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Unsteady Flow of Non- Newtonian Fluid in Pipe by Spectral Method

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Abstract: In the present investigation, the unsteady flow of upper- convected Maxwell fluid in a horizontal circular pipe is studied by spectral method. The unsteady problem is mathematically reduced to a partial differential equation of second order. By using spectral method the partial differential equation can be reduced to a system of ordinary differential equations for different terms of Chebyshev polynomials approximations. The ordinary differential equations are solved by the method of Laplace transform and the eigenvalue method that led to an analytical form of the solutions.

Key words: spectral method; unsteady flow; Chebyshev polynomial

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In 1930 s non- Newtonian fluid mechanics was developed rapidly as a new branch of fluid mechanics[1, 2]. The non - Newtonian fluid was studied originally because of its application in polymer processing. At present the principle of non - Newtonian fluid mechanics has application nearly in every field of industrial processes. The time dependent flow in a tube has received both theoretical and practical interest as an important type of flows encountered in industrial processes. Although it is usually solved with numerical method, the analytical approach is desirable for this problem. The variation method of Kantorovich[2,3] is one of analytical approaches. In the present paper a time dependent flow in a tube is analytically studied with spectral method for a non- Newtonian fluid, i. e. the upper- convected Maxwell fluid.

1 Governing Equations

The upper- convected Maxwell fluid as a non- Newtonian fluid model is used. The fluid is assumed to be incompressible. The cylindrical coordinate system $(r, \quad, z)$ is used. The velocity field is assumed to be of the following form:

$$V_r = 0, V \quad = 0, V_z = w(r, t). \tag{1}$$

The constitutive equation of the Maxwell fluid has the following form: $S^{ik} + \quad_1 S^{ik} = \quad_0 A^{ik}$, where S^{ik} is the contravariant components of the extra- stress tensor, A^{ik} is the contravariant components of the first order Rivlin - Ericksen tensor, $_1$ is relaxation time. The upper- convected derivative is used because the model with this derivative is in agreement with most of the experiments. This upper- convected derivative is defined by

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$$S^{ik} = \frac{S^{ik}}{t} + V^m \frac{S^{ik}}{X^m} - S^{im} \frac{V^k}{X^m} - S^{mk} \frac{V^i}{X^m}.$$

The equation of motion for the velocity field (1) is reduced as

$$\frac{W}{t} = - \frac{P}{Z} + \frac{1}{r} \frac{}{r}(rS_{rz}). \tag{2}$$

For the velocity field the constitutive equations are reduced as $S_r = S = S_{zz} = S_z = S_r = 0$, and

$$S_{rz} + {}_1 \frac{S_{rz}}{t} = {}_0 A^{rz}. \tag{3}$$

Using the constitutive eq. (3) and the eq. (2), one obtains a partial differential equation of second order for the velocity W which is given as

$$\frac{W}{t} + {}_1 \frac{^2 W}{t^2} - {}_0 \frac{^2 W}{r^2} - \frac{_0}{r} \frac{W}{r} + \frac{P}{Z} + {}_1 \frac{}{t}\left(\frac{P}{t}\right) = 0. \tag{4}$$

Introducing the following dimensionless variables, $= \frac{_0 W}{(P L) R^2}$, $= \frac{_0 t}{R^2}$, $= r R$. Eq. (4) is then reduced to the dimensionless form:

$$L(\ ) = \frac{}{} + H_a \frac{^2}{^2} - \frac{^2}{^2} - \frac{1}{} \frac{}{} - (\ ) = 0, \tag{5}$$

where $H_a = We\ Re$, $Re = R_1 V_0\ _0$, $We = {}_1 V_0\ R$, and

$$\bar{P} = P\ P, \bar{Z} = Z\ L, \quad (\ ) = - \frac{\bar{P}}{\bar{Z}} - H_a \frac{}{}\left(\frac{\bar{P}}{\bar{Z}}\right).$$

The boundary conditions are given as

$$\begin{cases} = 1, & (\ ,\ ) = (1,\ ) = 0, \\ = 0, & = 0. \end{cases} \tag{6}$$

