CONDITIONAL PROBABILITIES OF MULTIVARIATE POISSON DISTRIBUTIONS

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Abstract. Multivariate Poisson distributions have numerous applications. Fast computation of these distributions, holding constant a fixed set of linear combinations of these variables, has been explored by Sontag and Zeilberger [SZ10]. This elaborates on their work.

1. Introduction

Set $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Let $X_1, \ldots, X_n$ be independent Poisson random variables with $X_i \sim Poisson(\lambda_i)$. Let $\mathbf{A} \in \mathbf{Mat}_{m\times n}(\mathbb{N})$. Define new random variables $Y_1, \ldots, Y_m$ by taking linear combinations of the X_i:

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} = \mathbf{A} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

Then the joint p.d.f for $Y_1, \ldots, Y_m$ is given:

$$\begin{aligned} F_{\mathbf{Y}}(b_1, \ldots, b_m) &= \mathbb{P}(Y_1 = b_1, \ldots, Y_m = b_m) \\ &= \sum_{\substack{(k_1,\ldots,k_n)\in\mathbb{N}^n \\ (b_1,\ldots,b_m)^T = \mathbf{A}(k_1,\ldots,k_n)^T}} \frac{\lambda_1^{k_1}}{k_1!}\frac{\lambda_2^{k_2}}{k_2!}\cdots\frac{\lambda_n^{k_n}}{k_n!}\; e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)} \end{aligned}$$

Sontag and Zeilberger demonstrated that the multivariate generating function of $\mathbf{Y}$:

$$G_{\mathbf{Y}}(z_1, \ldots, z_m) = \sum_{(b_1,\ldots,b_m)\in\mathbb{N}^m} F_{\mathbf{Y}}(b_1, \ldots, b_m)\; z_1^{b_1} \cdots z_m^{b_m}$$

may be simplified [SZ10] (*cf.* [Bur12]):

$$G_{\mathbf{Y}}(z_1, \ldots, z_m) = \exp\left(\sum_{j=1}^{n} \lambda_j \prod_{i=1}^{m} z_i^{a_{ij}} \right)$$

Certain conditions can be placed on the matrix $\mathbf{A}$ to guarantee that the joint p.d.f. $F_{\mathbf{Y}}$ can be expressed as the sum over a single index (and thus practicably computed with Wilf-Zeilberger recurrences [WZ92, AZ06]). In particular, Sontag and Zeilberger established that $\mathbf{A} \in \mathbf{Mat}_{2\times n}(\{0, 1\})$ is a sufficient criterion for

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reduction to a single-index sum [SZ10]. This paper will explore another sufficient criterion on the matrix $\mathbf{A}$ to guarantee that the joint p.d.f. can be expressed as the sum over a single index.

2. Smith Normal Form

Let $\mathbf{A} \in \mathbf{Mat}_{m\times n}(\mathbb{N})$. Due to the nature of the problem, it may be assumed without loss that the rows of $\mathbf{A}$ are linearly independent. In particular, this imposes $m \leq n$.

Any matrix $\mathbf{A} \in \mathbf{Mat}_{m\times n}(\mathbb{Z})$ can be decomposed into its Smith normal form [Jac09, pp. 181-185]: i.e.: there exist invertible (over $\mathbb{Z}$) matrices $\mathbf{P} \in \mathbf{Mat}_{m\times m}(\mathbb{Z})$ and $\mathbf{Q} \in \mathbf{Mat}_{n\times n}(\mathbb{Z})$ such that:

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}_{m\times n} = \mathbf{PAQ}$$

where r is the rank of $\mathbf{A}$, $d_1, \dots, d_r \neq 0$, and $d_1 \mid \cdots \mid d_r$. These d_i, $1 \leq i \leq r$ are unique up to sign-change and are called elementary divisors. For $1 \leq i \leq r$, let Δ_i denote the g.c.d. of the i-rowed minors of $\mathbf{A}$. Then $d_1 = \pm\Delta_1$, $d_2 = \pm\Delta_2\Delta_1^{-1}$, ..., $d_r = \pm\Delta_r\Delta_{r-1}^{-1}$.

