A Class of Integrable Hamiltonian Systems Including Scattering of Particles on the Line with Repulsive Interactions

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Abstract. The main purpose of this paper is to introduce a new class of Hamiltonian scattering systems of the cone potential type that can be integrated via the asymptotic velocity. For a large subclass, the asymptotic data of the trajectories define a global canonical diffeomorphism $\mathcal{A}$ that brings the system into the normal form $\dot{P} = 0$, $\dot{Q} = P$.

The integrability theory applies for example to a system of n particles on the line interacting pairwise through rather general repulsive potentials. The inverse r-power potential for arbitrary $r > 0$ is included, the reduction to normal form being carried out for the exponents $r > 1$. In particular, the Calogero system is obtained for $r = 2$. The treatment covers also the nonperiodic Toda lattice.

The cone potentials that we allow can undergo small perturbations in any arbitrary compact set without losing the integrability and the reduction to normal form.

1. Introduction

An authonomous $2n$-dimensional Hamiltonian system is said to be integrable if there exist n smooth first integrals, independent and in involution (see e.g. [A] for details). The *scattering* systems provide natural constants of motion: the *asymptotic velocities*. Unfortunately, there is no obvious reason for them to be smooth functions of the initial data, and in fact they are sometimes not even continuous (see e.g. [GZ1], Section 3). Some work has been done to single out classes of scattering-type systems for which rigorous proofs of smoothness (C^k, $2 \leq k \leq +\infty$) of the asymptotic data and of integrability could be carried out.

In [GZ1] we investigated the *complete integrability* of Hamiltonian systems of the form

$$\dot{p} = -\nabla \mathcal{V}(q) \,, \quad \dot{q} = p \,, \qquad p, q \in \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{V}$ is a C^k ($3 \leq k \leq +\infty$) potential—defined on a domain that we can suppose for now to be simply $\mathbb{R}^n$—with the following basic properties:

CP$_1$ it is *bounded below* (say, $\mathcal{V} \geq 0$);

CP$_2$ it is a *cone potential*, which means the following. Let $\mathbb{C}$ be the convex cone in $\mathbb{R}^n$ generated by the forces $-\nabla \mathcal{V}$ (i.e., the set of all linear combinations of vectors of the form $-\nabla \mathcal{V}(q)$, with nonnegative coefficients). Then the closure of $\mathbb{C}$ is a *proper cone*, that is, it contains no straight line.

Condition CP$_2$ is equivalent to requiring that the *dual cone*

$$\mathcal{D} := \{ w \in \mathbb{R}^n \ : \ w \cdot v \geq 0 \quad \forall v \in \mathbb{C} \} \tag{1.2}$$

has nonempty interior: $\mathcal{D}^\circ \neq \gtreqless$.

The first simple consequences of CP$_1$ is that each solution of the system (1.1) is globally defined in time and has a bounded velocity (just remind that the energy $|p|^2/2 + \mathcal{V}(q)$ is conserved). On the other hand, because of CP$_2$, $\mathcal{D}$ contains a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$. The scalar product $p(t) \cdot b_i$ of the velocity of a trajectory with an element of the basis is monotonic in time (the derivative is $-\nabla \mathcal{V}(q(t)) \cdot b_i$, which is nonnegative for how $\mathcal{D}$ is defined). Altogether, we get the existence of the following finite limit:

$$p_\infty(\bar{p}, \bar{q}) = \lim_{t \to +\infty} p(t, \bar{p}, \bar{q}) \in \mathbb{R}^n \,, \tag{1.3}$$

the *asymptotic velocity*. We denote by $t \mapsto (p(t, \bar{p}, \bar{q}), q(t, \bar{p}, \bar{q}))$ the motion of the system (1.1) which has $(\bar{p}, \bar{q})$ as initial conditions.

Of course, p_∞ is a constant of motion. In [GZ1] we found sufficient conditions for the components of p_∞ to be smooth (C^k, $2 \leq k \leq +\infty$), independent and pairwise in involution first integrals of motion. This led to a large class of completely integrable systems of the form (1.1).

In the course of our investigations we figured out the graph of an ideal cone potential as a sort of smooth, convex "amphitheatre" surface, with hyperbolic-type level sets each asymptotic to the boundary of a translation of $\mathcal{D}$; the surface flattens down to zero in the "direction" of the dual cone $\mathcal{D}$ and steeps up in the opposite directions. Part of the job was imagining what pathologies may arise in the mere hypotheses a), b) and c), and finding further conditions that could rule them out.

The problem of the regularity of p_∞ was attacked through the following relation:

$$p_\infty(\bar p, \bar q) = \bar p + \int_0^{+\infty} -\nabla \mathcal{V}(q(s, \bar p, \bar q))\, ds\,, \tag{1.4}$$

which is an easy consequence of (1.1) and (1.3). By formally differentiating (1.4) n times with respect to the initial data $(\bar p, \bar q)$, we obtain formulas containing the differentials of $\mathcal{V}$, evaluated always in $q(s, \bar p, \bar q)$, and the derivatives of $q(s, \bar p, \bar q)$ again with respect to $(\bar p, \bar q)$.

If we wish to apply the theorems on the regular dependence of integrals from parameters, we seem to need the following:

1 information on how the trajectory $q(t, \bar p, \bar q)$ behaves as $t \to +\infty$, somehow locally independent of the initial data;
2 bounds on the growth of the derivatives of $q(t, \bar p, \bar q)$ with respect to $(\bar p, \bar q)$, as $t \to +\infty$.

The decay rate of $\mathcal{V}$ along the trajectories can then be chosen so as to compensate for the growth of the derivatives of $q(t, \bar p, \bar q)$ and yield the desired properties for formula (1.4) and the derived ones.

Point 1 is what the results of the present paper are all about. The techniques developed in [GZ1] to cope with problem 2 apply with no change to the new situations, and we are not going to dwell on them for longer than the following remarks. Within hypotheses of *exponential decay* of the derivatives of $\mathcal{V}$ in the direction of the dual cone $\mathcal{D}$, we could use a Gronwall argument to obtain an a priori less-than-exponential bound on the growth of the derivatives of $q(t, \bar p, \bar q)$ with respect to the initial data as $t \to +\infty$. Otherwise, much more general decays (such as inverse r-power for arbitrary $r > 0$) could be allowed if *convexity* and a kind of *monotonicity* in the Hessian matrix of $\mathcal{V}$ were imposed, in order to exploit certain Liapunov functions for the first variational equations.

In [GZ2] we somewhat specialize our hypotheses (the ones connected with point 2 only) to permit the reduction of (1.1) to the *normal form*

$$\dot P = 0\,, \quad \dot Q = P\,, \tag{1.5}$$

by means of the global canonical transformation $(P, Q) = \mathcal{A}(p, q)$ (*asymptotic map*) defined as

$$\mathcal{A}\colon (\bar p, \bar q) \mapsto \big(p_\infty(\bar p, \bar q), a_\infty(\bar p, \bar q)\big)\,, \tag{1.6}$$

where $a_\infty(\bar p, \bar q)$, the *asymptotic phase*, is given by

$$a_\infty(\bar p, \bar q) := \lim_{t\to+\infty} \big(q(t, \bar p, \bar q) - t\, p_\infty(\bar p, \bar q)\big)\,, \tag{1.7}$$

(this limit always exists within the assumptions of [GZ2]).

In the framework of [GZ1], the complete integrability has a certain property of "structural stability", in the precise sense that it is persistent under small perturbations of the potential $\mathcal{V}$ in an arbitrary compact set of $\mathbb{R}^n$. In [GZ2] the map $\mathcal{A}$ enjoys the same property.

* * *

The precise nature of the information we refer to in point 1 is the following:

AR we have global *asymptotic regularity*, that is, the asymptotic velocity p_∞ belongs to the *interior* of the dual cone $\mathcal{D}$ for *all* initial data: $p_\infty(\bar{p},\bar{q}) \in \mathcal{D}^\circ\ \forall(\bar{p},\bar{q})$; this ensures in particular that, for every single trajectory, the distance of $q(t,\bar{p},\bar{q})$ from the boundary $\partial\mathcal{D}$ of the dual cone $\mathcal{D}$ grows linearly as $t \to +\infty$;

UE we have *estimates* from below on the growth of $\operatorname{dist}(q(t,\bar{p},\bar{q}),\partial\mathcal{D})$ that are *locally uniform* in the initial data. More precisely, this distance is not less than γt for all times t larger than t_0, where $\gamma > 0$ and $t_0 \in \mathbb{R}$ are locally independent of $(\bar{p},\bar{q})$.

The basic reason why we use the quantity $\operatorname{dist}(q,\partial\mathcal{D})$ is that it has a very simple analytic expression (see Section 2 of [GZ1] and Proposition 2.2 of the present paper), in terms of relevant quantities of our problem. For example, the decay rate of $\mathcal{V}$ "at infinity" is naturally written in terms of this distance for all the examples that motivated our study.

In [GZ1] the two points AR and UE were treated in Section 4, Propositions 4.3 and 4.4 respectively, and they were tightly intertwined.

The result of Section 3 of this paper is the proof that, under mild conditions on $\mathcal{V}$ (not implied by [GZ1], Section 4), the two problems can be treated independently of each other. Namely, locally uniform estimates arise naturally in the invariant set $\mathcal{M}$ of the initial data which are *asymptotically regular*, that is, whose trajectories have asymptotic velocity in the interior of the dual cone:

$$\mathcal{M} := \{(\bar{p},\bar{q}) \ : \ p_\infty(\bar{p},\bar{q}) \in \mathcal{D}^\circ\}\,. \tag{1.8}$$

The set $\mathcal{M}$ turns out to be open and nonempty. The locally uniform estimates that we derive permit to apply the smoothness theory of [GZ1] and [GZ2] and obtain complete integrability and normal form for "restricted phase space" systems

$$\dot{p} = -\nabla\mathcal{V}\,, \quad \dot{q} = p\,, \qquad (p,q) \in \mathcal{M}\,. \tag{1.9}$$

Actually, the methods of Section 3 seem to be general enough to be possibly adapted to Hamiltonian systems where only subsets of the phase space are of cone potential type. For each of these subsets we can single out further subsets on which the system is integrable via asymptotic velocity.

One may wonder whether a cone potential system may be globally integrated via asymptotic velocity even if there is no global asymptotic regularity. The answer is "no" for all systems for which the range of p_∞ is contained in the closed set $\mathcal{D}$ (this property is an easy consequence of quite natural assumptions on the potential; see e.g. Proposition 4.10). In fact, if p_∞ takes some values on the boundary of $\mathcal{D}$, then it is not an open map. In particular, even if p_∞ were globally smooth, its components could not be independent at those points where $p_\infty \in \partial\mathcal{D}$.

