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Calculation of Some Expected Values for Parameterized Mean Model with Gaussian Noise

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Consider the measurement model

$$y = g(x) + v \tag{1}$$

where

- $x \in \mathbb{R}$ is the unknown scalar we would like to estimate described by the prior distribution $\mathcal{N}(x; 0; \sigma_x^2)$ where the notation $\mathcal{N}(\,\cdot\,; \bar{x}, \Sigma_x)$ denotes a (real-variate) Gaussian density with mean $\bar{x}$ and covariance Σ_x.
- $y \in \mathbb{R}^{n_y}$ is, in general, a complex measurement vector;
- $g(\,\cdot\,) : \mathbb{R} \to \mathbb{C}^{n_y}$ is, in general, a complex-valued observation function;
- $v \in \mathbb{C}^{n_y}$ is circular symmetric complex Gaussian measurement noise with zero-mean and covariance $\sigma_v^2 I_{n_y}$ where I_{n_y} denotes an identity matrix of size $n_y \times n_y$. The noise v is assumed independent of x.

In this document we are going to derive analytical formulae for the following functions.

$$\mu_{y,x}(s_1, s_2, h_1, h_2) \triangleq E\Big[\min(L^{s_1}(y, x+h_1, x), 1)\min(L^{s_2}(y, x+h_2, x), 1)\Big], \tag{2}$$

$$\mu_{y|x}(s_1, s_2, h_1, h_2, x) \triangleq E_y\Big[\min(L_1^{s_1}(y, x+h_1, x), 1)\min(L_1^{s_2}(y, x+h_2, x), 1)\Big| x\Big], \tag{3}$$

$$\mu_{x}(s_1, s_2, h_1, h_2) \triangleq E_x\Big[\min(L_2^{s_1}(x+h_1, x), 1)\min(L_2^{s_2}(x+h_2, x), 1)\Big]. \tag{4}$$

where $s_1, s_2 \in \mathbb{Z}$, $h_1, h_2 \in \mathbb{R}^{n_x}$ and

$$L(y, x, \xi) \triangleq \frac{p_{\tilde{y},\tilde{x}}(y, x)}{p_{\tilde{y},\tilde{x}}(y, \xi)}, \qquad L_1(y, x, \xi) \triangleq \frac{p_{\tilde{y}|\tilde{x}}(y|x)}{p_{\tilde{y}|\tilde{x}}(y|\xi)}, \qquad L_2(x, \xi) \triangleq \frac{p_{\tilde{x}}(x)}{p_{\tilde{x}}(\xi)}. \tag{5}$$

1 Calculation of $\mu_{y,x}(s_1, s_2, h_1, h_2)$

We first calculate $L(y, x+h, x)$ as

$$L(y, x+h, x) = \frac{\mathcal{CN}(y; g(x+h), \sigma_v^2 I_{n_y})\mathcal{N}(x+h; 0; \sigma_x^2)}{\mathcal{CN}(y; g(x), \sigma_v^2 I_{n_y})\mathcal{N}(x; 0; \sigma_x^2)} \tag{6}$$

$$= \exp\left[\frac{2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}} d(x, h)\right\} - b(x, h)\right] \tag{7}$$

where the notation $\mathcal{CN}(y; \bar{y}, \Sigma_y)$ denotes a circular symmetric complex Gaussian density with mean $\bar{y}$ and covariance Σ_y and

$$d(x, h) \triangleq g(x+h) - g(x), \tag{8}$$

$$b(x, h) \triangleq \frac{1}{\sigma_v^2}\|d(x, h)\|^2 + \frac{1}{\sigma_x^2}x^{\mathrm{T}} h + \frac{1}{2\sigma_x^2}\|h\|^2. \tag{9}$$

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Then we have

$$\begin{aligned}
&\mu_{y,x}(s_1,s_2,h_1,h_2)\\
&=\int\int \min\left(\exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\}-s_1b(x,h_1)\right],1\right)\\
&\quad\times\min\left(\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\}-s_2b(x,h_2)\right],1\right)p(v)\,\mathrm{d}v p(x)\,\mathrm{d}x. \qquad (10)
\end{aligned}$$

We are first going to handle the inner integral on the right hand side of (10) which we call as $\mathcal{I}_1$ as follows.

$$\begin{aligned}
\mathcal{I}_1 \triangleq &\int \min\left(\exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}-s_1b(x,h_1)\right],1\right)\\
&\times\min\left(\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}-s_2b(x,h_2)\right],1\right)p(v)\,\mathrm{d}v\\
=&\int_{\mathcal{V}_1}p(v)\,\mathrm{d}v\\
&+\int_{\mathcal{V}_2}\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}-s_2b(x,h_2)\right]p(v)\,\mathrm{d}v\\
&+\int_{\mathcal{V}_3}\exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}-s_1b(x,h_1)\right]p(v)\,\mathrm{d}v\\
&+\int_{\mathcal{V}_4}\exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}-s_1b(x,h_1)\right]\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}-s_2b(x,h_2)\right]p(v)\,\mathrm{d}v \qquad (11)
\end{aligned}$$

where the sets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ are defined as follows.

$$\mathcal{V}_1 \triangleq \left\{v\in\mathbb{C}^{n_y}\middle|\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}\geq\frac{\sigma_v^2b(x,h_1)}{2}\&\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}\geq\frac{\sigma_v^2b(x,h_2)}{2}\right\}, \qquad (12a)$$

$$\mathcal{V}_2 \triangleq \left\{v\in\mathbb{C}^{n_y}\middle|\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}\geq\frac{\sigma_v^2b(x,h_1)}{2}\&\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}<\frac{\sigma_v^2b(x,h_2)}{2}\right\}, \qquad (12b)$$

$$\mathcal{V}_3 \triangleq \left\{v\in\mathbb{C}^{n_y}\middle|\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}<\frac{\sigma_v^2b(x,h_1)}{2}\&\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}\geq\frac{\sigma_v^2b(x,h_2)}{2}\right\}, \qquad (12c)$$

$$\mathcal{V}_4 \triangleq \left\{v\in\mathbb{C}^{n_y}\middle|\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_1)\big\}<\frac{\sigma_v^2b(x,h_1)}{2}\&\mathcal{R}\big\{v^{\mathrm{H}}d(x,h_2)\big\}<\frac{\sigma_v^2b(x,h_2)}{2}\right\}. \qquad (12d)$$

