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Upper and Lower Solutions with Reversed Order for Differential Equations*

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Abstract: The method of upper and lower solutions with reversed ordering combined with monotone iterative technique is employed to the study of a class of differential equations with integral boundary conditions. Several new existence theorems are obtained.

Key words: upper and lower solution; maximal and minimal solution; differential equation; integral boundary condition
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1　Introduction

The purpose of this paper is to investigate the following differential problem

$$\begin{cases} x'(t) = f(t, x(t)) & t \in J = [0, T], T > 0, \\ x(0) + \mu \int_0^T x(s)\,\mathrm{d}s = \lambda x(T), \end{cases} \tag{1}$$

where $f \in C(J \times \mathbf{R}, \mathbf{R})$ and $\mu \geqslant 0$, $\lambda = 1$ or -1. Note that if $\mu = 0$, $\lambda = 1$, (1) is periodic boundary value problem. If $\mu = 0$, $\lambda = -1$, (1) reduces to anti-periodic boundary value problem. There are many existence results for ordinary differential equations with the above two boundary conditions (see references [1-5] and references therein).

It is well known that the monotone iterative technique is a powerful method used to approximate solutions of several problems[6-9]. In this paper, we consider (1) by using the method of upper and lower solutions method combined with monotone iterative technique. This technique plays important role in constructing monotone sequences which converge to the solutions of our problem. In presence of a lower solution α and an upper solution β with $\beta \leqslant \alpha$, we show the sequences converge to the solutions of (1) under suitable conditions. Two examples are given to illustrate the obtained results.

2　Case $\lambda = 1$

Definition 1　A function $\alpha \in C^1(J, \mathbf{R})$ is said to be a lower solution of (1) for $\lambda = 1$ if

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t)) & t \in J, \\ \alpha(0) + \mu \int_0^T \alpha(s)\,\mathrm{d}s \leqslant \alpha(T), \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Let $\Omega_{(u,v)} = \{y : u(t) \leqslant y(t) \leqslant v(t), t \in J\}$ if $u(t) \leqslant v(t)$ for $t \in J$. We introduce the following assump-

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tions:

(H_1) $\alpha_0,\beta_0\in C^1(J,\mathbf{R})$ are lower and upper solutions of (1) for $\lambda=1$ respectively, and $\beta_0(t)\leqslant\alpha_0(t)$ for $t\in J$;

(H_2) $f\in C(J\times\Omega_{(\beta_0,\alpha_0)},\mathbf{R})$;

(H_3) There exists $M>0$ such that $f(t,u)-f(t,v)\leqslant M(u-v)$ if $v\leqslant u$, $u,v\in\Omega_{(\beta_0,\alpha_0)}$, $t\in J$.

Lemma 1 Put $\lambda=1$, assume that (H_1), (H_2), (H_3) hold. If

$$\begin{cases} y'(t)=f(t,\alpha_0(t))+M(y(t)-\alpha_0(t)) & t\in J,\ \alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=y(T),\\ z'(t)=f(t,\beta_0(t))+M(z(t)-\beta_0(t)) & t\in J,\ \beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=z(T),\end{cases}$$

then

$$\beta_0(t)\leqslant z(t)\leqslant y(t)\leqslant\alpha_0(t)\qquad t\in J,\tag{2}$$

and y,z are lower and upper solutions of (1) respectively.

Proof Note that there exist unique solutions for y and z. Put $p=y-\alpha_0$, $q=\beta_0-z$, then

$$p(T)=y(T)-\alpha_0(T)\leqslant\alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s-\alpha_0(0)-\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=0,$$

$$q(T)=\beta_0(T)-z(T)\leqslant\beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s-\beta_0(0)-\mu\int_0^T\beta_0(s)\,\mathrm{d}s=0,$$

and

$$p'(t)\geqslant f(t,\alpha_0(t))+M(y(t)-\alpha_0(t))-f(t,\alpha_0(t))=Mp(t)\qquad t\in J,$$
$$q'(t)\geqslant f(t,\beta_0(t))-f(t,\beta_0(t))-M(z(t)-\beta_0(t))=Mq(t)\qquad t\in J.$$

