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# Optimal Sliding Mode Controller for a Class of Cascade Uncertain Nonlinear System

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**Abstract**: It is devoted to designing a kind of optimal sliding mode control method based on State Dependent Riccati Equation (SDRE) control for a class of cascade uncertain nonlinear system in this paper. This designed control method has two- loop control structure. The outer loop controller is designed by using a SDRE optimal control to generate an optimal sliding mode surface. The inner loop controller decreases sensitivity to parameter change by using sliding model control. Synchronously it can minish the influence coursed by model error and external disturbance of control system. Two methods are given to solve the state dependent riccati equation. The designed control method can make the system stable and robust. Finally, an example is given to demonstrate the availability of the proposed control method.

**Key words**: cascade system; nonlinear system; optimal control; sliding mode control
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## 1 Introduction

There are many control problems about cascade uncertain nonlinear system, so control of them is an active subject in the modern control area. The main control method is backstepping design method which first designs virtual control law then obtains the ultimate control law by successively deduced[1~3]. The controller designed by backstepping design method can make the nonlinear system have global robust performance. But it is only valid to a class of especial cascade uncertain nonlinear system and must choose an appropriate lyapunov function in each step for analyzing the stability of the system. So the application field of backstepping design method is small. The control method based on SDRE is one of the recently proposed nonlinear control methods. It is wide used in nonlinear system control problem due to its good real time performance and the flexible design[4~6]. The sliding mode control is an ordinary control method to control nonlinear system[7~9]. Its response velocity is quick and it is insensitive to parameter change. Moreover the sliding model control has good adaptability to unmodeled dynamic and external disturbance. So the sliding model control is widely used to control uncertain nonlinear system. But it is difficult to select the sliding mode surface in design sliding mode controller.

Ref. [4] has studied the design of filter for a rapid thermal processing system based on SDRE. Ref. [6] has studied nonlinear regulation design and nonlinear H Control design with SDRE. But the studied systems are not cascade nonlinear system with uncertainty. Ref. [9] has studied optimal sliding mode flight control based on SDRE controller and sliding mode controller. But it does not consider uncertain external disturbance.

A kind of optimal sliding mode control method based on SDRE controller is proposed for a class of cascade uncer-

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tain nonlinear system in this paper. This designed control method has two- loop control structure. The outer loop controller is designed using a SDRE optimal control to generate an optimal sliding mode surface. The inner loop controller can minish sensitivity to parameter change. Simultaneously it can minish model error and external disturbance of control system by using sliding mode control. Two new methods of solving state dependent riccati equation are proposed. For improving the control precision, the neural network is introduced to approximate the solution of state dependent riccati equation. The designed control method can make the system stable and robust. Finally, an example is given to demonstrate the availability of the proposed control method.

## 2 Problem Formulation

Considering the uncertain nonlinear cascade system

$$\begin{cases} \boldsymbol{x} = \boldsymbol{f}_1(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})z, \\ z = \boldsymbol{f}_2(z) + \quad \boldsymbol{f}(z,p,t) + [\boldsymbol{g}_2(z) + \quad \boldsymbol{g}_2(z,p,t)]x + \boldsymbol{d}_1(z)\boldsymbol{u} + \boldsymbol{d}(\boldsymbol{x},t), \end{cases} \tag{1}$$

where $\boldsymbol{x} \quad R^n, \boldsymbol{f}_1(\boldsymbol{x}) \quad R^n, \boldsymbol{g}_1(\boldsymbol{x}) \quad R^{n \quad m}, z \quad R^m, \boldsymbol{f}_2(z) \quad R^m, \quad \boldsymbol{f}(z,p,t) \quad R^m, \boldsymbol{g}_2(z) \quad R^{m \quad n}, \quad \boldsymbol{g}_2(z,p,t) \quad R^{m \quad n}, \boldsymbol{u} \quad R^m, \boldsymbol{d}_1(z) \quad R^{m \quad m}, \boldsymbol{d}(z) \quad R^m$. $\boldsymbol{x}, z$ are state vectors, $\boldsymbol{f}_1, \boldsymbol{g}_1, \boldsymbol{f}_2, \boldsymbol{g}_2, \boldsymbol{d}_1$ are known smooth functions, $\boldsymbol{f}(\boldsymbol{x},p,t), \quad \boldsymbol{g}_2(\boldsymbol{x},p,t)$ are bounded uncertain terms, namely system certainties, $p$ ($p \quad P$, $P$ is a compact set) is unknown parameter, $\boldsymbol{d}(\boldsymbol{x},t)$ is external disturbance of the system, $\boldsymbol{u}$ is control input of the system.