Theorem. *If the matrix $\mathbf{A} \in \mathbf{Mat}_{m\times n}(\mathbb{N})$ has rank $r = m = n-1$ and elementary divisors $d_i = \pm 1$ for all $1 \leq i \leq r$, then $F_{\mathbf{Y}}(k_1, \dots, k_m)$ can be written as the sum over a single index.*

Proof. Note that:

$$F_{\mathbf{Y}}(b_1, \dots, b_m) = \sum_{\substack{(k_1,\dots,k_n)\in\mathbb{N}^n \\ (b_1,\dots,b_m)^T = \mathbf{A}(k_1,\dots,k_n)^T}} \frac{\lambda_1^{k_1}}{k_1!}\frac{\lambda_2^{k_2}}{k_2!}\cdots\frac{\lambda_n^{k_n}}{k_n!}\, e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)}$$

By hypothesis, there exist invertible matrices $\mathbf{P} \in \mathbf{Mat}_{n-1\times n-1}(\mathbb{Z})$ and $\mathbf{Q} \in \mathbf{Mat}_{n\times n}(\mathbb{Z})$ such that:

$$\left(\mathbf{I}_{n-1}\middle|\mathbf{0}_{n\times 1}\right) = \mathbf{PAQ}$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)\times(n-1)$ identity matrix. Then $(b_1, \dots, b_m)^T = \mathbf{A}(k_1, \dots, k_n)^T$ may be rewritten:

$$\mathbf{P}^{-1}\left(\mathbf{I}_{n-1}\middle|\mathbf{0}_{n\times 1}\right)\mathbf{Q}^{-1}\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

so that the first $n-1$ entries of the vector $\mathbf{Q}^{-1}(k_1, \dots, k_m)^T$ agree with the $(n-1)\times 1$ vector $\mathbf{P}(b_1, \dots, b_m)^T$. In particular, by introducing $j \in \mathbb{N}$ as the index of summation it follows that the full set of solutions to $\mathbf{A}(k_1, \dots, k_n)^T = (b_1, \dots, b_m)^T$ for $(k_1, \dots, k_n) \in \mathbb{N}^n$ is given:

$$\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \mathbf{Q}\begin{pmatrix} P_{11} & \cdots & & P_{1n} \\ \vdots & & & \vdots \\ P_{m1} & \cdots & & P_{mn} \\ 0 & \cdots & 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ j \end{pmatrix}$$

Substituting $(k_1, \dots, k_n)$ as given above yields an expression for $F_{\mathbf{Y}}(b_1, \dots, b_m)$ as the summation over a single index j. □

Example. Let:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Then:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

So that $\mathbf{A}(k_1, \dots, k_n)^T = (b_1, \dots, b_m)^T$ has solutions:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \vec{b} \\ j \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ j \end{pmatrix} = \begin{pmatrix} b_1 - b_2 + 2j \\ j \\ b_2 - 2j \end{pmatrix}$$

So that:

$$F_{\mathbf{Y}}(b_1, b_2) = \sum_{\substack{(k_1,k_2,k_3) \in \mathbb{N}^3 \\ (b_1,b_2)^T = \mathbf{A}(k_1,k_2,k_3)^T}} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \frac{\lambda_3^{k_3}}{k_3!} \; e^{-(\lambda_1+\lambda_2+\lambda_3)}$$

$$= \sum_j \frac{\lambda_1^{b_1-b_2+2j} \lambda_2^j \lambda_3^{b_2-2j}}{(b_1 - b_2 + 2j)! j! (b_2 - 2j)!} e^{-(\lambda_1+\lambda_2+\lambda_3)}$$

Example. Let:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 5 & 16 & 12 & 17 \\ 3 & 16 & 21 & 56 \end{pmatrix}$$