Of course, a draw-back of the results of Section 3 is that the invariant set $\mathcal{M}$, on which we establish integrability, is defined *implicitly*, through asymptotic properties of the trajectories. It is hard in general to decide whether given initial data are asymptotically regular or not. The negative example of Section 5 goes the other way round, that is, it starts with a single trajectory with a suitable behaviour and builds upon it a potential that admits that trajectory as a non asymptotically regular solution. However, we already know from [GZ1], Section 4, a wide class of cone potentials for which $\mathcal{M}$ coincides with the whole configuration space. In Sections 6 and 7 of the present paper we will describe two new classes with that property, coming up with new *globally integrable systems*. These two classes are distinct from each other and from the one in [GZ1], although there are mutual overlappings.

The major shortcoming of the conditions stated in [GZ1], Section 4, is that they allow only cones $\mathbb{C}$ of width not larger than $\pi/2$:

$$\nabla\mathcal{V}(q') \cdot \nabla\mathcal{V}(q'') > 0 \quad \forall q', q''. \tag{1.10}$$

This restriction ruled out for example the well-known non-periodic Toda lattice system, the Calogero system and, more generally, the problem of n particles on the line with pairwise repulsive interactions, that had actually been the original motivation to introduce the concept of cone potential. The two new classes whose integrability we are going to prove here cover many "wide cone" systems.

We start in Section 4 with restricting our attention to the potentials with the particular *finite-sum* form:

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} f_\alpha(q \cdot v_\alpha) \tag{1.11}$$

($N \geq 1$, no relation to n), where the v_α are vectors of $\mathbb{R}^n$ and the f_α are smooth real functions of one variable. Since the force $-\nabla\mathcal{V}$ is

$$-\nabla\mathcal{V}(q) = \sum_{\alpha=1}^{N} -f'_\alpha(q \cdot v_\alpha) v_\alpha\,, \tag{1.12}$$

$\mathcal{V}$ is a bounded below cone potential whenever the functions f_α are bounded below and all (say) decreasing, and the convex cone generated by the vectors v_α is proper (this cone obviously contains the cone of the forces $\mathbb{C}$, and coincides with the closure $\bar{\mathbb{C}}$ under simple conditions on the functions f_α, see Proposition 4.2).

The potential of a system of n particles on the line interacting pairwise through the repulsive potentials f_{ij} is given by

$$\mathcal{V}(q_1, \ldots, q_n) = \sum_{1 \leq i < j \leq n} f_{ij}(q_i - q_j)\,, \tag{1.13}$$

and can be written in the form (1.11) by defining the vectors

$$v_{ij} := (\ldots, 0, \overset{(i)}{1}, 0, \ldots, 0, \overset{(j)}{-1}, 0, \ldots)\,, \quad 1 \leq i < j \leq n\,. \tag{1.14}$$

The cone generated by the v_{ij} is wider than $\pi/2$, since for example

$$v_{1,2} \cdot v_{2,3} = (1, -1, 0, 0, \ldots) \cdot (0, 1, -1, 0, \ldots) = -1 \tag{1.15}$$

and these two vectors form an angle of $2\pi/3$. The dual cone is

$$\{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \ : \ q_1 \leq q_2 \leq \ldots \leq q_n\}\,, \tag{1.16}$$

which has nonempty interior.

To be able to deal with wide cones we make heavy use of the finite-sum form of $\mathcal{V}$. We emphasize that the present results are distinct from the ones in [GZ1] and [GZ2], that allowed both finite-sum potentials that do not fit into the new classes and examples of a different, non finite-sum structure.

In Section 4 we establish sufficient conditions on the finite-sum potentials to give rise to "restricted phase space" integrability. Of course, the main conclusions rest upon the theory develped in [GZ1] and [GZ2]. We also provide a number of lemmas meant for the sequel.

In Section 5, as we already mentioned, we provide an example of a finite-sum potential admitting non asymptotically regular initial data. The counterexamples already given in [GZ1], Section 3, were, in a way or another, related to the presence of equilibria (even if not strictly speaking). The reason for the failed asymptotic regularity in the present case seems to be subtler. Figuratively speaking, there are two forces acting on $\mathbb{R}^2$, that largely oppose each other, because the angle between them is $2\pi/3$. One of the two decreases rapidly (exponentially) at infinity, while the other decays slowly (the potential is not integrable at infinity, behaving as $x \mapsto 1/x$). It is proved in Section 3 that all trajectories that start with a large enough speed, pointing into the interior of the dual cone $\mathcal{D}$, are asymptotically regular, because the particle escapes so fast that the ever dwindling forces never catch up to drive the velocity toward the boundary of $\mathcal{D}$. However, it happens that if the initial speed is too small, then the slow-decaying potential takes over the motion, the distance of the particle from the boundary of $q_0 + \mathcal{D}$ (all trajectories are contained in a set of this form) grows logarithmically, and the asymptotic velocity belongs to the boundary of $\mathcal{D}$.

In Section 6 we prove that if the functions f_α are *all equal*, or, at least, multiples of a single function: $f_\alpha = c_\alpha f$, $c_\alpha > 0$, plus a nonrestrictive technical assumption on f, then all initial data are asymptotically regular. Roughly speaking, none of the single forces can prevail, because they all decay the same way. The global complete integrability holds for all potentials in this class, whilst the reduction into normal form via asymptotic map is provided only if f is integrable, case which intersects with the next class.

In Section 7 we show that global asymptotic regularity holds if the functions f_α are *all integrable* at $+\infty$. In the language of the counterexample, the forces are too weak at infinity to bend enough the trajectories. For all the potentials in this class the asymptotic map is defined and smooth. The proof in this Section 7 is adapted from [Gu4], where the author also admits somewhat more general potentials than here, but does not address the problem of the smoothness of the asymptotic map (he calls "regular" the trajectories which are asymptotically regular in our sense and possess asymptotic phase).

The class of finite-sum potentials with global asymptotic regularity that was described in [GZ1], Section 10, corresponds to no special decay or similarity conditions on the functions f_α, but required instead that the cone $\mathbb{C}$ were not wider than $\pi/2$ ($v_\alpha \cdot v_\beta \geq 0\ \forall \alpha, \beta$). There the forces actually agree so much in direction that there is not enough competition.

Of course, any future attempt to find further classes of cone potentials with global asymptotic regularity must take into account the existence of counterexamples. Actually, there appears to be not much room for extensions within the finite-sum form.

* * *

In Section 8 we gather into one statement all the hypotheses on finite-sum potentials under which we will have proved global integrability via asymptotic velocity. We are now going to write down some explicit, remarkable examples with C^∞-integrability. In these examples $N \geq 1$ and the vectors $v_\alpha \in \mathbb{R}^n$, with $\alpha = 1, \ldots, N$, generate a proper cone (i.e., there exists $\bar{v} \in \mathbb{R}^n$ such that $\bar{v} \cdot v_\alpha > 0 \ \ \forall \alpha$).

a) Toda-like, or exponential potentials

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} c_\alpha e^{-q \cdot v_\alpha}\,, \qquad c_\alpha > 0, \quad q \in \mathbb{R}^n\,. \tag{1.17}$$

b) Finite sum of inverse powers:

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} \frac{1}{(q \cdot v_\alpha)^{r_\alpha}}\,, \qquad q \in \{\bar{q} \in \mathbb{R}^n : \bar{q} \cdot v_\alpha > 0 \ \forall \alpha\}\,, \tag{1.18}$$

in three different hypotheses:

b1) $r_\alpha > 0$ arbitrary but the vectors verify $v_\alpha \cdot v_\beta \geq 0$;
b2) v_α arbitrary but all the exponents r_α are > 1;
b3) v_α arbitrary but the exponents r_α are all equal to an $r > 0$.

In particular b3) includes the inverse r-power potential for arbitrary $r > 0$:

$$\mathcal{V}(q) := \sum_{1 \leq i < j \leq n} \frac{1}{(q_i - q_j)^r}\,, \qquad q \in \{(q_1, \ldots, q_n) \in \mathbb{R}^n \ : \ q_1 < q_2 < \ldots < q_n\}\,. \tag{1.19}$$

The case $r = 1$ is the Coulombian potential, whilst $r = 2$ is the Calogero potential. The exponent $r = 1$ divides what are usually called short range potentials ($r > 1$) from the long range ones ($0 < r \leq 1$).

c) Even longer range potentials are the inverse-logarithmic type:

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} \frac{1}{(\ln(1 + q \cdot v_\alpha))^r}\,, \qquad q \in \{\bar{q} \in \mathbb{R}^n : \bar{q} \cdot v_\alpha > 0 \ \forall \alpha\}\,, \tag{1.20}$$

for arbitrary $r > 0$.

For potentials a), b1) with $r_\alpha > 1$, and b2), there are both asymptotic velocities and phases, with consequent reduction to normal form. In case b3) with $0 < r \leq 1$ and c) we prove integrability through asymptotic velocities but we do not have the reduction to normal form.

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The earliest cone potential systems to be integrated, with a complete description of the motions, were some systems of particles on the line $\mathbb{R}^1$: most notably the non-periodic Toda lattice (*exponential* potentials) and the Calogero system (*inverse square* potential). The methods used in the proof were, most generally, Lax pairs and isospectral deformations (see e.g. [M]). As a side product, asymptotic velocities turned out to be a set of analytic, independent and pairwise in involution first integrals.

Gutkin in [Gu1] introduced the concept of cone potentials with the conjecture that the integrability of those special systems could be derived from a general theory. Gutkin carried on the study in some subsequent papers ([Gu2 to 4]). However, at the beginning of the present research, the earliest article we were aware of with a rigorous proof of C^∞-integrability via asymptotic velocities for cone potentials was the recent [OC], where Oliva and Castilla studied mainly finite-sum potentials for which the f_α decay exponentially at $+\infty$ and the vectors v_α generate a cone not wider than $\pi/2$, and a few special cases with wider cones. Their proof used strongly the finite-sum form of the potential to define a "compactifying" change of variable, and then applied a Lemma in Dynamical Systems on the differentiability of a foliation of invariant manifolds (developed and proved in the same paper, but of independent interest as well).