The system (1) consists of two subsystems, and the states of them are connected by each other. The control goal is to design controller which can make the system (1) has robust stability for the external disturbance and system uncertainty. Supposed that (1) satisfies the following assumptions:

(1) To $\quad \boldsymbol{x}$, state $\boldsymbol{x}$ is controllable and the system is stable when $\boldsymbol{x} = 0$;

(2) To $\quad z \quad R^m, p \quad P$, there are $\quad \boldsymbol{f}(z,p,t) \quad \bar{f}(z,t), \quad \boldsymbol{g}_2(z,p,t)\boldsymbol{x} \quad \bar{g}_2(\boldsymbol{x},z,t)$;

(3) To $\quad z \quad R^m$, the inversion of $\boldsymbol{d}_1(z)$ exist;

(4) To $\quad \boldsymbol{x} \quad R^n$, there is $\quad \boldsymbol{d}(\boldsymbol{x},t) \quad D(\boldsymbol{x})$.

## 3 Design of Optimal Sliding Mode Controller

The two- loop controller is designed to the cascade uncertain nonlinear system (1), the structure block diagram of the control system is shown in fig. 1.

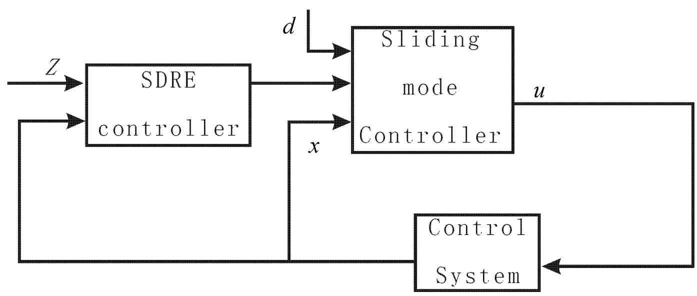


**Fig. 1 Block Diagram of Control System**

The outer loop controller is SDRE optimal control which is used to generate an optimal sliding surface for the following sliding mode controller. The inner controller is sliding mode controller which is used to minish sensitivity to parameter change and model error and external disturb of control system. Optimal control based on SDRE is a new control method for the nonlinear system which can treat as the linear LQR of the nonlinear system control. The SDRE controller can choose right $\boldsymbol{Q}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$ in order to obtain good control effect, where $\boldsymbol{Q}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$ are certain functions of states. Furthermore, the SDRE control method can easily transform the nonlinear system to the linear system, then design the optimal controller with linear LQR theory for the nonlinear system.

Considering the following nonlinear system

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \tag{2}$$

and determining the under performance index

$$J = \frac{1}{2} \,_{0} (\boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{x}) \boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}} \boldsymbol{R}(\boldsymbol{x}) \boldsymbol{u}) \mathrm{d}t, \tag{3}$$

where $\boldsymbol{x} \quad R^n$, $\boldsymbol{u} \quad R^m$, $\boldsymbol{f}(\boldsymbol{x}) \quad R^n$, $\boldsymbol{g}(\boldsymbol{x}) \quad R^{n \quad m}$, $\boldsymbol{Q}(\boldsymbol{x}) \quad 0 \quad R^{n \quad n}$, $\boldsymbol{R}(x) > 0 \quad R^{m \quad m}$.

Imitating the formulation of LQR optimal control of linear system, the equation (2) can be written as

$$\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{u}, \tag{4}$$

where $\boldsymbol{A}(\boldsymbol{x})$ satisfies $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{x}$. When the $\boldsymbol{f}(\boldsymbol{x})$ can not contain the independent state variable, we can divide the corresponding state variable in order to obtain $\boldsymbol{A}(\boldsymbol{x})$. For example, when $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1^2 + x_2^2 \\ 1 + x_2^2 \end{bmatrix}$, we can choose $\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} \frac{x_1^2 + x_2^2}{x_1} & 0 \\ \frac{1}{x_1} & x_2 \end{bmatrix}$. It is easily known that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{x}$. Apparently, the choice of $\boldsymbol{A}(\boldsymbol{x})$ is not single which can choose different expression according to different requirement. There is $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$. According to the LQR optimal control of linear system, we can construct the following state feedback control law

$$\boldsymbol{u}(\boldsymbol{x}) = -\boldsymbol{K}(\boldsymbol{x}) \boldsymbol{x} = -\boldsymbol{R}^{-1}(\boldsymbol{x}) \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{P}(\boldsymbol{x}) \boldsymbol{x}, \tag{5}$$

where $\boldsymbol{P}(\boldsymbol{x}) > 0$, which satisfies the following state dependent Riccati equation

$$\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{P}(\boldsymbol{x}) \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{Q}(\boldsymbol{x}) - \boldsymbol{P}(\boldsymbol{x}) \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{R}^{-1}(\boldsymbol{x}) \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{P}(\boldsymbol{x}) = 0. \tag{6}$$

