the following expressions for these polynomials

$$\alpha=4H_1H_2+4H_1c_2+b_2^2\,,\qquad \beta=4H_1H_2-4H_1c_1-b_1^2\,.$$

At this values of α and β the function f becomes a third order polynomial in momenta

$$\begin{aligned}H_3 &= \frac{1}{\sqrt{H_1}}\,\left(\alpha Z+\beta Z^{-1}\right)=\frac{1}{\sqrt{H_1}}\,\left(\alpha e^{2\sqrt{H_1}\omega_2}+\beta e^{-2\sqrt{H_1}\omega_2}\right)\\ &= \tfrac{8(p_1q_2-p_2q_1)(p_1^2+p_2^2+c_1+c_2)}{q_1^2+q_2^2}+\tfrac{4\left(2p_1q_1q_2-p_2(q_1^2-q_2^2)\right)b_1}{q_1^2+q_2^2}-\tfrac{4\left(2p_2q_1q_2+p_1(q_1^2-q_2^2)\right)b_2}{q_1^2+q_2^2}\,,\end{aligned}$$

such that

$$\{H_1,H_3\}=0\,.$$

In order to close the algebra of the polynomial integrals of motion H_1,H_2,H_3 we have to add one more polynomial generator

$$H_4=\{H_2,H_3\}=2\alpha e^{2\sqrt{H_1}\omega_2}-2\beta e^{-2\sqrt{H_1}\omega_2}\,.$$

by analogy with $\exp(\omega)$, $\sin(\omega)$ and $\cos(\omega)$ functions.

4.2 Case 2

In the second case the angle variable is equal to

$$\omega_2=\frac12\int^{q_1}\frac{\lambda d\lambda}{\sqrt{\lambda^4H_1+(H_2-b_1)\lambda^2-a_1}}-\frac12\int^{q_2}\frac{\lambda d\lambda}{\sqrt{\lambda^4H_1-(H_2+b_2)\lambda^2-a_2}}\,.$$

Changing variables $\mu=\lambda^2$ one gets second order polynomials

$$P_j=\mu^2H_1\pm(H_2\mp b_j)\mu-a_j$$

under the square root and desired sum of the logarithms

$$\omega_2=\frac{\ln\left(2q_1^2H_1+2q_1p_1\sqrt{H_1}+H_2-b_1)\right)}{4\sqrt{H_1}}-\frac{\ln\left(2q_2^2H_1+2q_2p_2\sqrt{H_1}-H_2-b_2)\right)}{4\sqrt{H_1}}\,.$$

The rational integral of motion (2.6) is equal to

$$Z=e^{4\sqrt{H_1}\omega_2}=\frac{2q_1^2H_1+2q_1p_1\sqrt{H1}+H_2-b_1}{2q_2^2H_1+2q_2p_2\sqrt{H_1}-H_2-b_2}\,.$$

As above we consider the expansion of the function

$$f=\frac{1}{\sqrt{H_1}}\,\left(\alpha Z+\beta Z^{-1}\right)$$

by momenta $p_{1,2}$. Equating first coefficient of this expansion to zero one gets polynomials α,β

$$\alpha=4H_1a_2+b_2^2+2b_2H_2+H_2^2\,,\qquad \beta=-4H_1a_1-b_1^2+2b_1H_2-H_2^2\,.$$

At this values of α and β the function f becomes a third order polynomial in momenta

$$H_3 = \frac{1}{\sqrt{H_1}}\left(\alpha Z + \beta Z^{-1}\right) = \frac{1}{\sqrt{H_1}}\left(\alpha e^{4\sqrt{H_1}\omega_2} + \beta e^{-4\sqrt{H_1}\omega_2}\right) = \frac{4(q_1p_1-q_2p_2)(q_2p_1-q_1p_2)^2}{q_1^2+q_2^2}$$

$$+\frac{4q_1q_2(q_2p_1-q_1p_2)(b_1+b_2)}{q_1^2+q_2^2} + \frac{4a_1q_2\left(q_2q_1p_1-(2q_1^2+q_2^2)p_2\right)}{q_1^2(q_1^2+q_2^2)} - \frac{4a_2q_1\left(q_2q_1p_2-(2q_2^2+q_1^2)p_1\right)}{q_2^2(q_1^2+q_2^2)}\,.$$

such that

$$\{H_1, H_3\} = 0\,.$$

In order to close the algebra of the polynomial integrals of motion H_1, H_2, H_3 we have to add one more polynomial generator

$$H_4 = \{H_2, H_3\} = 4\alpha e^{4\sqrt{H_1}\omega_2} - 4\beta e^{-4\sqrt{H_1}\omega_2}\,.$$

4.3 Case 3

In the third case the angle variable is equal to

$$\omega_2 = \int^{q_1} \frac{\lambda d\lambda}{\sqrt{-a_1\lambda^8 - b_1\lambda^6 + H_1\lambda^4 + H_2\lambda^2 - c_1}} - \int^{q_2} \frac{\lambda d\lambda}{\sqrt{-a_1\lambda^8 + b_1\lambda^6 + H_1\lambda^4 - H_2\lambda^2 - c_1}}\,.$$

Changing variables $\lambda = \mathrm{x}$ and $\lambda = i\sqrt{\mathrm{y}}$ at the first and second integral one gets the Euler addition theorem (3.3). In fact, this example has been considered in Euler's book [3] too.

Identifying quartic

$$P = -a_1\mu^4 - b_1\mu^3 + H_1\mu^2 + H_2\mu - c_1$$

with X (3.4) we can easily calculate the Euler integral of motion (3.6) in parabolic coordinates

$$H_3 = s = \frac{(q_1p_1-q_2p_2)(q_1p_2+q_2p_1)^2}{(q_1^2+q_2^2)^3} + \frac{a_1q_1q_2(2q_1^3p_2+q_2q_1^2p_1-q_1q_2^2p_2-2q_2^3p_1)}{q_1^2+q_2^2} + \frac{b_1q_1q_2(q_1p_2+q_2p_1)}{q_1^2+q_2^2}$$
$$+\frac{c_1(q_1p_1-q_2p_2)}{q_1^2q_2^2(q_1^2+q_2^2)}\,.$$

The algebra of the integrals of motion H_1, H_2, H_3 is more complicated then the algebra associated with the addition theorem for logarithms. In fact, in order to close this algebra we have to introduce the counterparts of the Jacobi elliptic functions $\mathrm{sn}(\omega)$, $\mathrm{cn}(\omega)$ and $\mathrm{dn}(\omega)$ instead of the trigonometric functions $\sin(\omega)$ and $\cos(\omega)$, which we used for the superintegrable systems associated with the addition theorem for logarithms.

5 Conclusion

It is known that orthogonal coordinate systems on Riemaniann manifolds can be viewed as an orthogonal sum of certain basic coordinate systems and these basic systems can be obtained from the elliptic coordinate system [5] using a degeneration procedure. This degeneration decreases the degree of polynomials standing under square roots into the angle variables (2.4). Thus, we have only one superintegrable systems separable in elliptic coordinates, whereas for degenerations we have a lot of different superintegrable systems. As usual, the addition theorem for logarithms allows us to get additional integrals of higher order in momenta [13].

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