Reducing Scattering Problems under Cone Potentials to Normal Form by Global Canonical Transformations

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Abstract. We introduce a class of Hamiltonian scattering systems which can be reduced to the *"normal form"* $\dot{P} = 0$, $\dot{Q} = P$, by means of a global *canonical transformation* $(P, Q) = \mathcal{A}(p, q)$, $p, q \in \mathbb{R}^n$, defined through *asymptotic* properties of the trajectories.

These systems are obtained requiring certain geometrical conditions on $\dot{p} = -\nabla\mathcal{V}(q)$, $\dot{q} = p$, where $\mathcal{V}$ is a bounded below *"cone potential"*, i.e., the force $-\nabla\mathcal{V}(q)$ always belongs to a closed convex cone which contains no straight lines.

We can deal with very different asymptotic behaviours of the potential and the potential can undergo small perturbations in any arbitrary compact set without losing the existence and the properties of $\mathcal{A}$.

1. Introduction

This paper presents new results within the theory developed in [3] on the Hamiltonian systems

$$\dot p = -\nabla\mathcal{V}(q)\,, \qquad \dot q = p\,, \qquad p,q \in I\!R^n\,. \tag{1.1}$$

$\mathcal{V}$ is assumed to be a *cone potential*, that is, the force $-\nabla\mathcal{V}(q)$ is always in a closed convex cone which contains no lines. If we also assume that $\mathcal{V}$ is bounded below (so that in particular the solutions globally exist), then the velocity p has a finite limit p_∞ as time goes to $+\infty$. This remarkably simple fact was showed by Gutkin in [4].

We denote by $t \mapsto (p(t,\bar p,\bar q), q(t,\bar p,\bar q))$ the solution to (1.1) with $(\bar p,\bar q)$ as initial data: $p(0,\bar p,\bar q) = \bar p$, $q(0,\bar p,\bar q) = \bar q$. The function

$$p_\infty(\bar p,\bar q) := \lim_{t\to+\infty} p(t,\bar p,\bar q) \in I\!R^n\,, \tag{1.2}$$

is trivially constant along the motions solving (1.1).

In [3] we found sufficient conditions on $\mathcal{V}$ for the components of p_∞ to be C^k ($2 \le k \le +\infty$) first integrals, independent and pairwise in involution. We could thereby construct a class of integrable Hamiltonian systems with cone potentials.

The present paper extends the investigation from the asymptotic velocities p_∞ to the limits, sometimes referred to as asymptotic phases,

$$a_\infty(\bar p,\bar q) := \lim_{t\to+\infty}\big(q(t,\bar p,\bar q) - t\,p(t,\bar p,\bar q)\big) \in I\!R^n\,. \tag{1.3}$$

We find sufficient conditions which, in particular, guarantee the existence and smoothness of these limits as functions of the initial data $(\bar p,\bar q)$ (Proposition 2.2).

A geometric interpretation of a_∞ follows from the fact that the limit of $(q(t) - t\,p_\infty)$ as $t \to +\infty$ is a_∞ too (Proposition 2.2), so that the motion $t \mapsto q(t,\bar p,\bar q)$, as $t \to +\infty$, is asymptotically rectilinear uniform:

$$q(t,\bar p,\bar q) = a_\infty(\bar p,\bar q) + t\,p_\infty(\bar p,\bar q) + o(1) \qquad \text{as } t \to +\infty,$$

and, in particular, the straight line $\{a_\infty(\bar p,\bar q) + \xi\,p_\infty(\bar p,\bar q)\ ;\ \xi \in I\!R\}$ is an *asymptote* for the trajectory.

We call $(p_\infty(\bar p,\bar q), a_\infty(\bar p,\bar q))$ *the asymptotic data*, and

$$\mathcal{A}\colon (\bar p,\bar q) \mapsto (p_\infty(\bar p,\bar q), a_\infty(\bar p,\bar q)) \tag{1.4}$$

the asymptotic map. Of course we can consider the corresponding *asymptotic map in the past*

$$\mathcal{A}_-\colon (\bar p,\bar q) \mapsto (p_{-\infty}(\bar p,\bar q), a_{-\infty}(\bar p,\bar q))$$

by taking the limits in (1.2) and (1.3) as time goes to $-\infty$, and finally the *scattering map* $\mathcal{A}\circ\mathcal{A}_-^{-1}$.

In this paper we strengthen the hypotheses of [3] and, as a main result (Theorem 2.3), we prove that:

a) *$\mathcal{A}$ is a global C^k-diffeomorphism ($2 \le k \le +\infty$), whose exact range has a simple geometric description;*

b) *$\mathcal{A}$ is a canonical transformation;*

c) *$\mathcal{A}$ transforms the Hamiltonian $\mathcal{H} := \frac{1}{2}|p|^2 + \mathcal{V}(q)$, which defines (1.1), into*

$$\mathcal{K}(P,Q) := \mathcal{H} \circ \mathcal{A}^{-1}(P,Q) = \frac{1}{2}|P|^2 , \tag{1.5}$$

(up to a trivial additive constant), that is, $\mathcal{A}$ transforms (1.1) into the simple linear form

$$\dot{P} = 0 , \qquad \dot{Q} = P , \tag{1.6}$$

which is precisely what we mean by "normal" in the title

(see Appendix I for the definition of canonical transformations and the properties that we use; remark that this definition is more restrictive than the one adopted by several authors).

