Liouville-Arnold Integrability for Scattering under Cone Potentials

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1. Introduction

The problem of scattering of particles on the line with repulsive interactions, gives rise to some well-known integrable Hamiltonian systems, for example, the nonperiodic Toda lattice or Calogero's system. The aim of this note is to outline our researches which proved the integrability of a much larger class of systems, including some that had never been considered, such as the scattering with very-long-range interaction potential. The integrability of all these systems survives any small enough perturbation of the potential in an arbitrary compact set. Our framework is based on the concept of cone potentials, as defined below, which include the scattering on the line as a particular case.

In Section 2 we present some remarkable examples that are covered by our theory. In Section 3 we discuss the related literature. Finally, in Section 4 we write down the statements of the results.

Let us consider a particle $q \in \mathbb{R}^n$ that moves in the field generated by a potential $\mathcal{V}$:

$$\dot{q} = p\,, \qquad \dot{p} = -\nabla\mathcal{V}(q)\,. \tag{1.1}$$

Suppose that $\mathcal{V} \in C^2(\mathbb{R}^n)$ and that

i) *$\mathcal{V}$ is bounded from below;*

ii) *the force $-\nabla\mathcal{V}(q)$ always lies in a closed, convex cone $\bar{\mathbb{C}}$ which is "proper", i.e., containing no straight lines ("cone potential").*

It is then well-known and easy to prove that all trajectories have a finite *asymptotic velocity* $p_\infty := \lim_{t\to+\infty} p(t)$, as we are going to see.

Observe first that the system admits the first integral of energy $\frac{1}{2}|p(t)|^2 + \mathcal{V}(q(t))$. From i) we see that every trajectory has a bounded speed, so that it is also globally defined in time.

On the other hand, property ii) is equivalent to the existence of a basis $\{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ such that $-\nabla\mathcal{V}(q) \cdot b_i \geq 0$ for all $q \in \mathbb{R}^n$ and all i (see e.g. Gorni, Zampieri [GZ1]). Hence for every trajectory $(p(t), q(t))$ the map $t \mapsto p(t) \cdot b_i$ is a monotone function for all b_i. Monotone bounded functions defined on $\mathbb{R}$ always admit a finite limit at $+\infty$, whence the claim is proved.

The existence of asymptotic velocities is a characteristic of scattering systems. Cone potential systems have actually been introduced as a generalization of the problem of a system of mutually repelling particles in one dimension.

The n components of p_∞ are trivially constants of motion. Some heuristics make them also likely candidates for being n independent first integrals in involution as called for by the theory of Integrable Hamiltonian Systems (see e.g. Arnold [A]). The conjecture, suggested by Gutkin in [Gu1], is false without further restrictions on the potential, though. In particular, the asymptotic velocity may not even be a continuous function of the initial data (see [GZ1], Section 3), and sometimes there are geometrical obstructions to the global independence of its components, even if they happened to be smooth (see [GZ3], Sections 1 and 5).

In the papers [GZ1,2,3] the authors develop a theory of the C^k ($2 \le k \le +\infty$) regularity of the asymptotic velocity and Liouville-Arnold integrability for cone potentials, whose main points we are going to sketch here.

Let $\mathbb{C}$ be the convex cone generated by the forces, that is, the smallest convex cone that contains all vectors $-\nabla\mathcal{V}(q)$ for q in the domain of $\mathcal{V}$ (the potentials in the examples below come with their natural domains; for simplicity think of the whole of $\mathbb{R}^n$ for now). Let us denote by $\mathcal{M}$ the subset of $\mathbb{R}^n \times \mathbb{R}^n$ of the initial data $(\text{velocity}, \text{position})$ that are *asymptotically regular*, i.e., that lead to an asymptotic velocity with *positive* scalar product with all vectors of the closure of $\mathbb{C}$:

$$p_\infty \cdot v > 0 \qquad \forall v \in \bar{\mathbb{C}} \tag{1.2}$$

(in the language of convex sets, this amounts to saying that p_∞ lies in the *interior* of the *dual cone* $\mathbb{C}^*$). Given these basic definitions, the theory splits into two distinct parts:

