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On the Cardinalities of Row Space of Some Special Boolean Matrices*

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Abstract: Let $B_{m\times n}$ be the set of all $m\times n$ Boolean matrices; $R(\boldsymbol{A})$ denote the row space of $\boldsymbol{A}\in B_n$, $|R(\boldsymbol{A})|$ denote the cardinality of $R(\boldsymbol{A})$, m, n be positive integers, and k be non negative integers. In this paper, we prove the following three results: (1) let $\boldsymbol{A}\in B_{n\times n}$, $\forall m$, (i) if $\boldsymbol{A}$ is the idempotent matrix, i. e., $\boldsymbol{A}^2=\boldsymbol{A}$, then $|R(\boldsymbol{A}^m)|=|R(\boldsymbol{A})|$; (ii) if $\boldsymbol{A}$ is the involutory matrix, i. e., $\boldsymbol{A}^2=\boldsymbol{I}$, then $|R(\boldsymbol{A}^m)|=|R(\boldsymbol{A})|$ when m is an odd number or $|R(\boldsymbol{A})|=2^n$ when m is an even number; (2) let $\boldsymbol{A}\in B_{m\times n}$ be k of the numbers of 1, $0\leqslant k\leqslant\min\{m,n\}$, and each row and column is at most one of the numbers of 1 in $\boldsymbol{A}$, then $|R(\boldsymbol{A})|=2^k$; (3) let $\boldsymbol{A}\in B_{n\times n}$ be the partitioned matrix as $\boldsymbol{A}=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}_1\end{pmatrix}$, $\boldsymbol{A}_1=(a_{ij})_{k\times k}$, $a_{ij}=0(i>j)$, $a_{ij}=1(i\leqslant j)$, $i,j=1,2,\cdots,k$, then $|R(\boldsymbol{A})|=k+1$.

Key words: Boolean matrix; row space; cardinality of a row space; permutation matrix

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1 Introduction

For any positive m, n, let $B_{m\times n}$ denote the set of all $m\times n$ Boolean matrices. While $m=n$, we write B_n. Then B_n is a semi-group with the ordinary matrix multiplication and entries using Boolean operation. Let $R(\boldsymbol{A})$ denote the row space of $\boldsymbol{A}\in B_n$, $|R(\boldsymbol{A})|$ denote the cardinality of $R(\boldsymbol{A})$. Then the cardinality is in $[1,2^n]$. The research of the distribution of cardinality of row space of matrix $\boldsymbol{A}\in B_n$ has a long time. In 1992, ref. [1] gave the distribution of cardinality of row space of matrix $\boldsymbol{A}\in B_n$ in interval $(2^{n-1},2^n]$. Simultaneously, he conjectured that there exists $\boldsymbol{A}\in B_n$ with $|R(\boldsymbol{A})|=m$ for any $m\in[1,2^{n-1}]$. In 1995, ref. [2- 6] showed some results on the cardinalities of row space of Boolean matrices. In this paper, by studying some special Boolean matrices, we obtain three results: (1) let, $\boldsymbol{A}\in B_{n\times n}$, $\forall m$, (i) if $\boldsymbol{A}$ is the idempotent matrix, i. e., $\boldsymbol{A}^2=\boldsymbol{A}$, then $|R(\boldsymbol{A}^m)|=R(\boldsymbol{A})|$; (ii) if $\boldsymbol{A}$ is the involutory matrix, i. e., $\boldsymbol{A}^2=\boldsymbol{I}$, then $|R(\boldsymbol{A}^m)|=|R(\boldsymbol{A})|$ when m is an odd number or $|R(\boldsymbol{A})|=2^n$ when m is an even number; (2) let $\boldsymbol{A}\in B_{m\times n}$ be k of the numbers of 1, $0\leqslant k\leqslant\min\{m,n\}$, and each row and column is at most one of the numbers of 1 in $\boldsymbol{A}$, then $|R(\boldsymbol{A})|=2^k$; (3) let $\boldsymbol{A}\in B_{n\times n}$ be the partitioned matrix as $\boldsymbol{A}=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}_1\end{pmatrix}$, $\boldsymbol{A}_1=(a_{ij})_{k\times k}$, a_{ij}

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$= 0 (i > j)$, $a_{ij} = 1 (i \leqslant j)$, $i, j = 1, 2, \cdots, k$, , then $|R(\boldsymbol{A})| = k + 1$.

2 Notions and Definitions

Let m, n, q, i be positive integers, and k be a nonnegative integer. By A_{i*} we denote the i-th row of $\boldsymbol{A} \in B_{m\times n}$, and by a_{ij} we denote the (i, j)-entry of $\boldsymbol{A}$. Let e_i be the n-tuple with 1 in the i-th coordinate and 0 in other coordinates, and θ be the n-tuple with 0 in all coordinates.