Substituting (9) and the identity $p(v)=\mathcal{CN}(v;0,\sigma_v^2I_{n_y})$ into (11), we get

$$\begin{aligned}
I_1 =&P\big(v\in\mathcal{V}_1\big|v\sim\mathcal{CN}(v;0,\sigma_v^2)\big)\\
&+\exp\left[\frac{s_2^2}{\sigma_v^2}\|d(x,h_2)\|^2-s_2b(x,h_2)\right]P\big(v\in\mathcal{V}_2\big|v\sim\mathcal{CN}(v;s_2d(x,h_2),\sigma_v^2)\big)\\
&+\exp\left[\frac{s_1^2}{\sigma_v^2}\|d(x,h_1)\|^2-s_1b(x,h_1)\right]P\big(v\in\mathcal{V}_3\big|v\sim\mathcal{CN}(v;s_1d(x,h_1),\sigma_v^2)\big)\\
&+\exp\left[\frac{1}{\sigma_v^2}\|s_1d(x,h_1)+s_2d(x,h_2)\|^2-s_1b(x,h_1)-s_2b(x,h_2)\right]\\
&\times P\big(v\in\mathcal{V}_4\big|v\sim\mathcal{CN}(v;s_1d(x,h_1)+s_2d(x,h_2),\sigma_v^2)\big). \qquad (13)
\end{aligned}$$

We now define the real scalars a_1 and a_2 as $a_1\triangleq\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\}$ and $a_2\triangleq\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\}$. Since each of the probabilities on the right hand side of (13) are conditioned on v being distributed with a circular symmetric complex Gaussian density and since a_1, a_2 are linearly dependent on v, we have the vector $a\triangleq[a_1,a_2]^{\mathrm{T}}$ distributed with a Gaussian density which gives

$$I_1 =P\left(\begin{smallmatrix}a_1\geq\frac{\sigma_v^2b(x,h_1)}{2}\\ a_2\geq\frac{\sigma_v^2b(x,h_2)}{2}\end{smallmatrix}\middle|a\sim\mathcal{N}\left(a;\bar{a}_1'(x,h_1,h_2),\frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\right)$$

$$+\exp\left[\frac{s_2^2}{\sigma_v^2}\|d(x,h_2)\|^2 - s_2 b(x,h_2)\right] P\left(\substack{a_1 \geq \frac{\sigma_v^2 b(x,h_1)}{2} \\ a_2 < \frac{\sigma_v^2 b(x,h_2)}{2}} \middle| a \sim \mathcal{N}\left(a; \bar{a}_2'(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\right)$$

$$+\exp\left[\frac{s_1^2}{\sigma_v^2}\|d(x,h_1)\|^2 - s_1 b(x,h_1)\right] P\left(\substack{a_1 < \frac{\sigma_v^2 b(x,h_1)}{2} \\ a_2 \geq \frac{\sigma_v^2 b(x,h_2)}{2}} \middle| a \sim \mathcal{N}\left(a; \bar{a}_3'(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\right)$$

$$+\exp\left[\frac{1}{\sigma_v^2}\|s_1 d(x,h_1) + s_2 d(x,h_2)\|^2 - s_1 b(x,h_1) - s_2 b(x,h_2)\right]$$

$$\times P\left(\substack{a_1 < \frac{\sigma_v^2 b(x,h_1)}{2} \\ a_2 < \frac{\sigma_v^2 b(x,h_2)}{2}} \middle| a \sim \mathcal{N}\left(a; \bar{a}_4'(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\right) \tag{14}$$

where

$$\bar{a}_1'(x,h_1,h_2) \triangleq [0,0]^{\mathrm{T}}, \tag{15a}$$

$$\bar{a}_2'(x,h_1,h_2) \triangleq s_2 \left[\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}, \|d(x,h_2)\|^2\right]^{\mathrm{T}}, \tag{15b}$$

$$\bar{a}_3'(x,h_1,h_2) \triangleq s_1 \left[\|d(x,h_1)\|^2, \mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\right]^{\mathrm{T}}, \tag{15c}$$

$$\bar{a}_4'(x,h_1,h_2) \triangleq \left[s_1\|d(x,h_1)\|^2 + s_2\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}, s_2\|d(x,h_2)\|^2 + s_1\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\right]^{\mathrm{T}}, \tag{15d}$$

$$\Gamma(x,h_1,h_2) \triangleq \begin{bmatrix} \|d(x,h_1)\|^2 & \mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\} \\ \mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\} & \|d(x,h_2)\|^2 \end{bmatrix}. \tag{16}$$

Each of the probabilities on the right hand side of (14) can be written using the cumulative distribution function $\mathcal{N}\mathrm{cdf}_2(\cdot,\cdot,\cdot)$ of a bivariate Gaussian random variable to give

$$I_1 = \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_1(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)$$

$$+\exp\left[\frac{s_2^2}{\sigma_v^2}\|d(x,h_2)\|^2 - s_2 b(x,h_2)\right]$$

$$\times \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_2(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)$$

$$+\exp\left[\frac{s_1^2}{\sigma_v^2}\|d(x,h_1)\|^2 - s_1 b(x,h_1)\right]$$

$$\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_3(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)$$

$$+\exp\left[\frac{1}{\sigma_v^2}\|s_1 d(x,h_1) + s_2 d(x,h_2)\|^2 - s_1 b(x,h_1) - s_2 b(x,h_2)\right]$$

$$\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_4(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right) \tag{17}$$

where

$$\bar{a}_1(x,h_1,h_2) \triangleq -\bar{a}_1'(x,h_1,h_2), \tag{18a}$$

$$\bar{a}_2(x,h_1,h_2) \triangleq \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bar{a}_2'(x,h_1,h_2), \tag{18b}$$

$$\bar{a}_3(x,h_1,h_2) \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{a}_3'(x,h_1,h_2), \tag{18c}$$

$$\bar{a}_4(x,h_1,h_2) \triangleq \bar{a}_4'(x,h_1,h_2), \tag{18d}$$

$$\overline{\Gamma}(x,h_1,h_2) \triangleq \begin{bmatrix} \|d(x,h_1)\|^2 & -\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\} \\ -\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\} & \|d(x,h_2)\|^2 \end{bmatrix}. \tag{19}$$