Hence $p(t)\leqslant e^{-M(T-t)}p(T)\leqslant 0$, $q(t)\leqslant e^{-M(T-t)}q(T)\leqslant 0$, $t\in J$, then $y(t)\leqslant\alpha_0(t)$, $\beta_0(t)\leqslant z(t)$, $t\in J$. Now let $p=z-y$, then

$$p(T)=\beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s-\alpha_0(0)-\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=$$
$$\beta_0(0)-\alpha_0(0)+\mu\int_0^T(\beta_0(s)-\alpha_0(s))\,\mathrm{d}s\leqslant 0.$$

Assumption (H_3)

$$p'(t)=f(t,\beta_0(t))-f(t,\alpha_0(t))+M(z(t)-\beta_0(t))-M(y(t)-\alpha_0(t))\geqslant$$
$$-M(\alpha_0(t)-\beta_0(t))+M(z(t)-y(t))+M(\alpha_0(t)-\beta_0(t))=Mp(t).$$

Hence $p(t)\leqslant e^{-M(T-t)}p(T)\leqslant 0$, $t\in J$ showing that $z(t)\leqslant y(t)$, $t\in J$. It proves that (2) holds. Now we need to show that y,z are lower and upper solutions of (1) respectively. Using again assumption (H_3), we have

$$y'(t)=f(t,\alpha_0(t))+M(y(t)-\alpha_0(t))-f(t,y(t))+f(t,y(t))\leqslant f(t,y(t))\qquad t\in J,$$
$$z'(t)=f(t,\beta_0(t))+M(z(t)-\beta_0(t))-f(t,z(t))+f(t,z(t))\geqslant f(t,z(t))\qquad t\in J,$$

and

$$y(0)+\mu\int_0^T y(s)\,\mathrm{d}s\leqslant\alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=y(T),$$
$$z(0)+\mu\int_0^T z(s)\,\mathrm{d}s\geqslant\beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=z(T).$$

It shows that y,z are lower and upper solutions of (1) respectively. The proof is complete.

Theorem 1 Put $\lambda=1$, suppose that (H_1), (H_2), (H_3) hold. Then there exist monotone sequences $\{\alpha_n,\beta_n\}$ such that $\alpha_n\to\alpha$, $\beta_n\to\beta$, $t\in J$ as $n\to\infty$ and this convergence is uniformly and monotonically on J. Moreover, α, β are maximal and minimal solutions of (1) in $[\beta_0,\alpha_0]=\{u\in C^1(J,\mathbf{R}):\beta_0\leqslant u\leqslant\alpha_0\}$.

Proof Let

$$\begin{cases} \alpha'_{n+1}(t) = f(t, \alpha_n(t)) + M(\alpha_{n+1}(t) - \alpha_n(t)) \quad t \in J, \\ \alpha_n(0) + \mu\int_0^T \alpha_n(s)\,\mathrm{d}s = \alpha_{n+1}(T), \\ \beta'_{n+1}(t) = f(t, \beta_n(t)) + M(\beta_{n+1}(t) - \beta_n(t)) \quad t \in J, \\ \beta_n(0) + \mu\int_0^T \beta_n(s)\,\mathrm{d}s = \beta_{n+1}(T), \end{cases} \tag{3}$$

for $n = 0, 1, 2, \cdots$. Lemma 1 shows $\beta_0(t) \leqslant \beta_1(t) \leqslant \alpha_1(t) \leqslant \alpha_0(t)$, $t \in J$, and α_1, β_1 are lower and upper solutions of (1) respectively. Assume that

$$\beta_0(t) \leqslant \beta_1(t) \leqslant \cdots \leqslant \beta_k(t) \leqslant \alpha_k(t) \leqslant \cdots \leqslant \alpha_1(t) \leqslant \alpha_0(t) \quad t \in J$$

for some $k \geqslant 1$ and let α_k, β_k be lower and upper solutions of (1) respectively. Then, using again lemma 1, we get $\beta_k(t) \leqslant \beta_{k+1}(t) \leqslant \alpha_{k+1}(t) \leqslant \alpha_k(t)$, $t \in J$, and α_{k+1}, β_{k+1} are lower and upper solutions of (1) respectively. By induction, we have

$$\beta_0(t) \leqslant \beta_1(t) \leqslant \cdots \leqslant \beta_n(t) \leqslant \alpha_n(t) \leqslant \cdots \leqslant \alpha_1(t) \leqslant \alpha_0(t) \quad t \in J \text{ for all } n.$$

Hence $\beta_n(t) \to \beta(t)$, $\alpha_n(t) \to \alpha(t)$, $t \in J$ if $n \to \infty$. Indeed, taking the limit $n \to \infty$ on both sides of (3), we know that α and β are solutions of (1).