The initial condition is given as

$$= 0, \quad (\ ,\ ) = (\ , 0) = 0. \tag{7}$$

2 Spectral Method

The Chebyshev polynomials are chosen to act as the radical of spectral method. The general solution of the time dependent eq. (9) is assumed to be of the form: $(\ ,\ ) = \sum_{i=0}^{n} f_i(\ ) T_i^*(\ )$. Consider that $(\ ,\ )$ can be expanded in a series of Chebyshev polynomials whose properties are outlined in reference[2, 5],

$$(\ ,\ ) = f_0(\ ) T_0^*(\ ) + f_1(\ ) T_1^*(\ ) + f_2(\ ) T_2^*(\ ) + \ ,$$

where $f_i(\ )$ are the unknown coefficients, $T_i^*(\ )$ are the Chebyshev polynomials that applies in the region 0 1 according to references[2 ~ 5],

$$T_0^*(\ ) = 1, T_1^*(\ ) = 2\ - 1, T_2^*(\ ) = 8\ ^2 - 8\ + 1,$$

$$T_3^*(\ ) = 32\ ^3 - 48\ ^2 + 18\ - 1, T_4^*(\ ) = 128\ ^4 - 256\ ^3 + 160\ ^2 - 32\ + 1. \tag{8}$$

2.1 The First Three Terms Approximation

For the first three terms,

$$(\ ,\ ) = f_0 + (2\ - 1) f_1 + (8\ ^2 - 8\ + 1) f_2. \tag{9}$$

According to the initial condition eq. (7), $f_0(0) = f_1(0) = f_2(0) = 0$. According to the boundary conditions eq. (6), $f_0 + f_1 + f_2 = 0, 2f_1 - 8f_2 = 0$. Thus

$$f_1 = 4f_2, f_0 = -5f_2. \tag{10}$$

Substituting eq. (10) into eq. (9) yields

$$(\ ,\ ) = -8f_2(1 - \ ^2). \tag{11}$$

The integral form is used to seek a solution to eq. (5), where 0 1. According to the orthogonality relationship,

$$\int_0^1 W_j L(\ )\mathrm{d}\ = \int_0^1 W_j[H_a \frac{^2}{^2} + \text{—} - \frac{^2}{^2} - \frac{1}{}\text{—} - (\ )]\mathrm{d}\ = 0, \qquad (12)$$

where W_j is a weighting function. Examining the classical theory of expanding a function in terms of Chebyshev polynomials, one finds that the appropriate form of W_j is

$$W_j = \frac{T_j^*}{\sqrt{\ (1-\ )}} \qquad j = 0, 1, 2 \ . \qquad (13)$$

Note the property

$$\int_0^1 \frac{T_i^* T_j^*}{\sqrt{\ (1-\ )}}\mathrm{d}\ = \begin{cases} & i = j = 0, \\ \overline{2} & i = j \quad 0, \\ 0 & i \quad j. \end{cases}$$

Substituting eq. (5) and (13) into eq. (12) yields

$$H_a \frac{\mathrm{d}^2 f_2}{\mathrm{d}\ ^2} + \frac{\mathrm{d}f_2}{\mathrm{d}\ } + \frac{32}{5} f_2 + \frac{1}{5}\ (\ ) = 0 \qquad j = 0, \qquad (14)$$

$$H_a \frac{\mathrm{d}^2 f_2}{\mathrm{d}\ ^2} = 0 \qquad j = 1, \qquad (15)$$

$$\frac{1}{2} H_a \frac{\mathrm{d}^2 f_2}{\mathrm{d}\ ^2} + \frac{1}{2}\frac{\mathrm{d}f_2}{\mathrm{d}\ } = 0 \qquad j = 2. \qquad (16)$$