Then:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 5 & 16 & 12 & 17 \\ 3 & 16 & 21 & 56 \end{pmatrix} \begin{pmatrix} 144 & -31 & 4 & 15 \\ -69 & 15 & -2 & -5 \\ 32 & -7 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So that $\mathbf{A}(k_1, \dots, k_4)^T = (b_1, \dots, b_3)^T$ has solutions:

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \mathbf{Q} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ j \end{pmatrix} = \begin{pmatrix} 144 & -31 & 4 & 15 \\ -69 & 15 & -2 & -5 \\ 32 & -7 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ j \end{pmatrix} = \begin{pmatrix} 144b_1 - 31b_2 + 4b_3 + 15j \\ -69b_1 + 15b_2 - 2b_3 - 5j \\ 32b_1 - 7b_2 + b_3 - j \\ j \end{pmatrix}$$

So that:

$$F_{\mathbf{Y}}(b_1, b_2, b_3) = \sum_{\substack{(k_1,k_2,k_3,k_4) \in \mathbb{N}^4 \\ (b_1,b_2,b_3)^T = \mathbf{A}(k_1,k_2,k_3,k_4)^T}} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \frac{\lambda_3^{k_3}}{k_3!} \frac{\lambda_4^{k_4}}{k_4!} \; e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$$

$$= \sum_j \frac{\lambda_1^{144b_1-31b_2+4b_3+15j} \lambda_2^{-69b_1+15b_2-2b_3-5j} \lambda_3^{32b_1-7b_2+b_3-j} \lambda_4^j e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}}{(144b_1 - 31b_2 + 4b_3 + 15j)!(-69b_1 + 15b_2 - 2b_3 - 5j)!(32b_1 - 7b_2 + b_3 - j)! j!}$$

3. Special Case: A is invertible

Proposition 1. *If the matrix* $\mathbf{A} \in \mathbf{Mat}_{n \times n}(\mathbb{Z}_{\geq 0})$ *is invertible, then* $F_{\mathbf{Y}}(\vec{b})$ *can be expressed explicitly.*

Proof. Note that:

$$F_{\mathbf{Y}}(\vec{b}) = \sum_{\substack{\vec{k} \in \mathbf{Mat}_{n\times 1}(\mathbb{Z}_{\geq 0}) \\ \mathbf{A}\vec{k} = \vec{b}}} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \cdots \frac{\lambda_n^{k_n}}{k_n!} \; e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)}$$

But this sum is taken over the singleton:

$$\{\vec{k} \in \mathbf{Mat}_{n\times 1}(\mathbb{Z}_{\geq 0}) : \mathbf{A}\vec{k} = \vec{b}\} = \{\vec{k} \in \mathbf{Mat}_{n\times 1} : \vec{k} = A^{-1}\vec{b}\}$$

□

Example. Let:

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 10 & 5 \\ 0 & 1 & 8 \end{pmatrix}$$

Then $\det(\mathbf{A}) = 1$ and for any $\vec{b} = (b_1, b_2, b_3)^T \in \mathbf{Mat}_{3\times 1}(\mathbb{Z}_{\geq 0})$, the unique solution to $\mathbf{A}\vec{k} = \vec{b}$ for $\vec{k} \in \mathbf{Mat}_{n\times 1}(\mathbb{Z}_{\geq 0})$ is given:

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \mathbf{A}^{-1}\vec{b} = \begin{pmatrix} 75 & -37 & -5 \\ -16 & 8 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 75b_1 - 37b_2 - 5b_3 \\ -16b_1 + 8b_2 + b_3 \\ 2b_1 - b_2 \end{pmatrix}$$

Thus the explicit probability density function for $\mathbf{Y}$ is given:

$$F_{\mathbf{Y}}(\vec{b}) = \frac{\lambda_1^{75b_1 - 37b_2 - 5b_3} \lambda_2^{-16b_1 + 8b_2 + b_3} \lambda_3^{2b_1 - b_2} e^{-(\lambda_1 + \lambda_2 + \lambda_3)}}{(75b_1 - 37b_2 - 5b_3)!(-16b_1 + 8b_2 + b_3)!(2b_1 - b_2)!}$$

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