Using (9) in (17), we get

$$\begin{aligned}
I_1 =& \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_1(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 - \frac{s_2}{\sigma_x^2}x^{\mathrm{T}}h_2 - \frac{s_2}{2\sigma_x^2}\|h_2\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_2(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 - \frac{s_1}{\sigma_x^2}x^{\mathrm{T}}h_1 - \frac{s_1}{2\sigma_x^2}\|h_1\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_3(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 + \frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{2s_1 s_2}{\sigma_v^2}\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\right. \\
&\left. - \frac{s_1 h_1 + s_2 h_2}{\sigma_x^2}x - \frac{s_1}{2\sigma_x^2}\|h_1\|^2 - \frac{s_2}{2\sigma_x^2}\|h_2\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_4(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right) \qquad (20) \\
=& \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_1(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{s_2^2 - s_2}{2\sigma_x^2}\|h_2\|^2\right]\exp\left[-\frac{s_2}{\sigma_x^2}x^{\mathrm{T}}h_2 - \frac{s_2^2}{2\sigma_x^2}\|h_2\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[-\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_2(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 + \frac{s_1^2 - s_1}{2\sigma_x^2}\|h_1\|^2\right]\exp\left[-\frac{s_1}{\sigma_x^2}x^{\mathrm{T}}h_1 - \frac{s_1^2}{2\sigma_x^2}\|h_1\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, -\frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_3(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right) \\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 + \frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{2s_1 s_2}{\sigma_v^2}\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\right] \\
&\times \exp\left[\frac{s_1^2 - s_1}{2\sigma_x^2}\|h_1\|^2 + \frac{s_2^2 - s_2}{2\sigma_x^2}\|h_2\|^2 + \frac{s_1 s_2}{\sigma_x^2}h_1^{\mathrm{T}}h_2\right]\exp\left[-\frac{s_1 h_1 + s_2 h_2}{\sigma_x^2}x - \frac{1}{2\sigma_x^2}\|s_1 h_1 + s_2 h_2\|^2\right] \\
&\times \mathcal{N}\mathrm{cdf}_2\left(\left[\frac{\sigma_v^2 b(x,h_1)}{2}, \frac{\sigma_v^2 b(x,h_2)}{2}\right]^{\mathrm{T}}; \bar{a}_4(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right). \qquad (21)
\end{aligned}$$

Substituting the result (21) into (10) and carrying out straightforward algebra, we obtain

$$\begin{aligned}
\mu_{y,x}(s_1,s_2,h_1,h_2) =& E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h_1)}{2} \\ -\frac{\sigma_v^2 b(x,h_2)}{2}\end{bmatrix}; \bar{a}_1(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\right] \\
&+ E_{\mathcal{N}(x;-s_2 h_2,\sigma_x^2)}\left[\exp\left[\frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{s_2^2 - s_2}{2\sigma_x^2}\|h_2\|^2\right]\right. \\
&\left.\times \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h_1)}{2} \\ \frac{\sigma_v^2 b(x,h_2)}{2}\end{bmatrix}; \bar{a}_2(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)\right]
\end{aligned}$$

$$+ E_{\mathcal{N}(x;-s_1h_1,\sigma_x^2)}\Bigg[\exp\left[\frac{s_1^2-s_1}{\sigma_v^2}\|d(x,h_1)\|^2+\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2\right]$$

$$\times\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}\frac{\sigma_v^2 b(x,h_1)}{2}\\ -\frac{\sigma_v^2 b(x,h_2)}{2}\end{bmatrix};\bar{a}_3(x,h_1,h_2),\frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)\Bigg]$$

$$+ E_{\mathcal{N}(x;-(s_1h_1+s_2h_2),\sigma_x^2)}\Bigg[\exp\Bigg[\frac{s_1^2-s_1}{\sigma_v^2}\|d(x,h_1)\|^2+\frac{s_2^2-s_2}{\sigma_v^2}\|d(x,h_2)\|^2$$

$$+\frac{2s_1s_2}{\sigma_v^2}\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\Bigg]\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2+\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2+\frac{s_1s_2}{\sigma_x^2}h_1^{\mathrm{T}}h_2\right]$$

$$\times\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}\frac{\sigma_v^2 b(x,h_1)}{2}\\ \frac{\sigma_v^2 b(x,h_2)}{2}\end{bmatrix};\bar{a}_4(x,h_1,h_2),\frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\Bigg]. \tag{22}$$

2 Calculation of $\mu_{y|x}(s_1,s_2,h_1,h_2,x)$ and $\mu_x(s_1,s_2,h_1,h_2)$

We first calculate $L_1(\cdot,\cdot,\cdot)$ and $L_2(\cdot,\cdot)$ as follows.