Of course we have symmetrical results for $\mathcal{A}_-$. Thus, the scattering map $\mathcal{A} \circ \mathcal{A}_-^{-1}$ is proved to be a canonical transformation too, and we can exhibit its exact domain and range (see the Remark following Theorem 2.3).

Let us also point out that the canonical variables (P,Q), introduced by the asymptotic map $\mathcal{A}$, are similar to the celebrated *action-angle variables* I, ϕ (ϕ defined mod 2π), which transform Hamilton equations into

$$\dot{I} = 0 , \qquad \dot{\phi} = \omega(I) . \tag{1.7}$$

—see [2], Chapter 4. Of course, equations (1.6) describe a scattering system (Q is not defined mod 2π), while (1.7) represents oscillations.

We think that a noteworthy (and probably new) property is the following: the complete integrability of our systems, as well as the the existence of the variables $(P,Q) = \mathcal{A}(p,q)$ yielding (1.6), are *persistent under any small perturbation of the potential in an arbitrary compact set* (Persistence Theorem 2.4).

* * *

Now, after the short rewiew of the results, let us briefly outline the nature of our hypotheses and highlight some of the crucial points of the proofs.

We always assume on the bounded below cone potential $\mathcal{V}$ some global properties, to be discussed later, and a certain "decay law" for the gradient $\nabla\mathcal{V}$ (Hypotheses 2.1, iv), whence the existence and continuity of the asymptotic map $\mathcal{A}$.

To achieve the differentiability of $\mathcal{A}$, we first consider potentials with *exponential* asymptotic behaviour of the partial derivatives (Hypotheses 2.1, v). This assumption, together with Gronwall a priori estimates on the solutions of the linear variational equations, permits to prove that the limits (1.2) and (1.3) are in the C^k norm, locally in $(\bar{p}, \bar{q})$.

As in [3], we can deal with *nonexponential* asymptotic decays of the derivatives of $\mathcal{V}$ by the use of the side hypotheses of *convexity* on $\mathcal{V}$ and a kind of *monotonicity* in the Hessian matrix of $\mathcal{V}$ (Hypotheses 2.1, vi). These assumptions permit the use of some *Liapunov functions* which yield a priori estimates much sharper than the mere Gronwall ones.

The *global* conditions that we mentioned (Hypotheses 2.1 ii,iii) are used in [3] first to guarantee that the asymptotic velocity always belongs to the *interior* of the convex closed cone $\mathcal{D}$ defined as the dual of the cone $\mathcal{C}$ spanned by the forces, i.e.,

$$\mathcal{C} := \left\{ -\sum_{\alpha\in I} \lambda_\alpha \, \nabla\mathcal{V}(q_\alpha) \ : \ \emptyset \neq I \text{ finite set}, \ \lambda_\alpha \geq 0, \ q_\alpha \in \mathbb{R}^n \ \forall \alpha \in I \right\}, \tag{1.8}$$

$$\mathcal{D} := \big\{ v \in \mathbb{R}^n \ : \ w \cdot v \geq 0 \quad \forall w \in \mathcal{C} \big\}, \tag{1.9}$$

and then to prove the following crucial *locally uniform* estimate on the trajectories: for every $(\bar p_0, \bar q_0)$ there exist $\gamma > 0$, $t_0 \in \mathbb{R}$ and a neighbourhood U of $(\bar p_0, \bar q_0)$ such that

$$p(t, \bar p, \bar q) \in \mathcal{D}^\circ \,, \qquad \operatorname{dist}\big(p(t,\bar p,\bar q), \partial\mathcal{D}\big) \geq \gamma \,, \tag{1.10}$$

for every $t \geq t_0$ and every $(\bar p, \bar q) \in U$.

These global conditions on the potential, in particular, require that the scalar product between any two forces is nonnegative. So, in our framework, *the cone $\mathcal{C}$ has width not larger than $\pi/2$.*

In Section 3 we give some examples where our theory applies. Corollary 3.3 is about potentials with inverse r-power for arbitrary $r > 1$. The complete integrability holds for arbitrary $r > 0$, as it is shown in [3] (in [3], Section 1, we also refer to some papers of Calogero, Marchioro and Moser who discovered an analytically integrable system where the potential has $r = 2$ and cone of the forces wider than $\pi/2$).

Our assumptions hold for a class of Toda-like systems (Corollary 3.4), which are defined through finite sums of exponentials. The complete integrability of these systems was proved in [9]. Both Corollaries 3.3 and 3.4 are consequences of Proposition 3.2, where more general functions are considered.

These examples of cone potentials, as well as all the ones considered in the literature (as far as we know), have polyhedrical cone $\mathcal{C}$ of the forces. Our Section 3 starts by studying a simple example where $\mathcal{C}$ is *not polyhedrical* (namely, it is circular; Proposition 3.1). In fact, the present approach does not exploit such additional structures of $\mathcal{V}$ as being finite sum of one-dimensional functions (in the sense of Proposition 3.2). Therefore it may be called theory of “cone” potential with width $\leq \pi/2$.

* * *

The complete solution of the scattering problem associated with the non-periodic Toda lattice, and many related topics, was given by Moser in [8] by methods different from ours. These systems are physically very interesting since they describe the dynamics of finitely many particles on the line under the influence of pairwise interactions with exponential potential. In such case the cone spanned by the forces is wider than $\pi/2$ and there is analytic integrability.