Definition 1 Let $\boldsymbol{A} \in B_{m\times n}$. The row space of $\boldsymbol{A}$, denoted by $R(\boldsymbol{A})$, is the span of the set of all row of $\boldsymbol{A}$. The set $R(\boldsymbol{A})$ consists of all sums of rows of $\boldsymbol{A}$, including the empty sum, which is the zero vector. The cardinality of $R(\boldsymbol{A})$, denoted by $|R(\boldsymbol{A})|$, is the number of the vectors of $R(\boldsymbol{A})$.

Definition 2 Let $\boldsymbol{A} \in B_{m\times n}$, $1 \leqslant q \leqslant m$, $1 \leqslant j \leqslant n$. Define

$$R^{+q}(\boldsymbol{A}) = \{A_{i_1*} + A_{i_2*} + \cdots + A_{i_k*} \mid k \geqslant 1, 1 \leqslant i_1 < \cdots < i_k \leqslant m, \exists j, i_j = q\},$$
$$R^{-q}(\boldsymbol{A}) = \{A_{i_1*} + A_{i_2*} + \cdots + A_{i_k*} \mid k \geqslant 0, 1 \leqslant i_1 < \cdots < i_k \leqslant m, \forall j, i_j \neq q\}.$$

3 Lemmas and Main Results

Lemma 1[3] Let $\boldsymbol{A} \in B_{mn}$, $1 \leqslant q \leqslant m$. If $R^{-q}(\boldsymbol{A}) \cap R^{+q}(\boldsymbol{A}) = \varnothing$, then

$$|R(\boldsymbol{A})| = |R^{-q}(\boldsymbol{A})| + |R^{+q}(\boldsymbol{A})|.$$

Lemma 2[2] Let $\boldsymbol{A} \in B_{m\times n}$. If there exist two permutation matrices $\boldsymbol{P}, \boldsymbol{Q}$ such that, then $\boldsymbol{B} = \boldsymbol{PAQ}$, then $|R(\boldsymbol{A})| = |R(\boldsymbol{B})|$.

Theorem 1 Let $\boldsymbol{A} \in B_n$, $\forall m$, (i) if $\boldsymbol{A}$ is the idempotent matrix, i.e., $\boldsymbol{A}^2 = \boldsymbol{A}$, then $|R(\boldsymbol{A}^m)| = |R(\boldsymbol{A})|$; (ii) if $\boldsymbol{A}$ is the involutory matrix, i.e., $\boldsymbol{A}^2 = \boldsymbol{I}$, then $|R(\boldsymbol{A}^m)| = |R(\boldsymbol{A})|$ when m is an odd number or $|R(\boldsymbol{A})| = 2^n$ when m is an even number.

By the hypothesis, it is easy to show theorem 1.

Theorem 2 Let $\boldsymbol{A} \in B_{m\times n}$ be k of the numbers of 1, $0 \leqslant k \leqslant \min\{m, n\}$, and each row and column is at most one of the numbers of 1 in $\boldsymbol{A}$, then $|R(\boldsymbol{A})| = 2^k$.

Proof For $\boldsymbol{A}$, there exist two permutation matrices $\boldsymbol{P}, \boldsymbol{Q}$ such that $\boldsymbol{PAQ} = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} = \boldsymbol{B}$, then $|R(\boldsymbol{A})| = |R(\boldsymbol{B})| = 2^k$.

Corollary 1 $\boldsymbol{A} \in B_n$, $i = 1, 2, \cdots, k$,

$$A_{k*} = \begin{cases} e_{i+k-1} & k = 1, 2, \cdots, n - i + 1, \\ \theta & k = n - i + 2, n - i + 3, \cdots, n, \end{cases}$$

then $|R(\boldsymbol{A})| = 2^{n-i+1}$.

Proof Clearly, $\boldsymbol{A}$ satisfies the conditions of theorem 2, $k = n - i + 1$. So,

$$|R(\boldsymbol{A})| = 2^{n-i+1}.$$

Corollary 2 $\boldsymbol{A} \in B_n$, $i = 2, 3, \cdots, n$,

$$A_{k*} = \begin{cases} e_{1+k-i} & k = i, i + 1, \cdots, n, \\ \theta & k = 1, 2, \cdots, i - 1, \end{cases}$$

then $|R(\boldsymbol{A})| = 2^{n-i+1}$.

By using the similar proof as corollary 1, we can show $|R(\boldsymbol{A})| = 2^{n-i+1}$.