$$L_1(y,x+h,x)=\frac{\exp\left[-\frac{1}{\sigma_v^2}\|y-g(x+h)\|^2\right]}{\exp\left[-\frac{1}{\sigma_v^2}\|y-g(x)\|^2\right]} \tag{23}$$

$$=\exp\left[\frac{2}{\sigma_v^2}\mathcal{R}\left\{y^{\mathrm{H}}d(x,h)\right\}-\frac{1}{\sigma_v^2}\|g(x+h)\|^2+\frac{1}{\sigma_v^2}\|g(x)\|^2\right] \tag{24}$$

$$=\exp\left[\frac{2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h)\right\}-\frac{1}{\sigma_v^2}\|d(x,h)\|^2\right] \tag{25}$$

$$=\exp\left[\frac{2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h)\right\}-b_1(x,h)\right], \tag{26}$$

$$L_2(x+h,x)=\frac{\exp\left[-\frac{1}{2\sigma_x^2}\|x+h\|^2\right]}{\exp\left[-\frac{1}{2\sigma_x^2}\|x\|^2\right]} \tag{27}$$

$$=\exp\left[-\frac{1}{\sigma_x^2}x^{\mathrm{T}}h-\frac{1}{2\sigma_x^2}\|h\|^2\right]=\exp\left[-b_2(x,h)\right], \tag{28}$$

where

$$b_1(x,h)\triangleq\frac{1}{\sigma_v^2}\|d(x,h)\|^2, \tag{29}$$

$$b_2(x,h)\triangleq\frac{1}{\sigma_x^2}x^{\mathrm{T}}h+\frac{1}{2\sigma_x^2}\|h\|^2. \tag{30}$$

Then the function $\mu_{y|x}(\cdot,\cdot,\cdot,\cdot,\cdot)$ is given as

$$\mu_{y|x}(s_1,s_2,h_1,h_2,x)$$

$$=\int\min\left(\exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\}-s_1b_1(x,h_1)\right],1\right)$$

$$\times\min\left(\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\}-s_2b_1(x,h_2)\right],1\right)p(v)\,\mathrm{d}v \tag{31}$$

$$=\int_{\mathcal{V}_1}p(v)\,\mathrm{d}v$$

$$+\int_{\mathcal{V}_2}\exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\}-s_2b_1(x,h_2)\right]p(v)\,\mathrm{d}v$$

$$+\int_{\mathcal{V}_3} \exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} - s_1 b_1(x,h_1)\right] p(v)\,\mathrm{d}v$$

$$+\int_{\mathcal{V}_4} \exp\left[\frac{2s_1}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} - s_1 b_1(x,h_1)\right] \exp\left[\frac{2s_2}{\sigma_v^2}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} - s_2 b_1(x,h_2)\right] p(v)\,\mathrm{d}v \quad (32)$$

where the sets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ are defined as follows.

$$\mathcal{V}_1 \triangleq \left\{v \in \mathbb{C}^{n_y} \middle| \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} \geq \tfrac{\sigma_v^2 b_1(x,h_1)}{2} \,\&\, \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} \geq \tfrac{\sigma_v^2 b_1(x,h_2)}{2}\right\}, \quad (33a)$$

$$\mathcal{V}_2 \triangleq \left\{v \in \mathbb{C}^{n_y} \middle| \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} \geq \tfrac{\sigma_v^2 b_1(x,h_1)}{2} \,\&\, \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} < \tfrac{\sigma_v^2 b_1(x,h_2)}{2}\right\}, \quad (33b)$$

$$\mathcal{V}_3 \triangleq \left\{v \in \mathbb{C}^{n_y} \middle| \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} < \tfrac{\sigma_v^2 b_1(x,h_1)}{2} \,\&\, \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} \geq \tfrac{\sigma_v^2 b_1(x,h_2)}{2}\right\}, \quad (33c)$$

$$\mathcal{V}_4 \triangleq \left\{v \in \mathbb{C}^{n_y} \middle| \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} < \tfrac{\sigma_v^2 b_1(x,h_1)}{2} \,\&\, \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} < \tfrac{\sigma_v^2 b_1(x,h_2)}{2}\right\}. \quad (33d)$$

Substituting (29) and the identity $p(v) = \mathcal{CN}(v; 0, \sigma_v^2 I_{n_y})$ into (32), we get

$$\begin{aligned}
&\mu_{y|x}(s_1,s_2,h_1,h_2,x)\\
&= P\left(\begin{matrix}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} \geq \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} \geq \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{matrix} \middle| v \sim \mathcal{CN}(v; 0, \sigma_v^2 I_{n_y})\right) && (34)\\
&+ \exp\left[\frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2\right] P\left(\begin{matrix}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} \geq \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} < \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{matrix} \middle| v \sim \mathcal{CN}(v; s_2 d(x,h_2), \sigma_v^2 I_{n_y})\right)\\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2\right] P\left(\begin{matrix}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} < \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} \geq \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{matrix} \middle| v \sim \mathcal{CN}(v; s_1 d(x,h_1), \sigma_v^2 I_{n_y})\right)\\
&\quad + \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 + \frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{2s_1 s_2}{\sigma_v^2}\mathcal{R}\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\}\right]\\
&\quad \times P\left(\begin{matrix}\mathcal{R}\left\{v^{\mathrm{H}}d(x,h_1)\right\} < \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \mathcal{R}\left\{v^{\mathrm{H}}d(x,h_2)\right\} < \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{matrix} \middle| v \sim \mathcal{CN}(v; s_1 d(x,h_1) + s_2 d(x,h_2), \sigma_v^2 I_{n_y})\right). && (35)
\end{aligned}$$

Continuing in the same way as in Section 1, we get

$$\begin{aligned}
\mu_{y|x}(s_1,s_2,h_1,h_2,x) =& \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h_1)}{2}\\ -\frac{\sigma_v^2 b_1(x,h_2)}{2}\end{bmatrix}; \bar{a}_1(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right)\\
&+ \exp\left[\frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2\right]\\
&\times \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{bmatrix}; \bar{a}_2(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)\\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2\right.\\
&\times \left.\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ -\frac{\sigma_v^2 b_1(x,h_2)}{2}\end{bmatrix}; \bar{a}_3(x,h_1,h_2), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h_1,h_2)\right)\right]\\
&+ \exp\left[\frac{s_1^2 - s_1}{\sigma_v^2}\|d(x,h_1)\|^2 + \frac{s_2^2 - s_2}{\sigma_v^2}\|d(x,h_2)\|^2 + \frac{2s_1 s_2}{\sigma_v^2}\mathcal{R}\left\{d^{\mathrm{H}}(x,h_1)d(x,h_2)\right\}\right]\\
&\times \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h_1)}{2}\\ \frac{\sigma_v^2 b_1(x,h_2)}{2}\end{bmatrix}; \bar{a}_4(x,h_1,h_2), \frac{\sigma_v^2}{2}\Gamma(x,h_1,h_2)\right). \quad (36)
\end{aligned}$$