Gutkin in [5] and [6] considered generically the problem of the existence of the limits in (1.3), and he studied the global existence (i.e. for every $(\bar{p},\bar{q})$) in [7], specially in connection with the problem of the scattering of particles in the line with pairwise interactions. In the title of [7], as well as in the whole paper, a "regular trajectory" is a trajectory for which $p_\infty \in \mathcal{D}^\circ$ and the limit in (1.3) exists. In Gutkin's framework the cone spanned by the forces has arbitrary width $< \pi$, but no estimates similar to (1.10) are given. Whether they hold anyway, and whether the methods of the present paper can be adapted to the systems considered in [7] are open problems. We guess they are worth studying even in very particular contexts like low dimensions and special forms of the potential.

2. Existence and Regularity of the Asymptotic Map

Given a smooth function $\mathcal{V}: \mathbb{R}^n \to \mathbb{R}$, we will denote by $\nabla\mathcal{V}$ its gradient, as a column vector, and $\mathsf{D}^m\mathcal{V}$ will be its m-th differential, regarded as a multilinear map from $(\mathbb{R}^n)^{m-1}$ into $\mathbb{R}^n$, endowed with the norm

$$\|\,\mathsf{D}^m\mathcal{V}(q)\| := \sup\{|\,\mathsf{D}^m\mathcal{V}(q)(x^{(1)},\dots,x^{(m-1)})| \;:\; x^{(i)} \in \mathbb{R}^n,\ |x^{(i)}| \le 1\}.$$

Throughout this section the function $\mathcal{V}$ is assumed to be defined on all of $\mathbb{R}^n$, but only for convenience of notation. Everything runs just as well if the domain is a set of the form $q + \mathcal{D}^\circ$. Remind that, given a cone $\mathcal{C} \subset \mathbb{R}^n$, we define the dual cone $\mathcal{D}$ as

$$\mathcal{D} := \left\{ v \in \mathbb{R}^n \;:\; w \cdot v \ge 0 \quad \forall w \in \mathcal{C} \right\}.$$

Hypotheses 2.1 *$\mathcal{V}$ is a function in $C^{m+1}(\mathbb{R}^n)$, $m \ge 2$. Let $\mathcal{C}$ be the the convex cone generated by $-\nabla\mathcal{V}$, and $\mathcal{D}$ be the dual cone of $\mathcal{C}$. We assume i) to iv):*

i) $\inf \mathcal{V} > -\infty$;

ii) for each $E > 0$ there exists a $q_E \in \mathbb{R}^n$ such that

$$q \in \mathbb{R}^n \backslash (q_E + \mathcal{D}) \quad \Rightarrow \quad \mathcal{V}(q) \ge E\,;$$

iii) for each q', $q'' \in \mathbb{R}^n$ such that $q'' \in q' + \mathcal{D}$, and for each $v \in \bar{\mathcal{C}} \backslash \{0\}$ there exists $\varepsilon > 0$ such that

$$\Big(q \in q' + \mathcal{D} \quad \text{and} \quad q \cdot v \le q'' \cdot v \Big) \quad \Rightarrow \quad -\nabla\mathcal{V}(q) \cdot v \ge \varepsilon\,;$$

iv) there exist a vector $q_0 \in \mathbb{R}^n$ and a weakly decreasing function $h_0: \mathbb{R}_+ \to \mathbb{R}$ such that $\int_0^{+\infty} x h_0(x)\,dx < +\infty$ and

$$q \in q_0 + \mathcal{D} \quad \Rightarrow \quad |\nabla\mathcal{V}(q)| \le h_0\Big(\operatorname{dist}\big(q, q_0 + \partial\mathcal{D}\big) \Big);$$

and either one of v) and vi):

v) there exist $q_1, \dots, q_m \in \mathbb{R}^n$, $A_1, \dots, A_m \ge 0$, $\lambda_1, \dots, \lambda_m > 0$ such that

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \|\,\mathsf{D}^{i+1}\mathcal{V}(q)\,\| \le A_i \exp\Big(-\lambda_i \operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big) \Big),$$

vi) there exist $q_1, \dots, q_m \in \mathbb{R}^n$ and weakly decreasing functions $h_1, \dots h_m: \mathbb{R}_+ \to \mathbb{R}$ such that

- *$\mathcal{V}$ is convex on $q_1 + \mathcal{D}$;*
- *for all q', $q'' \in q_1 + \mathcal{D}$ and all $z \in \mathbb{R}^n$ we have*

$$q'' \in q' + \mathcal{D} \quad \Rightarrow \quad \mathsf{D}^2\mathcal{V}(q'')z \cdot z \le \mathsf{D}^2\mathcal{V}(q')z \cdot z;$$

- *for all i, $\int_0^{+\infty} x^{i+1} h_i(x)dx < +\infty$ and*

$$q \in q_i + \mathcal{D} \quad \Rightarrow \quad \|\,\mathsf{D}^{i+1}\mathcal{V}(q)\| \le h_i\Big(\operatorname{dist}\big(q, q_i + \partial\mathcal{D}\big) \Big);$$

