**Article ID:** 1007- 2985(2004)04- 0062- 04

# The Stability of One- Leg Methods for Nonlinear Multi- Delay Differential Equations*

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**Abstract:** A sufficient condition of GR- stability was given for nonlinear multi- delays differential equations and then, the author proved that any A- stable one- leg method $(\rho, \sigma)$ is GR- stable.

**Key words:** multi- delay differential equations; one- leg methods; GR- stable; A- stable; G- stable
**CLC number:** O175.4　　**Document code:** A

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In recent years, many papers discussed numerical methods for the solution of delay differential equations. The concepts of GR- stability, GAR- stability and weak GAR- stability were introduced. Some good conclusion about the ability of DDEs were offered and proved. In this paper, after modifying the condition of the system in[1], we extend the concepts to MDDEs and gain some similar results.

## 1　The Asymptotic Stability of Nonlinear MDDEs

Let $\langle \cdot , \cdot \rangle$ be an inner product on $C^N$ and $\|\cdot\|$ be the corresponding norm. Consider the following initial problem of nonlinear MDDEs:

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2)) & t \geqslant 0, \\ y(t) = \varphi_1(t) & t \leqslant 0, \end{cases} \tag{1}$$

where $\tau_1$, $\tau_2$ are positive delay terms and $\tau_1 \geqslant \tau_2 < 0$, $\varphi_1$ is a continuous function, and $f: [0, +\infty) C^N \times C^N \times C^N \to C^N$ is a given mapping which satisfies the following conditions:

$$\mathrm{Re}\, \langle y - z, f(t, y, u_1, v_1) - f(t, z, u_2, v_2) \rangle \leqslant \alpha(t) \|y - z\|^2 + \gamma_1(t) \|u_1 - u_2\|^2 + \gamma_2(t) \|v_1 - v_2\|^2 \quad t \geqslant 0, y, z, u_1, v_1, u_2, v_2 \in C^N, \tag{2}$$

where $\alpha(t)$, $\gamma_1(t)$ and $\gamma_2(t)$ are bounded functions; furthermore, $\gamma_1(t)$ and $\gamma_2(t)$ is not always less than 0, and there exists a function $\eta(t)$ satisfying the following condition:

$$\eta(t)\alpha(t) + \gamma_1(t) + \gamma_2(t) = 0 \qquad 0 \leqslant \eta(t) \leqslant 1, \alpha(t) \leqslant 0, t \geqslant 0. \tag{3}$$

In order to discuss the stability and asymptotic stability of the system (1), we introduce another system, defined by the same function, but with another initial condition:

$$\begin{cases} z'(t) = f(t, z(t), z(t-\tau_1), z(t-\tau)) & t \geqslant 0, \\ z(t) = \varphi_2(t) & t \leqslant 0, \ \tau_1 \geqslant \tau_2 > 0. \end{cases} \tag{4}$$

**Definition 1**[2]　If $y(t)$ and $z(t)$ are the solutions of (1) and (4) respectively, they satisfy,

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* **Received date:** 2004- 10- 18
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$$\|y(t)-z(t)\| \leqslant \max_{x\leqslant 0}\|\varphi_1(x)-\varphi_2(x)\| \qquad t\geqslant 0, \tag{5}$$

we called the system (1) stable, asymptotic stability is defined by further satisfying $\lim_{t\to\infty}\|y(t)-z(t)\|=0$.

**Theorem 1**[3] If (3) is true then (1) is stable.

**Theorem 2** If (3) is satisfied and $\eta=\sup_{t\geqslant 0}\eta(t)<1$ then (1) is asymptotic stable.

**Proof** We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)-z(t)\|^2 = 2\mathrm{Re}\langle y'(t)-z'(t), y(t)-z(t)\rangle = 2\mathrm{Re}\langle f(t,y(t),y(t-\tau_1),y(t-\tau_2))-f(t,z(t),z(t-\tau_1),z(t-\tau_2)), y(t)-z(t)\rangle \leqslant 2\alpha(t)\|y(t)-z(t)\|^2 + 2\gamma_1(t)\|y(t-\tau_1)-z(t-\tau_1)\|^2 + 2\gamma_2(t)\|y(t-\tau_2)-z(t-\tau_2)\|^2.$$

Let $Y(t):=\|y(t)-z(t)\|^2$, $A(t)=\int_0^t 2\alpha(s)\mathrm{d}s$ combined with (3), we can obtain

$$Y'(t)\leqslant 2\alpha(t)Y(t)+[2\gamma_1(t)+2\gamma_2(t)]\sup_{x\leqslant t-\tau_1}Y(x) = 2\alpha(t)Y(t)-2\eta(t)\alpha(t)\sup_{x\leqslant t-\tau_1}Y(x).$$