Similarly the function $\mu_x(\cdot,\cdot,\cdot,\cdot)$ is given as

$$\begin{aligned}
&\mu_x(s_1,s_2,h_1,h_2)\\
&=\int \min\left(\exp\left[-s_1 b_2(x,h_1)\right],1\right)\min\left(\exp\left[-s_2 b_2(x,h_2)\right],1\right)p(x)\ \mathrm{d}x \qquad (37)\\
&=\int_{\substack{b_2(x,h_1)<0\\ b_2(x,h_2)<0}} p(x)\ \mathrm{d}x+\int_{\substack{b_2(x,h_1)<0\\ b_2(x,h_2)\geq 0}}\exp\left[-s_2 b_2(x,h_2)\right]p(x)\ \mathrm{d}x\\
&\quad+\int_{\substack{b_2(x,h_1)\geq 0\\ b_2(x,h_2)<0}}\exp\left[-s_1 b_2(x,h_1)\right]p(x)\ \mathrm{d}x+\int_{\substack{b_2(x,h_1)\geq 0\\ b_2(x,h_2)\geq 0}}\exp\left[-s_1 b_2(x,h_1)\right]\exp\left[-s_2 b_2(x,h_2)\right]p(x)\ \mathrm{d}x \qquad (38)\\
&=P\left(\begin{smallmatrix} b_2(x,h_1)<0\\ b_2(x,h_2)<0\end{smallmatrix}\middle| x\sim\mathcal{N}(x;0;\sigma_x^2)\right)\\
&\quad+\exp\left[\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2\right]P\left(\begin{smallmatrix} b_2(x,h_1)<0\\ b_2(x,h_2)\geq 0\end{smallmatrix}\middle| x\sim\mathcal{N}(x;-s_2h_2,\sigma_x^2)\right)\\
&\quad+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2\right]P\left(\begin{smallmatrix} b_2(x,h_1)\geq 0\\ b_2(x,h_2)<0\end{smallmatrix}\middle| x\sim\mathcal{N}(x;-s_1h_1,\sigma_x^2)\right)\\
&\quad+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2+\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2+\frac{s_1s_2}{\sigma_x^2}h_1^{\mathrm{T}}h_2\right]P\left(\begin{smallmatrix} b_2(x,h_1)\geq 0\\ b_2(x,h_2)\geq 0\end{smallmatrix}\middle| x\sim\mathcal{N}(x;-s_1h_1-s_2h_2,\sigma_x^2)\right) \qquad (39)\\
&=P\left(\begin{matrix} a_1<0\\ a_2<0\end{matrix}\middle| a\sim\mathcal{N}\left(a;\tilde{a}_1'(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)\right)\\
&\quad+\exp\left[\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2\right]P\left(\begin{matrix} a_1<0\\ a_2\geq 0\end{matrix}\middle| a\sim\mathcal{N}\left(a;\tilde{a}_2'(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)\right)\\
&\quad+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2\right]P\left(\begin{matrix} a_1\geq 0\\ a_2<0\end{matrix}\middle| a\sim\mathcal{N}\left(a;\tilde{a}_3'(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)\right)\\
&\quad+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2+\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2+\frac{s_1s_2}{\sigma_x^2}h_1^{\mathrm{T}}h_2\right]\\
&\quad\times P\left(\begin{matrix} a_1\geq 0\\ a_2\geq 0\end{matrix}\middle| a\sim\mathcal{N}\left(a;\tilde{a}_4'(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)\right) \qquad (40)
\end{aligned}$$

where

$$\tilde{a}_1'(h_1,h_2)\triangleq\frac{1}{2\sigma_x^2}\left[\|h_1\|^2,\|h_2\|^2\right]^{\mathrm{T}}, \qquad (41a)$$

$$\tilde{a}_2'(h_1,h_2)\triangleq\frac{1}{2\sigma_x^2}\left[-2s_2h_1^{\mathrm{T}}h_2+\|h_1\|^2,(1-2s_2)\|h_2\|^2\right]^{\mathrm{T}}, \qquad (41b)$$

$$\tilde{a}_3'(h_1,h_2)\triangleq\frac{1}{2\sigma_x^2}\left[(1-2s_1)\|h_1\|^2,-2s_1h_1^{\mathrm{T}}h_2+\|h_2\|^2\right]^{\mathrm{T}}, \qquad (41c)$$

$$\tilde{a}_4'(h_1,h_2)\triangleq\frac{1}{2\sigma_x^2}\begin{bmatrix}(1-2s_1)\|h_1\|^2-2s_2h_1^{\mathrm{T}}h_2\\ -2s_1h_1^{\mathrm{T}}h_2+(1-2s_2)\|h_2\|^2\end{bmatrix}, \qquad (41d)$$

$$\Lambda(h_1,h_2)\triangleq\begin{bmatrix}\|h_1\|^2 & h_1^{\mathrm{T}}h_2\\ h_1^{\mathrm{T}}h_2 & \|h_2\|^2\end{bmatrix}. \qquad (42)$$