For a potential $\mathcal{V}$ satisfying the previous assumptions, we consider the Hamiltonian system

$$\dot{p} = -\nabla \mathcal{V}(q)\,, \qquad \dot{q} = p\,, \tag{2.1}$$

the associated flow

$$\Phi^t(\bar{p}, \bar{q}) = \begin{pmatrix} p(t, \bar{p}, \bar{q}) \\ q(t, \bar{p}, \bar{q}) \end{pmatrix}. \tag{2.2}$$

and the mappings

$$\mathcal{A}_t(\bar{p}, \bar{q}) := \begin{pmatrix} p(t, \bar{p}, \bar{q}) \\ q(t, \bar{p}, \bar{q}) - t\, p(t, \bar{p}, \bar{q}) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -tI_n & I_n \end{pmatrix} \Phi^t(\bar{p}, \bar{q})\,. \tag{2.3}$$

We are interested in the asymptotic map

$$\begin{pmatrix} p_\infty(\bar{p}, \bar{q}) \\ a_\infty(\bar{p}, \bar{q}) \end{pmatrix} = \mathcal{A}(\bar{p}, \bar{q}) := \lim_{t \to +\infty} \mathcal{A}_t(\bar{p}, \bar{q})\,. \tag{2.4}$$

Proposition 2.2 *Suppose that the Hypotheses 2.1 hold. Then the the asymptotic map $\mathcal{A}$ exists, and the convergence is locally in the C^m norm. The functions p_∞ and a_∞ verify*

$$\begin{aligned} p_\infty(\Phi^t(\bar{p}, \bar{q})) &= p_\infty(\bar{p}, \bar{q})\,, \\ a_\infty(\Phi^t(\bar{p}, \bar{q})) &= a_\infty(\bar{p}, \bar{q}) + t\, p_\infty(\bar{p}, \bar{q})\,, \end{aligned} \tag{2.5}$$

that is to say,

$$\mathcal{A} \circ \Phi^t = \begin{pmatrix} I_n & 0 \\ tI_n & I_n \end{pmatrix} \mathcal{A}\,. \tag{2.6}$$

Moreover,

$$\begin{aligned} a_\infty(\bar{p}, \bar{q}) &= \lim_{t \to +\infty} \big(q(t, \bar{p}, \bar{q}) - t\, p(t, \bar{p}, \bar{q})\big) = \\ &= \lim_{t \to +\infty} \big(q(t, \bar{p}, \bar{q}) - t\, p_\infty(\bar{p}, \bar{q})\big). \end{aligned} \tag{2.7}$$

Proof. Most of the arguments needed for the proof are an adaptation of the ones given in [3]. The starting point is the identity

$$\mathcal{A}_t(\bar{p}, \bar{q}) = \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix} + \int_0^t \begin{pmatrix} -I_n \\ sI_n \end{pmatrix} \nabla \mathcal{V}(q(s, \bar{p}, \bar{q}))\, ds\,, \tag{2.8}$$

which follows from the Hamilton equations. The reasoning of [3], Sections 4 and 5, proves that we can go to the limit in the formula, and the convergence is locally uniform. If we let $\tau \to +\infty$ in the next identity

$$\begin{pmatrix} I_n & 0 \\ -\tau I_n & I_n \end{pmatrix} \Phi^\tau \circ \Phi^t = \begin{pmatrix} I_n & 0 \\ -\tau I_n & I_n \end{pmatrix} \Phi^{\tau+t} = \begin{pmatrix} I_n & 0 \\ tI_n & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -(\tau+t)I_n & I_n \end{pmatrix} \Phi^{\tau+t}$$

we get formulas (2.6) and (2.5).
The proofs of [3], Sections 6, 7 and 8, can be adjusted to obtain that the convergence in formula (2.4) is locally C^m.
To end with, remark the two following identities:

$$
\begin{aligned}
q(t,\bar p,\bar q) - t\,p(t,\bar p,\bar q) &= \bar q + \iint_{0\le r\le s\le t} \nabla\mathcal{V}(q(s,\bar p,\bar q))\,dr\,ds\,,\\
q(t,\bar p,\bar q) - t\,p_\infty(\bar p,\bar q) &= \bar q + \iint_{\substack{0\le r\le t\\ r\le s}} \nabla\mathcal{V}(q(s,\bar p,\bar q))\,dr\,ds\,.
\end{aligned}
$$

The proof of the existence of $\mathcal{A}$ actually amounted to showing that the function $(r,s)\mapsto |\nabla\mathcal{V}(q(s,\bar p,\bar q))|$ is integrable on the set $\{(r,s)\in I\!R^2 \;:\; 0\le r\le s\}$, whence the equality (2.7).

$\diamondsuit$

Theorem 2.3 *Suppose that Hypotheses 2.1 hold, namely, i) to iv) and either v) or vi). Then the asymptotic map $\mathcal{A}$ exists, it is a C^m diffeomorphism of $I\!R^n\times I\!R^n$ onto $\mathcal{D}^\circ\times I\!R^n$ and it is a canonical transformation. The transformed Hamiltonian is*

$$
\mathcal{K}(P,Q) := \mathcal{H}\circ\mathcal{A}^{-1}(P,Q) = \frac{1}{2}|P|^2 + \inf\mathcal{V}\,, \tag{2.9}
$$