Multiply both sides by $e^{-A(t)}$, we have

$$e^{-A(t)}Y'(t)\leqslant 2\alpha(t)e^{-A(t)}Y(t)-2\eta\alpha(t)e^{-A(t)}\sup_{x\leqslant t-\tau_1}Y(x),$$

$$e^{-A(t)}Y'(t)-2\alpha(t)e^{-A(t)}Y(t)\leqslant -2\eta\alpha(t)e^{-A(t)}\sup_{x\leqslant t-\tau_1}Y(x),$$

$$(e^{-A(t)}Y(t))'\leqslant -2\eta\alpha(t)e^{-A(t)}\sup_{x\leqslant t-\tau_1}Y(x).$$

Integral the sides from 0 to $t$, we can obtain

$$e^{-A(t)}Y(t)\leqslant y(0)+\eta(-1+e^{A(t)})\sup_{x\leqslant t-\tau_1}Y(x),$$

$$Y(t)\leqslant e^{A(t)}Y(0)+\eta(1-e^{A(t)})\sup_{x\leqslant t-\tau_1}Y(x).$$

According to $\eta<1$ and $e^{A(t)}\to 0, (t\to+\infty)$, so there exists a constant $b(\eta\leqslant b<1)$, which satisfies $Y(t)\leqslant b\sup_{x\leqslant t-\tau_1}Y(x)$, combining with Theorem 1, we can easily obtain $\lim_{t\to\infty}Y(t)=0$. The proof of Theorem 2 is over.

## 2 The Stability of One- Leg Methods of Nonlinear MDDEs

Apply the one- leg $k$- step method $(\rho,\sigma)$ to system (1)[4]

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n, y_{1n}, y_{2n}) \qquad n=0,1,2,\cdots, \tag{6}$$

where $h>0$ is the stepsize, $E$ is the translation operator: $Ey_n=y_{n+1}$, $y_n$ is an approximation to the exact solution $y(t_n)$ with $t_n=nh$, $\rho(x)=\sum_{j=0}^{k}\alpha_j x^j$ and $\sigma(x)=\sum_{j=0}^{k}\beta_j x^j$ are generating polynomials, which are assumed to have real coefficients and no common divisor. We assume $\rho(1)=0$, $\rho'(1)=\sigma(1)=1$, $y_{in}(i=1,2)$ denotes an approximation to $y(\sigma(E)t_n)\tau_i)$ that is obtained by a specific interpolation at the point $t_i=\sigma(E)t_n-\tau_i$ using $\{y_j\}_{j\leqslant n+k}$, process (6) is defined completely by the one- leg method $(p,\sigma)$ and the interpolation procedure for $y_{in}$.

As we know, any $A$- stable one- leg method for ODEs has order at most 2. So we can use the linear interpolation procedure for $y_{in}$. Let $\tau_i=(m_i-\delta_i)h(i=1,2)$ with integer $m_i\geqslant 1$ and $\delta_i\in[0,1)$. We define

$$y_{in}=\delta_i\sigma(E)y_{n-m_i+1}+(1-\delta_i)\sigma(E)y_{n-m} \qquad i=1,2, \tag{7}$$

$y_l=\varphi_1(lh)$ for $l\leqslant 0$.

Apply the same method $(p,\sigma)$ to MDDEs (4), we have

$$\rho(E)z_n = hf(\sigma(E)t_n, \sigma(E)z_n, z_{1n}, z_{2n} \qquad n=0,1,2, \tag{8}$$

$$z_{in}=\delta_i\sigma(E)z_{n-m_i+1}+(1-\delta_i)\sigma(E)z_{n-m_i} \qquad i=1,2, \tag{9}$$

$z_l=\varphi_2(lh)$ for $l\leqslant 0$.

In the following discussion, we define $\gamma_i=\sup_{t\geqslant 0}\gamma_i(t)(i=1,2)$, and $\alpha=\sup_{t\geqslant 0}\alpha(t)$.

**Definition 2** A numerical method for MDDEs is called GR- stable, if under the condition $\gamma_1+\gamma_2\leqslant-\alpha$, there exists a constant $C$ which only depends on the method, $\tau_1$, $\tau_2$, $\gamma_1$ and $\gamma_2$, the numerical approximations $y_n$ and $z_n$ to the solutions $y(t)$ and $z(t)$ of any given system (1) and (4), satisfy the following inequality: $\|y_n-z_n\|\leqslant C(\max_{0\leqslant j\leqslant k-1}\|y_j-z_j\|+\max_{t\leqslant 0}\|\varphi_1(t)-\varphi_2(t)\|)$ for every $n\geqslant k$ and stepsize $h>0$.

