第 31 卷 第 6 期 吉首大学学报(自然科学版) Vol. 31 No. 6
2010 年 11 月 Journal of Jishou University (Natural Science Edition) Nov. 2010

Article ID: 1007 - 2985(2010)06 - 0009 - 05

Positive Solutions of Third-Order Periodic Boundary Value Problems with Variable Coefficient*

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Abstract: The positive solutions of a nonlinear third-order periodic boundary value problem with variable coefficient are studied. The nonlinear term may be singular with respect to the space variable. By making use of a suitable transformation, the problem is transformed into a Hammerstein integral equation. By applying Guo-Krasnosel'skii fixed point theorem on cone, the existence of one and two po sitive solutions is proved.

Key words: periodic boundary value problem; positive solution; fixed point theorem on cone
CLC number: O175. 8 **Document code**: A

1 Introduction

In this paper, we study positive solutions of the following nonlinear third-order periodic boundary value problem with variable coefficient

$$\begin{cases} u'''(t) + h(t)u(t) = f(t, u(t)) & 0 \leqslant t \leqslant 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi) & i = 0, 1, 2, \end{cases} \tag{1}$$

where $h \in C[0, 2\pi]$ is a nonnegative function and $0 < \max\limits_{0 \leqslant t \leqslant 2\pi} h(t) < \dfrac{1}{3\sqrt{3}}$. Here, the function $u^* \in C^2[0, 1]$ is called positive solution of the problem (1) if $u^*(t)$ satisfies (1) and $u^*(t) > 0, 0 \leqslant t \leqslant 2\pi$.

When $h(t) \equiv \rho^3$ is a positive constant and $0 < \rho < \dfrac{1}{\sqrt{3}}$, the problem (1) has the special form

$$\begin{cases} u'''(t) + \rho^3 u(t) = f(t, u(t)) & 0 \leqslant t \leqslant 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi) & i = 0, 1, 2. \end{cases}$$

The positive solutions of nonlinear third-order periodic boundary value problem with constant coefficient ρ^3 have been studied by some authors. For example, see ref. [1 - 4].

To our best knowledge, the positive solutions of problem (1) have not been considered by any authors when $h(t)$ is not a constant.

Throughout this paper, we always assume that $f: [0, 2\pi] \times (0, +\infty) \to [0, +\infty)$ is a continuous function. The assumption implies that the nonlinear term $f(t, u)$ may be singular at $u = 0$.

The purpose of this paper is to establish the existence of one and two positive solutions for the problem (1). In the study, we do not require the existence of upper and lower solutions and do not impose any monotonicity

* **Received date**: 2010 - 06 - 13
Foundation item: National Natural Science Foundation of China(11071109)
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conditions on the nonlinear term $f(t,u)$. By making use of the ingenious transformation constructed in ref. [1], we will change the problem (1) into a Hammerstein integral equation. By applying Guo-Krasnosel'skii fixed point theorem on cone, we will establish two local existence theorems of positive solutions. Finally, we will illustrate that this paper presents new results by an example. For other nonlinear periodic boundary value problems, we refer readers to the ref. [5 - 8] and the references therein.

2 Preliminaries

Let $\rho=\sqrt[3]{\max\limits_{0\leqslant t\leqslant 2\pi} h(t)}$ and let

$$m_1=\frac{1}{e^{2\rho\pi}-1},M_1=\frac{e^{2\rho\pi}}{e^{2\rho\pi}-1},m_2=\frac{2\sin(\sqrt{3}\rho\pi)}{\sqrt{3}\rho(e^{\rho\pi}+1)^2},M_2=\frac{2}{\sqrt{3}\rho\sin(\sqrt{3}\rho\pi)},\sigma_1=\frac{1}{e^{2\rho\pi}},\sigma_2=\frac{\sin^2(\sqrt{3}\rho\pi)}{(e^{\rho\pi}+1)^2}.$$

So $\sigma_1=\frac{m_1}{M_1},\sigma_2=\frac{m_2}{M_2}$ and $\sigma_1,\sigma_2\in(0,1)$.