When we had already completed the bulk of the present work, we came to know Hubacher's paper [Hu], whose results overlap partially with ours. It is concerned with systems of mutually repulsive particles on the line, that is, with potentials of the form (1.13). To start with, it gives results of what we call global asymptotic regularity—see AR above—analogous to the ones in Section 6 and 7 of the present paper, with different proofs, tailored to the structure of the mutually repelling particles on the line. It also provides a counterexample to global asymptotic regularity that reminds ours: the potential is the sum of exponentials and an inverse power (the latter is however $x \mapsto x^{-\alpha}$ with $\alpha = 1/2$, and not -1 as here; -1 is a limit exponent, because the case $\alpha > 1$ is short-range and global regularity holds, see Section 7). Then Hubacher restricts her attention to the systems with short range repulsive equal potentials

$$\begin{gathered}\mathcal{V}(q_1, \dots, q_n) = \sum_{1 \le i < j \le n} f(q_i - q_j)\,, \\ |f'(x)| \le \frac{M}{x^{2+\alpha}}\,, \quad |f''(x)| \le \frac{M}{x^{3+\alpha}}\,, \quad M > 0,\ \alpha > 0\,,\end{gathered} \tag{1.21}$$

and states their integrability through asymptotic data by invoking the results of Simon [S] and Herbst [He]. Those authors had studied the scattering of n mutually repulsive particles in $\mathbb{R}^n$, through an approach totally different from ours. The proof of smoothness was there obtained by solving a kind of Cauchy problem at infinity with the asymptotic data playing the role of initial data. The method is remarkably simple for short range potentials, but it gets into complications for long range potentials $x \mapsto 1/x^\alpha$ when $0 < \alpha \le 1$, and does not seem to cover for example the inverse logarithm case (1.20).

Let us also say that Hubacher does not adopt the mere existence of n independent first integrals in involution as definition of integrability. Among the systems with potential (1.21), she prefers to reserve the name "integrable" to those which satisfy certain conditions including a suitable preservation property between the asymptotic velocities of the particles as $t \to -\infty$ and the ones as $t \to +\infty$. For these integrable systems she obtains an interesting result (Theorem 4 of [Hu]). However, that kind of integrability is very special and totally outside the spirit of the present paper.

Moauro, Negrini and Oliva [MNO] have recently obtained a proof of *analytic* integrability for the systems of cases a) and b) above. They exploited the geometric techniques introduced in [OC], with crucial changes in the time variable as in [MN], and, strongly, the theory of invariant manifolds for fixed points of analytic diffeomorphisms. To their goal they also used Propositions 6.2 and 7.2 of the present paper.

Finally, we remark that the problem of the mere existence of asymptotic velocities and phases for scattering systems of mutually repelling particles in $\mathbb{R}^n$ was studied by various authors in the sixties and seventies, see for instance [Ga] and the references contained therein.

2. Preliminaries

The starting assumption on *cone potentials* are as follows.

Hypotheses 2.1 *The potential $\mathcal{V}$ is a C^2 real function defined on a nonempty, open subset* $\operatorname{dom}\mathcal{V}$ *of $\mathbb{R}^n$. Moreover:*

CP_0 *for every $E \in \mathbb{R}$, the closure in $\mathbb{R}^n$ of the set $\{q \in \operatorname{dom}\mathcal{V} \ : \ \mathcal{V}(q) \leq E\}$ is contained in* $\operatorname{dom}\mathcal{V}$*;*

CP_1 *$\mathcal{V}$ is bounded below (it makes no harm to assume $\mathcal{V} \geq 0$);*

CP_2 *the convex cone*

$$\mathcal{D} := \{w \in \mathbb{R}^n \ : \ -\nabla\mathcal{V}(q) \cdot w \geq 0 \ \forall q \in \operatorname{dom}\mathcal{V}\} \tag{2.1}$$

has nonempty interior (to avoid trivialities we can assume that $\mathcal{D}$ is not all of $\mathbb{R}^n$, i.e., $\mathcal{V}$ is nonconstant).

From Hypotheses 2.1 it follows in particular that $q \in \operatorname{dom}\mathcal{V} \Rightarrow q + \mathcal{D} \subset \operatorname{dom}\mathcal{V}$. In fact, for all $w \in \mathcal{D}$, the function $t \mapsto \mathcal{V}(q + tw)$ is (weakly) decreasing, and hence $\operatorname{dom}\mathcal{V}$ must contain all the half-line $\{q + tw \ : \ t \geq 0\}$.

From the conservation of energy it is easy to see that, for all initial data $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \operatorname{dom}\mathcal{V}$, the Hamiltonian system

$$\dot{p} = -\nabla\mathcal{V}(q)\,, \quad \dot{q} = p\,, \qquad p \in \mathbb{R}^n\,,\ q \in \operatorname{dom}\mathcal{V} \tag{2.2}$$

has a unique solution $t \mapsto (p(t,\bar{p},\bar{q}), q(t,\bar{p},\bar{q}))$, which is globally defined in time, and with bounded velocity: $|p(t,\bar{p},\bar{q})|^2 \leq |\bar{p}|^2 + 2\mathcal{V}(\bar{q})$. Next, $\mathcal{D}$, having nonempty interior, contains a basis $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$. We have

$$\dot{p}(t,\bar{p},\bar{q}) \cdot b_i = -\nabla\mathcal{V}(q(t,\bar{p},\bar{q})) \cdot b_i \geq 0\,, \tag{2.3}$$

so that each function $t \mapsto p(t,\bar{p},\bar{q}) \cdot b_i$ is monotone. Being also bounded, it has a finite limit as $t \to +\infty$. This proves that, under Hypotheses 2.1, the *asymptotic velocity*

$$p_\infty(\bar{p},\bar{q}) := \lim_{t\to+\infty} p(t,\bar{p},\bar{q}) \tag{2.4}$$

exists for all initial data $(\bar{p},\bar{q}) \in \mathbb{R}^n \times \operatorname{dom}\mathcal{V}$.

We remind that by *cone* in $\mathbb{R}^n$ we mean a nonempty subset C of $\mathbb{R}^n$ such that $\lambda \geq 0 \Rightarrow \lambda C \subset C$. All the cones we consider are convex. For a cone C we define its *dual cone* C^* by

$$C^* := \{v \in \mathbb{R}^n \ : \ v \cdot w \geq 0 \ \forall w \in C\}\,. \tag{2.5}$$

Some properties of cones are listed in Section 2 of [GZ1]. We only report here the following statement, concerning the *distance* of a point of a convex cone from the boundary.

Proposition 2.2 *Let C be a convex cone in $\mathbb{R}^n$, not reduced to $\{0\}$, and let D be its dual. If $w \in D$ then*

$$\operatorname{dist}(w, \partial D) = \min_{\substack{v\in\bar{C}\\ |v|=1}} \ w \cdot v\,. \tag{2.6}$$

3. Integrability on an Invariant Set

We are going to show how a very simple geometric condition on the cone potential $\mathcal{V}$ guarantees that the set $\mathcal{M}$ of the asymptotically regular initial data is nonempty and open, and that locally uniform estimates hold as needed to apply the smoothness theory of [GZ1] and [GZ2].

In the sequel, $\mathbb{C}$ will be the convex cone generated by the force $-\nabla\mathcal{V}$ of a cone potential, and $\mathcal{D}$ will be the dual of $\mathbb{C}$.

Hypothesis 3.1 *There exists $q_0 \in \operatorname{dom}\mathcal{V}$ and a nonnegative, weakly decreasing and integrable function $h_0\colon [0,+\infty[\to\mathbb{R}$ such that*

$$q \in q_0 + \mathcal{D} \quad \Rightarrow \quad |\nabla\mathcal{V}(q)| \le h_0\big(\operatorname{dist}(q, q_0 + \partial\mathcal{D})\big)\,. \tag{3.1}$$

We start with proving that there are asymptotically regular initial data.

Proposition 3.2 *Assume Hypotheses 2.1 and 3.1. Then, for any $\gamma > 0$ there exists $q_\gamma \in q_0 + \mathcal{D}$ such that*

$$\left.\begin{array}{l} \bar p \in \mathcal{D}^\circ\,, \\ \operatorname{dist}(\bar p, \partial\mathcal{D}) \ge 2\gamma\,, \\ \bar q \in q_\gamma + \mathcal{D}\,, \end{array}\right\} \quad \Rightarrow \quad \forall t \ge 0 \quad \left\{\begin{array}{l} p(t,\bar p,\bar q) \in \mathcal{D}^\circ\,, \\ \operatorname{dist}\big(p(t,\bar p,\bar q), \partial\mathcal{D}\big) \ge \gamma\,, \\ q(t,\bar p,\bar q) \in q_\gamma + \mathcal{D}\,, \\ \operatorname{dist}\big(q(t,\bar p,\bar q), q_\gamma + \partial\mathcal{D}\big) \ge \gamma t\,. \end{array}\right. \tag{3.2}$$

Proof. Fix $\gamma > 0$ and and pick $b_\gamma > 0$ such that

$$\int_0^{+\infty} h_0(\gamma t + b_\gamma)\, dt < \gamma\,. \tag{3.3}$$

Let $q_\gamma \in q_0 + \mathcal{D}$ be such that $\operatorname{dist}(q_\gamma, q_0 + \partial\mathcal{D}) \ge b_\gamma$. Let the initial data $(\bar p, \bar q)$ verify

$$\bar p \in \mathcal{D}\,, \quad \operatorname{dist}(\bar p, \partial\mathcal{D}) \ge 2\gamma\,, \quad \bar q \in q_\gamma + \mathcal{D}\,. \tag{3.4}$$

For all $t \ge 0$ such that $q(t,\bar p,\bar q) \in q_\gamma + \mathcal{D}$ we define the function

$$\psi(t) := \operatorname{dist}\big(q(t,\bar p,\bar q), q_\gamma + \partial\mathcal{D}\big) = \min_{\substack{v\in\bar{\mathbb{C}} \\ |v|=1}} \big(q(t,\bar p,\bar q) - q_\gamma\big)\cdot v\,. \tag{3.5}$$