Each of the probabilities on the right hand side of (40) can be written using the cumulative distribution function $\mathcal{N}\mathrm{cdf}_2(\cdot,\cdot,\cdot)$ as

$$\begin{aligned}
&\mu_x(s_1,s_2,h_1,h_2)\\
&=\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix};\tilde{a}_1(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)
\end{aligned}$$

$$+\exp\left[\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2\right]\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}0\\0\end{array}\right];\tilde{a}_2(h_1,h_2),\frac{1}{\sigma_x^2}\overline{\Lambda}(h_1,h_2)\right)$$
$$+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2\right]\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}0\\0\end{array}\right];\tilde{a}_3(h_1,h_2),\frac{1}{\sigma_x^2}\overline{\Lambda}(h_1,h_2)\right)$$
$$+\exp\left[\frac{s_1^2-s_1}{2\sigma_x^2}\|h_1\|^2+\frac{s_2^2-s_2}{2\sigma_x^2}\|h_2\|^2+\frac{s_1s_2}{\sigma_x^2}h_1^{\mathrm{T}}h_2\right]\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}0\\0\end{array}\right];\tilde{a}_4(h_1,h_2),\frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right) \quad (43)$$

where

$$\tilde{a}_1(h_1,h_2)\triangleq\tilde{a}_1'(h_1,h_2), \quad (44a)$$
$$\tilde{a}_2(h_1,h_2)\triangleq\left[\begin{array}{cc}1&0\\0&-1\end{array}\right]\tilde{a}_2'(h_1,h_2), \quad (44b)$$
$$\tilde{a}_3(h_1,h_2)\triangleq\left[\begin{array}{cc}-1&0\\0&1\end{array}\right]\tilde{a}_3'(h_1,h_2), \quad (44c)$$
$$\tilde{a}_4(h_1,h_2)\triangleq-\tilde{a}_4'(h_1,h_2), \quad (44d)$$

$$\overline{\Lambda}(h_1,h_2)\triangleq\left[\begin{array}{cc}\|h_1\|^2&-h_1^{\mathrm{T}}h_2\\-h_1^{\mathrm{T}}h_2&\|h_2\|^2\end{array}\right]. \quad (45)$$

3 Special Cases

In this section, we are going to find the expressions for the following quantities one by one: $\mu_{y,x}(1,1,h,h)$, $\mu_{y,x}(1,0,h,h)$, $\mu_{y|x}(1,1,h,h,x)$, $\mu_{y|x}(1,0,h,h,x)$, $\mu_x(1,1,h,h)$, $\mu_x(1,0,h,h)$.

- $\mu_{y,x}(1,1,h,h)$: Substituting $s_1=s_2=1$ and $h_1=h_2=h$ in (22), we get

$$\begin{aligned}\mu_{y,x}(1,1,h,h)=&E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}-\frac{\sigma_v^2b(x,h)}{2}\\-\frac{\sigma_v^2b(x,h)}{2}\end{array}\right];\bar{a}_1^{1,1}(x,h,h),\frac{\sigma_v^2}{2}\Gamma(x,h,h)\right)\right]\\&+E_{\mathcal{N}(x;-h,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}-\frac{\sigma_v^2b(x,h)}{2}\\\frac{\sigma_v^2b(x,h)}{2}\end{array}\right];\bar{a}_2^{1,1}(x,h,h),\frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\right]\\&+E_{\mathcal{N}(x;-h,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}\frac{\sigma_v^2b(x,h)}{2}\\-\frac{\sigma_v^2b(x,h)}{2}\end{array}\right];\bar{a}_3^{1,1}(x,h,h),\frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\right]\\&+E_{\mathcal{N}(x;-2h,\sigma_x^2)}\left[\exp\left[\frac{2}{\sigma_v^2}\|d(x,h)\|^2\right]\exp\left[\frac{1}{\sigma_x^2}\|h\|^2\right]\right.\\&\left.\times\mathcal{N}\mathrm{cdf}_2\left(\left[\begin{array}{c}\frac{\sigma_v^2b(x,h)}{2}\\\frac{\sigma_v^2b(x,h)}{2}\end{array}\right];\bar{a}_4^{1,1}(x,h,h),\frac{\sigma_v^2}{2}\Gamma(x,h,h)\right)\right]\end{aligned} \quad (46)$$

where

$$\bar{a}_1^{1,1}(x,h,h)\triangleq[0,0]^{\mathrm{T}}, \quad (47a)$$
$$\bar{a}_2^{1,1}(x,h,h)\triangleq\left[-\|d(x,h)\|^2,\|d(x,h)\|^2\right]^{\mathrm{T}}, \quad (47b)$$
$$\bar{a}_3^{1,1}(x,h,h)\triangleq\left[\|d(x,h)\|^2,-\|d(x,h)\|^2\right]^{\mathrm{T}}, \quad (47c)$$
$$\bar{a}_4^{1,1}(x,h,h)\triangleq2\left[\|d(x,h)\|^2,\|d(x,h)\|^2\right]^{\mathrm{T}}, \quad (47d)$$

and

$$\Gamma(x,h,h)\triangleq\left[\begin{array}{cc}\|d(x,h)\|^2&\|d(x,h)\|^2\\\|d(x,h)\|^2&\|d(x,h)\|^2\end{array}\right], \quad (48a)$$
$$\overline{\Gamma}(x,h,h)\triangleq\left[\begin{array}{cc}\|d(x,h)\|^2&-\|d(x,h)\|^2\\-\|d(x,h)\|^2&\|d(x,h)\|^2\end{array}\right]. \quad (48b)$$

Noting that $\Gamma(x,h,h)$ and $\overline{\Gamma}(x,h,h)$ are singular, the quantities related to the bivariate cumulative distribution function $\mathcal{N}\mathrm{cdf}_2(\,\cdot\,)$ can be written as

$$\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h)}{2} \\ -\frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_1(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right) = \mathcal{N}\mathrm{cdf}_1\left(-\frac{\sigma_v^2 b(x,h)}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right), \tag{49a}$$

$$\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h)}{2} \\ \frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_2(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right) = 0, \tag{49b}$$

$$\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b(x,h)}{2} \\ -\frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_3(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right) = 0, \tag{49c}$$

$$\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b(x,h)}{2} \\ \frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_4(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right) = \mathcal{N}\mathrm{cdf}_1\left(\frac{\sigma_v^2 b(x,h)}{2}; 2\|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right), \tag{49d}$$

where $\mathcal{N}\mathrm{cdf}_1(\,\cdot\,)$ is the cumulative distribution function for the univariate Gaussian density, which gives

$$\begin{aligned}\mu_{y,x}(1,1,h,h) =& E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_1\left(-\frac{\sigma_v^2 b(x,h)}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right)\right] \\ &+ E_{\mathcal{N}(x;-2h,\sigma_x^2)}\left[\exp\left[\frac{2}{\sigma_v^2}\|d(x,h)\|^2\right]\exp\left[\frac{1}{\sigma_x^2}\|h\|^2\right]\right. \\ &\left.\times\,\mathcal{N}\mathrm{cdf}_1\left(\frac{\sigma_v^2 b(x,h)}{2}; 2\|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right)\right].\end{aligned} \tag{50}$$

