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# Critical Group of the Edge Corona $T_m \diamondsuit S_n$ *

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**Abstract**: The structures of the critical group on the edge corona $\boldsymbol{T}_m \diamondsuit \boldsymbol{S}_n$ of trees and stars are determined and it is shown that the Smith normal forms of critical group of the edge corona $\boldsymbol{T}_m \diamondsuit \boldsymbol{S}_n$ are the direct sum of $(n-2)m$ cyclic groups. At the same time the number of spanning trees in $\boldsymbol{T}_m \diamondsuit \boldsymbol{S}_n$ is given.

**Key words**: Laplacian matrix; critical group; Smith normal form; edge corona
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## 1 Introduction

Let $\boldsymbol{G}=(\boldsymbol{V},\boldsymbol{E})$ be a finite connected graph without self-loops, but with multiple edges allowed. Thinking of Laplacian matrix $\boldsymbol{L}(\boldsymbol{G})$ as representing an abelian group homomorphism $\boldsymbol{L}(\boldsymbol{G}):\boldsymbol{Z}^{|V|}\to\boldsymbol{Z}^{|V|}$, the critical group is the unique finite abelian group such that the cokernel can be expressed: $\boldsymbol{Z}^{|V|}/\mathrm{im}\,\boldsymbol{L}(\boldsymbol{G})\cong\boldsymbol{Z}\oplus\boldsymbol{K}(\boldsymbol{G})$, where $\boldsymbol{K}(\boldsymbol{G})$ is defined to be the critical group (also called Jacobian group[1], Picard group[2], or sandpile group[3]) on $\boldsymbol{G}$ in the sense of isomorphism. It follows from Kirchhoff's matrix-tree theorem that the order $\boldsymbol{K}(\boldsymbol{G})$ is known to be $\boldsymbol{\kappa}(\boldsymbol{G})$, the number of spanning trees in $\boldsymbol{G}$.

Recently, there are very few interesting infinite families of graphs for which the critical group structure has been completely determined, such as the wheel graphs[4], the Möbius ladder[5], the square of a cycle $\boldsymbol{C}_n^2$[6] etc. In this paper, the structure of the critical group of the corona of trees and paths is determined and the explicit expression of the Smith normal form of the critical group on the corona of trees and paths are given.

The main tools used in this paper are the computation for Smith normal form of an integer matrix. Two integral matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are unimodular equivalent (written by $\boldsymbol{A}\sim\boldsymbol{B}$) if there exist unimodular matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ such that $\boldsymbol{B}=\boldsymbol{PAQ}$. It can be se en easily that $\boldsymbol{A}\sim\boldsymbol{B}$ implies that coker $\boldsymbol{A}\cong$ coker $\boldsymbol{B}$, and if $\boldsymbol{A}=\mathrm{diag}(s_{11},s_{22},\cdots,s_{n-1,n-1},0)$, then coker $\boldsymbol{A}\cong\boldsymbol{Z}_{s_{11}}\oplus\boldsymbol{Z}_{s_{22}}\oplus\cdots\oplus\boldsymbol{Z}_{s_{n-1,n-1}}$, where $\boldsymbol{Z}_s=\boldsymbol{Z}/s\boldsymbol{Z}$ (Of course, $\boldsymbol{Z}_1$ is the trivial group and $\boldsymbol{Z}_0=\boldsymbol{Z}$).

## 2 Basic Definition and Lemmas

In this paper, let $\boldsymbol{T}_m$ be any tree with $m$ vertices. Let one vertex be connected to the rest of the vertices, and then it is known as $n$ order Stars, denoted by $\boldsymbol{S}_n$. The basic definition and lemmas are as follows.

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**Definition** 1[7] Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs on disjoint sets of $V_1$ and $V_2$ vertices, $E_1$ and $E_2$ edges, respectively. The edge corona $G_1\diamondsuit G_2=(V,E)$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $E_1$ copies of $G_2$, and then joining two end-vertices of the $i$-th edge of $G_1$ to every vertex in the $i$-th copy of $G_2$.

By definition 1, the edge corona $T_m\diamondsuit S_n$ is defined as the graph obtained by taking one copy of $T_m$ and $n-1$ copies of $S_n$, and then joining two end-vertices of the $i$-th edge of $T_m$ to every vertex in the $i$-th copy of $S_n$. The following is the diagram of the edge corona of tree $T_7$ and stars $S_4$ (see fig. 1 and fig. 2).

**Lemma** 1[8] (Kirchoff's matrix-tree theorem) Let $G=(V,E)$ be connected graph with $n$ vertices, $\kappa(G)=\{(-1)^{i+j}\det\,(L(G))_{ij}\}$, where $L\,(G)_{ij}$ is a reduced Laplacian matrix obtained from $L(G)$ by deleting row $i$ and column $j$.