Since $\bar p \in \mathcal{D}^\circ$, the trajectory $q(t,\bar p,\bar q)$ belongs to $q_\gamma + \mathcal{D}$ for all $t \ge 0$ small enough. For each $v \in \bar{\mathbb{C}}$, $|v| = 1$, the function

$$t \mapsto \big(q(t,\bar p,\bar q) - q_\gamma\big)\cdot v \tag{3.6}$$

is Lipschitz with constant $\sup_t |p(t,\bar p,\bar q)| < +\infty$. Hence ψ is Lipschitz too, with the same constant (see the Appendix), and in particular it is absolutely continuous.
In an interval contained in the domain of ψ, let $t_0 < t$ and write

$$q(t,\bar p,\bar q) - q(t_0,\bar p,\bar q) = (t-t_0)\, p(t_0,\bar p,\bar q) + (t-t_0)\varepsilon(t)\,, \tag{3.7}$$

with $\varepsilon(t) \to 0$ as $t \searrow t_0$. Let $v_t \in \bar{\mathbb{C}}$, $|v_t| = 1$ be such that

$$\psi(t) = \big(q(t,\bar p,\bar q) - q_\gamma\big)\cdot v_t\,. \tag{3.8}$$

Then

$$\begin{aligned}
\psi(t) - \psi(t_0) &= \big(q(t,\bar p,\bar q) - q_\gamma\big)\cdot v_t - \min_{\substack{v\in\bar{\mathbb{C}}\\|v|=1}} \big(q(t_0,\bar p,\bar q) - q_\gamma\big)\cdot v \geq \\
&\geq \big(q(t,\bar p,\bar q) - q_\gamma\big)\cdot v_t - \big(q(t_0,\bar p,\bar q) - q_\gamma\big)\cdot v_t = \\
&= \big(q(t,\bar p,\bar q) - q(t_0,\bar p,\bar q)\big)\cdot v_t = \\
&= (t-t_0)\, p(t_0,\bar p,\bar q)\cdot v_t + (t-t_0)\,\varepsilon(t)\cdot v_t \geq \\
&\geq (t-t_0) \min_{\substack{v\in\bar{\mathbb{C}}\\|v|=1}} p(t_0,\bar p,\bar q)\cdot v - (t-t_0)|\varepsilon(t)| = \\
&= (t-t_0)\,\mathrm{dist}\big(p(t_0,\bar p,\bar q),\partial\mathcal{D}\big) - (t-t_0)|\varepsilon(t)|\,.
\end{aligned} \tag{3.9}$$

If we apply this inequality first for $t_0 = 0$, reminding that $\mathrm{dist}(\bar p,\partial\mathcal{D}) \geq 2\gamma$, we obtain

$$\psi(t) > \gamma t \quad \text{for all } t>0 \text{ small enough.} \tag{3.10}$$

On the other hand, for all $t_0 \geq 0$ where ψ is differentiable, we can write

$$\psi'(t_0) \geq \mathrm{dist}\big(p(t_0,\bar p,\bar q),\partial\mathcal{D}\big)\,. \tag{3.11}$$

For almost all t in any interval where ψ is defined we have

$$\begin{aligned}
\psi'(t) &\geq \mathrm{dist}\big(p(t,\bar p,\bar q),\partial\mathcal{D}\big) = \\
&= \min_{\substack{v\in\bar{\mathbb{C}}\\|v|=1}} \Big(\bar p\cdot v + \int_0^t -\nabla\mathcal{V}(q(s,\bar p,\bar q))\cdot v\, ds\Big) \geq \\
&\geq \min_{\substack{v\in\bar{\mathbb{C}}\\|v|=1}} \bar p\cdot v - \int_0^t h_0\big(\mathrm{dist}(q(s,\bar p,\bar q), q_0+\partial\mathcal{D})\big)\, ds \geq \\
&\geq 2\gamma - \int_0^t h_0\big(\psi(s)+b_\gamma\big)\, ds\,.
\end{aligned} \tag{3.12}$$

Integrating the inequality we get (recalling that $\psi(0)\geq 0$):

$$\psi(t) \geq 2\gamma t - \int_0^t (t-s)\, h_0\big(\psi(s)+b_\gamma\big)\, ds\,. \tag{3.13}$$

For the function $\tilde{\psi}(t) := \gamma t$ we have

$$\begin{aligned} 2\gamma t - \int_0^t (t-s)\, h_0\big(\tilde{\psi}(s) + b_\gamma\big)\, ds = \\ = 2\gamma t - \int_0^t (t-s)\, h_0\big(\gamma s + b_\gamma\big)\, ds \geq \\ \geq 2\gamma t - t \int_0^t h_0(\gamma s + b_\gamma)\, ds \geq \\ \geq 2\gamma t - t \int_0^{+\infty} h_0(\gamma s + b_\gamma)\, ds > 2\gamma t - \gamma t = \gamma t = \tilde{\psi}(t)\,. \end{aligned} \tag{3.14}$$

Since $\psi(t) > \tilde{\psi}(t)$ for small enough $t > 0$, a standard argument in integral inequalities (see e.g. [LL], Theorem 5.5.1) yields

$$\psi(t) \geq \gamma t \quad \forall t \geq 0\,. \tag{3.15}$$

The relation $\operatorname{dist}(p(t,\bar{p},\bar{q}), \partial\mathcal{D})) \geq \gamma$ comes from

$$\operatorname{dist}\big(p(t,\bar{p},\bar{q}), \partial\mathcal{D}\big) \geq 2\gamma - \int_0^t h_0\big(\psi(s) + b_\gamma\big)\, ds \geq \gamma\,. \tag{3.16}$$

□

Proving that $\mathcal{M}$ is open is easy now.

Proposition 3.3 *Suppose that Hypotheses 2.1 and 3.1 are verified. Then, for each $(\bar{p}_0, \bar{q}_0) \in \mathbb{R}^n \times \operatorname{dom}\mathcal{V}$ whose asymptotic velocity $p_\infty(\bar{p}_0, \bar{q}_0)$ belongs to the interior of the dual cone $\mathcal{D}$ and each $q' \in \operatorname{dom}\mathcal{V}$, there exist $\gamma > 0$, $t_0 \in \mathbb{R}$ and a bounded neighbourhood U of $(\bar{p}_0, \bar{q}_0)$ in $\mathbb{R}^n \times \operatorname{dom}\mathcal{V}$ such that, for all $t \geq t_0$ and $(\bar{p}, \bar{q}) \in U$ we have*

$$\begin{aligned} &p(t,\bar{p},\bar{q}) \in \mathcal{D}^\circ, && \operatorname{dist}\big(p(t,\bar{p},\bar{q}), \partial\mathcal{D}\big) \geq \gamma, \\ &q(t,\bar{p},\bar{q}) \in q' + \mathcal{D}, && \operatorname{dist}\big(q(t,\bar{p},\bar{q}), q' + \partial\mathcal{D}\big) \geq \gamma(t - t_0). \end{aligned} \tag{3.17}$$

In particular, $p_\infty(\bar{p}, \bar{q}) \in \mathcal{D}^\circ$ for all $(\bar{p}, \bar{q}) \in U$, so that $\mathcal{M}$ is open in $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $4\gamma := \operatorname{dist}(p_\infty(\bar{p}_0, \bar{q}_0), \partial\mathcal{D}) > 0$. The trajectory $q(t,\bar{p},\bar{q})$ eventually enters all sets of the form $q + \mathcal{D}^\circ$. Let $t_0 \in \mathbb{R}$ be such that

$$\operatorname{dist}(p(t_0, \bar{p}_0, \bar{q}_0), \partial\mathcal{D}) \geq 3\gamma\,, \quad q(t_0, \bar{p}_0, \bar{q}_0) \in (q_\gamma + \mathcal{D}^\circ) \cap (q' + \mathcal{D}^\circ)\,. \tag{3.18}$$

Let U be a bounded neighbourhood of $(\bar{p}_0, \bar{q}_0)$ such that, for all $(\bar{p}, \bar{q}) \in U$ we have

$$p(t_0,\bar{p},\bar{q}) \in \mathcal{D}^\circ\,, \quad \operatorname{dist}\big(p(t_0,\bar{p},\bar{q}), \partial\mathcal{D}\big) \geq 2\gamma\,, \quad q(t_0,\bar{p},\bar{q}) \in (q_\gamma + \mathcal{D}^\circ) \cap (q' + \mathcal{D}^\circ)\,. \tag{3.19}$$

To conclude we only need to apply Proposition 3.2. □

Proposition 3.4 *Suppose that the Hypotheses 1.1 and 3.1 are verified. Then the mapping $(\bar p, \bar q) \mapsto p_\infty(\bar p, \bar q)$ is continuous from $\mathcal{M}$ onto all of $\mathcal{D}^\circ$. If, moreover, the function h_0 is integrable on $[0, +\infty[$, then the asymptotic map $\mathcal{A}$ defined in (1.5) exists on $\mathcal{M}$, it is continuous and its range is all of $\mathcal{D}^\circ \times \mathbb{R}^n$.*

Proof. See [GZ1], Section 5 and Proposition 7.5 and [GZ2], Section 2. □

We are now going to write the two sets of assumptions on the potential $\mathcal{V}$ that in [GZ1] and [GZ2] were shown to imply the smoothness of the asymptotic data in $\mathcal{M}$.

We will denote by $\mathsf{D}^k\mathcal{V}$ the k-th differential of $\mathcal{V}$, regarded as a multilinear map from $(\mathbb{R}^n)^{k-1}$ into $\mathbb{R}^n$, endowed with the norm

$$\| \mathsf{D}^k\mathcal{V}(q)\| := \sup\{| \mathsf{D}^k\mathcal{V}(q)(x^{(1)}, \dots, x^{(k-1)})| \ : \ x^{(j)} \in \mathbb{R}^n, \ |x^{(j)}| \le 1\}.$$

Hypotheses 3.5 *The potential $\mathcal{V}$ is a C^{m+1}, $m \ge 2$, function. For all $1 \le i \le m$ there exist $q_i \in \mathbb{R}^n$, $A_i \ge 0$, $\lambda_i > 0$ such that*

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \| \mathsf{D}^{i+1}\mathcal{V}(q) \| \le A_i \exp\Big(-\lambda_i \operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big)\Big), \tag{3.21}$$

Hypotheses 3.6 *The potential $\mathcal{V}$ is a C^{m+1}, $m \ge 2$, function. For all $1 \le i \le m$ there exist $q_i \in \mathbb{R}^n$, and a weakly decreasing function h_i: $[0, +\infty[\to \mathbb{R}$ such that*

1) $\mathcal{V}$ is convex on $q_1 + \mathcal{D}$;

2) for all q', $q'' \in q_1 + \mathcal{D}$ and all $z \in \mathbb{R}^n$ we have

$$q'' \in q' + \mathcal{D} \quad \Rightarrow \quad \mathsf{D}^2\mathcal{V}(q'')z \cdot z \le \mathsf{D}^2\mathcal{V}(q')z \cdot z;$$

3) for all i, $\int_0^{+\infty} x^i \, h_i(x)dx < +\infty$ and

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \| \mathsf{D}^{i+1}\mathcal{V}(q)\| \le h_i\Big(\operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big)\Big);$$

Here is the precise statement of the integrability.