Let $C[0,2\pi]$ be the Banach space with the norm $\|u\|=\max\limits_{0\leqslant t\leqslant 2\pi}|u(t)|$,

$$K_1=\{u\in C[0,2\pi]:u(t)\geqslant\sigma_1\|u\|,0\leqslant t\leqslant 2\pi\},$$
$$K_2=\{u\in C[0,2\pi]:u(t)\geqslant\sigma_2\|u\|,0\leqslant t\leqslant 2\pi\}.$$

Then K_1 and K_2 are cone of nonnegative functions in $C[0,2\pi]$. For $0<r_1<r_2$, write $K_i[r_1,r_2]=\{u\in K_i:r_1\leqslant\|u\|\leqslant r_2\},i=1,2$.

Define the operator J as follows

$$(Ju)(t)=\int_0^{2\pi}g(t,s)u(s)\mathrm{d}s\qquad 0\leqslant t\leqslant 2\pi,$$

where

$$g(t,s)=\begin{cases}\dfrac{e^{\rho(s-t)}}{e^{2\rho\pi}-1} & 0\leqslant t\leqslant s\leqslant 2\pi,\\[2mm] \dfrac{e^{\rho(2\pi+s-t)}}{e^{2\rho\pi}-1} & 0\leqslant s\leqslant t\leqslant 2\pi.\end{cases}$$

Simple calculations give that

$$\max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi}g(t,s)\mathrm{d}s=\int_0^{2\pi}g(t,s)\mathrm{d}s\equiv\rho^{-1}\qquad 0\leqslant t\leqslant 2\pi.$$

For $u\in K_2\backslash\{0\}$, define the operator T as follows

$$(Tu)(t)=\int_0^{2\pi}G(t,s)[f(s,(Ju)(s))+(\rho^3-h(s))(Ju)(s)]\mathrm{d}s\qquad 0\leqslant t\leqslant 2\pi,$$

where

$$G(t,s)=\begin{cases}\dfrac{2e^{(\rho/2)(2\pi+t-s)}[\sin\frac{\sqrt{3}}{2}\rho(s-t)+e^{-\rho\pi}\sin\frac{\sqrt{3}}{2}\rho(2\pi-s+t)]}{\sqrt{3}\rho(e^{\rho\pi}+e^{-\rho\pi}-2\cos\sqrt{3}\rho\pi)} & 0\leqslant t\leqslant s\leqslant 2\pi,\\[4mm] \dfrac{2e^{(\rho/2)(t-s)}[\sin\frac{\sqrt{3}}{2}\rho(2\pi-t+s)+e^{-\rho\pi}\sin\frac{\sqrt{3}}{2}\rho(t-s)]}{\sqrt{3}\rho(e^{\rho\pi}+e^{-\rho\pi}-2\cos\sqrt{3}\rho\pi)} & 0\leqslant s\leqslant t\leqslant 2\pi.\end{cases}$$

Lemma 1 $m_1\leqslant g(t,s)\leqslant M_1,m_2\leqslant G(t,s)\leqslant M_2,t,s\in[0,2\pi]$.

Proof $m_1\leqslant g(t,s)\leqslant M_1$ is direct. For the proof of $m_2\leqslant G(t,s)\leqslant M_2$, see lemma 3 in ref. [1].

Lemma 2 For any $u\in K_2,Ju\in K_1$ and $\sigma_2\rho^{-1}\|u\|\leqslant\|Ju\|\leqslant\rho^{-1}\|u\|$.

Proof Let $u\in K_2$. Then $\sigma_2\|u\|\leqslant u(t)\leqslant\|u\|,0\leqslant t\leqslant 2\pi$. Since $g:[0,2\pi]\times[0,2\pi]\to[0,+\infty)$ is continuous, $Ju\in C[0,2\pi]$ and $\|Ju\|=\max\limits_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi}g(t,s)u(s)\mathrm{d}s$. Applying lemma 1, we get that

$$(Ju)(t)\geqslant\frac{m_1}{M_1}\int_0^{2\pi}M_1u(s)\mathrm{d}s\geqslant\sigma_1\max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi}g(t,s)u(s)\mathrm{d}s=\sigma_1\|Ju\|,$$

$$\| Ju \| \geqslant \sigma_2 \ \| u \| \max_{0\leqslant t\leqslant 2\pi} \int_0^{2\pi} g(t,s)\mathrm{d}s = \sigma_2 \rho^{-1} \| u \|,$$

$$\| Ju \| \leqslant \| u \| \max_{0\leqslant t\leqslant 2\pi} \int_0^{2\pi} g(t,s)\mathrm{d}s = \rho^{-1} \| u \|.$$

Lemma 3 $T:K_2[r_1,r_2]\to K_2$ is completely continuous.