The following lemmas are crucial for us to obtain the critical group of the corona graph in section 3.

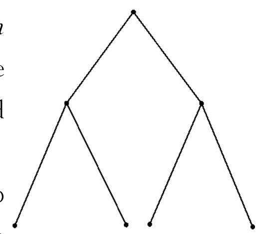

**Fig. 1 Tree $T_7$**

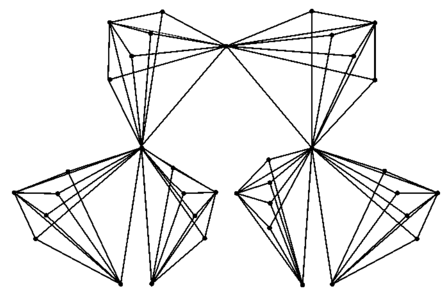

**Fig. 2 Edge Corona $T_7\diamondsuit S_4$**

**Lemma** 2[4] Let the blocks of $G$ be $G_1, G_2,\cdots,G_r$, then $K(G)\cong K(G_1)\oplus K(G_2)\oplus\cdots\oplus K(G_r)$.

**Lemma** 3 Let the blocks of $G$ be $G_1,G_2,\cdots,G_r$ and $H$ be any graph, then the critical group of the edge corona $G\diamondsuit H$ is $K(G\diamondsuit H)\cong K(G_1\diamondsuit H)\oplus K(G_2\diamondsuit H)\oplus\cdots\oplus K(G_r\diamondsuit H)$.

**Proof** For the blocks of the graph $G$ are $G_1,G_2,\cdots,G_r$, and by definition 1, the blocks of the edge corona $G\diamondsuit H$ are $G_1\diamondsuit H, G_2\diamondsuit H,\cdots,G_r\diamondsuit H$. By lemma 2, then the critical group of the edge corona $G\diamondsuit H$ is $K(G\diamondsuit H)\cong K(G_1\diamondsuit H)\oplus K(G_2\diamondsuit H)\oplus\cdots\oplus K(G_r\diamondsuit H)$.

By lemma 3, this paper holds that the blocks of the graph are $K_2$ two special cases, which leads to the following lemma.

**Lemma** 4 Let $T_m$ be any tree with $m$ vertices and $H$ be any graph, then the critical group of the edge corona $T_m\diamondsuit H$ is $(K(K_2\diamondsuit H))^{m-1}$.

**Proof** For tree $T_m$ with $m$ vertices, then there exist $m-1$ edges. By definition 1, the blocks of the edge corona $T_m\diamondsuit H$ are $m-1$ $K_2\diamondsuit H$. By lemma 3, we have $K(T\diamondsuit H)\cong(K(K_2\diamondsuit H))^{m-1}$.

## 3 Critical Group of Edge Corona $T_m\diamondsuit S_n$

This paper mainly discusses the abstract structure of the critical group of the edge corona of trees and paths. By lemma 4, in order to determine the structure of the critical group of $T_m\diamondsuit S_n$, it suffices to compute the Smith normal form of the graph $H=K_2\diamondsuit S_n$ (see fig. 3).

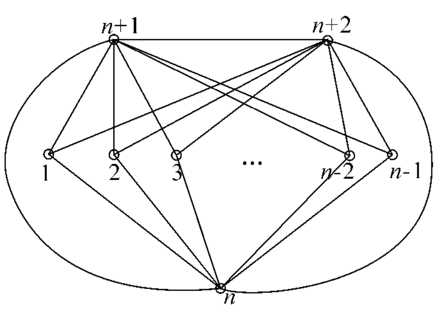


**Fig. 3 Graph $H=K_2\diamondsuit S_n$**

In order to obtain the critical group of graph $T_m\diamondsuit S_n$, it is sufficient to compute the Smith normal form of the reduced Laplacian matrix $L(G)_{ij}$. By lemma 4, now we work on the system of relations of the critical group of $H=K_2\diamondsuit S_n$.

**Lemma** 5 For $n\geqslant 3, m\geqslant 2$, *the critical group of* $H=K_2\diamondsuit S_n$ is

$$K(K_2\diamondsuit S_n)\cong\begin{cases}(Z_3)^{n-3}\oplus Z_{n+3}\oplus Z_{3(n+3)} & \text{if } 3\mid n+3,\\ (Z_3)^{n-3}\oplus(Z_{3(n+3)})^2 & \text{if } 3\nmid n+3.\end{cases}$$