Clearly, eq. (15) and (16) are not valid. This is due to the error created by truncating the terms. The ordinary differential eq. (14) is solved by Laplace transform,

$$f_2(\ ) = -\frac{1}{5H(r_1 - r_2)}\int_0 (\ - u)(\mathrm{e}^{r_1 u} - \mathrm{e}^{r_2 u})\mathrm{d}u + \frac{A}{r_1 - r_2}(\mathrm{e}^{r_1} - \mathrm{e}^{r_2}), \qquad (17)$$

where $f_2(0) = 0, f_2(0) = A$, r_1 and r_2 are tow roots of the algebraic equation, $r^2 + \frac{1}{H_a} r + \frac{32}{5H_a} = 0$, and

$$r_{1,2} = \frac{1}{2H_a} \quad \frac{1}{2H_a}\sqrt{1 - \frac{128}{5}H_a}. \qquad (18)$$

It is assumed that $f_2(\ )$ can be expanded into Taylor series for $= 0$. The small higher order variables $o(\ ^2)$ are omitted. When is very small, $f_2(\ )$ can be considered to be a linear function for . From eq. (14),

$$f_2(0) = A = -\frac{1}{5}\ (0) = -\frac{1}{5}[-\frac{P}{Z} - H_a \text{—}(-\frac{P}{Z})]\ |_{\ =0}.$$

In this paper, only the flow under the condition of constant pressure gradient is presented, that is, $P \quad Z$ is a constant. Because the coefficient of the expression of velocity in steady axial flow is 1 4, we choose $P \quad Z = -4$,

$$f_2(0) = -\frac{1}{5}\ (0) = -\frac{4}{5}. \qquad (19)$$

Substituting eq. (19) into eq. (11) yields

$$(\ ) = 4. \qquad (20)$$

Substituting eq. (20) and $A = -4 \ 5$ into eq. (17) yields

$$f_2(\ ) = -\frac{1}{8} - \frac{4}{5(r_1 - r_2)}(\mathrm{e}^{r_1} - \mathrm{e}^{r_2}) - \frac{1}{8(r_1 - r_2)}(r_2 \mathrm{e}^{r_1} - r_1 \mathrm{e}^{r_2}). \qquad (21)$$

Substituting eq. (21) into eq. (11) yields

$$(\ ,\ ) = [1 + \frac{32}{5(r_1 - r_2)}(\mathrm{e}^{r_1} - \mathrm{e}^{r_2}) + \frac{1}{r_1 - r_2}(r\mathrm{e}^{r_1} - \mathrm{e}^{r_2})](1 - \ ^2), \qquad (22)$$

and

$$_m = \quad (0, \quad) = [1 + \frac{32}{5(r_1 - r_2)}(e^{r_1} - e^{r_2}) + \frac{1}{r_1 - r_2}(re^{r_1} - e^{r_2})], \tag{23}$$

where (0,) is the velocity of the center of the tube.

2.2 The First Four Terms Approximation

For the first four terms,

$$(\ ,\ ) = f_0 T_0^* + f_1 T_1^* + f_2 T_2^* + f_3 T_3^* = f_0 + (2 \ - 1) f + (8 \ ^2 - 8 \ + 1) f_2 + (32 \ ^2 - 48 \ ^2 + 18 \ - 1) f_3. \tag{24}$$

According to the initial condition (14) and boundary conditions (12) and (13),

$$f_0(0) = f_1(0) = f_2(0) = f_3(0),$$

$$f_0 + f_1 + f_2 + f_3 = 0,$$

and

$$2f_1 - 8f_2 + 18f_3 = 0. \tag{25}$$

Thus $f_0 = -5f_2 + 8f_3$, and

$$f_1 = 4f_2 - 9f_3. \tag{26}$$

Substituting eq. (25) and (26) into eq. (24) yields

$$(\ ,\ ) = -8f_2(1 - \ )^2 + 16f_3(2 \ ^3 - 3 \ ^2 + 1). \tag{27}$$

Substituting eq. (27) into eq. (12) yields

$$\int_0^1 W_j [H_a \frac{^2}{^2} + \frac{}{} - \frac{^2}{^2} - \frac{1}{}\frac{}{} - \ (\ )] d \ = 0.$$