Proof Let $u\in K_2[r_1,r_2]$. Then $r_1\leqslant \| u \| \leqslant r_2$. By lemma 2, $Ju\in K_1$ and $\sigma_2\rho^{-1}r_1\leqslant \sigma_2\rho^{-1} \| u \| \leqslant \| Ju \| \leqslant \rho^{-1} \| u \| \leqslant \rho^{-1}r_2$. It follows that $J:K_2[r_1,r_2]\to K_1[\sigma_2\rho^{-1}r_1,\rho^{-1}r_2]$. Now, let

$$(Fu)(t) = f(t,u(t)) + (\rho^3 - h(t))u(t),\ (Su)(t) = \int_0^{2\pi} G(t,s)u(s)\mathrm{d}s.$$

Obviously,

$$J:K_2[r_1,r_2]\to K_1[\sigma_2\rho^{-1}r_1,\rho^{-1}r_2],\ F:K_1[\sigma_2\rho^{-1}r_1,\rho^{-1}r_2]\to C[0,2\pi]$$

are continuous. Moreover, $S:C[0,2\pi]\to C[0,2\pi]$ is completely continuous by Arzela-Ascoli theorem.

Since $T=S\circ F\circ J$, we see that $T:K_2[r_1,r_2]\to C[0,2\pi]$ is completely continuous. On the other hand, by lemma 1,

$$\begin{aligned}(Tu)(t) = &\int_0^{2\pi} G(t,s)[f(s,(Ju)(s)) + (\rho^3 - h(s))(Ju)(s)]\mathrm{d}s \geqslant \\ &m_2\int_0^{2\pi}[f(s,(Ju)(s)) + (\rho^3 - h(s))(Ju)(s)]\mathrm{d}s = \\ &\sigma_2\int_0^{2\pi} M_2[f(s,(Ju)(s)) + (\rho^3 - h(s))(Ju)(s)]\mathrm{d}s \geqslant \\ &\sigma_2 \max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} G(t,s)[f(s,(Ju)(s)) + (\rho^3 - \\ &h(s))(Ju)(s)]\mathrm{d}s = \sigma_2 \| Tu \|.\end{aligned}$$

Consequently, $T:K_2[r_1,r_2]\to K_2$.

Lemma 4 If $\bar{u}\in K_2[r_1,r_2]$ is a fixed point of the operator T, then $u^* = J\bar{u}\in K_1$ is a positive solution of the problem (1).

The proof is direct.

In order to prove main results, we need the following Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type.

Lemma 5 Let X be a Banach space, $K\subset X$ be a cone and Ω_1,Ω_2 be two bounded open subsets of K with $0\in\Omega_1\subset\bar{\Omega}_1\subset\Omega_2$. Assume that $F:\bar{\Omega}_2\setminus\Omega_1\to K$ is a completely continuous operator and one of the following conditions is satisfied:

(i) $\| Fu \| \leqslant \| u \|, u\in\partial\Omega_1$ and $\| Fu \| \geqslant \| u \|, u\in\partial\Omega_2$.

(ii) $\| Fu \| \geqslant \| u \|, u\in\partial\Omega_1$ and $\| Fu \| \leqslant \| u \|, u\in\partial\Omega_2$.

Then F has a fixed point in $\bar{\Omega}_2\setminus\Omega_1$.

3 Main Results

For $r>0$, we introduce the following control functions

$$\varphi(t,r) = \max\{f(t,u) + (\rho^3 - h(t))u : u\in[\sigma_2\rho^{-1}r,\rho^{-1}r]\},$$

$$\psi(t,r) = \min\{f(t,u) + (\rho^3 - h(t))u : u\in[\sigma_2\rho^{-1}r,\rho^{-1}r]\}.$$

In geometry, $\varphi(t,r)$ and $\psi(t,r)$ are maximum height function and minimum height function of function $f(t,u)+(\rho^3-h(t))u$ on the set $[0,2\pi]\times[\sigma_2\rho^{-1}r,\rho^{-1}r]$ respectively.

Write $\Omega(r)=\{u\in K_2: \| u \| <r\}$, $\partial\Omega(r)=\{u\in K_2: \| u \| =r\}$. Then $K_2[r_1,r_2]=\overline{\Omega(r_2)}\setminus\Omega(r_1)$.