- $\mu_{y,x}(1,0,h,h)$: Substituting $s_1 = 1$, $s_2 = 0$ and $h_1 = h_2 = h$ in (22), we get

$$\begin{aligned}\mu_{y,x}(1,0,h,h) =& E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h)}{2} \\ -\frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_1^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right)\right] \\ &+ E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b(x,h)}{2} \\ \frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_2^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\right] \\ &+ E_{\mathcal{N}(x;-h,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b(x,h)}{2} \\ -\frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_3^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\right] \\ &+ E_{\mathcal{N}(x;-h,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b(x,h)}{2} \\ \frac{\sigma_v^2 b(x,h)}{2} \end{bmatrix}; \bar{a}_4^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right)\right]\end{aligned} \tag{51}$$

where

$$\bar{a}_1^{1,0}(x,h,h) \triangleq [0,0]^{\mathrm{T}}, \tag{52a}$$

$$\bar{a}_2^{1,0}(x,h,h) \triangleq [0,0]^{\mathrm{T}}, \tag{52b}$$

$$\bar{a}_3^{1,0}(x,h,h) \triangleq \left[\|d(x,h)\|^2, -\|d(x,h)\|^2\right]^{\mathrm{T}}, \tag{52c}$$

$$\bar{a}_4^{1,0}(x,h,h) \triangleq \left[\|d(x,h)\|^2, \|d(x,h)\|^2\right]^{\mathrm{T}}. \tag{52d}$$

Again due to the singularity of $\Gamma(x,h,h)$ and $\overline{\Gamma}(x,h,h)$, we have

$$\begin{aligned}\mu_{y,x}(1,0,h,h) =& E_{\mathcal{N}(x;0,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_1\left(-\frac{\sigma_v^2 b(x,h)}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right)\right] \\ &+ E_{\mathcal{N}(x;-h,\sigma_x^2)}\left[\mathcal{N}\mathrm{cdf}_1\left(\frac{\sigma_v^2 b(x,h)}{2}; \|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right)\right].\end{aligned} \tag{53}$$

- $\mu_{y|x}(1,1,h,h,x)$: Substituting $s_1 = s_2 = 1$ and $h_1 = h_2 = h$ in (36), we get

$$\begin{aligned}\mu_{y|x}(1,1,h,h,x) =&\mathcal{N}\text{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h)}{2} \\ -\frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_1^{1,1}(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right) \\ &+ \mathcal{N}\text{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h)}{2} \\ \frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_2^{1,1}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right) \\ &+ \mathcal{N}\text{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h)}{2} \\ -\frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_3^{1,1}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\Bigg] \\ &+ \exp\left[\frac{2}{\sigma_v^2}\|d(x,h)\|^2\right]\mathcal{N}\text{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h)}{2} \\ \frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_4(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right). \end{aligned} \tag{54}$$

Using the singularity of $\Gamma(x,h,h)$ and $\overline{\Gamma}(x,h,h)$, we obtain

$$\begin{aligned}\mu_{y|x}(1,1,h,h,x) =&\mathcal{N}\text{cdf}_1\left(-\frac{\sigma_v^2 b_1(x,h)}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right) \\ &+ \exp\left[\frac{2}{\sigma_v^2}\|d(x,h)\|^2\right]\mathcal{N}\text{cdf}_1\left(\frac{\sigma_v^2 b_1(x,h)}{2}; 2\|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right). \end{aligned} \tag{55}$$

- $\mu_{y|x}(1,0,h,h,x)$: Substituting $s_1 = 1$, $s_2 = 0$ and $h_1 = h_2 = h$ in (36), we get

$$\begin{aligned}\mu_{y|x}(1,0,h,h,x) =&\mathcal{N}\text{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h)}{2} \\ -\frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_1^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right) \\ &+ \mathcal{N}\text{cdf}_2\left(\begin{bmatrix} -\frac{\sigma_v^2 b_1(x,h)}{2} \\ \frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_2^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right) \\ &+ \mathcal{N}\text{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h)}{2} \\ -\frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_3^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\overline{\Gamma}(x,h,h)\right)\Bigg] \\ &+ \mathcal{N}\text{cdf}_2\left(\begin{bmatrix} \frac{\sigma_v^2 b_1(x,h)}{2} \\ \frac{\sigma_v^2 b_1(x,h)}{2}\end{bmatrix}; \bar{a}_4^{1,0}(x,h,h), \frac{\sigma_v^2}{2}\Gamma(x,h,h)\right). \end{aligned} \tag{56}$$

Using the singularity of $\Gamma(x,h,h)$ and $\overline{\Gamma}(x,h,h)$, we obtain

$$\begin{aligned}\mu_{y|x}(1,0,h,h,x) =&\mathcal{N}\text{cdf}_1\left(-\frac{\sigma_v^2 b_1(x,h)}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right) \\ &+ \mathcal{N}\text{cdf}_1\left(\frac{\sigma_v^2 b_1(x,h)}{2}; \|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right). \end{aligned} \tag{57}$$