Now, let's focus on the stability analysis of A- stable one- leg methods with respect to nonlinear MDDEs.

Let $y_n, z_n\in C^N$, $\omega_n=[(y_n-z_n)^{\mathrm T}, (y_{n+1}-z_{n+1})^{\mathrm T}, \cdots, (y_{n+k-1}-z_{n+k-1})^{\mathrm T}]^{\mathrm T}$, and for a real symmetric positive definite matrix $G=[g_{ij}]_{k\times k}$, the norm $\|\cdot\|_G$ is defined by

$$\|U\|_G=(\sum_{i,j=1}^{k}g_{ij}\langle u_i,u_j\rangle)^{1/2},\ U=(u_1^{\mathrm T},u_2^{\mathrm T},\cdots,u_k^{\mathrm T})^{\mathrm T}\in C^{k\times N}.$$

**Theorem 3** Assume that the one- leg $k$- step method $(\rho,\sigma)$ is G- stable[4] for a real symmetric positive definite matrix $G$, then

$$\|\omega_{n+1}\|_G\leqslant\|\omega_0\|_G^2+2h\sum_{j=0}^{n}[\alpha\|\sigma(E)(y_j-z_j)\|^2+\sum_{i=1}^{2}\delta_i\gamma_i\|\sigma(E)(y_{j-m_i+1}-z_{j-m_i+1}\|^2+\sum_{i=1}^{2}(1-\delta_i)\gamma_i\|\sigma(E)(y_{j-m_i}-z_{j-m_i})\|^2]. \tag{10}$$

**Proof** Assume that the method is G- stable for G, then for all real $a_0, a_1, \cdots, a_n$ we have $A_1^{\mathrm T}GA_1-A_0^{\mathrm T}GA_0\leqslant 2\sigma(E)a_0\rho a_0$ where $A_i=(a_i,a_{i+1},\cdots,a_{i+k-1})^{\mathrm T}$, $i=0,1$, we can obtain:

$$\begin{aligned}\|\omega_{n+1}\|_G^2-\|\omega_n\|_G^2&\leqslant 2\mathrm{Re}\langle\sigma(E)(y_n-z_n),\rho(E)(y_n-z_n)\rangle=\\&2\mathrm{Re}\langle\sigma(E)(y_n-z_n),h[f(\sigma(E)t_n,\sigma(E)y_n,y_{1n},y_{2n})-\\&f(\sigma(E)t_n,\sigma(E)z_n,\sigma(E)z_n,z_{1n},z_{2n})]\rangle\leqslant\\&2h\alpha(t)\|\sigma(E)(y_n-z_n)\|^2+2h\gamma_1(t)\|y_{1n}-z_{1n}\|^2+\\&2h\gamma_2(t)\|y_{2n}-z_{2n}\|^2.\end{aligned}$$

According to (7) and (9), we have,

$$\|y_{in}-z_{in}\|^2\leqslant\delta_i\|\sigma(E)y_{n-m_i+1}-z_{n-m_i+1})\|^2+(1-\delta_i)\|\sigma(E)(y_{n-m_i}-z_{n-m_i})\|^2,$$

hence

$$\|\omega_{n+1}\|_G^2-\|\omega_n\|_G^2\leqslant 2\alpha h\|\sigma(E)(y_n-z_n)\|^2+2h\sum_{i=1}^{2}[\delta_i\gamma_i\|\sigma(E)(y_{n-m_i+1}-z_{n-m_i+1})\|^2+(1-\delta_i)\gamma_i\|\sigma(E)(y_{n-m_i}-z_{n-m_i})\|^2].$$

By induction, we have (10). The proof of Theorem 3 is over.