So the SDRE controller can make the close- loop system matrix of (4) has expression

$$\boldsymbol{A}_c = \boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{K}(\boldsymbol{x}). \tag{7}$$

From the foregoing description, we can see that $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{B}(\boldsymbol{x})$ change along with the change of state vector. So the system (4) can approximatively denote the system (2) and can choose different $\boldsymbol{Q}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$ according to the different requirement.

According to the foregoing method, the system (1) can be written as

$$\begin{cases} \boldsymbol{x} = \boldsymbol{A}_x \boldsymbol{x} + \boldsymbol{B}_x \boldsymbol{z}, \\ \boldsymbol{z} = \boldsymbol{f}_2(\boldsymbol{z}) + \quad \boldsymbol{f}(\boldsymbol{z}, p, t) + [\boldsymbol{g}_2(z) + \quad \boldsymbol{g}_2(z, p, t)] \boldsymbol{x} + \\ \quad [\boldsymbol{d}_1(\boldsymbol{z}) + \quad \boldsymbol{g}(\boldsymbol{z}, p, t)] \boldsymbol{u} + \boldsymbol{d}(\boldsymbol{x}, t), \end{cases} \tag{8}$$

where $\boldsymbol{A}_x \boldsymbol{x} = \boldsymbol{f}_1(\boldsymbol{x})$, $\boldsymbol{B}_x = \boldsymbol{g}_1(\boldsymbol{x})$.

By the LQR theory based on SDRE, the optimal control law of the first subsystem in the system (8) is

$$\boldsymbol{z} = -\boldsymbol{R}^{-1} \boldsymbol{B}_x^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x}. \tag{9}$$

When the system is low dimension system, we can solve eq. (6) using analytical method. But it is too difficult to solve the eq. (6) when the system is a high dimension system. We can use an ordinary computational method in engineering. First suppose an initialization state $\boldsymbol{x}(0)$ and solve the $\boldsymbol{P}(\boldsymbol{x}_0)$. We can obtain the control law $\boldsymbol{z}(0)$ according to (9), then the control law keeps invariable till the next computational time (the time interval is $T$). Before the next computation, we compute the state $\boldsymbol{x}_{0+\ T}$ and make it as initialization state at the time. Under the $\boldsymbol{x}_{0+\ T}$, according to (6) compute $\boldsymbol{P}(\boldsymbol{x}_{0+\ T})$. By eq. (9), we can obtain control law $\boldsymbol{z}_{0+\ T}$ and keep invariable in $T$. Making circular computation, we can obtain the corresponding control law $\boldsymbol{z}$ at different time. For keeping the real time ability and improving the control precision we can choose a small $T$.

In the modern control area the neural network is widely used due to its self- study ability, so it can be used to approximate the solution $\boldsymbol{P}(x)$ of the equation eq. (6). If the system has required high control precision, the neural network can be introduced to approximate $\boldsymbol{P}(\boldsymbol{x})$. Choose the different state vector $\boldsymbol{x}$ as the input of neural networks, the corresponding $\boldsymbol{P}(\boldsymbol{x})$ which obtains from the eq. (6) is output of neural networks. By training, the output of neural network can approximate $\boldsymbol{P}(\boldsymbol{x})$ with a small error.

Under the foregoing optimal control law design the sliding mode controller. At a time, $\boldsymbol{R}(\boldsymbol{x})$, $\boldsymbol{B}(\boldsymbol{x})$, $\boldsymbol{P}(\boldsymbol{x})$ are constant due to the real time ability of SDRE. Choose the follow sliding surface

$$ = z + \boldsymbol{R}^{-1}\boldsymbol{B}_x^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}. \quad (10)$$

From the above equation yields

$$ = z + \boldsymbol{R}^{-1}\boldsymbol{B}_x^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}. \quad (11)$$

Substituting the eq. (8) into eq. (11), there yields

$$ = \boldsymbol{f}_2(z) + \ \boldsymbol{f}(z, p, t) + [\boldsymbol{g}_2(z) + \ \boldsymbol{g}_2(z, p, t)]\boldsymbol{x} + \boldsymbol{d}_1(z)\boldsymbol{u} + \boldsymbol{d}(\boldsymbol{x}, t) + \boldsymbol{R}^{-1}\boldsymbol{B}_x^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}_x\boldsymbol{x} + \boldsymbol{B}_x z). \quad (12)$$