If we now substitute $b_1(\cdot,\cdot)$ from (29) into (57), we get

$$\begin{aligned}\mu_{y|x}(1,0,h,h,x) =&\mathcal{N}\text{cdf}_1\left(-\frac{\|d(x,h)\|^2}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right) \\ &+ \mathcal{N}\text{cdf}_1\left(\frac{\|d(x,h)\|^2}{2}; \|d(x,h)\|^2, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right) && (58) \\ =&2\mathcal{N}\text{cdf}_1\left(-\frac{\|d(x,h)\|^2}{2}; 0, \frac{\sigma_v^2}{2}\|d(x,h)\|^2\right) && (59) \\ =&1 - \text{erf}\left(\frac{\|d(x,h)\|}{2\sigma_v}\right) && (60) \end{aligned}$$

where we used the identity

$$\mathcal{N}\text{cdf}_1\left(\xi; \bar{\xi}, \sigma_\xi^2\right) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\xi - \bar{\xi}}{\sqrt{2}\sigma_\xi}\right)\right). \tag{61}$$

- $\mu_x(1,1,h,h)$: Substituting $s_1 = s_2 = 1$ and $h_1 = h_2 = h$ in (43), we get

$$\begin{aligned}\mu_x(1,1,h,h) =& \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_1^{1,1}(h,h), \frac{1}{\sigma_x^2}\Lambda(h_1,h_2)\right)\\ &+ \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_2^{1,1}(h,h), \frac{1}{\sigma_x^2}\overline{\Lambda}(h,h)\right)\\ &+ \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_3^{1,1}(h,h), \frac{1}{\sigma_x^2}\overline{\Lambda}(h,h)\right)\\ &+ \exp\left[\frac{1}{\sigma_x^2}\|h\|^2\right]\mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_4^{1,1}(h,h), \frac{1}{\sigma_x^2}\Lambda(h,h)\right)\end{aligned} \tag{62}$$

where

$$\tilde{a}_1^{1,1}(h,h) \triangleq \frac{1}{2\sigma_x^2}\left[\|h\|^2, \|h\|^2\right]^{\mathrm{T}}, \tag{63a}$$

$$\tilde{a}_2^{1,1}(h,h) \triangleq \frac{1}{2\sigma_x^2}\left[-\|h\|^2, \|h\|^2\right]^{\mathrm{T}}, \tag{63b}$$

$$\tilde{a}_3^{1,1}(h,h) \triangleq -\frac{1}{2\sigma_x^2}\left[\|h\|^2, -\|h\|^2\right]^{\mathrm{T}}, \tag{63c}$$

$$\tilde{a}_4^{1,1}(h,h) \triangleq \frac{3}{2\sigma_x^2}\left[\|h\|^2, \|h\|^2\right]^{\mathrm{T}}, \tag{63d}$$

and

$$\Lambda(h,h) \triangleq \begin{bmatrix}\|h\|^2 & \|h\|^2\\ \|h\|^2 & \|h\|^2\end{bmatrix}, \tag{64a}$$

$$\overline{\Lambda}(h,h) \triangleq \begin{bmatrix}\|h\|^2 & -\|h\|^2\\ -\|h\|^2 & \|h\|^2\end{bmatrix}. \tag{64b}$$

Using the singularity of $\Lambda(h,h)$ and $\overline{\Lambda}(x,h,h)$, we obtain

$$\mu_x(1,1,h,h) = \mathcal{N}\mathrm{cdf}_1\left(0; \frac{\|h\|^2}{2\sigma_x^2}, \frac{\|h\|^2}{\sigma_x^2}\right) + \exp\left[\frac{1}{\sigma_x^2}\|h\|^2\right]\mathcal{N}\mathrm{cdf}_1\left(0; \frac{3\|h\|^2}{2\sigma_x^2}, \frac{\|h\|^2}{\sigma_x^2}\right). \tag{65}$$

- $\mu_x(1,0,h,h)$: Substituting $s_1 = 1$, $s_2 = 0$ and $h_1 = h_2 = h$ in (43), we get

$$\begin{aligned}\mu_x(1,0,h,h) =& \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_1^{1,0}(h,h), \frac{1}{\sigma_x^2}\Lambda(h,h)\right) + \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_2^{1,0}(h,h), \frac{1}{\sigma_x^2}\overline{\Lambda}(h,h)\right)\\ &+ \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_3^{1,0}(h,h), \frac{1}{\sigma_x^2}\overline{\Lambda}(h,h)\right) + \mathcal{N}\mathrm{cdf}_2\left(\begin{bmatrix}0\\0\end{bmatrix}; \tilde{a}_4^{1,0}(h,h), \frac{1}{\sigma_x^2}\Lambda(h,h)\right)\end{aligned} \tag{66}$$

where

$$\tilde{a}_1^{1,0}(h,h) \triangleq \frac{1}{2\sigma_x^2}\left[\|h\|^2, \|h\|^2\right]^{\mathrm{T}}, \tag{67a}$$

$$\tilde{a}_2^{1,0}(h,h) \triangleq \frac{1}{2\sigma_x^2}\left[-\|h\|^2, \|h\|^2\right]^{\mathrm{T}}, \tag{67b}$$

$$\tilde{a}_3^{1,0}(h,h) \triangleq -\frac{1}{2\sigma_x^2}\left[\|h\|^2, -\|h\|^2\right]^{\mathrm{T}}, \tag{67c}$$

$$\tilde{a}_4^{1,0}(h,h) \triangleq \frac{1}{2\sigma_x^2}\left[\|h\|^2, \|h\|^2\right]^{\mathrm{T}}. \tag{67d}$$

Using the singularity of $\Lambda(h,h)$ and $\overline{\Lambda}(x,h,h)$, we obtain

$$\mu_x(1,0,h,h) = \mathcal{N}\mathrm{cdf}_1\left(0; \frac{\|h\|^2}{2\sigma_x^2}, \frac{\|h\|^2}{\sigma_x^2}\right) + \mathcal{N}\mathrm{cdf}_1\left(0; \frac{\|h\|^2}{2\sigma_x^2}, \frac{\|h\|^2}{\sigma_x^2}\right) \tag{68}$$

$$= 2\mathcal{N}\mathrm{cdf}_1\left(0; \frac{\|h\|^2}{2\sigma_x^2}, \frac{\|h\|^2}{\sigma_x^2}\right) \tag{69}$$

$$= 1 - \mathrm{erf}\left(\frac{\|h\|}{2\sqrt{2}\sigma_x}\right). \tag{70}$$