**Theorem 4** Any A- stable[4] one- leg method $(\rho,\sigma)$ is GR- stable.

**proof** Assume that the method is A- stable. Then the method is G- stable. Applying Theorem 3 and combining with the condition $0\leqslant\gamma_1+\gamma_2\leqslant-\alpha$, we have

$$\begin{aligned}\|\omega_{n+1}\|_G^2\leqslant&\|\omega_0\|_G^2+2\alpha h\sum_{j=1}^{n}\|\sigma(E)(y_j-z_j)\|^2+2h\sum_{j=0}^{n}[\delta_1\gamma_1\|\sigma(E)(y_{j-m_1+1}-z_{j-m_1+1})\|^2+\\&(1-\delta_1)\gamma_1\|\sigma(E)(y_{j-m_1}-z_{j-m_1})\|^2]+2h\sum_{j=0}^{n}[\delta_2\gamma_2\|\sigma(E)(y_{j-m_2+1}-z_{j-m_2+1})\|^2+\\&(1-\delta_2)\gamma_2\|\sigma(E)(y_{j-m_2}-z_{j-m_2})\|^2].\end{aligned}\tag{11}$$

(1) When $m_i\geqslant 2$, $i=1,2$, we have

$$\begin{aligned}\|\omega_{n+1}\|_G^2\leqslant&\|\omega_0\|_G^2+2\delta_1\gamma_1 h\sum_{j=-m_1+1}^{-1}\|\sigma(E)(y_j-z_j)\|^2+2(1-\delta_1)\gamma_1 h\sum_{j=-m_1}^{-1}\|\sigma(E)(y_j-z_j)\|^2+\\&2\delta_2\gamma_2 h\sum_{j=-m_2+1}^{-1}\|\sigma(E)(y_j-z_j)\|^2+2(1-\delta_2)\gamma_2 h\sum_{j=-m_2}^{-1}\|\sigma(E)(y_j-z_j)\|^2\leqslant\\&\|\omega_0\|_G^2+(2\gamma_1\tau_1+2\gamma_2\tau_2)\max_{-m_2\leqslant j\leqslant -1}\|\sigma(E)(y_j-z_j)\|^2.\end{aligned}\tag{12}$$

(2) When $m_1 = m_2 = 1$, we have

$$\|\omega_{n+1}\|_G^2 \leqslant \|\omega_0\|_G^2 + 2(\gamma_1\tau_1 + \gamma_2\tau_2)\|\sigma(E)(y_{-1} - z_{-1})\|^2. \quad (13)$$

(3) When $m_1 = 1$ and $m_2 > 1$, we have

$$\|\omega_{n+1}\|_G^2 \leqslant \|\omega_0\|_G^2 + 2\gamma_1\tau_1\|\sigma(E)(y_{-1} - z_{-1}\|^2 + 2\gamma_2\tau_2 \max_{-m_2 \leqslant j \leqslant -1}\|\sigma(E)(y_j - z_j)\|^2. \quad (14)$$

Combining (12), (13) with (14), we have

$$\|\omega_{n+1}\|_G^2 \leqslant \|\omega_0\|_G^2 + 2(\gamma_1\tau_1 + \gamma_2\tau_2)\max_{-m_2 \leqslant j \leqslant -1}\|\sigma(E)(y_j - z_j)\|^2 \qquad n \geqslant 0, m \geqslant 1. \quad (15)$$

Assume $\lambda_1$ and $\lambda_2$ are the maximum and minimum eigenvalue of the matrix $G$ respectively, we can obtain

$$\lambda_2 \cdot \|y_{n+k} - z_{n+k}\|^2 \leqslant \lambda_1 \cdot \sum_{i=0}^{k-1}\|y_i - z_i\|^2 + 2(\gamma_1\tau_1 + \gamma_2\tau_2)\max_{-m_2 \leqslant j \leqslant -1}\|\sigma(E)(y_j - z_j)\|^2 \qquad n \geqslant 0,$$

$$\|y_{n+k} - z_{n+k}\|^2 \leqslant \frac{k\lambda_1}{\lambda_2}\max_{0 \leqslant i \leqslant k-1}\|y_i - z_i\|^2 + \frac{2(\gamma_1\tau_1 + \gamma_2\tau_2)}{\lambda_2}\max_{-m_2 \leqslant j \leqslant -1}\|\sigma(E)(y_j - z_j)\|^2 \qquad n \geqslant 0.$$

This shows that the method is GR- stable. The proof of Theorem 4 is over.

**Remark:** According to the procedure of the proofs, we know that the conclusions of the paper are true to the delay differential equation systems with many delays.

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# 非线性多延迟微分方程单支方法的稳定性

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摘 要: 本文对[1]中初值问题条件改造后, 给出了非线性多延迟微分方程的单支方法 GR- 稳定的一个充分条件, 并将[1]的部分工作推广到了多延迟的情形, 获得了较好的结论.

关键词: 多延迟微分方程; 单支方法; GR- 稳定; A- 稳定; G- 稳定
中图分类号: O175.14 文献标识码: A