I. $\mathcal{M}$ is clearly an *invariant* subset of the phase space. Under very general and natural hypotheses on the potential (made explicit in [GZ3]), $\mathcal{M}$ is also *open* and *nonempty*. The bulk of [GZ1] is devoted to proving that the asymptotic velocity p_∞ is a smooth (C^k, $2 \le k \le +\infty$) function of the initial data on $\mathcal{M}$, if suitable additional hypotheses are made on $\mathcal{V}$. The starting point is the integral expression for the asymptotic velocity: $p_\infty = p(0) - \int_0^{+\infty} \nabla\mathcal{V}(q(t))dt$. Two basic ways are found to apply to this formula the usual theorems of continuity and differentiability of integrals depending on parameters. If $\mathcal{V}$ decays *exponentially* at infinity (see a later section for precise statements), we give Gronwall-like estimates on the growth of the derivatives of q with respect to the initial data. If instead $\mathcal{V}$ is assumed *convex*, then a simple *monotonicity* condition on the Hessian matrix of $\mathcal{V}$ enables us to build a Liapunov function that checks the growth of the derivatives of q without special decay requirements on $\mathcal{V}$. In both settings it is proved, again in [GZ1], that the components of p_∞, in addition to being differentiable, are independent and in involution on $\mathcal{M}$, and the range of p_∞ (on $\mathcal{M}$) is exactly the interior of the dual cone $\mathbb{C}^*$. In particular, the Hamiltonian system is integrable if it is restricted to the open, nonempty, invariant set $\mathcal{M}$.

II. In [GZ1,3] we provide examples where $\mathcal{M}$ does not coincide with the whole phase space $\mathbb{R}^n \times \mathbb{R}^n$. This can happen either for a trivial reason such as the presence of an equilibrium ($\nabla\mathcal{V}(q) = 0$), or for subtler reasons. In those same papers we present three different hypotheses on the potential $\mathcal{V}$ that guarantee that all initial data are asymptotically regular. Two of them have close parallels in Gutkin [Gu2] and Hubacher [Hu].

If we collect together the hypotheses of I with those of II, we come up with wide classes of *completely integrable systems.*

The paper [GZ2] shows that if we restrict to potentials with fast enough decay at infinity, then the motion of the particle q is asymptotically rectilinear uniform: $q(t) = a_\infty + p_\infty t + o(1)$ as $t \to +\infty$, and the *asymptotic data* (p_∞, a_∞), as functions of the initial data, define a *global canonical diffeomorphism ("asymptotic map")* $\mathcal{A}\colon (p, q) \mapsto (P, Q)$, which brings the original Hamiltonian system (1.1) into the *normal form*:

$$\dot{P} = 0\,, \qquad \dot{Q} = P\,. \tag{1.3}$$

A noteworthy property of the systems introduced in [GZ1,2,3] is that the integrability, and (when it is the case) the reduction to normal form via asymptotic data, endure any small enough perturbation of the potential $\mathcal{V}$ in any arbitrary compact set of $\mathbb{R}^n$. Such "structural stability" property seems to be unusual in the literature on integrable Hamiltonian systems, that have been studied mostly by algebraic techniques which are rather "rigid".

2. Examples

Some remarkable examples of cone potential systems that are covered by our theory of C^∞-integrability are the following. Here $N \geq 1$ and the vectors $v_\alpha \in \mathbb{R}^n$, with $\alpha = 1, \ldots, N$, generate a proper cone (i.e., there exists $\bar{v} \in \mathbb{R}^n$ such that $\bar{v} \cdot v_\alpha > 0 \;\; \forall\alpha$).

a) Toda-like, or exponential potentials

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} c_\alpha e^{-q \cdot v_\alpha}\,, \qquad c_\alpha > 0, \quad q \in \mathbb{R}^n\,. \tag{2.1}$$

b) Finite sum of inverse powers:

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} \frac{1}{(q \cdot v_\alpha)^{r_\alpha}}\,, \qquad q \in \{\bar{q} \in \mathbb{R}^n : \bar{q} \cdot v_\alpha > 0 \; \forall\alpha\}\,, \tag{2.2}$$

in three different hypotheses:

b1) $r_\alpha > 0$ arbitrary but the vectors satisfy $v_\alpha \cdot v_\beta \geq 0$;

b2) v_α arbitrary but all the exponents satisfy $r_\alpha > 1$;

b3) v_α arbitrary but the exponents r_α are all equal to an $r > 0$.

In particular b3) includes the scattering of particles in one dimension with inverse r-power potential, for arbitrary $r > 0$:

$$\mathcal{V}(q) := \sum_{1 \leq i < j \leq n} \frac{1}{(q_i - q_j)^r}\,, \qquad q \in \{(q_1, \ldots, q_n) \in \mathbb{R}^n : q_1 < q_2 < \ldots < q_n\}\,. \tag{2.3}$$

The case $r = 1$ is the Coulombian potential, whilst $r = 2$ is the Calogero potential. The exponent $r = 1$ divides what are usually called short range potentials ($r > 1$) from the long range ones ($0 < r \leq 1$).

c) Even longer range potentials are the inverse-logarithmic type:

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} \frac{1}{(\ln(1 + q \cdot v_\alpha))^r}, \qquad q \in \{\bar{q} \in \mathbb{R}^n \,:\, \bar{q} \cdot v_\alpha > 0 \;\forall \alpha\}, \tag{2.4}$$

for arbitrary $r > 0$.