Accomplishing this definite integral yields

$$-5H_a f_2 + 8H_a f_3 - 5f_2 + 8f_3 - 32f_2 + 48f_3 - \ (\ ) = 0 \qquad j = 0, \tag{28}$$

$$2H_a f_2 - \frac{9}{2} H_a f_3 + 2f_2 - \frac{9}{2} f_3 - 72f_3 = 0 \qquad j = 1. \tag{29}$$

According to eq. (28) and (29),

$$H_a f_2 + f_2 = -\frac{288}{13} f_2 - \frac{720}{13} f_3 - \frac{9}{13} \quad , H_a f_3 + f_3 = -\frac{128}{13} f_2 - \frac{528}{13} f_2 - \frac{4}{13} \quad . \tag{30}$$

The system of ordinary differential equations can be reduced to an ordinary differential equation of fourth order,

$$H_a^2 f_2^{(4)} + 2H_a f_2^{(3)} + (\frac{816}{13} H_a + 1) f_2 + \frac{816}{13} f_2 + \frac{4\,608}{13} f_2 + \ (\ ) = 0, \tag{31}$$

where $(\ ) = \frac{9}{13}(H_a \quad + \quad) + \frac{144}{13}$. The eigenequation of eq. (31) is given by

$$H_a \ ^4 + 2H_a \ ^3 + (\frac{816}{13} H_a + 1) \ ^2 + \frac{816}{13} \ + \frac{4\,608}{13} = 0. \tag{32}$$

The roots of eq. (32) are given as follows:

$$_{1,2} = \frac{-1 \quad \sqrt{1 - \frac{96}{13} H_a (17 + \sqrt{185})}}{2H_a}, \tag{33}$$

$$_{3,4} = \frac{-1 \quad \sqrt{1 - \frac{96}{13} H_a (17 - \sqrt{185})}}{2H_a}. \tag{34}$$

For the first four terms approximation the general solution of eq. (31) is given by

$$f_2 = D_1 e^{\ 1} + D_2 e^{\ 2} + D_3 e^{\ 3} + D_4 e^{\ 4} + f_2^*, \tag{35}$$

where f_2^* is one of the special solutions of eq. (31). Constant D_1, D_2, D_3 and D_4 are determined by the initial condition. The constant pressure gradient − P Z are considered. Pressure gradient is considered as constant gradient, it can be assumed to be of the form:

$$- \frac{\partial P}{\partial Z} = \frac{L}{} \frac{P}{Z} = 4, \tag{36}$$

and $(\ ) = - \frac{\partial P}{\partial Z} - H_a \frac{}{}(\frac{\partial P}{\partial Z}) = 4.$

In this case, the special solution $f_2^*(\ )$ of eq. (31) is assumed to as

$$f_2^*(\ ) = K, \tag{37}$$

where K is a constant which is not determined.

Substituting eq. (37) into eq. (31) yields

$$K = - \frac{13}{4\,608}(\ ) = - \frac{1}{8}. \tag{38}$$

It is assumed that $f_2(\ )$ and $f_3(\ )$ can be expanded in Taylor series at $= 0$. When is very small, we consider that $f_2(\ )$ and $f_3(\ )$ are linear functions for . Substituting this assumption into eq. (30) yields

$$f_2(0) = - 36\ 13, f_3(0) = - 16\ 13. \tag{39}$$

Substituting eq. (37) and (38) into eq. (35) yields

$$f_2(\ ) = \sum_{i=1}^{4} D_i e^{\ i} - \frac{1}{8}. \tag{40}$$

Substituting eq. (40) into eq. (30) yields

$$f_3(\ ) = - \frac{13}{720}[H_a f_2 + f_2 + \frac{288}{13} f_2 + \frac{36}{13}] = \sum_{i=1}^{4} [- \frac{13}{720}(H_a \ _i^2 + \ _i) - \frac{2}{5}] D_i e^{\ i}. \tag{41}$$