Next, we are going to show that α, β are maximal and minimal solutions of (1) in $[\beta_0, \alpha_0]$. To do it, we need to show that if $w(t)$ is any solution of (1) such that $\beta_0(t) \leqslant w(t) \leqslant \alpha_0(t)$, $t \in J$, then $\beta_0(t) \leqslant \beta(t) \leqslant w(t) \leqslant \alpha(t) \leqslant \alpha_0(t)$, $t \in J$. Assume that for some k, $\beta_k(t) \leqslant w(t) \leqslant \alpha_k(t)$, $t \in J$. Let $p = \beta_{k+1} - w$, $q = w - \alpha_{k+1}$. Then

$$p(T) = \beta_{k+1}(T) - w(T) = \beta_k(0) - w(0) + \mu\int_0^T [\beta_k(s) - w(s)]\,\mathrm{d}s \leqslant 0,$$

$$q(T) = w(T) - \alpha_{k+1}(T) = w(0) - \alpha_k(0) + \mu\int_0^T [w(s) - \alpha_k(s)]\,\mathrm{d}s \leqslant 0.$$

From assumption (H_3), we have

$$\begin{aligned} p'(t) &= f(t, \beta_k(t)) - f(t, w(t)) + M(\beta_{k+1}(t) - \beta_k(t)) \geqslant \\ &\quad - M(w(t) - \beta_k(t)) + M(\beta_{k+1}(t) - \\ &\quad \beta_k(t)) = Mp(t) \quad t \in J, \\ q'(t) &= f(t, w(t)) - f(t, \alpha_k(t)) - M(\alpha_{k+1}(t) - \alpha_k(t)) \geqslant \\ &\quad - M(\alpha_k(t) - w(t)) - M(\alpha_{k+1}(t) - \\ &\quad \alpha_k(t)) = Mq(t) \quad t \in J. \end{aligned}$$

Hence $p(t) \leqslant \mathrm{e}^{-M(T-t)} p(T) \leqslant 0$, $q(t) \leqslant \mathrm{e}^{-M(T-t)} q(T) \leqslant 0$, $t \in J$ showing that $\beta_{k+1}(t) \leqslant w(t) \leqslant \alpha_{k+1}(t)$, $t \in J$. It proves, by induction, that $\beta_n(t) \leqslant w(t) \leqslant \alpha_n(t)$, $t \in J$ for all n. Taking the limit $n \to \infty$, we have $\beta_0(t) \leqslant \beta(t) \leqslant w(t) \leqslant \alpha(t) \leqslant \alpha_0(t)$, $t \in J$, so the assertion of theorem 1 is true. The proof is complete.

3 Case $\lambda = -1$

Definition 2 Functions $\alpha, \beta \in C^1(J, \mathbf{R})$ are coupled lower and upper solutions of (1) for $\lambda = -1$ if

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t)) \quad t \in J, \ \alpha(0) + \mu\int_0^T \alpha(s)\,\mathrm{d}s \leqslant -\beta(T), \\ \beta'(t) \geqslant f(t, \beta(t)) \quad t \in J, \ \beta(0) + \mu\int_0^T \beta(s)\,\mathrm{d}s \geqslant -\alpha(T). \end{cases}$$

(H_4) $\alpha_0, \beta_0 \in C^1(J, \mathbf{R})$ are coupled lower and upper solutions of (1) for $\lambda = -1$ and $\beta_0(t) \leqslant \alpha_0(t)$ for $t \in J$.

(H_5) $f_x \in C(J \times \Omega_{(\beta_0, \alpha_0)}, \mathbf{R})$ and $1 + \mu\int_0^T \mathrm{e}^{\int_0^t f_x(s, \xi(s))\,\mathrm{d}s}\,\mathrm{d}t - \mathrm{e}^{\int_0^T f_x(s, \xi(s))\,\mathrm{d}s} \neq 0$ for any $\xi \in \Omega_{(\beta_0, \alpha_0)}$, $\mu \geqslant 0$.

Lemma 2 Put $\lambda = -1$, assume that (H_4), (H_2) and (H_3) hold. If

$$\begin{cases} y'(t)=f(t,\alpha_0(t))+M(y(t)-\alpha_0(t)) & t\in J,\ \beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=-y(T),\\ z'(t)=f(t,\beta_0(t))+M(z(t)-\beta_0(t)) & t\in J,\ \alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=-z(T),\end{cases}$$

then

$$\beta_0(t)\le z(t)\le y(t)\le \alpha_0(t)\qquad t\in J, \tag{4}$$

and y,z are coupled lower and upper solutions of (1).