**Proof** The vertex corona $K_2\diamondsuit S_n$ is the graph of $n+2$ vertices. First properly sorting the remaining vertices, the Laplacian matrix is

$$\boldsymbol{L}(\boldsymbol{K}_2\Diamond\boldsymbol{S}_n)=\begin{pmatrix} n+2 & -1 & -1 & -1 & -1 & \cdots & -1\\ -1 & n+2 & -1 & -1 & -1 & \cdots & -1\\ -1 & -1 & n+2 & -1 & -1 & \cdots & -1\\ -1 & -1 & -1 & 3 & 0 & \cdots & 0\\ -1 & -1 & -1 & 0 & 3 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & -1 & 0 & 0 & \cdots & 3 \end{pmatrix}$$

$$\sim\begin{pmatrix} n+3 & 0 & 0 & 0 & \cdots & 0\\ 0 & -1 & -1 & -1 & \cdots & -1\\ 0 & -1 & 3 & 0 & \cdots & 0\\ 0 & -1 & 0 & 3 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -1 & 0 & 0 & \cdots & 3 \end{pmatrix}=\begin{pmatrix} n+3 & 0\\ 0 & \boldsymbol{B}\end{pmatrix}.$$

Before computing the Smith normal form of Laplacian matrix $\boldsymbol{L}(\boldsymbol{K}_2\Diamond\boldsymbol{S}_n)$, we first give the Smith normal form $\boldsymbol{S}(\boldsymbol{B})=\mathrm{diag}(s_{11}',s_{22}',\cdots,s_{nn}')$ of the integer matrix $\boldsymbol{B}$. Now we give in turn the invariant factors of matrix $\boldsymbol{B}$.

$s_{11}'=1$ , $s_{22}'=\left(\begin{vmatrix}-1 & -1\\ -1 & 3\end{vmatrix},\begin{vmatrix}-1 & -1\\ 3 & 0\end{vmatrix},\begin{vmatrix}-1 & 3\\ -1 & 0\end{vmatrix},\begin{vmatrix}3 & 0\\ 0 & 3\end{vmatrix}\right)=(-5,3,9)=1.$

If $i\geqslant 3$, all the $i$-th determinantal divisors of the integer matrix $\boldsymbol{B}$ which are non-zero and not equal have the following form

$$\boldsymbol{M}_1=\begin{pmatrix}-1 & -1 & \cdots & -1\\ -1 & 3 & & \\ \vdots & & \ddots & \\ -1 & & & 3\end{pmatrix},\ \boldsymbol{M}_2=\begin{pmatrix}-1 & -1 & \cdots & -1\\ 3 & 0 & & \\ & & \ddots & \\ & & 3 & 0\end{pmatrix},\ \boldsymbol{M}_3=\begin{pmatrix}3 & & \\ & 3 & & \\ & & \ddots & \\ & & & 3\end{pmatrix}.$$

The corresponding determinantal divisor is $|\boldsymbol{M}_1|=-(i+2)3^{i-2}$, $|\boldsymbol{M}_2|=-3^{i-1}$, $|\boldsymbol{M}_3|=3^i$. Then the greatest common divisor of all $i$-th determinantal divisors of matrix $\boldsymbol{B}$ is $s_{11}'s_{22}'\cdots s_{ii}'=\gcd(-(i+2)3^{i-2},-3^{i-1},3^i)=3^{i-2}$. We have $s_{33}'=s_{44}'=\cdots=s_{nn}'=3$ . And $|\boldsymbol{B}|=(n+3)3^{n-1}$, then $s_{n+1,n+1}'=3(n+3)$ .

Let the Smith normal form of matrix $\boldsymbol{L}(\boldsymbol{K}_2\Diamond\boldsymbol{S}_n)$ be $\boldsymbol{S}(\boldsymbol{L}(\boldsymbol{K}_2\boldsymbol{S}_n))=(s_{11},s_{22},\cdots,s_{nn})$. The following two cases are invariant factors of matrix $\boldsymbol{L}(\boldsymbol{K}_2\Diamond\boldsymbol{S}_n)$.

If $3\mid(n+3)$ :

(ⅰ) $s_{11}=(3,n+3,3(n+3))=3$;

(ⅱ) $s_{11}s_{22}=(3^2,3(n+3),3(n+3)^2,9(n+3))=3^2$, then $s_{22}=3$;

(ⅲ) if $3\leqslant i\leqslant n-2$ , $s_{11}s_{22}\cdots s_{ii}=(3^{i-1}(n+3),3^i,3^i(n+3),3^{i-1}(n+3)^2)=3^i$, then $s_{33}=s_{44}=\cdots=s_{n-2,n-2}=3$;

(ⅳ) $s_{11}s_{22}\cdots s_{n-1,n-1}=(3^{n-2}(n+3),3^{n-1}(n+3),3^{n-2}(n+3)^2)=3^{n-2}(n+3)$ , then $s_{n-1,n-1}=n+3$ , $s_{nn}=\dfrac{3^{n-1}(n+3)^2}{3^{n-2}(n+3)}=3(n+3)$.

