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# Upper Embeddablity of Graphs and 2-Factor*

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**Abstract**: This paper discusses the upper embeddablity of some 3-connected or $k$-regular graphs with 4-polygon 2-factor, and obtains some classes of upper embeddable graphs.

**Key words**: 2-factor; maximum genus; upper embeddable

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## 1 Introduction

Graphs considered here are undirect, finite and connevted. A graph is called simple if it has no multiple edges and loops. For graphical notation and terminology without explanation, we refer to ref. [1].

By a surface, we shall mean a closed compact and connected 2-manifold. Let $G=(V,E)$ be a connected graph. The maximum genus[2], denoted by $\gamma_M(G)$, of a graph $G$ is defined to be the maximum integer $k$ such that $G$ admits a cellular embedding in an orientable surface $S$ of genus $k$. Since any cellular embedding of a graph into a surface has at least one face, the Euler polyhedral equation gives the following upper bound on the maximum genus of a graph $G$ (for any integer $x$, $[x]$ denotes the greatest integer no more than $x$) $\gamma_M(G)\leqslant[\frac{\beta(G)}{2}]$, where $\beta(G)=|E(G)|-|V(G)|+1$ is known as the circle rank of $G$ (noting that the circle rank of a graph is also commonly known as the cycle rank, cyclotomic number, or the better number). A graph $G$ is called upper embeddable if $\gamma_M(G)=[\frac{\beta(G)}{2}]$.

Maximum genus of graphs has been an interesting topic in the topological graph theory since the introductory investigation by Nordhaus in ref. [3]. It is seen that to show a graph is upper embeddable is equivalent to deriving a lower bound $\gamma_M(G)=[\frac{\beta(G)}{2}]$ for the maximum genus. Combined with one or more invariants, many papers give some classes of upper embeddable graphs. For example, using 2-factor, the two papers [3－4] discussed the upper embeddable graphs with the same value of degree of vertex under modulo 4.

Inspired by the article [4], the author also use 2-factor to study the upper embeddality of some graphs in this paper.

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This paper is organized as follows. Section 2 explains some definitions and notations. Section 3 gives some basic theorems and Section 4 uses 2-factor to study the upper embeddality of some graphs and gives the main results.

## 2 Some Definitions and Notations

Let $T$ be a spanning tree of graph $G$. Define the deficiency $\xi(G,T)$ of a spanning tree $T$ in a graph $G$ is as the number of components of $G\backslash E(T)$ which have an odd number of edges. The Betti deficiency $\xi(G)$ of the graph $G$ is defined to be the minimum of $\xi(G,T)$ over all spanning tree $T$ of $G$. Note that $\xi(G)=\beta(G)(\bmod 2)$.

For a subset $A\subseteq E(G)$, $c(G\backslash A)$ denotes the number of all components of $G\backslash A$, and $b(G\backslash A)$ denotes the number of components of $G\backslash A$ with odd circle rank.

$G$ will be called $k$-regular graph when its every vertex with $k$-degree.

A 2-factor $F$, of a graph $G$, is a spanning subgraph of $G$ such that $d_F(v)=2$ for any $v\in V(F)$. It is obvious that each 2-factor of $G$ is a vertex-disjoin union of circuits. If the length of every circuit of a 2-factor of $G$ is $k$, it is called that $G$ contains a $k$-polygon 2-factor.

## 3 Some Basic Theorems

For the upper embeddability of graphs, we now state two characterizations whose proofs may be seen in ref. [2] or [5]. They can be expressed in the following theorems.

**Theorem A** Let $G$ be a graph.

(1) $\gamma_M(G)=\frac{1}{2}(\beta(G)-\xi(G))$;

(2) $G$ is upper embeddable if and only if $\xi(G)\leqslant 1$.

It is clear from theorem A(1) that the maximum genus of a graph $G$ is mainly determined by the Betti deficiency $\xi(G)$, for which Nebesky in ref. [6] has given another combinatorial expression.

**Theorem B** Let $G$ be a graph. Then $\xi(G)=\max\limits_{A\subseteq E(G)}\{c(G\backslash A)+b(G\backslash A)-|A|-1\}$.

For two subgraphs $F$ and $H$ of a graph $G$, let $E(F,H)$ denote edges one of whose end vertices are in $V(F)$ and the others in $V(H)$. And let $E(F,G)$ denote edges one of whose end vertices are in $V(F)$ and the others not in $V(F)$.