Theorem 3 Let $\boldsymbol{A} \in B_n$ be the partitioned matrix as $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_1 \end{pmatrix}$, $\boldsymbol{A}_1 = (a_{ij})_{k\times k}$, $0 \leqslant k \leqslant n$, $a_{ij} = 0 (i > j)$, $a_{ij} = 1 (i \leqslant j)$, $i, j = 1, 2, \cdots, k$, then $|R(\boldsymbol{A})| = k + 1$.

Proof Clearly, $|R(\boldsymbol{A})| = |R(\boldsymbol{A}_1)|$, $R^{+1}(\boldsymbol{A}_1) \cap R^{-1}(\boldsymbol{A}_1) = \varnothing$.

So, by lemma 1, $|R(\boldsymbol{A}_1)| = |R^{+}(\boldsymbol{A}_1)| + |R^{-1}(\boldsymbol{A}_1)|$, $|R^{+1}(\boldsymbol{A}_1)| = 1$, $|R^{-1}(\boldsymbol{A}_1)| = k$. We show $|R(\boldsymbol{A})|$

$|=k+1$.

Corollary 3 Let $\boldsymbol{B}=(b_{ij})_n$, $b_{ij}=0(i<j)$, $b_{ij}=1(i\geqslant j)$, then $i,j=1,2,\cdots,n$, then $|R(\boldsymbol{B})|=n+1$.

By using the similar argument of theorem 3, we can show it.

Corollary 4 $\boldsymbol{A}\in B_n$. If there exist one row and one column in $\boldsymbol{A}$, its numbers of 1 is i, $i=1,2,\cdots,k$, $1\leqslant k\leqslant n$, then $|R(\boldsymbol{A})|=k+1$.

For example, $\boldsymbol{A}=\begin{pmatrix}0&0&1&1&0\\0&0&0&0&0\\0&0&0&1&0\\0&1&1&1&0\\0&0&0&0&0\end{pmatrix}$, $n=5$, $i=1,2,3$, $k=3$. $|R(\boldsymbol{A})|=4$.

Proof Because for $\boldsymbol{A}$, there exist two permutation matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, such that

$$\boldsymbol{PAQ}=\begin{pmatrix}\boldsymbol{O}&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{A}_1\end{pmatrix}=\boldsymbol{B},\ \boldsymbol{A}_1=(a_{ij})_k,\ a_{ij}=0(i>j),\ a_{ij}=1(i\leqslant j),\ i,j=1,2,\cdots,k.$$

$$|R(\boldsymbol{A})|=|R(\boldsymbol{B})|=k+1.$$

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一些特殊结构的布尔矩阵行空间基数

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摘 要: 设 $B_{m\times n}$ 是所有 $m\times n$ 布尔矩阵的集合, $R(\boldsymbol{A})$ 为 $\boldsymbol{A}\in B_n$ 的行空间, $|R(\boldsymbol{A})|$ 表示行空间 $R(\boldsymbol{A})$ 的基数, m,n 是正整数, k 为非负整数. 证明了如下 3 个结果: (1) 设 $\boldsymbol{A}\in B_{m\times n}$, $\forall m$, (ⅰ) 如果 $\boldsymbol{A}$ 是幂等矩阵, 即 $\boldsymbol{A}^2=\boldsymbol{A}$, 那么 $|R(\boldsymbol{A}^m)|=|R(\boldsymbol{A})|$; (ⅱ) 如果 $\boldsymbol{A}$ 是对合矩阵, 即 $\boldsymbol{A}^2=\boldsymbol{I}$, 那么当 m 是奇数时, $|R(\boldsymbol{A}^m)|=|R(\boldsymbol{A})|$, 当 m 是偶数时 $|R(\boldsymbol{A})|=2^n$. (2) 设 $\boldsymbol{A}\in B_{m\times n}$, $\boldsymbol{A}$ 含 1 的元素个数为 k, $0\leqslant k\leqslant\min\{m,n\}$, 且 $\boldsymbol{A}$ 的每行每列元素中 1 的元素个数最多为 1, 那么 $|R(\boldsymbol{A})|=2^k$. (3) 若 $\boldsymbol{A}\in B_{m\times n}$ 是形如 $\boldsymbol{A}=\begin{pmatrix}\boldsymbol{O}&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{A}_1\end{pmatrix}$ 的分块矩阵, $\boldsymbol{A}_1=(a_{ij})_{k\times k}$, $a_{ij}=0(i>j)$, $a_{ij}=1(i\leqslant j)$, $i,j=1,2,\cdots,k$, 则 $|R(\boldsymbol{A})|=k+1$.

关键词: 布尔矩阵; 行空间; 行空间基数; 置换矩阵

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