Proposition 3.7 *Assume Hypotheses 2.1, 3.1 and either 3.5 or 3.6. Then the components of the asymptotic velocity p_∞ are C^m first integrals, independent and in involution on the set $\mathcal{M}$ of the asymptotically regular initial data. If, moreover, the functions h_i of Hypotheses 3.1 and 3.6 verify $\int_0^{+\infty} x^{i+1} \, h_i(x) \, dx < +\infty$ for all $0 \le i \le m+1$, then the asymptotic map $\mathcal{A}$ exists on $\mathcal{M}$, it is a global canonical C^m diffeomorphism from $\mathcal{M}$ onto $\mathcal{D}^\circ \times \mathbb{R}^n$, and it brings the restricted phase space system (1.8) into the normal form $\dot P = 0$, $\dot Q = P$.*

Let a potential $\mathcal{V}$ verify all the assumptions of the previous proposition, and let us modify it smoothly *outside a set of the form $q + \mathcal{D}$*. Then all that happens to the thesis of the proposition is that the set $\mathcal{M}$ has changed. We will see in the next section how sufficiently small perturbations of $\mathcal{V}$ may not even alter $\mathcal{M}$.

4. Finite-sum Potentials

In this Section we concentrate our attentions on the potentials which can be written as finite sums of one-dimentional functions ("finite-sum potentials"):

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} f_\alpha(q \cdot v_\alpha)\,, \tag{4.1}$$

where $v_1, \ldots, v_N$ are given nonzero vectors in $\mathbb{R}^n$ ($N \geq 1$, no relation to n), and the functions $f_1, \ldots, f_N$ are real functions of one variable.

Sufficient conditions, under which Proposition 3.7 applies to the potential $\mathcal{V}$ with its associated Hamiltonian system, are as follows.

Hypotheses 4.1 *The f_α are C^{m+1} ($m \geq 2$) functions, whose domains* $\operatorname{dom} f_\alpha$ *are each either $\mathbb{R}$ or the interval $]0, +\infty[$, and*

$$\sup f_\alpha = +\infty\,, \qquad \inf f_\alpha = 0\,, \tag{4.2}$$

$$f'_\alpha(x) < 0 \quad \forall x \in \operatorname{dom}\, f_\alpha\,, \tag{4.3}$$

$$f^{(k)}_\alpha(x) \begin{cases} > 0 & \text{if } k \text{ is even,} \\ < 0 & \text{if } k \text{ is odd,} \end{cases} \qquad \forall x \geq a\,, \tag{4.4}$$

$$f^{(m+1)}_\alpha \text{ is monotone on } [a, +\infty[, \tag{4.5}$$

where $a > 0$ is a constant. Finally, the vectors $v_1, \ldots, v_N$ are nonzero and the cone generated by them is proper.

The domain of $\mathcal{V}$ is either $\mathbb{R}^n$ or the set

$$\{q \in \mathbb{R}^n \ : \ q \cdot v_\alpha \in \operatorname{dom} f_\alpha \quad \forall \alpha = 1, \ldots, N\}\,. \tag{4.6}$$

As we already observed in the Introduction, the convex cone $\mathbb{C}$ generated by the forces $-\nabla\mathcal{V}(q)$ is contained in the one generated by the vectors $v_1, \ldots, v_N$, and for our potentials the two have the same closure.

Proposition 4.2 *If Hypotheses 4.1 hold, the closure $\bar{\mathbb{C}}$ of $\mathbb{C}$ coincides with the convex cone generated by $v_1, \ldots, v_N$:*

$$\bar{\mathbb{C}} = \Big\{ -\sum_{\alpha=1}^{N} \lambda_\alpha v_\alpha \ : \ \lambda_\alpha \geq 0 \Big\}. \tag{4.7}$$

Proof. See [GZ1], Lemma 10.2. □

The dual cone $\mathcal{D}$ of $\bar{\mathbb{C}}$ is then

$$\mathcal{D} := \{w \in \mathbb{R}^n \ : \ w \cdot v \geq 0 \quad \forall v \in \mathbb{C}\} = \{w \in \mathbb{R}^n \ : \ w \cdot v_\alpha \geq 0 \quad \forall \alpha = 1, \ldots, N\}\,. \tag{4.8}$$

The set in formula (4.6) is simply the interior of $\mathcal{D}$.

Hypotheses 2.1 are certainly verified for our potentials. Also Hypothesis 3.1 holds with q_0 any point in $\mathcal{D}^\circ$ whose distance from $\partial\mathcal{D}$ is $\geq a$, and h_0 given by

$$h_0(x) := \sum_{\alpha=1}^{N} |v_\alpha| \left| f_\alpha(x \min_\beta |v_\beta| + a) \right| . \tag{4.9}$$

Hypotheses 3.6 hold as well and the verification can be found in [GZ1], Section 10. We can sum up with the following statement.

Proposition 4.3 *If Hypotheses 4.1 hold, then the first part of Proposition 3.7 applies to the potential (4.1). If moreover all the functions f_α have finite integrals on $[a, +\infty[$, then also the second part applies.*

To study the effects of small perturbations of the potential $\mathcal{V}$, we start with some technical Lemmas.

Lemma 4.4 *If the nonzero vectors $v_1, \ldots, v_N$ generate a proper cone C in $\mathbb{R}^n$, (i.e., the dual D of C has nonempty interior), then there exists a vector $\bar{v} \in D^\circ$ of the form*

$$\bar{v} = \sum_{\alpha=1}^{N} \mu_\alpha v_\alpha \qquad \textit{with } \mu_\alpha > 0 \textit{ for all } \alpha. \tag{4.10}$$

Proof. Let us first prove that $C \cap D^\circ$ is nonempty. In fact, if that were not the case, we could separate C from D, i.e., there would exist $\bar{w} \neq 0$ such that

$$\begin{aligned} &\bar{w} \cdot v_\alpha \geq 0 \qquad \forall \alpha = 1, \ldots, N \,, \\ &\bar{w} \cdot v \leq 0 \qquad \forall v \in D \,. \end{aligned} \tag{4.11}$$

The first relation is equivalent to $\bar{w} \in D$, and this, together with the second one, would yield $|\bar{w}|^2 = \bar{w} \cdot \bar{w} \leq 0$, that contradicts $\bar{w} \neq 0$.
Let us now pick $\bar{v}' \in C \cap D^\circ$. In particular, $\bar{v}'$ is of the form

$$\bar{v}' = \sum_{\alpha=1}^{N} \lambda_\alpha v_\alpha \,, \qquad \lambda_\alpha \geq 0 \,, \tag{4.12}$$

and $\bar{v}'$ belongs to the *open* set D°. Choose $\tau > 0$ small enough for the vector

$$\bar{v} := \bar{v}' + \tau \sum_{\alpha=1}^{N} v_\alpha = \sum_{\alpha=1}^{N} (\lambda_\alpha + \tau) v_\alpha \tag{4.13}$$

to belong to D°. This is what we were looking for. □

Lemma 4.5 *Let $v_1, \ldots, v_N$, C, D, $\bar{v}$ be as in Lemma 4.4. Then, for all $\bar{q} \in \mathbb{R}^n$ and $c > 0$, each of the functions $q \mapsto q \cdot v_\alpha$ is bounded on the set*

$$\{q \in \mathbb{R}^n \ : \ q \in \bar{q} + D, \quad q \cdot \bar{v} \leq c\}\,. \tag{4.14}$$

Proof. From $q \in \bar{q} + D$ we see that $q \cdot v_\alpha$ is bounded from below: $\bar{q} \cdot v_\alpha \leq q \cdot v_\alpha$. The condition $q \cdot \bar{v} \leq c$ is

$$\sum_{\alpha=1}^{N} \mu_\alpha q \cdot v_\alpha \leq c\,. \tag{4.15}$$

Each term $\mu_\alpha q \cdot v_\alpha$ is bounded from below, and their sum is bounded from above. Hence all of them are bounded from above, too. Remind finally that $\mu_\alpha > 0$ for all α. □

Lemma 4.6 *In the Hypotheses 4.1, let the vector $\bar{v}$ be supplied by Lemma 4.4. Then, for all $\bar{q} \in \operatorname{dom} \mathcal{V}$ and $c > 0$ we have*

$$\inf\{-\nabla\mathcal{V}(q) \cdot \bar{v} \ : \ q \in \bar{q} + \mathcal{D}\,,\ q \cdot \bar{v} \leq c\} > 0\,. \tag{4.16}$$

Proof. Let b be an upper bound for $q \cdot v_{\alpha_0}$ on $\{q \ : \ q \in \bar{q} + \mathcal{D},\ q \cdot \bar{v} \leq c\}$. Then, for q in that set,

$$\begin{aligned} -\nabla\mathcal{V}(q) \cdot \bar{v} = \ & -\sum_{\alpha=1}^{N} c_\alpha f'_\alpha(q \cdot v_\alpha) v_\alpha \cdot \bar{v} \geq -c_{\alpha_0} f'_{\alpha_0}(q \cdot v_{\alpha_0}) v_{\alpha_0} \cdot \bar{v} \geq \\ & \geq c_{\alpha_0} v_{\alpha_0} \cdot \bar{v}\ \min\{-f'_{\alpha_0}(x) \ : \ \bar{q} \cdot v_{\alpha_0} \leq x \leq b\} > 0\,. \end{aligned} \tag{4.17}$$

□

Lemma 4.7 *Assume Hypotheses 4.1. Then for all $E > 0$ there exists $q_E \in \operatorname{dom} \mathcal{V}$ such that*

$$\mathcal{V}(q) \leq E \quad \Rightarrow \quad q \in q_E + \mathcal{D}\,. \tag{4.18}$$

Proof. The function $x \mapsto f_\alpha(x)$ is strictly decreasing and onto $]0, +\infty[$. Define $q_E := \theta\bar{v}$, where

$$\theta := \min_\alpha \frac{f_\alpha^{-1}(E)}{\bar{v} \cdot v_\alpha}. \tag{4.19}$$

We have $q_E \cdot v_\alpha \leq f_\alpha^{-1}(E)$ for all α. If $q \in \operatorname{dom} \mathcal{V}$ is such that $q \notin q_E + \mathcal{D}$, then there exists α for which $q \cdot v_\alpha < q_E \cdot v_\alpha$ and hence

$$\mathcal{V}(q) \geq f_\alpha(q \cdot v_\alpha) \geq f_\alpha(q_E \cdot v_\alpha) \geq E\,. \tag{4.20}$$

□

Here is what we can say at this point about the asymptotic behaviour of the trajectory of a slightly perturbed system.