Proof Note that there exist unique solutions for y and z. Put $p=y-\alpha_0$, $q=\beta_0-z$, then

$$p(T)=y(T)-\alpha_0(T)\le-\beta_0(0)-\mu\int_0^T\beta_0(s)\,\mathrm{d}s+\beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=0,$$

$$q(T)=\beta_0(T)-z(T)\le-\alpha_0(0)-\mu\int_0^T\alpha_0(s)\,\mathrm{d}s+\alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=0,$$

and

$$p'(t)\ge f(t,\alpha_0(t))+M(y(t)-\alpha_0(t))-f(t,\alpha_0(t))=Mp(t)\qquad t\in J,$$

$$q'(t)\ge f(t,\beta_0(t))-f(t,\beta_0(t))-M(z(t)-\beta_0(t))=Mq(t)\qquad t\in J.$$

Hence $p(t)\le e^{-M(T-t)}p(T)\le 0$, $q(t)\le e^{-M(T-t)}q(T)\le 0$, $t\in J$, then $y(t)\le\alpha_0(t)$, $\beta_0(t)\le z(t)$, $t\in J$. Now let $p=z-y$, then

$$p(T)=-\alpha_0(0)-\mu\int_0^T\alpha_0(s)\,\mathrm{d}s+\beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=$$

$$\beta_0(0)-\alpha_0(0)+\mu\int_0^T(\beta_0(s)-\alpha_0(s))\,\mathrm{d}s\le 0.$$

Assumption (H_3) yields

$$\begin{aligned} p'(t)&=f(t,\beta_0(t))-f(t,\alpha_0(t))+M(z(t)-\beta_0(t))-M(y(t)-\alpha_0(t))\ge\\ &\quad -M(\alpha_0(t)-\beta_0(t))+M(z(t)-y(t))+\\ &\quad M(\alpha_0(t)-\beta_0(t))=Mp(t).\end{aligned}$$

Hence $p(t)\le e^{-M(T-t)}p(T)\le 0$, $t\in J$ showing that $z(t)\le y(t)$, $t\in J$. It proves that (4) holds. Now we need to show that y,z are coupled lower and upper solutions of (1) for $\lambda=-1$. Using again assumption (H_3) we have

$$y'(t)=f(t,\alpha_0(t))+M(y(t)-\alpha_0(t))-f(t,y(t))+f(t,y(t))\le f(t,y(t))\qquad t\in J,$$

$$z'(t)=f(t,\beta_0(t))+M(z(t)-\beta_0(t))-f(t,z(t))+f(t,z(t))\ge f(t,z(t))\qquad t\in J,$$

and

$$y(0)+\mu\int_0^T y(s)\,\mathrm{d}s\le \alpha_0(0)+\mu\int_0^T\alpha_0(s)\,\mathrm{d}s=-z(T),$$

$$z(0)+\mu\int_0^T z(s)\,\mathrm{d}s\ge \beta_0(0)+\mu\int_0^T\beta_0(s)\,\mathrm{d}s=-y(T).$$

It shows that y,z are coupled lower and upper solutions of (1) respectively. The proof is complete.

Theorem 2 Put $\lambda=-1$, suppose that (H_2), (H_3), (H_4) and (H_5) hold. Then there exist monotone sequences $\{\alpha_n,\beta_n\}$ such that $\alpha_n\to\alpha$, $\beta_n\to\beta$, $t\in J$ as $n\to\infty$ and this convergence is uniformly and monotonically on J. Moreover, (1) has a unique solution $x\in[\beta_0,\alpha_0]=\{u\in C^1(J,\mathbf{R}):\beta_0\le u\le\alpha_0\}$ and $x=\alpha=\beta$.