The sliding mode controller goal is to achieve $= 0$, under no uncertainty and external disturbance, there can obtain the equivalent control law

$$\boldsymbol{u}_{eq} = -\boldsymbol{d}_1^{-1}(z)[\boldsymbol{f}_2(z) + \boldsymbol{g}_2(z)\boldsymbol{x} + \boldsymbol{R}^{-1}\boldsymbol{B}_x^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_x\boldsymbol{x} + \boldsymbol{R}^{-1}\boldsymbol{B}_x^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_x z)]. \quad (13)$$

Design the control law $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{u}_{eq} + \boldsymbol{u}_c, \quad (14)$$

$$\boldsymbol{u}_c = - \ \mathrm{sgn}(\ ), \quad (15)$$

where $\mathrm{sgn}(\ ) = [\mathrm{sgn}(\ _1), \ , \mathrm{sgn}(\ _m)]^{\mathrm{T}}$, $\boldsymbol{u}_c$ is sliding mode controller. Substituting eq. (14) and eq. (15) into eq. (12), there is

$$^{\mathrm{T}} \ = \ ^{\mathrm{T}}[\ \boldsymbol{f}(z) + \ \boldsymbol{g}_2(z)\boldsymbol{x} + \boldsymbol{d}(\boldsymbol{x})] - \ ^{\mathrm{T}}\boldsymbol{d}_1(z) \ \mathrm{sgn}(\ ). \quad (16)$$

If there is

$$ = \boldsymbol{d}_1^{-1}(z)[\bar{f}(z, t) + \bar{g}_2(\boldsymbol{x}, z, t) + D(\boldsymbol{x}, t) + k], \quad (17)$$

where $k$ is an arbitrary positive parameter. Substituting eq. (17) into eq. (16) and considering the assumption (2), (3), (4) can yield

$$^{\mathrm{T}} \ - k \ < 0. \quad (18)$$

So the system satisfies the condition of existing sliding mode.

So the control law of the system can be chosen as

$$\boldsymbol{u} = \boldsymbol{u}_{eq} - \ \mathrm{sgn}(\ ), \quad (19)$$

where $\boldsymbol{u}_{eq}$ is defined in eq. (13), is defined in eq. (17), is defined in eq. (10).

**Theorem 1** For the cascade uncertain nonlinear system (1), under the assumption (1), (2), (3), (4), the control law (19) can make the system (1) globally stable.

**Proof** Choosing Lyapunov function for the system (1)

$$V = m + \frac{1}{2} \ ^{\mathrm{T}} \ , \quad (20)$$

where $m > 0$, is defined in eq. (10).

Calculating the time derivative of $V$ can yield

$$V = \ ^{\mathrm{T}} \ . \quad (21)$$

From the eq. (18) we can know $V < 0$, so under the assumption (1), (2), (3), (4), the control law (19) can make the system (1) globally stable.

Note that the sliding mode control term can introduce a high frequency signal to the system which may excite unmodeled dynamics causing unforeseen instability. To avoid this, we use a smoothed sliding mode control term, so the control law can written as

$$\boldsymbol{u} = \boldsymbol{u}_{eq} - \ \mathrm{sat}(\ / \ ), \quad (22)$$

where $> 0$ and

$$\mathrm{sat}(u_i) = \begin{cases} 1 & u_i \quad 1, \\ u_i & |u_i| < 1, \\ -1 & u_i \quad -1. \end{cases} \quad (23)$$

# 4　Simulation Example

The system (1) is widely used in the field of aeronautics and astronautics. Specially, in fighter control system and missile control system[9], the states of fighter or missile are divided into quick change loop and slow change loop according to the time scale separation principle. The output of slow loop is the input of quick loop, and that the quick change state variable control the slow change loop. But the simulation about them is very complicated, so we use a simple example to demonstrate the availability of the proposed control method.

Considering a simple uncertain cascade nonlinear system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_1 x_2 - x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} z,$$

$$z = z^2 + \sin zt + \begin{bmatrix} z & 1+z \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + zu + \sin x_1(t).$$

According to the foregoing SDRE control theory, we can choose $\boldsymbol{A}_x = \begin{bmatrix} -1 & x_1 \\ x_2 & -1 \end{bmatrix}$, $\boldsymbol{B}_x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{Q}(\boldsymbol{x}) = \begin{bmatrix} x_1^2 & 1 \\ 0 & 1+x_2^2 \end{bmatrix}$, $\boldsymbol{R}(\boldsymbol{x}) = 1 + x_1^2 + x_2^2$. Apparently when $\boldsymbol{x} = 0$, $\boldsymbol{A}_x$ is a stable system matrix. Because the system is a low dimension system, we can solve the equation (6) by using analytical method to obtain $\boldsymbol{P}(\boldsymbol{x})$. By eq. (10) and eq. (14), the control law can obtain. The simulation results are shown in fig. 2 to fig. 5.