Proposition 4.8 *Suppose that Hypotheses 4.1 hold for the potential (4.1). Let K be a compact set contained in the domain of $\mathcal{V}$. Then there exists an $\varepsilon > 0$ with the following property. Let V be a C^{m+1} real function defined in* $\operatorname{dom}\mathcal{V}$*, vanishing outside K, and such that*

$$\sup_K |\nabla V| \leq \varepsilon . \tag{4.21}$$

Then each motion of the Hamiltonian system associated with the perturbed potential $\mathcal{V}+V$ coincides on an interval $[t_0, +\infty[$ with a motion of the original system.

Proof. Let $\bar{q} \in \operatorname{dom}\mathcal{V}$ be such that $K \subset \bar{q} + \mathcal{D}$. Let $\bar{v}$ be given by Lemma 4.4. We claim that the thesis holds for any ε such that

$$0 < \varepsilon < \inf\{-\nabla\mathcal{V}(q) \cdot \bar{v} \;:\; q \in K\} , \tag{4.22}$$

(the infimum is > 0 because of Lemma 4.6). In fact, let $(\tilde{p}(t), \tilde{q}(t))$ be a trajectory of the perturbed system. We will show that $\tilde{q}(t) \cdot \bar{v} \to +\infty$ as $t \to +\infty$, so that $\tilde{q}(t)$ eventually quits the compact set K forever.
There certainly exists $q_E \in \mathbb{R}^n$ such that $\tilde{q}(t) \in q_E + \mathcal{D}$ for all $t \in \mathbb{R}$ because the conservation of energy and the thesis of Lemma 4.7 still hold for $\mathcal{V} + V$.
The function $t \mapsto \tilde{q}(t) \cdot \bar{v}$ is convex: $\ddot{\tilde{q}}(t) \cdot \bar{v} = -\nabla(\mathcal{V} + V)(\tilde{q}(t)) \cdot \bar{v} > 0$ for all $t \in \mathbb{R}$ because of (4.21) and (4.22). If it did not tend to $+\infty$, as claimed, then it would be decreasing, and $\tilde{q}(t)$ would belong to $\{q \in \mathbb{R}^n \;:\; q \in q_E + \mathcal{D},\; q \cdot \bar{v} \leq \tilde{q}(0) \cdot \bar{v}\}$ for all $t \geq 0$.
Let

$$\varepsilon' := \inf\bigl\{-\nabla\mathcal{V}(q) \cdot \bar{v} \;:\; q \in q_E + \mathcal{D} ,\; q \cdot \bar{v} \leq \tilde{q}(0) \cdot \bar{q}\bigr\} > 0 \tag{4.23}$$

(see Lemma 4.6). We can write

$$\Bigl(\, q \in q_E + \mathcal{D} \text{ and } q \cdot \bar{v} \leq \tilde{q}(0) \cdot \bar{q} \,\Bigr) \quad \Rightarrow \quad -\nabla(\mathcal{V} + V)(q) \cdot \bar{v} \geq \min\{\varepsilon - \sup|\nabla V| \,,\, \varepsilon'\} > 0 . \tag{4.24}$$

But then $t \mapsto \tilde{q}(t) \cdot \bar{v}$ cannot be decreasing, because its second derivative is greater than a positive constant for all $t \geq 0$. ☐

Suppose that for a finite-sum potential verifying Hypotheses 4.1 the set $\mathcal{M}$ of the asymptotically regular initial data coincides with the whole phase space. Then the associated Hamiltonian system is globally integrable *together with all slightly perturbed systems* as specified by the previous proposition.

The paper [GZ1] provides a class of globally integrable finite-sum potential systems. At this point it is easily described:

Proposition 4.9 *Assume the Hypotheses 4.1 and that the vectors $v_1, \ldots, v_N$ verify*

$$v_\alpha \cdot v_\beta \geq 0 \qquad \forall \alpha, \beta = 1, \ldots, N . \tag{4.25}$$

Then for the associated Hamiltonian System all initial data are asymptotically regular.

The proof of Proposition 4.8 contains also the proof of the following one.

Proposition 4.10 *Assume the Hypotheses 4.1. Then, for all initial data* $(\bar{p}, \bar{q})$,

$$p_\infty(\bar{p}, \bar{q}) \in \mathcal{D}\,. \tag{4.26}$$

Moreover, let the vector $\bar{v}$ *be given by Lemma 4.4. Then*

$$p_\infty(\bar{p}, \bar{q}) \cdot \bar{v} > 0\,. \tag{4.27}$$

In particular, the asymptotic velocity never vanishes.

Proof. Formula (4.26) comes from the fact that $q(t, \bar{p}, \bar{q})$ belongs to a set of the form $q_{\scriptscriptstyle E} + \mathcal{D}$ for all t. Formula (4.27) comes along with the argument used in the proof of Proposition 4.8, assuming $V \equiv 0$. $\square$

5. A Counterexample to Global Asymptotic Regularity

Let us start with a finite-sum potential in $\mathbb{R}^2$ of the form

$$\mathcal{V}(x,y) := 2e^{-x+y} + f(y)\,, \qquad (x,y) \in \mathbb{R}^2, \tag{5.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function to be determined. We impose that the following trajectory

$$\begin{cases} x(t) := 4t - 2\ln t\,, \\ y(t) := 4t - 4\ln t \end{cases} \tag{5.2}$$

be a solution, for large t, of the Hamiltonian system associated to $\mathcal{V}$. We are going to see that this is indeed possible with $\mathcal{V}$ satisfying Hypotheses 4.1. The cone of the forces will turn out to be

$$\mathbb{C} = \{(x,y) \in \mathbb{R}^2 \ :\ y > 0,\ x + y > 0\} \tag{5.3}$$

which is generated by the vectors $(1,-1)$ and $(0,1)$, and is *wider than* $\pi/2$. The dual cone will be

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \ :\ y \geq 0,\ x - y \geq 0\}\,. \tag{5.4}$$

The asymptotic velocity for the trajectory (5.2) is the vector $(4,4) \in \mathbb{R}^2$, which is *on the boundary of* $\mathcal{D}$.

The acceleration of the trajectory is

$$\ddot{x}(t) = \frac{2}{t^2}\,, \qquad \ddot{y}(t) = \frac{4}{t^2}\,. \tag{5.5}$$

The force $-\nabla\mathcal{V}$ has components

$$-\frac{\partial\mathcal{V}}{\partial x}(x,y) = 2e^{-x+y}, \qquad -\frac{\partial\mathcal{V}}{\partial y}(x,y) = -2e^{-x+y} - f'(y)\,. \tag{5.6}$$

We must impose the equality $(\ddot{x}, \ddot{y}) = -\nabla\mathcal{V}$. For the x component this is already true:

$$\frac{2}{t^2} = \ddot{x}(t) = -\frac{\partial\mathcal{V}}{\partial x}(x(t), y(t)) = 2e^{-x(t)+y(t)} = 2e^{-2\ln t}. \tag{5.7}$$

For the y component we get the following condition on f:

$$\frac{4}{t^2} = \ddot{y}(t) = -\frac{\partial\mathcal{V}}{\partial y}(x(t), y(t)) = -\frac{2}{t^2} + f'(y(t))\,, \tag{5.8}$$

that is,

$$f'(4t - 4\ln t) = -\frac{6}{t^2}\,. \tag{5.9}$$

Upon multiplication by $4 - 4/t$ and integration we get

$$f(4t - 4\ln t) = \frac{24}{t} - \frac{12}{t^2} + c\,. \tag{5.10}$$

The function $t \mapsto 4t - 4\ln t$ is a C^∞ diffeomorphism between the intervals, say, $[2, +\infty[$ and $[8 - 4\ln 2, +\infty[$. So (selecting the constant $c = 0$) there exists a C^∞ function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\sup f = +\infty\,, \qquad \inf f = 0\,, \tag{5.11}$$

$$f'(y) < 0 \quad \forall y \in \mathbb{R}\,, \tag{5.12}$$

$$f(4t - 4\ln t) = \frac{24}{t} - \frac{12}{t^2} \qquad \forall t \geq 2\,. \tag{5.13}$$

The function f is *not integrable at* $+\infty$:

$$\int_{8-4\ln 2}^{4t-4\ln t} f(y)\,dy = \int_2^t f(4s - 4\ln s)\Big(4 - \frac{4}{s}\Big)ds = 96\int_2^t \Big(\frac{1}{s} - \frac{3}{2s^2} + \frac{1}{2s^3}\Big)ds\,, \tag{5.14}$$

which diverges as $t \to +\infty$. Of course, f is not a multiple of an exponential function, not even asymptotically. The cones $\mathbb{C}$ and $\mathcal{D}$ are easily verified to be as announced.

We are only left to prove that the derivatives of f have alternate signs:

$$f^{(k)}(y) \begin{cases} > 0 & \text{if } k \text{ is even,} \\ < 0 & \text{if } k \text{ is odd} \end{cases} \tag{5.15}$$

for all k and all large y (possibly depending on k). The inequalities already hold, globally, for $k = 0$, 1. Differentiating (5.9) we get

$$f^{(k)}(4t - 4\ln t) = \varphi_k(t)\,, \tag{5.16}$$

where the function φ_k is defined recursively as

$$\varphi_1(t) := -\frac{6}{t^2}\,, \qquad \varphi_{k+1}(t) := \varphi_k'(t)\frac{t}{4t - 4}\,. \tag{5.17}$$

It is easy to see that φ_k is a rational function and that the degree of the denominator exceedes (by $k + 1$) the degree of the numerator. Then φ_k is monotone and infinitesimal at $+\infty$, so that $\varphi_k'(t)$, and hence $\varphi_{k+1}(t)$ too, has the opposite sign of $\varphi_k(t)$ for all large t.

We can conlude with the following statement: *For the Hamiltonian system associated with this cone potential $\mathcal{V}$, the two components of the asymptotic velocity are C^∞ integrals of motion, independent and in involution on the nonempty, open, invariant set $\mathcal{M}$ of the asymptotically regular initial data. However, $\mathcal{M}$ does not coincide with the whole phase space.*

6. The Case of All Equal Functions

In this Section we make the following assumptions on the functions f_α of Section 4.