and the associated system is

$$
\dot P = 0\,, \qquad \dot Q = P\,. \tag{2.10}
$$

Proof. We refer again to the results and methods developed in [3]. Proposition 4.3 of that paper shows in particular that the range of $\mathcal{A}$ is contained in $\mathcal{D}^\circ\times I\!R^n$.
As in [3], Propositions 5.3, 6.4 and 7.5, given γ, there exists $\tilde q_\gamma\in I\!R^n$ such that, if we define the set A_γ as

$$
A_\gamma := \{(\bar p,\bar q) \;:\; \bar p\in\mathcal{D}^\circ,\ \mathrm{dist}(\bar p,\partial\mathcal{D})>\gamma\,,\ \bar q\in\tilde q_\gamma+\mathcal{D}\}\,,
$$

then

$$
(\bar p,\bar q)\in A_\gamma \quad\Rightarrow\quad \left|\mathcal{A}(\bar p,\bar q)-\begin{pmatrix}\bar p\\ \bar q\end{pmatrix}\right| + \left\|\mathsf{D}\mathcal{A}(\bar p,\bar q)-I_{2n}\right\| \le \frac{1}{2}\,. \tag{2.11}
$$

Let $(\bar p_0,\bar q_0)\in\mathcal{D}^\circ\times I\!R^n$, $t_0\in I\!R$ be such that $(\bar p_0,\bar q_0+t_0\bar p_0)\in A_\gamma$, $\mathrm{dist}(\bar q_0+t_0\bar p_0,\tilde q_\gamma+\partial\mathcal{D})\ge 1/2$, where $2\gamma:=\mathrm{dist}(\bar p_0,\partial\mathcal{D})$. Then the mapping

$$
\begin{pmatrix}\bar p\\ \bar q\end{pmatrix} \quad\mapsto\quad \begin{pmatrix}\bar p_0\\ \bar q_0+t_0\bar p_0\end{pmatrix} + \begin{pmatrix}\bar p\\ \bar q\end{pmatrix} - \mathcal{A}(\bar p,\bar q)
$$

is a contraction of the closed ball

$$
\{(\bar p,\bar q) \;:\; \left|(\bar p,\bar q)-(\bar p_0,\bar q_0+t_0\bar p_0)\right|\le 1/2\}
$$

into itself. The corresponding fixed point $(\bar{p}_0', \bar{q}_0')$ verifies

$$\mathcal{A}(\bar{p}_0', \bar{q}_0') = \begin{pmatrix} \bar{p}_0 \\ \bar{q}_0 + t_0 \bar{p}_0 \end{pmatrix}$$

and therefore

$$\mathcal{A}\big(\Phi^{-t_0}(\bar{p}_0', \bar{q}_0')\big) = \begin{pmatrix} I_n & 0 \\ -t_0 I_n & I_n \end{pmatrix} \begin{pmatrix} \bar{p}_0 \\ \bar{q}_0 + t_0 \bar{p}_0 \end{pmatrix} = \begin{pmatrix} \bar{p}_0 \\ \bar{q}_0 \end{pmatrix}.$$

This proves that $\mathcal{A}$ is onto $\mathcal{D}^\circ \times \mathbb{R}^n$.

To prove that $\mathcal{A}$ is a local diffeomorphism, we start remarking that $\mathcal{A}_t$ is canonical. In fact, it is the composition of the transformations

$$\begin{pmatrix} I_n & 0 \\ -t I_n & I_n \end{pmatrix} \quad \text{and} \quad \Phi^t$$

which are canonical (the latter through a general theorem and the former with a direct computation—see the Appendix I). From the identity (see (4.2) in Appendix I)

$$(\mathsf{D}\mathcal{A}_t) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} (\mathsf{D}\mathcal{A}_t)^T = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

going to the limit as $t \to +\infty$, we obtain

$$(\mathsf{D}\mathcal{A}) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} (\mathsf{D}\mathcal{A})^T = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

which at once gives that $\mathsf{D}\mathcal{A}$ is nonsingular, and will yield also that $\mathcal{A}$ is canonical, as soon as we prove that it is one-to-one.

To show this, we can start by noticing that $\mathcal{A}$ is certainly one-to-one on A_γ for any $\gamma > 0$, because on the *convex* set A_γ the symmetric part of the Jacobian matrix $\mathsf{D}\mathcal{A}$ is positive definite (for the proof of this simple and well-known global injectivity result, see the Appendix II). But for any couple of initial data $x_0 \neq x_1 \in \mathbb{R}^n \times \mathbb{R}^n$, the points $\Phi^t(x_0)$ and $\Phi^t(x_1)$ belong to the set A_γ from a certain time t_1 on, where 2γ is the smaller between the distance of $p_\infty(x_0)$ and $p_\infty(x_1)$ from the boundary of $\mathcal{D}$. Since Φ^t is always one-to-one, we surely have $\Phi^{t_1}(x_0) \neq \Phi^{t_1}(x_1)$, and consequently $\mathcal{A}(\Phi^{t_1}(x_0)) \neq \mathcal{A}(\Phi^{t_1}(x_1))$. The conclusion $\mathcal{A}(x_0) \neq \mathcal{A}(x_1)$ comes from the identity (2.6) rewritten in the form

$$\mathcal{A} = \begin{pmatrix} I_n & 0 \\ t_1 I_n & I_n \end{pmatrix}^{-1} \mathcal{A} \circ \Phi^{t_1}. \tag{2.12}$$

Finally, (2.10) comes from (2.5) by derivation in the time, reminding that $(P, Q) := \mathcal{A} = (a_\infty, p_\infty)$. The Hamiltonian of the transformed system (2.10) is $\mathcal{K}(P, Q) = \frac{1}{2}|P|^2$ up to a trivial additive constant. Actually this constant is $\inf \mathcal{V}$ by means of Corollary 4.6 in [3].