For potentials a), b1) with $r_\alpha > 1$, and b2), there are both asymptotic velocities and phases, with consequent reduction to normal form. In case b3) with $0 < r \leq 1$ and c) we prove integrability through asymptotic velocities but we do not have the reduction to normal form through asymptotic map.

All the previous examples share the "finite sum" form, that gives rise to *polyhedral* cones $\bar{\mathbb{C}}$ and $\mathbb{C}^*$. This is not an essential feature of the theory, as we show in [GZ2], where we prove the C^∞-integrability of the system in three dimensions whose potential is

$$\mathcal{V}(x, y, z) := \frac{2z}{z^2 - x^2 - y^2} \exp\Big(-\frac{z^2 - x^2 - y^2}{2z}\Big) \tag{2.5}$$

defined on the set $\mathcal{D}^\circ = \{(x, y, z) \in \mathbb{R}^3 \;:\; z > \sqrt{x^2 + y^2}\}$. Here the cone $\mathbb{C}$ and the interior of the dual turn out to coincide with the domain of $\mathcal{V}$, which is a *circular* cone.

3. Related papers

Special instances of integrable systems with cone potentials, such as the non-periodic Toda lattice and Calogero's system, have been well-known since the seventies. The techniques of integration did not make any use of the asymptotic velocity (see e.g. Moser [M]).

Gutkin in [Gu1] introduced the concept of cone potentials with the conjecture that the integrability of those special systems could be derived from a general theory. Stimulated by Gutkin's idea, Oliva and Castilla in [OC] gave a rigorous proof of C^∞-integrability via asymptotic velocities for a class of finite-sum potentials which decay exponentially using Dynamical Systems techniques, very different from our approach.

The results of Hubacher's paper [Hu] overlap partially with the ones of [GZ3]. It is concerned with systems of mutually repulsive particles on the line, that fall under what we call "finite-sum form" potential, as in formula (2.3). Hubacher states the integrability through asymptotic data in the case of "short range" potentials (roughly corresponding to $r > 1$ in (2.3)) by invoking the results of Simon [S] and Herbst [He]. Those authors had studied the scattering of n mutually repulsive particles in $\mathbb{R}^n$, through an approach totally different from ours. The proof of smoothness was obtained there by solving a kind of Cauchy problem at infinity with the asymptotic data playing the role of initial data. The method is remarkably simple for short range potentials, but it gets into complications for long range potentials ($0 < r \leq 1$), and does not seem to cover for example the inverse-logarithm case (2.4). (Incidentally, Hubacher also obtains another interesting result outside the spirit of the present paper).

Moauro, Negrini and Oliva [MNO] recently gave a proof of *analytic* integrability for the systems of cases a) and b) in Section 2 by means of Dynamical Systems techniques.

Finally, we remark that the problem of the mere existence of asymptotic velocities and phases for scattering systems of mutually repelling particles in $\mathbb{R}^n$ was studied by various authors in the sixties and seventies, see for instance [Ga] and the references contained therein.

4. Statements of Results

Given a smooth function $\mathcal{V}\colon \mathbb{R}^n \to \mathbb{R}$, we will denote by $\nabla\mathcal{V}$ its gradient, as a column vector, and $\mathsf{D}^m\mathcal{V}$ will be its m-th differential, regarded as a multilinear map from $(\mathbb{R}^n)^{m-1}$ into $\mathbb{R}^n$, endowed with the norm

$$\| \mathsf{D}^m\mathcal{V}(q)\| := \sup\{| \mathsf{D}^m\mathcal{V}(q)(x^{(1)},\ldots,x^{(m-1)})| \ : \ x^{(i)} \in \mathbb{R}^n, \ |x^{(i)}| \le 1\}. \tag{4.1}$$

In the sequel, $\mathbb{C}$ will be the convex cone generated by the force $-\nabla\mathcal{V}$ of a cone potential, and $\mathcal{D}$ will be the dual of $\mathbb{C}$:

$$\mathcal{D} := \left\{ v \in \mathbb{R}^n \ : \ w \cdot v \ge 0 \quad \forall w \in \mathbb{C} \right\}. \tag{4.2}$$