The following theorem in ref. [7] provides a structural characterization for a non-upper embeddable graph, i. e. graph $G$ with $\xi(G)\geqslant 2$, and plays a fundamental role throughout this paper.

**Theorem C** Let $G$ be a graph. If $G$ is not upper embeddable, i. e. , $\xi(G)\geqslant 2$, there exists an edge subset $A$ of $G$ satisfying the following properties:

(1) $c(G\backslash A)\geqslant 2$, and furthermore for any component $F$ of $G\backslash A$, $\beta(F)\equiv 1(\bmod 2)$;

(2) for any component $F$ of $G\backslash A$, $F$ is a vertex-induced subgraph of $G$;

(3) for any $k(\geqslant 2)$ different connected components $F_1,F_2,\cdots,F_k$, then $|E_G(F_1,F_2,\cdots,F_k)|\leqslant 2k-3$, when $k=2$, $|E_G(F_1,F_2)\leqslant 1$, specially;

(4) $\xi(G)=2c(G\backslash A)-|A|-1$.

Suppose $A$ is such a chosen edge subset of $G$ as in theorem C above. With the help of this notion, we have the following result, a continuation of theorem C.

**Theorem D** Under the conditions and the conclusions of theorem C, we have

(1) for any connected component $F$ of $G\backslash A$, let $G$ be a graph with $k$-connectivity ($k\geqslant 1$), then $|E(G,F)|\geqslant k$;

(2) $|A|=\frac{1}{2}\sum_{F}(E(G\backslash F,F)$ ,where the sum is taken over all connected components $F$ of $G\backslash A$.

## 4 Main Results

In this section, the author will investigate the upper embeddability of graphs with 3-connected.

Paper [4] demonstrates that a graph with 3-connectivity can be upper embeddable when its all vertices with even degree. Naturally, we will think of the upper-embeddablity of a graph with 3-connectivity when all its vertices with odd degree. Generally, such graphs would not be all upper embeddable. In fact, there exist a number of graphs with 3-connectivity and 3-regular not being upper embeddable. But, if we add the condition of "having 4-polygon 2-factor", we can obtain the following theorem:

**Theorem 1** Let $G$ be a 3-connectivity graph having 4-polygon 2-factor and for any $v\in V(G)$, $d_G(v)\equiv 1(\bmod 2)$, graph $G$ is upper embeddable.

**Proof** Suppose $G$ not be upper embeddable, i. e. $\xi(G)\geqslant 2$. By theorem C, we can know that there exists an edge-subset $A$ of $G$ making $G\backslash A$ satisfy all the properties (1)～(4) of theorem C. Let $H_1, H_2, \cdots, H_k(k\geqslant 2)$ be all connected components of $G\backslash A$, where $l=c(G\backslash A)\geqslant 2$. We only need to prove that $|E(G,H_i)|\geqslant 4$ for any $H_i(1\leqslant i\leqslant l)$. First, because of the graph with 3-connectivity, we can easily obtain that $|E(G,H_i)|\geqslant 3$ from theorem D(1). Then we only need to prove that $|E(G,H_i)|\neq 3$. If not, we suppose that $E(G,H_i)=\{e_1,e_2,e_3\}$ and let $x,y,z$ be the endpoints of $e_1,e_2,e_3$ respectively, which belong to $H_i$ and the three endpoints $x,y$ and $z$ are different because of the graph $G$ with 3-connectivity. According to the graph $G$ having 4-polygon 2-factor, one of the following two situations are sure to occurr: (a) there are two edges of $e_1,e_2$ and $e_3$ belonging to some 4-circle, while the third edge not belonging to any 4-circle; (b) any edge of $e_1,e_2$ and $e_3$ does not belong to any 4-circle. Noting the difference among $x,y$ and $z$, we can obtain the following conclusions: if the case (a) happens, the number of vertexes in $H_i$ can be expressed as $4n+2(n\geqslant 1)$; if the case (b) happened, the number of vertexes in $H_i$ can be expressed as $4n(n\geqslant 1)$. No matter what circumstances, the number of vertexes in $H_i$ is even. And we can easily know that there are only three vertexes $x$, $y$and $z$ with even degree, so the components $H_i$ has odd vertexes with odd degree. That is a clear contradiction. Then we can obtain that for any components $H_i(1\leqslant i\leqslant l)$, the inequality $|E(G,H_i)|\geqslant 4$ comes into existence. It is known from theorem D(2) and theorem C(4) respectively, we can obtain that $|A|\geqslant 2l$ and $\xi(G)\leqslant -1$, which contradics with $\xi(G)\geqslant 2$. Thus the proof of the theorem is finished.