$\diamondsuit$

Remark. *Let $\mathcal{A}_-$ be defined as $\mathcal{A}$ but with the limits as $t \to -\infty$ instead of $t \to +\infty$. Then, by reversing the time, we obtain for $\mathcal{A}_-$ similar results. So the scattering map $\mathcal{A} \circ \mathcal{A}_-^{-1}$ exists and it is a canonical transformation with domain $-\mathcal{D}^\circ \times \mathbb{R}^n$ and range $\mathcal{D}^\circ \times \mathbb{R}^n$.*

The next theorem uses the same arguments as for Theorem 9.3 of [3], so that we omit the proof.

Theorem 2.4 (Persistence) *Suppose that $\mathcal{V}$ verifies Hypotheses 2.1. Let $K \subset \mathbb{R}^n$ be compact. Then there exists an $\varepsilon > 0$ with the following property. If $f: \mathbb{R}^n \to \mathbb{R}$ is a C^{m+1} function with support in K and*

$$\sup |\nabla f| < \varepsilon \,, \tag{2.13}$$

then for the system whose Hamiltonian H is

$$H(p,q) := \frac{1}{2}|p|^2 + \mathcal{V}(q) + f(q) \tag{2.14}$$

all the claims of Proposition 2.2 and Theorem 2.3 hold.

3. Examples

Consider the closed convex cone in $I\!R^3$ defined by

$$\mathcal{D} := \{(x,y,z) \in I\!R^3 \ : \ z \geq \sqrt{x^2+y^2}\}\,. \tag{3.1}$$

The function

$$f(x,y,z) := \frac{z^2 - x^2 - y^2}{2z} \tag{3.2}$$

is continuous and positive on the interior of $\mathcal{D}$, and its level sets are single sheets of two-sheeted hyperboloids of revolution:

$$\frac{z^2 - x^2 - y^2}{2z} = c \qquad \Longleftrightarrow \qquad (z-c)^2 - x^2 - y^2 = c^2, \quad z \neq 0.$$

The function f is closely related to the distance of a point of $\mathcal{D}^\circ$ from the boundary of $\mathcal{D}$:

$$0 < f(x,y,z) \leq z - \sqrt{x^2+y^2} = \sqrt{2}\ \mathrm{dist}\big((x,y,z), \partial\mathcal{D}\big) \leq 2f(x,y,z)\,. \tag{3.3}$$

In particular, f vanishes on the boundary of $\mathcal{D}$. The gradient of f

$$\nabla f(x,y,z) = \Big(-\frac{x}{z}\,,\, -\frac{y}{z}\,,\, \frac{1}{2} + \frac{x^2+y^2}{2z^2}\Big) \tag{3.4}$$

spans a cone $\mathcal{C}$ which happens to coincide with the interior of $\mathcal{D}$. The n-th differential $\mathsf{D}^n f(x,y,z)$ is homogeneous of degree $1-n$ with respect to (x,y,z).

The one-variable function

$$\varphi(r) := \frac{e^{-r}}{r}\,, \qquad r > 0\,, \tag{3.5}$$

is positive, $\lim_{r\to 0+}\varphi(r) = +\infty$, $\lim_{r\to+\infty}\varphi(r) = 0$, and the n-th derivative $\varphi^{(n)}$ can be expressed as

$$\varphi^{(n)}(r) = (-1)^n P_n\Big(\frac{1}{r}\Big) e^{-r}, \tag{3.6}$$

where P_n is a nonzero polinomial with nonnegative coefficients. In particular

$$r \geq 1 \quad \Rightarrow \quad |\varphi^{(n)}(r)| \leq P_n(1) e^{-r}. \tag{3.7}$$

Let us now define on $\mathcal{D}^\circ$ the following potential

$$\mathcal{V}(x,y,z) := \varphi(f(x,y,z))\,. \tag{3.8}$$

This is a C^∞ function, and $-\nabla\mathcal{V}$ spans the cone $\mathcal{C}$. We are going to verify the Hypotheses 2.1 i) to v).

It is obvious that $\mathcal{V} > 0$. Next, for a given $E > 0$, choose $\varepsilon > 0$ such that $0 < r \le \varepsilon \Rightarrow \varphi(r) \ge E$. Then

$$(x,y,z) \in \mathcal{D}^\circ \setminus \big((0,0,\varepsilon) + \mathcal{D}\big) \quad \Rightarrow \quad 0 < f(x,y,z) \le \varepsilon \quad \Rightarrow \quad \mathcal{V}(x,y,z) \ge E\,, \tag{3.9}$$

and ii) is settled. Verifying iii) is more complicated. Because of the symmetry with respect to the z-axis, instead of a generic vector $v \in \bar{\mathcal{C}} \setminus \{0\}$, we can just take $(\alpha, 0, 1)$, with $-1 \le \alpha \le 1$. What we need is a positive lower bound on

$$-\nabla \mathcal{V}(x,y,z) \cdot (\alpha,0,1) = -\varphi'(f(x,y,z)) \nabla f(x,y,z) \cdot (\alpha,0,1) \tag{3.10}$$