Throughout this section the function $\mathcal{V}$ can be assumed to be defined either on all of $\mathbb{R}^n$ or on a set of the form $q + \mathcal{D}^\circ$. In addition to the asymptotic velocity, we are also interested in the existence and the smoothness of the *asymptotic map*

$$\begin{pmatrix} p_\infty(\bar p, \bar q) \\ a_\infty(\bar p, \bar q) \end{pmatrix} = \mathcal{A}(\bar p, \bar q) := \lim_{t\to+\infty} \mathcal{A}_t(\bar p, \bar q)\,. \tag{4.3}$$

We start with the assumptions corresponding to point I of the Introduction, that is, those leading to integrability on the set $\mathcal{M}$.

Hypothesis 4.1 *There exists $q_0 \in \operatorname{dom}\mathcal{V}$ and a nonnegative, weakly decreasing and integrable function $h_0\colon [0,+\infty[\to \mathbb{R}$ such that*

$$q \in q_0 + \mathcal{D} \quad \Rightarrow \quad |\nabla\mathcal{V}(q)| \le h_0\big(\operatorname{dist}(q, q_0 + \partial\mathcal{D})\big)\,. \tag{4.4}$$

Hypotheses 4.2 *The potential $\mathcal{V}$ is a C^{m+1}, $m \ge 2$, function. For all $1 \le i \le m$ there exist $q_i \in \mathbb{R}^n$, $A_i \ge 0$, $\lambda_i > 0$ such that*

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \| \mathsf{D}^{i+1}\mathcal{V}(q) \| \le A_i \exp\Big(-\lambda_i \operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big)\Big)\,. \tag{4.5}$$

Hypotheses 4.3 *The potential $\mathcal{V}$ is a C^{m+1}, $m \ge 2$, function. For all $1 \le i \le m$ there exist $q_i \in \mathbb{R}^n$, and a weakly decreasing function $h_i\colon [0,+\infty[\to \mathbb{R}$ such that*

1) *$\mathcal{V}$ is convex on $q_1 + \mathcal{D}$;*
2) *for all q', $q'' \in q_1 + \mathcal{D}$ and all $z \in \mathbb{R}^n$ we have*

$$q'' \in q' + \mathcal{D} \quad \Rightarrow \quad \mathsf{D}^2\mathcal{V}(q'')z\cdot z \le \mathsf{D}^2\mathcal{V}(q')z\cdot z; \tag{4.6}$$

3) *for all i, $\int_0^{+\infty} x^i\, h_i(x)dx < +\infty$ and*

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \| \mathsf{D}^{i+1}\mathcal{V}(q)\| \le h_i\Big(\operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big)\Big). \tag{4.7}$$

Here is the precise statement of the integrability on the set $\mathcal{M}$.

Proposition 4.4 *Assume that $\mathcal{V}$ is a cone potential and Hypotheses 4.1 and either 4.2 or 4.3. Then the components of the asymptotic velocity p_∞ are C^m first integrals, independent and in involution on the set $\mathcal{M}$ of the asymptotically regular initial data. If, moreover, the functions h_i of Hypotheses 4.1 and (when the case) 4.3 verify $\int_0^{+\infty} x^{i+1}\, h_i(x)\, dx < +\infty$ for all $0 \le i \le m+1$, then the asymptotic map $\mathcal{A}$ exists on $\mathcal{M}$, it is a global canonical C^m diffeomorphism from $\mathcal{M}$ onto $\mathcal{D}^\circ \times \mathbb{R}^n$, and it brings the restricted phase space system*

$$\dot{p} = -\nabla\mathcal{V}(q)\,, \quad \dot{q} = p\,, \qquad p, q \in \mathcal{M} \tag{4.8}$$

into the normal form

$$\dot{P} = 0\,, \quad \dot{Q} = P\,, \qquad (P, Q) \in \mathcal{D}^\circ \times \mathbb{R}^n\,. \tag{4.9}$$

Next we present the assumptions under which we can prove that the set $\mathcal{M}$ coincides with the whole phase space. The first one guarantees that all trajectories are contained in some translation of $\mathcal{D}$, so that the asymptotic velocity lies always in the closed set $\mathcal{D}$, although possibly on the boundary.

Hypotheses 4.5 *For each $E > 0$ there exists a $q_E \in \mathbb{R}^n$ such that*

$$q \in \mathbb{R}^n \backslash (q_E + \mathcal{D}) \quad \Rightarrow \quad \mathcal{V}(q) \ge E\,. \tag{4.10}$$

The following hypothesis is not so transparent at first view, but one of its points is that the cone of the forces $\mathbb{C}$ should be of width not larger than $\pi/2$. No finite-sum form of the potential is called for, though. The circular cone example of the previous Section falls into this category.