If $3\nmid(n+3)$:

(ⅰ) $s_{11}=(3,n+3,3(n+3))=1$;

(ⅱ) $s_{11}s_{22}=(3^2,3(n+3),3(n+3)^2,9(n+3))=3$ , then $s_{22}=3$;

(ⅲ) if $3\leqslant i\leqslant n-2$ , $s_{11}s_{22}\cdots s_{ii}=(3^{i-1}(n+3),3^i,3^i(n+3),3^{i-1}(n+3)^2)=3^{i-1}$, then $s_{33}=s_{44}=\cdots=s_{n-2,n-2}=3$;

(ⅳ) $s_{11}s_{22}\cdots s_{n-1,n-1}=(3^{n-2}(n+3),3^{n-1}(n+3),3^{n-2}(n+3)^2)=3^{n-2}(n+3)$ , then $s_{n-1,n-1}=\dfrac{3^{n-2}(n+3)}{3^{n-3}}=3(n+3)$ , $s_{nn}=\dfrac{3^{n-1}(n+3)^2}{3^{n-2}(n+3)}=3(n+3)$.

Hence, the Smith normal form of the critical group of the edge corona $\boldsymbol{K}_2\diamondsuit \boldsymbol{S}_n$ is

$$\boldsymbol{K}(\boldsymbol{K}_2\diamondsuit \boldsymbol{S}_n)\cong\begin{cases}(\boldsymbol{Z}_3)^{n-3}\oplus\boldsymbol{Z}_{n+3}\oplus\boldsymbol{Z}_{3(n+3)} & \text{if } 3\mid n+3,\\(\boldsymbol{Z}_3)^{n-3}\oplus(\boldsymbol{Z}_{3(n+3)})^2 & \text{if } 3\nmid n+3.\end{cases}$$

Now, the proof of lemma 5 is completed.

By lemma 4 and lemma 5, we get the main theorem of this section.

**Theorem** 1 For $n\geqslant 3, m\geqslant 2$, the critical group of $\boldsymbol{T}_m\diamondsuit \boldsymbol{S}_n$ is

$$\boldsymbol{K}(\boldsymbol{T}_m\diamondsuit \boldsymbol{S}_n)\cong\begin{cases}(\boldsymbol{Z}_3)^{(n-3)(m-1)}\oplus(\boldsymbol{Z}_{n+3})^{m-1}\oplus(\boldsymbol{Z}_{3(n+3)})^{m-1} & \text{if } 3\mid n+3,\\(\boldsymbol{Z}_3)^{(n-3)(m-1)}\oplus(\boldsymbol{Z}_{3(n+3)})^{2(m-1)} & \text{if } 3\nmid n+3.\end{cases}$$

**Example** 1 To give an illustration of theorem 1, we consider the two graphs $\boldsymbol{T}_3\diamondsuit \boldsymbol{S}_3$, $\boldsymbol{T}_3\diamondsuit \boldsymbol{S}_4$ and $\boldsymbol{T}_4\diamondsuit \boldsymbol{S}_3$. By theorem 1 we have that $\boldsymbol{K}(\boldsymbol{T}_3\diamondsuit \boldsymbol{S}_3)\cong(\boldsymbol{Z}_6)^2\oplus(\boldsymbol{Z}_{18})^2$, $\boldsymbol{K}(\boldsymbol{T}_3\diamondsuit \boldsymbol{S}_4)\cong(\boldsymbol{Z}_6)^3\oplus(\boldsymbol{Z}_{18})^3$ and $\boldsymbol{K}(\boldsymbol{T}_4\diamondsuit \boldsymbol{S}_3)\cong(\boldsymbol{Z}_3)^2\oplus(\boldsymbol{Z}_{21})^4$. Example gives the identical result.

An immediate consequence of theorem 1 is the following corollary:

**Corollary** 1 The number of spanning trees in $\boldsymbol{T}_m\diamondsuit \boldsymbol{S}_n$ is

$$\kappa(\boldsymbol{T}_m\diamondsuit \boldsymbol{S}_n)\cong\begin{cases}3^{(n-2)(m-1)}(n+3)^{2(m-1)} & \text{if } 3\mid(n+3),\\3^{(n-1)(m-1)}(n+3)^{2(m-1)} & \text{if } 3\nmid(n+3).\end{cases}$$

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# 边冠图 $T_m\diamondsuit S_n$ 的临界群

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摘 要: 确定了任意树与星的边冠图 $T_m\diamondsuit S_n$ 的临界群的代数结构, 证明了边冠图 $T_m\diamondsuit S_n$ 的临界群的 Smith 标准型为 $(n-2)m$ 个循环群的直和, 同时给出了图 $T_m\diamondsuit S_n$ 的生成树数目.

关键词: Laplacian 矩阵; 临界群; 群的 Smith 标准型; 边冠图

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