In this paper, we obtain the following local existence theorems. These theorems show that the existence of positive solutions is independent of the state of nonlinear term $f(t,u)$ near $u=0$.

Theorem 1 Assume that there exist two positive numbers $a<b$ such that one of the following condi-

tions is satisfied:

(ⅰ) $\int_0^{2\pi}\varphi(t,a)\mathrm{d}t \leqslant M_2^{-1}a, \int_0^{2\pi}\psi(t,b)\mathrm{d}t \geqslant m_2^{-1}b$;

(ⅱ) $\int_0^{2\pi}\psi(t,a)\mathrm{d}t \geqslant m_2^{-1}a, \int_0^{2\pi}\varphi(t,b)\mathrm{d}t \leqslant M_2^{-1}b$.

Then problem (1) has at least one positive solution $u^* \in K_1$ and $\sigma_2\rho^{-1}a \leqslant \|u^*\| \leqslant \rho^{-1}b$.

Proof Without loss of generality, we only prove the case (ⅰ).

Let $u \in \partial\Omega(a)$. Then $\|u\| = a$ and $\sigma_2 a \leqslant u(t) \leqslant a, 0 \leqslant t \leqslant 2\pi$. So for any $0 \leqslant t \leqslant 2\pi$,

$$\sigma_2 a\rho^{-1} \leqslant \sigma_2 a\int_0^{2\pi} g(t,s)\mathrm{d}s \leqslant \int_0^{2\pi} g(t,s)u(s)\mathrm{d}s \leqslant a\int_0^{2\pi} g(t,s)\mathrm{d}s \leqslant a\rho^{-1}.$$

This implies that $\sigma_2\rho^{-1}a \leqslant (Ju)(t) \leqslant \rho^{-1}a, 0 \leqslant t \leqslant 2\pi$. By the definition of $\varphi(t,a)$,

$$f(t,(Ju)(t)) + (\rho^3 - h(t))(Ju)(t) \leqslant \varphi(t,a) \qquad 0 \leqslant t \leqslant 2\pi.$$

By lemma 1, one has

$$\|Tu\| = \max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} G(t,s)[f(s,(Ju)(s)) + (\rho^3 - h(s))(Ju)(s)]\mathrm{d}s \leqslant$$

$$M_2\int_0^{2\pi}\varphi(s,a)\mathrm{d}s \leqslant M_2 \cdot M_2^{-1}a = a = \|u\|.$$

Let $u \in \partial\Omega(b)$. Then $\|u\| = a$. So $\sigma_2 b \leqslant u(t) \leqslant b$ and $\sigma_2 b\rho^{-1} \leqslant (Ju)(t) \leqslant b\rho^{-1}, 0 \leqslant t \leqslant 2\pi$. By the definition of $\psi(t,b)$,

$$f(t,(Ju)(t)) + (\rho^3 - h(t))(Ju)(t) \geqslant \psi(t,b) \qquad 0 \leqslant t \leqslant 2\pi.$$

It follows that

$$\|Tu\| \geqslant \min_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} G(t,s)[f(s,(Ju)(s)) + (\rho^3 - h(s))(Ju)(s)]\mathrm{d}s \geqslant$$

$$m_2\int_0^{2\pi}\psi(s,b)\mathrm{d}s \geqslant m_2 \cdot m_2^{-1}b = b = \|u\|.$$

By lemma 3 and lemma 5, we assert that the operator T has a fixed point $\bar{u} \in \overline{\Omega(b)}\backslash\Omega(a) = K_2[a,b]$. By lemma 2 and lemma 4, the problem (1) has one positive solution $u^* = J\bar{u} \in K_1$ and $\sigma_2\rho^{-1}a \leqslant \|u^*\| \leqslant \rho^{-1}b$.

Theorem 2 Assume that there exist four positive numbers $a < b < c < d$ such that $b < \sigma_2 c$ and one of the following conditions is satisfied:

(ⅰ) $\int_0^{2\pi}\varphi(t,a)\mathrm{d}t \leqslant M_2^{-1}a, \int_0^{2\pi}\psi(t,b)\mathrm{d}t \geqslant m_2^{-1}b, \int_0^{2\pi}\psi(t,c)\mathrm{d}t \geqslant m_2^{-1}c, \int_0^{2\pi}\varphi(t,d)\mathrm{d}t \leqslant M_2^{-1}d$;

(ⅱ) $\int_0^{2\pi}\psi(t,a)\mathrm{d}t \geqslant m_2^{-1}a, \int_0^{2\pi}\varphi(t,b)\mathrm{d}t \leqslant M_2^{-1}b, \int_0^{2\pi}\varphi(t,c)\mathrm{d}t \leqslant M_2^{-1}c, \int_0^{2\pi}\psi(t,d)\mathrm{d}t \geqslant m_2^{-1}d$.