Hypothesis 6.1 *All the functions f_α are multiples of a single smooth function f:*

$$f_\alpha = c_\alpha f\,, \quad c_\alpha > 0\,, \tag{6.1}$$

defined on either $\mathbb{R}$ or on $]0,+\infty[$, and such that $f'(x) < 0$ for all x and

$$x \mapsto x|f'(x)| \text{ is weakly decreasing on } [a,+\infty[\tag{6.2}$$

for some $a > 0$.

Proposition 6.2 *Suppose that Hypotheses 4.1 and 6.1 hold. Then, for the Hamiltonian system associated to the potential $\mathcal{V}$, all initial data $(\bar{p},\bar{q}) \in \operatorname{dom}\mathcal{V}$ are asymptotically regular, i.e., the asymptotic velocity always belongs to the interior of the dual cone $\mathcal{D}$:*

$$p_\infty(\bar{p},\bar{q}) \cdot v_\alpha > 0 \qquad \forall(\bar{p},\bar{q}) \in \operatorname{dom}\mathcal{V}\,, \quad \forall\alpha = 1,\ldots,N\,. \tag{6.3}$$

The condition that $x \mapsto x|f'(x)|$ be monotone is not very restrictive. With only Hypotheses 4.1, the derivative $|f'|$ is monotone and integrable on $[a,+\infty[$, and this already implies that $x|f'(x)| \to 0$ as $x \to +\infty$ (this is elementary; see [GZ1], Lemma 10.3). Examples of functions f that verify our requirements are

$$f(x) := e^{-x}, \qquad x \in \mathbb{R}\,; \tag{6.4}$$

$$f(x) := \frac{1}{x^r}\,, \qquad x > 0\,, \quad r > 0\,; \tag{6.5}$$

$$f(x) := \frac{1}{(\ln(1+x))^r}\,, \qquad x > 0\,, \quad r > 0\,. \tag{6.6}$$

Lemma 6.3 *Suppose that f verifies the Hypothesis 6.1. Let $\varphi: \mathbb{R} \to \operatorname{dom} f \subset \mathbb{R}$ be a function such that*

$$\inf \varphi = L > \inf \operatorname{dom} f\,, \qquad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = 0\,. \tag{6.7}$$

Then, for any $\gamma > 0$,

$$\lim_{t\to+\infty} \frac{f'(\gamma t)}{f'(\varphi(t))} = 0\,. \tag{6.8}$$

Proof. For all large t we have $\gamma t \geq \varphi(t)$ and $\gamma t \geq a$. Then, using the monotonicity of $x \mapsto x|f'(x)|$:

$$0 < |f'(\gamma t)| = \frac{1}{\gamma t}\gamma t|f'(\gamma t)| \leq \theta(t)|f'(\varphi(t))|\,, \tag{6.9}$$

where $\theta(t)$ is defined as

$$\theta(t) := \begin{cases} \dfrac{\varphi(t)}{\gamma t} & \text{if } \varphi(t) \geq a, \\[2ex] \dfrac{a|f'(a)|}{\gamma t \min\{|f'(x)| \;:\; L \leq x \leq a\}} & \text{if } \varphi(t) \leq a. \end{cases} \tag{6.10}$$

It is clear that $\theta(t) \to 0$ as $t \to +\infty$. $\square$

Proof of Proposition 6.2 We suppress the initial conditions $(\bar p, \bar q)$ from the notation, because we are only interested in single trajectories. So, let $(p(t), q(t))$ be a motion of the system, with asymptotic velocity p_∞. We already know (Lemma 4.7) that $q(t) \in q_E + \mathcal{D}$ for all t, with $q_E \in \operatorname{dom} \mathcal{V}$, and that $0 \neq p_\infty \in \mathcal{D}$ (Proposition 4.10). What we are left to prove is that $p_\infty \notin \partial\mathcal{D}$.
Suppose the contrary. Let I_0 be the subset of $\{1, \dots, N\}$ where $p_\infty \cdot v_\alpha = 0$ and I_1 be the complement, i.e., where $p_\infty \cdot v_\alpha > 0$. The set I_0 is nonempty because $p_\infty \in \partial\mathcal{D}$. The complement I_1 is nonempty because $0 \neq p_\infty \in \mathcal{D}$.
With Lemma 4.4 applied to $\{v_\alpha \;:\; \alpha \in I_0\}$, we can find a vector $\bar v$ such that

$$\begin{aligned} &\bar v = \sum_{\alpha \in I_0} \rho_\alpha v_\alpha \qquad \rho_\alpha > 0\,, \\ &\bar v \cdot v_\alpha > 0 \qquad \forall \alpha \in I_0\,. \end{aligned} \tag{6.11}$$

In particular, $\bar v$ is orthogonal to p_∞.
Define the function

$$g(t) := q(t) \cdot \bar v\,. \tag{6.12}$$

We have $g'(t) = p(t) \cdot \bar v \to p_\infty \cdot \bar v = 0$. We will reach a contradiction by showing that $g''(t) \geq \text{constant} > 0$ for large t.
Let us split $\mathcal{V}$ into the sum of two potentials

$$\mathcal{V}_0(q) := \sum_{\alpha \in I_0} c_\alpha f(q \cdot v_\alpha)\,, \qquad \mathcal{V}_1(q) := \sum_{\alpha \in I_1} c_\alpha f(q \cdot v_\alpha)\,. \tag{6.13}$$

The gradient of $\mathcal{V}_0$ is orthogonal to p_∞, whilst p_∞ is in the interior of the dual of the cone spanned by $-\nabla\mathcal{V}_1$.
Let

$$2\gamma := \min_{\alpha \in I_1} p_\infty \cdot v_\alpha > 0\,. \tag{6.14}$$

We get $q(t) \cdot v_\alpha \geq \gamma t$ for all large t and all $\alpha \in I_1$, so that

$$\begin{aligned} |\nabla\mathcal{V}_1(q(t)) \cdot \bar v| \leq &\sum_{\alpha \in I_1} c_\alpha |\bar v \cdot v_\alpha| \big| f'(q(t) \cdot v_\alpha) \big| \leq \\ \leq &\sum_{\alpha \in I_1} c_\alpha |\bar v \cdot v_\alpha| \; \big| f'(\gamma t) \big| := c |f'(\gamma t)| \end{aligned} \tag{6.15}$$

for all large t, because $|f'|$ is decreasing on $[a, +\infty[$.
Let us consider the second derivative of g:

$$\begin{aligned} g''(t) = &-\nabla\mathcal{V}(q(t)) \cdot \bar v = -\nabla\mathcal{V}_0(q(t)) \cdot \bar v - \nabla\mathcal{V}_1(q(t)) \cdot \bar v = \\ = &-\sum_{\alpha \in I_0} c_\alpha f'(q(t) \cdot v_\alpha)\, v_\alpha \cdot \bar v - \nabla\mathcal{V}_1(q(t)) \cdot \bar v \geq \\ \geq &-\sum_{\alpha \in I_0} c_\alpha f'(q(t) \cdot v_\alpha)\, v_\alpha \cdot \bar v - c\, |f'(\gamma t)| \end{aligned} \tag{6.16}$$

for all large t.

For any $\alpha \in I_0$, the function $\varphi(t) := q(t) \cdot v_\alpha$ is bounded from below by $q_E \cdot v_\alpha$ and

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = \lim_{t\to+\infty} \varphi'(t) = p_\infty \cdot v_\alpha = 0 \tag{6.17}$$

with L'Hôpital's rule. We can apply Lemma 6.3 and get that

$$|f'(\gamma t)| = o(-f'(q(t) \cdot v_\alpha)) \quad \text{as } t \to +\infty, \tag{6.18}$$

and, since $-f' > 0$,

$$g''(t) > 0 \quad \text{for all large } t. \tag{6.19}$$

But we know that $g'(t) \to 0$ as $t \to +\infty$, so that g turns out to be decreasing for large t:

$$q(t) \cdot \bar{v} \leq q(t_0) \cdot \bar{v} \quad \text{for all } t \geq t_0. \tag{6.20}$$

Next, Lemma 4.6, applied to $\mathcal{V}_0$ and $\bar{v}$, together with formula (6.16), yields

$$-\nabla\mathcal{V}_0(q(t)) \cdot \bar{v} \geq \inf\{-\nabla\mathcal{V}_0(q) \cdot \bar{v} \ :\ (q - q_E) \cdot v_\alpha \geq 0 \ \forall\alpha \in I_0,\ q \cdot \bar{v} \leq q(t_0) \cdot \bar{v}\} > 0 \tag{6.21}$$

for all $t \geq t_0$, so that

$$g''(t) \geq \text{constant} > 0\,, \tag{6.22}$$

for all large t, which is the desired contradiction. □

Note. After reading Theorem 1 of [Hu], we realized that condition (6.2) can be dropped. In fact, although equation (6.18) may no longer be true, it is still possible to write

$$\int_t^{+\infty} |f'(\gamma s)|\, ds = o\Big(\int_t^{+\infty} \big|f'\big(q(s) \cdot v_\alpha\big)\big|\, ds\Big) \quad \text{as } t \to +\infty, \tag{6.23}$$

and apply it to the integrated version of (6.16). However, we have chosen to retain our original proof, because it seems to use more "mechanical" quantities, and it applies to all relevant examples we know of.

7. The Case of all Integrable Functions

In this Section the assumptions on f_α, in addition to Hypotheses 4.1, are of fast decay type. Namely, they are integrable at $+\infty$.

Hypothesis 7.1 *For all $\alpha = 1, \ldots, N$, the function f_α is such that*

$$\int_a^{+\infty} f_\alpha(x)\, dx < +\infty\,. \tag{7.1}$$

Proposition 7.2 *Suppose that Hypotheses 4.1 and 7.1 hold. Then, for the Hamiltonian system associated to the potential $\mathcal{V}$, all initial data $(\bar{p}, \bar{q}) \in \operatorname{dom} \mathcal{V}$ are asymptotically regular, i.e., the asymptotic velocity always belongs to the interior of the dual cone $\mathcal{D}$:*

$$p_\infty(\bar{p}, \bar{q}) \cdot v_\alpha > 0 \qquad \forall (\bar{p}, \bar{q}) \in \operatorname{dom} \mathcal{V}\,, \quad \forall \alpha = 1, \ldots, N\,. \tag{7.2}$$

Lemma 7.3 *If the Hypotheses 1.1 and 3.1 hold, then the function h_0 of formula (4.9) verifies*

$$\int_0^{+\infty} x\, h_0(x)\, dx < +\infty\,. \tag{7.3}$$

The statement of the following lemma is a little awkward, because it is going to be applied not to the original potential $\mathcal{V}$.