Proof Let

$$\begin{cases}\alpha'_{n+1}(t)=f(t,\alpha_n(t))+M(\alpha_{n+1}(t)-\alpha_n(t)) & t\in J,\\ \beta_n(0)+\mu\int_0^T\beta_n(s)\,\mathrm{d}s=-\alpha_{n+1}(T),\\ \beta'_{n+1}(t)=f(t,\beta_n(t))+M(\beta_{n+1}(t)-\beta_n(t)) & t\in J,\\ \alpha_n(0)+\mu\int_0^T\alpha_n(s)\,\mathrm{d}s=-\beta_{n+1}(T),\end{cases}\tag{5}$$

for $n = 0, 1, 2, \cdots$. Lemma 2 shows $\beta_0(t) \le \beta_1(t) \le \alpha_1(t) \le \alpha_0(t)$, $t \in J$, and α_1, β_1 are coupled lower and upper solutions of (1) respectively. Assume that

$$\beta_0(t) \le \beta_1(t) \le \cdots \le \beta_k(t) \le \alpha_k(t) \le \cdots \le \alpha_1(t) \le \alpha_0(t) \quad t \in J,$$

for some $k \ge 1$ and let α_k, β_k be coupled lower and upper solutions of (1) respectively. Then, using again lemma 2, we get $\beta_k(t) \le \beta_{k+1}(t) \le \alpha_{k+1}(t) \le \alpha_k(t)$, $t \in J$, and α_{k+1}, β_{k+1} are coupled lower and upper solutions of (1) respeetively. By induction, we have

$$\beta_0(t) \le \beta_1(t) \le \cdots \le \beta_n(t) \le \alpha_n(t) \le \cdots \le \alpha_1(t) \le \alpha_0(t) \quad t \in J \text{ for all } n.$$

Hence $\beta_n(t) \to \beta(t)$, $\alpha_n(t) \to \alpha(t)$, $t \in J$ if $n \to \infty$. Indeed, taking the limit $n \to \infty$ on both sides of (5), we know that (α, β) is a solution of the following system:

$$\begin{cases} \alpha'(t) = f(t, \alpha(t)) \quad t \in J, \ \beta(0) + \int_0^T \beta(s)\,ds = -\alpha(T), \\ \beta'(t) = f(t, \beta(t)) \quad t \in J, \ \alpha(0) + \int_0^T \alpha(s)\,ds = -\beta(T). \end{cases}$$

Note that if we show that $\alpha = \beta$, then $\alpha = \beta$ is a solution of (1) for $\lambda = -1$. Put $P = \beta - \alpha$, then we have

$$P(T) = P(0) + \int_0^T P(s)\,ds, \tag{6}$$

and

$$P'(t) = f(t, \beta(t)) - f(t, \alpha(t)) = f_x(t, \xi(t))P(t) \quad t \in J,$$

where ξ is between β and α. It yields

$$P(T) = e^{\int_0^T f_x(s,\xi(s))ds}P(0), \ P(t) = e^{\int_0^t f_x(s,\xi(s))ds}P(0) \quad t \in J. \tag{7}$$

From (6) and (7), we obtain

$$e^{\int_0^T f_x(s,\xi(s))ds}P(0) = P(0) + \int_0^T e^{\int_0^t f_x(s,\xi(s))ds}dt\,P(0).$$

Hence, $P(0) = 0$, by assumption (H_5), and finally, $P(t) = 0$ on J, by (7). It proves that $\alpha = \beta$, so $\alpha = \beta$ is a solution of (1) for $\lambda = -1$.

Next, we are going to show that $\alpha = \beta$ is a unique solution of (1) in $[\beta_0, \alpha_0]$. To do it, we need to show that if $w(t)$ is any solution of (1) such that $\beta_0(t) \le w(t) \le \alpha_0(t)$, $t \in J$, then $\beta_0(t) \le \beta(t) \le w(t) \le \alpha(t) \le \alpha_0(t)$, $t \in J$. Assume that for some k, $\beta_k(t) \le w(t) \le \alpha_k(t)$, $t \in J$. Let $p = \beta_{k+1} - w$, $q = w - \alpha_{k+1}$. Then

$$p(T) = \beta_{k+1}(T) - w(T) = w(0) - \alpha_k(0) + \int_0^T [w(s) - \alpha_k(s)]\,ds \le 0,$$

$$q(T) = w(T) - \alpha_{k+1}(T) = \beta_k(0) - w(0) + \int_0^T [\beta_k(s) - w(s)]\,ds \le 0.$$

From assumption (H_3), we have

$$\begin{aligned} p'(t) &= f(t, \beta_k(t)) - f(t, w(t)) + M(\beta_{k+1}(t) - \beta_k(t)) \ge \\ &\quad - M(w(t) - \beta_k(t)) + M(\beta_{k+1}(t) - \\ &\quad \beta_k(t)) = Mp(t) \quad t \in J, \\ q'(t) &= f(t, w(t)) - f(t, \alpha_k(t)) - M(\alpha_{k+1}(t) - \alpha_k(t)) \ge \\ &\quad - M(\alpha_k(t) - w(t)) - M(\alpha_{k+1}(t) - \\ &\quad \alpha_k(t)) = Mq(t) \quad t \in J. \end{aligned}$$