Then problem (1) has at least two positive solutions $u_1^*, u_2^* \in K_1$ and

$$\sigma_2\rho^{-1}a \leqslant \|u_1^*\| \leqslant \rho^{-1}b, \sigma_2\rho^{-1}c \leqslant \|u_2^*\| \leqslant \rho^{-1}d.$$

Proof By theorem 1, the problem (1) has two positive solutions $u_1^*, u_2^* \in K_1$ satisfying $\sigma_2\rho^{-1}a \leqslant \|u_1^*\| \leqslant \rho^{-1}b, \sigma_2\rho^{-1}c \leqslant \|u_2^*\| \leqslant \rho^{-1}d$. Since $b < \sigma_2 c$, we have $\rho^{-1}b < \sigma_2\rho^{-1}c$. Therefore, $\|u_2^*\| > \|u_1^*\|$ and $u_2^* \neq u_1^*$.

4 An Example

The following example illustrates that main results are applicable to more general cases. Therefore, our results extend and improve the existing results.

Consider the third-order periodic problem

$$\begin{cases} u'''(t) + \frac{1}{54}(1 + \sin\frac{t}{2})u(t) = \frac{1 + \sin u(t)}{u(t)} & 0 \leqslant t \leqslant 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi) & i = 0,1,2. \end{cases}$$

Here, $h(t)=\frac{1}{54}(1+\sin\frac{t}{2})$ and the nonlinear term $f(t,u)=\frac{1+\sin u}{u}$ which is singular at $u=0$.

Let $\rho=\frac{1}{3}=\sqrt[3]{\frac{1}{54}\max_{0\leqslant t\leqslant 2\pi}(1+\sin\frac{t}{2})}$. Then $M_2\approx 3.5690$. Moreover,

$$f(t,u)+(\rho^3-h(t))u=\frac{1+\sin u}{u}+\frac{1}{54}(1-\sin\frac{t}{2})u\qquad 0\leqslant t\leqslant 2\pi, 0<u<+\infty.$$

Applying the expression we get that, for any $r>0$,

$$\int_0^{2\pi}\varphi(t,r)\mathrm{d}t\leqslant\frac{4\pi}{3\sigma_2 r}+\frac{2(\pi-1)r}{18},\int_0^{2\pi}\psi(t,r)\mathrm{d}t\geqslant\frac{2\pi}{3r}.$$

It follows that

$$\limsup_{r\to+\infty}\frac{1}{r}\int_0^{2\pi}\varphi(t,r)\mathrm{d}t\leqslant\limsup_{r\to+\infty}\left[\frac{4\pi}{3\sigma_2 r^2}+\frac{2(\pi-1)}{18}\right]=\frac{2(\pi-1)}{18}\approx\frac{1}{4.2025}<M_2^{-1},$$

$$\liminf_{r\to+0}\frac{1}{r}\int_0^{2\pi}\psi(t,r)\mathrm{d}t\geqslant\liminf_{r\to+0}\frac{2\pi}{3r^2}=0.$$

This shows that there exist $0<a<b$ such that

$$\int_0^{2\pi}\psi(t,a)\mathrm{d}t\geqslant m_2^{-1}a,\int_0^{2\pi}\varphi(t,b)\mathrm{d}t\leqslant M_2^{-1}b.$$

By theorem 1 (ii), the problem has a positive solution $u^*\in K_1$.

Because the coefficient $h(t)$ is not a constant, the conclusion can not be derived from ref. [1-4].

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变系数三阶周期边值问题的正解

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摘　要:研究了变系数非线性三阶周期边值问题的正解.非线性项可以关于空间变元奇异.利用适当的变换此问题被转换为一个 Hammerstein 积分方程,利用锥上的 Guo-Krasno-sel'ski 不动点定理获得了 1～2 个正解的存在性.

关键词:周期边值问题;正解;锥上的不动点定理

中图分类号:O175.8　　**文献标识码**:A

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