when (x,y,z) belongs to the set

$$M := \{(x,y,z) \in I\!\!R^3 \ : \ a + \sqrt{x^2+y^2} \le z \le b - \alpha x\} \tag{3.11}$$

for $0 < a \le b$. The y variable can be dispensed with, first noticing that M is contained in $\{a + |x| \le z \le b - \alpha x\}$. Next we can compute

$$\begin{aligned} \nabla f(x,y,z) \cdot (\alpha,0,1) = \nabla f(x,y,z) \cdot \Big(\frac{1-\alpha}{2}(-1,0,1) + \frac{1+\alpha}{2}(1,0,1)\Big) \ge \\ \ge \frac{1-\alpha}{4}\Big(\frac{x}{z}+1\Big)^2 + \frac{1+\alpha}{4}\Big(\frac{x}{z}-1\Big)^2. \end{aligned} \tag{3.12}$$

We can write

$$(x,y,z) \in M \quad \Rightarrow \quad \begin{cases} \frac{x}{z} + 1 \ge \frac{a}{b} > 0 & \text{if } \alpha \le 0, \\ \frac{x}{z} - 1 \le -\frac{a}{b} < 0 & \text{if } \alpha \ge 0, \end{cases} \tag{3.13}$$

so that

$$(x,y,z) \in M \quad \Rightarrow \quad \nabla f(x,y,z) \cdot (\alpha,0,1) \ge \frac{a^2}{4b^2} > 0\,. \tag{3.14}$$

Finally, noticing that $(x,y,z) \in M \Rightarrow a/2 \le f(x,y,z) \le b$,

$$(x,y,z) \in M \quad \Rightarrow \quad -\nabla \mathcal{V}(x,y,z) \cdot (\alpha,0,1) \ge \frac{a^2}{4b^2} \inf\{-\varphi'(r) \ : \ a/2 \le r \le b\} > 0\,. \tag{3.15}$$

All we are left to do is giving an exponential bound on $\| \mathsf{D}^n \mathcal{V}(x,y,z)\|$ in terms of the distance of (x,y,z) from the boundary of $\mathcal{D}$. But $\mathsf{D}^n \mathcal{V}(x,y,z)$ is a linear combination of n-th order objects extracted from f (which are homogeneous of degree $1-n$ and continuous on the half space $z > 0$, hence bounded on $\mathcal{D} \cap \{z \ge 2\}$), with $\varphi^{(i)}(f(x,y,z))$ as coefficients. We can then estimate, using (3.7),

$$\begin{aligned} &(x,y,z) \in (0,0,2) + \mathcal{D} \ \Rightarrow \ f(x,y,z) \ge 1 \Rightarrow \\ &\qquad \Rightarrow \ \| \mathsf{D}^n \mathcal{V}(x,y,z)\| \le A_n e^{-f(x,y,z)} \le A_n \exp\Big(-\frac{\sqrt{2}}{2} \mathrm{dist}\big((x,y,z), \partial\mathcal{D}\big)\Big). \end{aligned} \tag{3.16}$$

Summing up:

Proposition 3.1 *The conclusions of Proposition 2.2 are true for the Hamiltonian system with potential*

$$\mathcal{V}(x,y,z) := \frac{2z}{z^2-x^2-y^2}\exp\Big(-\frac{z^2-x^2-y^2}{2z}\Big) \tag{3.17}$$

defined on the set $\mathcal{D}^\circ = \{(x,y,z) \in I\!R^3 \;:\; z > \sqrt{x^2+y^2}\}$. *The corresponding asymptotic map* $\mathcal{A}$ *is* C^∞ *and the system is* C^∞*-integrable.*

The following results are adaptations of the ones given in [3], Section 10.

Proposition 3.2 *Let the vectors* $v_1,\dots,v_N \in I\!R^n$ *be such that*

$$v_\alpha \cdot v_\beta \geq 0 \qquad \forall \alpha,\beta\,. \tag{3.18}$$

Let the C^{m+1} *real functions* $f_1,\dots,f_N$ *be defined either on the interval* $]0,+\infty[$ *or on all of* $I\!R$, *where* $\sup f_\alpha = +\infty$, $f_\alpha > 0$, $f'_\alpha < 0$. *Suppose moreover that the* f_α *are integrable on an interval of the form* $[a,+\infty[$, *and that, for* $0 \leq i \leq m+1$ *and* $x \geq a$,

$$f_\alpha^{(i)}(x)\begin{cases} <0 & \text{if } i \text{ is odd,} \\ >0 & \text{if } i \text{ is even.}\end{cases} \tag{3.19}$$

Assume finally that the $|f_\alpha^{(m+1)}|$ *are monotone on* $[a,+\infty[$. *Then the conclusions of Proposition 2.2 hold true for the Hamiltonian system with the potential*

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} f_\alpha(q\cdot v_\alpha)\,, \tag{3.20}$$

defined on either the set $\mathcal{D}^\circ = \{q \in I\!R^n \;:\; q\cdot v_\alpha > 0 \;\forall\alpha\}$ *or on all of* $I\!R^n$.