Hypotheses 4.6 *For each q', $q'' \in \mathbb{R}^n$ such that $q'' \in q' + \mathcal{D}$, and for each $v \in \bar{\mathbb{C}}\backslash\{0\}$ there exists $\varepsilon > 0$ such that*

$$\Big(q \in q' + \mathcal{D} \quad \text{and} \quad q \cdot v \le q'' \cdot v \Big) \quad \Rightarrow \quad -\nabla\mathcal{V}(q) \cdot v \ge \varepsilon\,. \tag{4.11}$$

The last assumptions concern the potentials which can be written as finite sums of one-dimentional functions ("finite-sum potentials"):

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} f_\alpha(q \cdot v_\alpha)\,, \tag{4.12}$$

where $v_1, \dots, v_N$ are given nonzero vectors in $\mathbb{R}^n$ ($N \ge 1$, no relation to n), and the functions $f_1, \dots, f_N$ are real functions of one variable, whose domains are each either $\mathbb{R}$ or the interval $]0, +\infty[$. The potential $\mathcal{V}$ is accordingly defined either on $\mathbb{R}^n$ or on $\mathcal{D}^\circ = \{q \in \mathbb{R}^n \ : \ q \cdot v_\alpha > 0\ \forall \alpha\}$.

Hypotheses 4.7 *The vectors $v_1, \ldots, v_N$ are nonzero and the cone generated by them is proper. The f_α are C^{m+1} ($m \geq 2$) functions and*

$$\sup f_\alpha = +\infty\,, \qquad \inf f_\alpha = 0\,, \tag{4.13}$$

$$f'_\alpha(x) < 0 \quad \forall x \in \mathrm{dom}\ f_\alpha\,, \tag{4.14}$$

$$f_\alpha^{(k)}(x) \begin{cases} > 0 & \text{if } k \text{ is even,} \\ < 0 & \text{if } k \text{ is odd,} \end{cases} \qquad \forall x \geq a\,, \tag{4.15}$$

$$f_\alpha^{(m+1)} \text{ is monotone on } [a, +\infty[, \tag{4.16}$$

where $a \geq 0$ is a constant. Moreover, whichever of the three following conditions i, ii, iii holds:

i) the vectors v_α verify $v_\alpha \cdot v_\beta \geq 0\ \forall \alpha, \beta$;

ii) all the functions f_α are multiples of a single function f:

$$f_\alpha = c_\alpha f\,, \quad c_\alpha > 0\,, \tag{4.17}$$

such that

$$x \mapsto x|f'(x)| \text{ is weakly decreasing on } [a, +\infty[\,; \tag{4.18}$$

iii) for all $\alpha = 1, \ldots, N$, the function f_α is such that

$$\int_a^{+\infty} f_\alpha(x)\, dx < +\infty\,. \tag{4.19}$$

Now we are going to write down the statement of global integrability. We skip for simplicity the intermediate results concerning the mere point II of the Introduction.

Theorem 4.8 *Assume either one of the two following sets of hypotheses:*

a. the ones of Proposition 4.3 plus Hypotheses 4.5 and 4.6;

b. Hypotheses 4.7 for a potential of the form (4.12).

Then the Hamiltonian system

$$\dot{p} = -\nabla \mathcal{V}\,, \quad \dot{q} = p\,, \tag{4.20}$$

is C^m-completely integrable via asymptotic velocities on all of the phase space. If moreover we are either in Hypotheses 4.7, case iii, or the functions h_i of Hypotheses 4.1 and 4.3 verify $\int_0^{+\infty} x^{i+1} h_i(x)\, dx < +\infty$ for all $0 \leq i \leq m+1$, then the asymptotic map $\mathcal{A}$ exists globally and it performs the reduction to normal form.

Here is what we can say about the integrability of a slightly perturbed system.

Theorem 4.9 *Suppose that we are in such hypotheses on $\mathcal{V}$ that we can apply Theorem 4.8. Let K be a compact set contained in the domain of $\mathcal{V}$. Then there exists an $\varepsilon > 0$ with the following property. Let V be a C^{m+1} real function defined in $\mathrm{dom}\,\mathcal{V}$, vanishing outside K, and such that*

$$\sup_K |\nabla V| \leq \varepsilon\,. \tag{4.21}$$

Then the thesis of Theorem 4.8 applies to the system with potential $\mathcal{V} + V$.

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