A corollary can be obtained from theorem 1. We will introduce a definition before giving the corollary.

A graph $G$ is called circle-k-edge-connected if any two different circles of $G$ need removal of at least $k$ edges from$G$.

**Corollary 1** Every circle-3-edge-connected graph $G$, if it has a 4-polygon 2-factor; and for any $v\in V(G)$, $d_G(v)\equiv 1(\bmod 2)$, the graph $G$ is upper embeddable.

**Proof** Suppose $G$ not be upper embeddable, i. e. $\xi(G)\geqslant 2$. By theorem C, we can know that there exists an edge-subset $A$ of $G$ making $G\backslash A$ satisfy all the properties (1)～(4) of theorem C. Let $H_1, H_2, \cdots, H_k(k\geqslant 2)$ be all connected components of $G\backslash A$, where $l=c(G\backslash A)\geqslant 2$. We only need to prove that $|E(G,H_i)|\geqslant 4$ for any $H_i(1\leqslant i\leqslant l)$. From theorem C, for any connected component $F$ of $G\backslash A$, $F$ has a circle too; and also because of $G$ being a circle-3-edge-connected graph, we have $|E(G,H_i)|\geqslant 3$. Next we only need to prove that $|E(G,H_i)|\neq 3$. The remaining process is similar to theorem 1.

Then, we will use 2-factor to study the upper embeddability of graphs with even-regular.

**Theorem 2** Let $G$ be an even-regular connected graph with no cut-vertex, if it has a 4-polygon 2-factor, then the graph $G$ is upper embeddable.

**Proof** Suppose $G$ not be upper embeddable, i. e. $\xi(G)\geqslant 2$. By the theorem C, we can know that there exists an edge-subset $A$ of $G$ making $G\backslash A$ satisfy all the properties (1)～(4) of theorem C. Let $H_1$, $H_2,\cdots,H_k(k\geqslant 2)$ be all connected components of $G\backslash A$, where $l=c(G\backslash A)\geqslant 2$. From theorem D, every $H_i$ has at least a circle, where $(1\leqslant i\leqslant l)$. Let $F$ be a 4-polygon 2-factor, we will prove that $|E(G,H_i)|\geqslant 4$ for any $(1\leqslant i\leqslant l)$. Fistly, because the graph $G$ is a connected graph, $|E(G,H_i)|\geqslant 1$ from theorem C. And also we have $|E(G,H_i)|\neq 1,3$ due to the Euler model of graph $G$.

Now we only need to prove $|E(G,H_i)|\neq 2$ for any $(1\leqslant i\leqslant l)$. Otherwise, let $E(G,H_i)=\{e_1,e_2\}$ whose endpoints in $V(H_i)$ are $x$ and $y$ respectively. Because the graph $G$ has no vertices, then $x\neq y$. According to the graph $G$ having 4-polygon 2-factor, one of t he following two situations are sure to occurr: (a) the two edges of $e_1$ and $e_2$ belong to some 4-circle; (b) both of the two edges $e_1$ and $e_2$ do not belong to any 4-circle. Noting the difference between $x$ and $y$, we can obtain the following conclusions: if the case (a) happens, the number of vertexes in $H_i$ can be expressed as $4n+2(n\geqslant 1)$; if the case (b) happened, the number of vertexes in $H_i$ can be expressed as $4n(n\geqslant 1)$. No matter what circumstances, the number of vertexes in $H_i$ is even. Because $G$ is even-regular graph, $d_G(v)=2k$ for any $v\in V(G)$. We have

$$|E(H_i)|=\frac{1}{2}(\sum_{v\in V(H_i)}d_G(v)-2)=\frac{1}{2}2k|V(H_i)|-1=k|V(H_i)|-1,$$

and

$$\beta(H_i)=|E(H_i)|-|V(H_i)|+1=(k-1)|V(H_i)|=0(\mathrm{mod}\ 2),$$

it is a clear contradiction to theorem C(1). Same to theorem 1, we have $|A|\geqslant 2l$ and $\xi(G)\leqslant -1$ which contradics with the assumption of $\xi(G)\geqslant 2$. Then theorem 2 can be obtained.

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# 图的上可嵌入性与 2 -因子

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**摘 要**: 对含有 4 -边形 2 -因子的 3 -连通图和 $k$ -正则图的上可嵌入性进行了讨论, 得到了一些上可嵌入图类.

**关键词**: 2 -因子; 最大亏格; 上可嵌入性关

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