Corollary 3.3 *Let* $v_1,\dots,v_N \in I\!R^n\backslash\{0\}$ *be such that* $v_\alpha\cdot v_\beta \geq 0$ *for all* α,β. *Let* $r>1$ *and define the potential*

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} \frac{1}{(q\cdot v_\alpha)^r}$$

on the set $\mathcal{D}^\circ = \{q\in I\!R^n : q\cdot v_\alpha > 0\;\forall\alpha\}$. *Then the conclusions of Proposition 2.2 hold for the associated Hamiltonian system, and the asymptotic map* $\mathcal{A}$ *is* C^∞.

Corollary 3.4 *Let* $v_1,\dots,v_N \in I\!R^n\backslash\{0\}$ *be such that* $v_\alpha\cdot v_\beta \geq 0$ *for all* α,β, *and let* $c_\alpha > 0$. *Define the potential*

$$\mathcal{V}(q) := \sum_{\alpha=1}^{N} c_\alpha\, e^{-q\cdot v_\alpha}$$

on $I\!R^n$. *Then the conclusions of Proposition 2.2 hold for the associated Hamiltonian system, and the asymptotic map* $\mathcal{A}$ *is* C^∞.

4. Appendix I

We consider time-independent C^2 Hamiltonian functions defined in an open domain Ω of $\mathbb{R}^n \times \mathbb{R}^n$. Such a (real) function $(p,q) \mapsto H(p,q)$ defines the Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}\,, \qquad \dot{q} = \frac{\partial H}{\partial p}\,. \tag{4.1}$$

In this paper *canonical transformations* are central. We adopt the following definition (remark that in the literature also more general concepts are denoted with the same name, and the following transformations are sometimes called "completely canonical"). The C^2-diffeomorphism $Y : \Omega \to Y(\Omega)$ is said a canonical transformation if, for *any* Hamiltonian H as above, the system (4.1) is transformed into the Hamiltonian system associated to the tranformed Hamiltonian $K = H \circ Y^{-1}$. We see at once that this is equivalent to the validity of the following equality at any point of Ω:

$$(\mathsf{D}Y)\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}(\mathsf{D}Y)^T = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\,, \tag{4.2}$$

where $(\mathsf{D}Y)$ is the Jacobian matrix, 0 is the zero $n \times n$ matrix, I_n is the unit $n \times n$ matrix, and the exponent T means transposition.

The canonical transformations constitute a *group*.

Let us introduce the notation $(P,Q) \in \mathbb{R}^n \times \mathbb{R}^n$ for the transformation Y. An easy theorem says that the preceding condition is equivalent to requiring that the differential form

$$P\,dQ - p\,dq$$

be closed in Ω, or, equivalently, locally exact at every point $(p,q) \in \Omega$:

$$\sum_{i=1}^n P_i \frac{\partial Q_i}{\partial p_j} = \frac{\partial S}{\partial p_j}\,, \qquad \sum_{i=1}^n P_i \frac{\partial Q_i}{\partial q_j} - p_j = \frac{\partial S}{\partial q_j}\,, \qquad \forall j = 1, \ldots, n\,, \tag{4.3}$$

for some function S defined near (p,q).

In the present paper we use the following fact: *a locally C^2 limit of canonical transformations is canonical*, provided it is a diffeomorphism. This follows from (4.2).

Let us conclude with the important theorem stating that, for any fixed value of the time, *the phase flow gives a canonical transformation.*

Let $H \in C^k$, with $k \geq 3$. Then, for any $\lambda \in \mathbb{R}$, the map

$$(\bar{p},\bar{q}) \mapsto \big(P^\lambda(\bar{p},\bar{q}), Q^\lambda(\bar{p},\bar{q})\big) := (p(\lambda,\bar{p},\bar{q}), q(\lambda,\bar{p},\bar{q})) \tag{4.4}$$

is a C^{k-1} canonical transformation.

The proof is just verifying that the following function works for the condition in (4.3):

$$S(\bar{p},\bar{q}) = \int_0^\lambda \left(\sum_i p_i(\xi,\bar{p},\bar{q})\,\dot{q}_i(\xi,\bar{p},\bar{q}) - H\big(p(\xi,\bar{p},\bar{q}), q(\xi,\bar{p},\bar{q})\big) \right) d\xi\,.$$

5. Appendix II

In this paper we use the following known

Proposition. *Let $f\colon \Omega \to I\!\!R^n$ be a $C^1(\Omega)$ map with $\Omega \subseteq I\!\!R^n$ open and convex. If the quadratic form $\xi \mapsto \xi^T \,\mathsf{D}f(x)\, \xi$ (defined by the Jacobian matrix) is positive definite at any $x \in \Omega$, then f is injective.*

Proof. Let $x \neq y$ be two points of Ω, and s the segment which joins them

$$s\colon [0,1] \to \Omega, \quad \theta \mapsto \theta x + (1-\theta) y\,.$$

Moreover, consider the map

$$g\colon [0,1] \to \Omega\,, \quad \theta \mapsto (x-y)\cdot \bigl(f(s(\theta)) - f(y)\bigr)\,.$$

The derivative at any θ is strictly positive:

$$g'(\theta) = (x-y)^T \,\mathsf{D}f(s(\theta))(x-y) > 0\,,$$

by the hypothesis of the Proposition. Therefore the map g is strictly increasing and, in particular, $g(1) \neq g(0) = 0$. Finally

$$g(1) = (x-y)\cdot \bigl(f(x) - f(y)\bigr) \neq 0$$

implies $f(x) \neq f(y)$.

$\diamondsuit$

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