Lemma 7.4 *Let $\tilde{q}\colon \mathbb{R} \to \mathbb{R}^n$ be a C^1 function such that*

$$\lim_{t \to +\infty} \frac{d\tilde{q}}{dt}(t) = \tilde{p}_\infty \in \mathcal{D}^\circ\,. \tag{7.4}$$

Then the function $(r, s) \mapsto |\nabla \mathcal{V}(\tilde{q}(s))|$ is integrable on $\{(r, s) \in \mathbb{R}^2 \ :\ 0 \le r \le s < +\infty\}$. In particular, the following integrals converge absolutely:

$$\int_t^{+\infty} dr \int_r^{+\infty} \nabla \mathcal{V}(\tilde{q}(s))\, ds = \int_t^{+\infty} (s - t) \nabla \mathcal{V}(\tilde{q}(s))\, ds\,. \tag{7.5}$$

Proof. If $2\gamma := \operatorname{dist}(\tilde{p}_\infty, \partial \mathcal{D})$, then the trajectory satisfies

$$\operatorname{dist}\bigl(\tilde{q}(t), q(t_0) + \partial \mathcal{D}\bigr) \ge \gamma t \quad \text{for all } t \ge t_0 \tag{7.6}$$

for some t_0. We can conclude using Lemma 7.3. $\square$

Proof of Proposition 7.2 We suppress the initial conditions $(\bar p, \bar q)$ from the notation, because we are only interested in single trajectories. So, let $(p(t), q(t))$ be a motion of the system, with asymptotic velocity p_∞. We already know (Lemma 4.7) that $q(t) \in q_E + \mathcal{D}$ for all t, with $q_E \in \operatorname{dom} \mathcal{V}$, and that $0 \neq p_\infty \in \mathcal{D}$ (Proposition 4.10). What we are left to prove is that $p_\infty \notin \partial\mathcal{D}$.

Suppose the contrary. Let I_0 be the subset of $\{1, \ldots, N\}$ where $p_\infty \cdot v_\alpha = 0$ and I_1 be the complement, i.e., where $p_\infty \cdot v_\alpha > 0$. The set I_0 is nonempty because $p_\infty \in \partial\mathcal{D}$. The complement I_1 is nonempty because $0 \neq p_\infty \in \mathcal{D}$.

With Lemma 4.4 applied to $\{v_\alpha \;:\; \alpha \in I_0\}$, we can find a vector $\bar v$ such that

$$\begin{aligned} &\bar v = \sum_{\alpha \in I_0} \rho_\alpha v_\alpha \qquad \rho_\alpha > 0\,, \\ &\bar v \cdot v_\alpha > 0 \qquad \forall \alpha \in I_0\,. \end{aligned} \tag{7.7}$$

In particular, $\bar v$ is orthogonal to p_∞.

Define the function

$$g(t) := q(t) \cdot \bar v\,. \tag{7.8}$$

We have $g'(t) = p(t) \cdot \bar v \to p_\infty \cdot \bar v = 0$. We will reach a contradiction by showing that $g''(t) \geq \text{constant} > 0$ for large t.

Let us split $\mathcal{V}$ into the sum of two potentials

$$\mathcal{V}_0(q) := \sum_{\alpha \in I_0} f_\alpha(q \cdot v_\alpha)\,, \qquad \mathcal{V}_1(q) := \sum_{\alpha \in I_1} f_\alpha(q \cdot v_\alpha)\,. \tag{6.13}$$

The gradient of $\mathcal{V}_0$ is orthogonal to p_∞, whilst p_∞ is in the interior of the dual of the cone spanned by $-\nabla\mathcal{V}_1$.

So far the proof was just the same as for Proposition 6.2. Let us write $g(t)$ this way:

$$\begin{aligned} g(t) &= \Big(q(t) + \int_t^{+\infty} dr \int_r^{+\infty} \nabla\mathcal{V}_1(q(s))\, ds\Big) \cdot \bar v - \Big(\int_t^{+\infty} dr \int_r^{+\infty} \nabla\mathcal{V}_1(q(s))\, ds\Big) \cdot \bar v := \\ &:= g_0(t) - g_1(t)\,. \end{aligned} \tag{6.14}$$

The integral converges because of Lemma 7.4 applied to $\mathcal{V}_1$. It is obvious that $g_1(t) \to 0$ as $t \to +\infty$, and in particular g_1 is bounded as $t \to +\infty$.

We are going to show that g_0 is bounded too. In fact

$$\begin{aligned} &g_0'(t) = \Big(p(t) - \int_t^{+\infty} \nabla\mathcal{V}_1(q(s))\, ds\Big) \cdot \bar v \;\to\; p_\infty \cdot \bar v = 0 \qquad \text{as } t \to +\infty, \\ &g_0''(t) = \big(\dot p(t) + \nabla\mathcal{V}_1(q(t))\big) \cdot \bar v = -\nabla\mathcal{V}_0(q(t)) \cdot \bar v > 0 \quad \forall t\,, \end{aligned} \tag{6.15}$$

so that g_0 is decreasing. It is bounded from below because the integral is infinitesimal and $q(t) \cdot \bar v \geq q_E \cdot \bar v$ since $q(t) \in q_E + \mathcal{D}$ (see Lemma 4.7).

Having proved that $g(t)$ is bounded as $t \to +\infty$, we get in particular that each $q(t) \cdot v_\alpha$ is bounded for each $\alpha \in I_0$. We can write

$$g''(t) = -\sum_{\alpha \in I_0} f'_\alpha(q(t) \cdot v_\alpha)\, v_\alpha \cdot \bar{v} - \nabla \mathcal{V}_1(q(t))\,. \tag{6.16}$$

The last term is infinitesimal as $t \to +\infty$, whilst each $-f'_\alpha(q(t) \cdot v_\alpha)$ is bounded below by the positive constant

$$\min\{-f'_\alpha(x) \;:\; q_E \cdot v_\alpha \leq x \leq \sup_{t \geq 0} q(t) \cdot v_\alpha\} \tag{6.17}$$

and $v_\alpha \cdot \bar{v} > 0$ for all $\alpha \in I_0$.
We get finally $g''(t) \geq$ constant > 0 as $t \to +\infty$, which contradicts the boundedness of g.
$\square$

8. Global Integrability for Finite-sum Potentials

We are going to gather here the statements of the global integrability results, whose proofs are scattered in the previous Sections, concerning the potentials which can be written as finite sums of one-dimentional functions ("finite-sum potentials"):

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} f_\alpha(q \cdot v_\alpha)\,, \tag{8.1}$$

where $v_1, \ldots, v_N$ are given nonzero vectors in $\mathbb{R}^n$ ($N \geq 1$, no relation to n), and the functions $f_1, \ldots, f_N$ are real functions of one variable, whose domains are each either $\mathbb{R}$ or the interval $]0, +\infty[$. The potential $\mathcal{V}$ is itself defined on the set

$$\{q \in \mathbb{R}^n \ : \ q \cdot v_\alpha \in \operatorname{dom} f_\alpha \quad \forall \alpha = 1, \ldots, N\}\,. \tag{8.2}$$

Hypotheses 8.1 *The vectors $v_1, \ldots, v_N$ are nonzero and the cone generated by them is proper. The f_α are C^{m+1} ($m \geq 2$) functions and*

$$\sup f_\alpha = +\infty\,, \qquad \inf f_\alpha = 0\,, \tag{8.3}$$

$$f'_\alpha(x) < 0 \quad \forall x \in \operatorname{dom}\, f_\alpha\,, \tag{8.4}$$

$$f_\alpha^{(k)}(x) \begin{cases} > 0 & \text{if } k \text{ is even,} \\ < 0 & \text{if } k \text{ is odd,} \end{cases} \quad \forall x \geq a\,, \tag{8.5}$$

$$f_\alpha^{(m+1)} \text{ is monotone on } [a, +\infty[, \tag{8.6}$$

where $a \geq 0$ is a constant. Moreover, whichever one of the three following conditions i), ii), iii) holds:

i) the vectors v_α verify $v_\alpha \cdot v_\beta \geq 0$ $\forall \alpha, \beta$;

ii) all the functions f_α are multiples of a single function f:

$$f_\alpha = c_\alpha f\,, \quad c_\alpha > 0\,, \tag{8.7}$$

such that

$$x \mapsto x|f'(x)| \text{ is weakly decreasing on } [a, +\infty[\,; \tag{8.8}$$

iii) for all $\alpha = 1, \ldots, N$, the function f_α is such that

$$\int_a^{+\infty} f_\alpha(x)\, dx < +\infty\,. \tag{8.9}$$

Theorem 8.2 *Assume Hypotheses 8.1. Then the Hamiltonian system*

$$\dot{p} = -\nabla \mathcal{V}\,, \quad \dot{q} = p\,,$$

where $\mathcal{V}$ is given by (8.1), is C^m-completely integrable.

9. Appendix

We provide here the detailed proof of a simple fact needed in Section 3.

Proposition 9.1 *Let $\{f_i : i \in I\}$ be a nonempty family of functions $\mathbb{R} \to \mathbb{R}$, all of them Lipschitz with the same constant M:*

$$\big|f_i(x) - f_i(y)\big| \le M|x - y| \qquad \forall x, y \in \mathbb{R}\,, \quad \forall i \in I\,. \tag{9.1}$$

Let $f(x) := \inf\{f_i(x) : i \in I\}$ be the pointwise infimum of the family. If f is finite at a point $x_0 \in \mathbb{R}$, then it is finite everywhere and Lipschitz with constant M.

Proof. Let $y \in \mathbb{R}$, $i \in I$. Then

$$f_i(y) \ge f_i(x_0) - M|y - x_0| \ge f(x_0) - M|y - x_0|\,, \tag{9.2}$$

so that

$$f(y) \ge f(x_0) - M|y - x_0| > -\infty\,. \tag{9.3}$$

The finiteness of f and half of the Lipschitz property are settled (x_0 becomes generic now). Let $\varepsilon > 0$ and $i \in I$ such that $f_i(x_0) \le f(x_0) + \varepsilon$. Then, for any $y \in \mathbb{R}$,

$$f_i(y) \le f_i(x_0) + M|y - x_0| \le f(x_0) + \varepsilon + M|y - x_0|\,, \tag{9.4}$$

so that

$$f(y) \le f(x_0) + \varepsilon + M|y - x_0| \qquad \forall \varepsilon > 0\,, \tag{9.5}$$

and the proof is complete. $\square$

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