According to initial conditions $f_2(0) = f_3(0) = 0$ and eq. (39),

$$\begin{cases} \sum_{i=1}^{4} D_i - \frac{1}{8} = 0, \\ \sum_{i=1}^{4} [- \frac{13}{720}(H_a \ _i^2 + \ _i) - \frac{2}{5}] D_i = 0, \\ \sum_{i=1}^{4} D_i \ _i = - \frac{36}{13}, \\ \sum_{i=1}^{4} [- \frac{13}{720}(H_a \ _i^3 + \ _i^2) - \frac{2}{5} \ _i] D_i = \frac{16}{13}. \end{cases} \tag{42}$$

A system of algebraic equations for D_1, D_2, D_3 and D_4 are composed of eq. (42). D_1, D_2, D_3 and D_4 are determined by this system of equations. Substituting eq. (40) and (41) into eq. (24) yields

$$(\ ,\ ) = [1 + \sum_{i=1}^{4}(- 8 D_i e^{\ i})](1 - \ ^2) + \left\{ \sum_{i=1}^{4} 16[- \frac{13}{720} \frac{2}{5}](H_a \ _i^2 + \ _i) - D_i e^{\ i} \right\}(2\ ^3 - 3\ ^2 + 1). \tag{43}$$

The dimensionless velocity of the center of the tube is given by

$$(0,\ ) = \ _m = 1 + \sum_{i=1}^{4}(- 8 D_i e^{\ i}) + \sum_{i=1}^{4} 16[- \frac{13}{720}(H_a \ _i^2 + \ _i) - \frac{2}{5}] D_i e^{\ i}. \tag{44}$$

The first five terms and more terms approximations can be deduced in the same way.

3 Results and Discussion

In the present paper the time dependent flow in a pipe is investigated for the upper - convected Maxwell fluid. This problem is reduced mathematically to a partial differential equation of second order for the dimensionless velocity. The spectral method is used to solve the problem. The partial differential equation is reduced to a system of ordinary differential equations with the proper terms for a desired approximation. For each approximation the analytical solution, i. e. the eigensolution is found. The numerical results of the problem and the analytical solution of the first three and four terms approximations are given in fig. 1 to fig. 5 for comparison.