Hence $p(t) \le e^{-M(T-t)}p(T) \le 0$, $q(t) \le e^{-M(T-t)}q(T) \le 0$, $t \in J$ showing that $\beta_{k+1}(t) \le w(t) \le \alpha_{k+1}(t)$, $t \in J$. It proves, by induction, that $\beta_n(t) \le w(t) \le \alpha_n(t)$, $t \in J$ for all n. Taking the limit $n \to \infty$, we have $\alpha(t) = w(t) = \beta(t)$, $t \in J$, so the assertion of theorem 2 is true. The proof is complete.

4 Examples

Example 1 Consider the equation

$$\begin{cases} xc(t) = (x(t) + 5 - e^{-2t})\sin x(t) - \dfrac{t}{100} & 0 < t < 1, \\ x(0) + \dfrac{1}{4Q_0} \overset{1}{} x(s)\,ds = x(1). \end{cases} \quad (8)$$

It is easy to see that $A_0 = t + 1$, $B_0 = 0$ are lower and upper solution of (8) respectively. $f(t, u) = (u(t) + 5 - e^{-2t})\sin u - \frac{t}{100}$, $f_u = \sin u + (u(t) + 5 - e^{-2t})\cos u < 9$ for $0 [u [t + 1, 0 [t [1$. $f(t, y) - f(t, x) < 9(y - x)$ for $0 [x [y [t + 1$. (H_3) is satisfied. By theorem 1, (8) has at least a solution $x(t): 0 [x(t) [t + 1$.

Example 4.2 Consider the equation

$$\begin{cases} xc(t) = x^2(t) - 1 + t\sin x(t) & 0 < t < 1, \\ x(0) = -x(1). \end{cases} \quad (9)$$

$A_0 = 1$, $B_0 = -1$ are coupled lower and upper solution of (9) respectively. $f(t, u) = u^2 - 1 + t\sin u$, $f_u = 2u + t\cos u [3$ for $-1 [u [1, 0 [t [1$, $f(t, y) - f(t, x) [3(y - x)$ for $-1 [x [y [1$. (H_3) is satisfied. Further, $1 - e^{Q f_u(s, N(s))\,ds} \; x \; 0$, (H_5) is satisfied. By theorem 2, (9) has at least a solution $x(t): -1 [x(t) [1$.

References:

[1] LAKSHMIKANTHAM V, SHAHZAD N, NIETO J J. Methods of Generalized Quasilinearization for Geriodic Boundary Value Groblems [J]. Nonlinear Anal., 1996, 27: 143- 151.

[2] NIETO J J, ALVAREZ-NORIEGA N. Periodic Boundary Problems for Nonlinear First Order Ordinary Differential Equations [J]. Acta. Math. Hunfar, 1996, 71: 49- 58.

[3] YIN Y. Monotone Iterative Technique and Quasilinearization for Some Anti-Periodic Problems [J]. Nonlinear World, 1996, 3: 253- 266.

[4] CHEN Y Q. On Masserac s Theorem for Anti-Periodic Solution [J]. Acdv. Math. Sci. Appl., 1999, 9: 125- 128.

[5] FRANCO D, NIETO J J, Oc REGAN D. Anti-Periodic Boundary Value Problem for Nonlinear First Order Ordinary Differential Equations [J]. Math. Inequal. Appl., 2003, 6: 477- 485.

[6] LADDE G S, LAKSHMIKANTHAM V, VATSALA A S. Monotone Iterative Techniques for Nonlinear Differential Equations [M]. Pitman, London, 1985.

[7] ZUO W, JIANG D, Oc REGAN D, et al. Optimal Existence Conditions for the Periodic Delay j-Laplace Equation with Upper and Lower Solutions in the Reverse Order [J]. Results Math., 2003, 44: 375- 385.

[8] HE Z, HE X. Monotone Iterative Technique for Impulsive Integro-Differential Equations with Periodic Boundary Conditions [J]. Comput. Math. Appl., 2004, 48: 73- 84.

[9] RACHUNKOVA I, TVRDY M. Non-Order Lower and Upper Function in Second Order Impulsive Periodic Problems [J]. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 2005, 12: 397- 415.

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