The simulation results demonstrate the availability of the proposed control method.

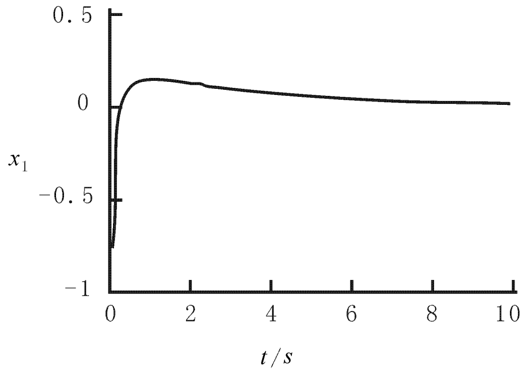


**Fig. 2　Response Plot of State $x_1$**

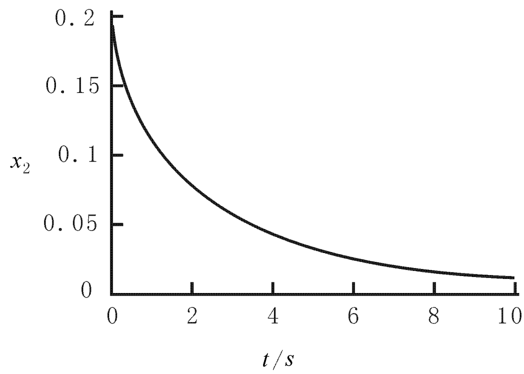


**Fig. 3　Response Plot of State $x_2$**

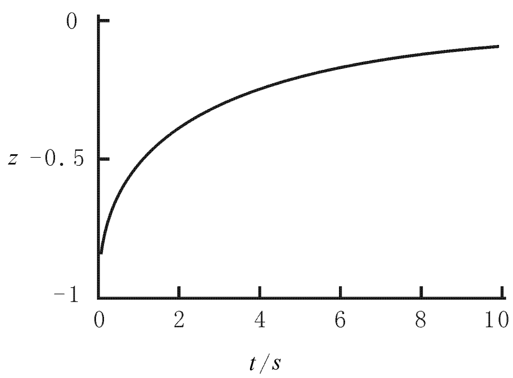


**Fig. 4　Response Plot of State $z$**

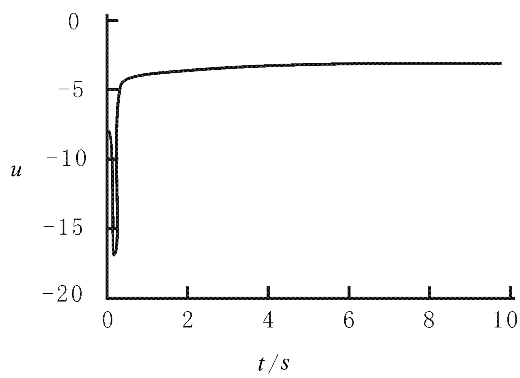


**Fig. 5　Control Input $u$**

# 5　Conclusion

The optimal sliding mode control method is proposed for the cascade uncertain nonlinear system which is widely

used in aeronautics and astronautics field. The controller consists of optimal controller and sliding mode controller. The outer loop is designed using a SDRE optimal controller to produce an optimal sliding mode surface. The inner loop decrease sensitivity to parameter change and model error and external disturb of control system using sliding mode control. Finally, an example is given to demonstrate the availability of the proposed control method.

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# 一类串级非线性系统的最优滑模控制器设计

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摘 要: 对一类串级不确定非线性系统提出了一种基于SDRE控制的最优滑模控制方法. 该方法采用2环控制结构, 外环控制器的设计采用基于依赖状态的Riccati方程最优控制器, 用以产生最优滑模面; 内环控制器的设计采用滑动模控制以减小控制系统对参数变化、模型误差、外部干扰的敏感. 设计的最优滑模控制器能使一类串级不确定系统具有鲁棒稳定性. 同时, 提出了2种求解依赖于状态的Riccati方程的方法. 最后, 通过仿真实例验证了该控制方法的有效性.

关键词: 串级系统; 非线性系统; 最优控制; 滑模控制

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