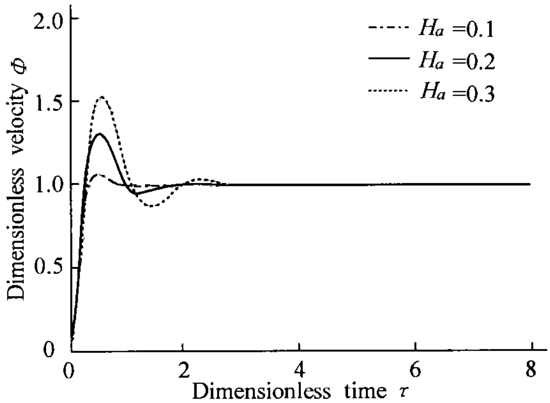


Fig. 1 $H_a = 0.1 \sim 0.3$, **The Change of the Velocity With Time**

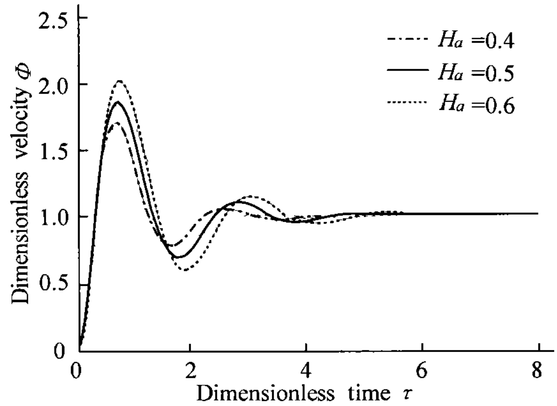


Fig. 2 $H_a = 0.4 \sim 0.6$, **The Change of the Velocity With Time**

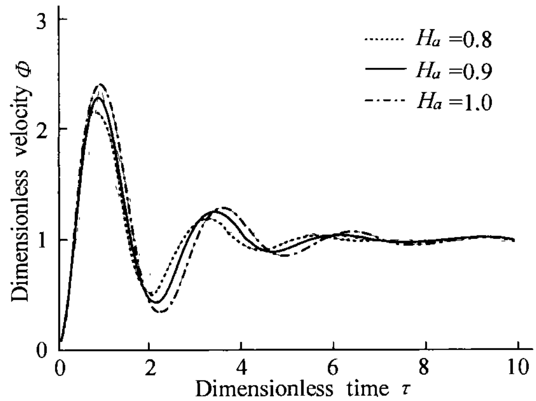


Fig. 3 $H_a = 0.8 \sim 1.0$, **The Change of the Velocity With Time**

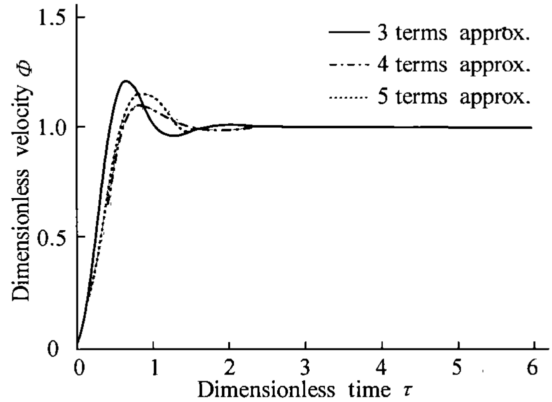
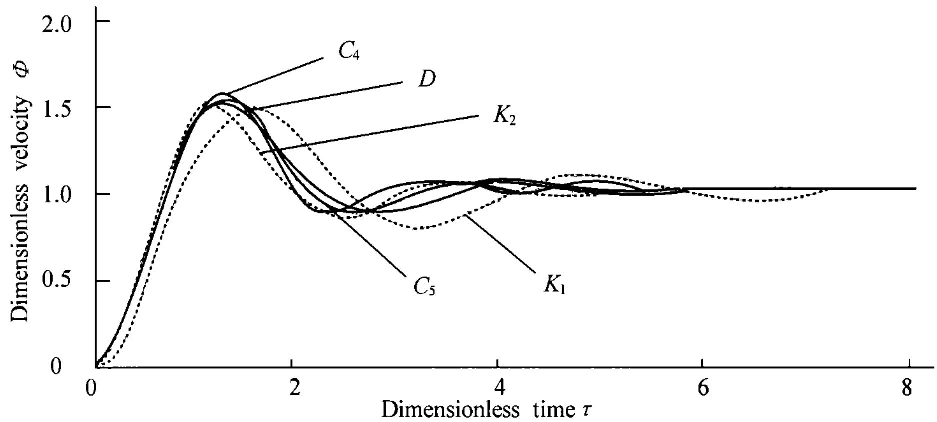


Fig. 4 $H_a = 0.2$, **Comparison of Results**



K_1) the first approx. of Kantorovich; K_2) the second approx. of Kantorovich; D) the definite difference; C_4) the first four terms approx. of spectral method; C_5) the first five terms approx. of spectral method

Fig. 5 $H_a = 0.8$, **in Comparison With Results**

In fig. 1 to fig. 3 the change of the dimensionless velocity at the center of the tube with time is shown for the parameter values $H_a = 0.2 \sim 1.0$ are chosen. In fig. 4 for $H_a = 0.2$ the results of the first three, four and five terms approximations are compared with that of the difference. In fig. 5 for $H_a = 0.8$ the results of the first four terms and five terms approximations compare with that of definite difference and variation approach of Kantorovich.The change of the velocity with time has some special character because it is caused by both unsteady behavior and the viscoelasticity of the fluid.

A new analytical method for solving the time dependent flow of non－ Newtonian fluids is put forward. In comparison with results of the definite difference and variation approach of Kantorovich, the results of spectral method are content. The results show that the spectral method is suitable for the study of time dependent flow of non－ Newtonian fluids.

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