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Curvature tensors, gauge field are actually curl field of gradient*

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Abstract: It is introduced the concept of absolute integral in Riemannian spaces and in fibre bundle space (Gauge field), with respect to frames at every point. This is just the inverse of the absolute differential. After discussing the exterior differential $d(dx^i) \neq 0$, rewrote the exterior differential form into symmetric form and established the relation between the exterior differential form and the absolute differential. By the aid of the absolute integral, it is improved Stokes' formula: The strict definitions of the circulation, the curl and the divergence were obtained (usually, they were obtained only by analogy in Euclid-space, unable to apply here). It had been proved that the curvature tensor is a curl (grad), not zero except in Eucldean space, and so discovered the essence of Bianchi identity: div(rot(grad)) = 0, the curvature, forming tube field, is invariant along the tube, i.e. pointed out that Gauge fields are curl fields of gradients and so on. By the way, it is obtained the torsion tensor is rot of base of frame also.

Key words: riemannian space; gauge field; absolute integral; curvature tensor; curl
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Edward Witten, the preeminent superstring theorist, has proclaimed that progress in that field is most likely to come about by unearthing what he calls the theory's "core geometrical ideas"[1]. What is the "core"? It is mysterious that the curvature almost is closely related to every aspect of Riemannian spaces, fibre bundle spaces (Gauge fields). As [2] describe: Large amount of work on this subject have been done, yet the result obtained are still unsatisfactory. I think the curvature is just the element of the "core", have been waiting for human being to unearth for over hundred years. In this paper, we see from the angle of calculus, by introducing the concept of the absolute integral (inverse of the absolute differential) in these spaces (Gauge field) and discussing of exterior differential $d(dx^i)$, rewrote exterior differential form into symmetrical form with respect to the frame at every point. Thus a relation between the exterior differential form and the absolute differential is established. As Prof. Chern[1] pointed out: the general method (tensor analysis) in differential geometry was disadvantages more than advantages usually, some improvements were obtained (tensor and frame be used at same time, i.e. we use the linear combination of base of frame). In view of absolute integral, the expressions of circulation, curl (rotation) and that of divergence are obtained. They were puzzled before, because they were obtained only in Euclid-space or analogy to Euclid-space. Consequently we obtained an important result that in non-Euclidean space the curvature tensor is the curl of the gradient of base (not zero), and hence the essence of the Bianchi identity is div rot = 0 really, and so on. The curvature tensor was induced by Riemann firstly and proved that a Riemannian space is Euclidean if and only if its curvature tensor is identically equal to zero his-

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torically. Einstein then reintroduced by parallel translation of vector[3] along a closed (especially infinitesimal) contour[4,5]. In this paper, we shall introduce the curvature tensor by the aid of curl (rotation), independent of parallel translate. Let us see the detail. Let M is Riemannian space. A point $p \in M$, may also be written as $p(x^v)$ with respect to certain coordinate system. The distance ds between the point $p(x^v)$ and a neighboring point $p'(x^v + dx^v)$, which is invariant under transformation of coordinates, is given by

$$ds^2 = \sum_{v\mu} e_v e_\mu dx^v dx^\mu = \sum_{v\mu} g_{v\mu} dx^v dx^\mu = \sum_{v\mu} (g_{v\mu} e^v \otimes e^\mu)(e_v dx^v \otimes e_\mu dx^\mu).$$

(because scalar product: $e_v e_\mu = g_{v\mu}, e^v e_v = 1$).

Where dx^v is differential scalar, e_v is corresponding frame and consequently $dx^v e_v$ corresponds common differential, only added the e_v. Besides, $\otimes$ stands for the tensor product and $e^v = e^v(p)$, $e_v = e_v(p)$ and $g_{v\mu} = g_{v\mu}(p)$ are the quantities with respect to dual natural base at the point p.

Remark. In this paper, the only symbol used for summation is $\sum$. That is, without the presence of the symbol $\sum$, it does not mean "to take sum", even if a same letter occurred both in superscript and subscript simultaneously. Use the notation $\partial_i(\cdot)e^i$ similar to $dx^v e_v$.

The Riemann connections are: $\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$,

To a given a covariant vector A_i, the absolute gradient and the absolute differential are respectively

$$\nabla_j A_i = \partial_j A i - \sum_k \Gamma^k_{ji} A_k,$$

and
$$\delta A_i = \sum_j \nabla_j A_i dx^j = \sum_j (\partial_j A_i - \sum_k \Gamma^k_{ji} A_k) dx^j.$$

Let $A = e^i$, which only the A_i is one and others are zero, i.e. it is base. we have

$$\partial_j e^i = \nabla_j e^i = -\sum_k \Gamma^i_{jk} e^k, \tag{1}$$

$$\delta A_s = \sum_j -\Gamma^i_{js} dx^j, \qquad \nabla_\mu(0\cdots 0,1,0\cdots 0) = (-\Gamma^i_{\mu 1}, -\Gamma^i_{\mu 2}, \cdots, -\Gamma^i_{\mu n}).$$

Thus the connection $-\Gamma^i_{jk}$ is a gradient of the base e^i, and $-\Gamma^i_{jk}$ is the j-th component of partial derivative of e^i with respect to k-th coordinate. For a fixed i, $-\Gamma^i_{jk}$ is a tensor of order two. For three indices, it is not a tensor. More over by mapping the vector and the frame to a Euclidean space (in small scope), the differential of the component of the vector and the frame, i.e, the so-called absolute differential. As a result of this, people may define the absolute integral as a common Stieltjes integral of a vector and the corresponding frame at every point (we shall drop the word 'absolute' usually).

Definition 1 (absolute integral) An absolute integral is a Stieltjes integral, forming by Riemann integral, in which the bases of the frame at every point are adhered to the integrand function (vector, tensor) and the integral variable.

In Euclid space: $\int_a^b f(x_1)e^2 d(x_1 e^1) = (\int_a^b f(x_1)dx_1)e^2 e^1$, $x_2 = f(x_1)$, $(\cdots)$ is common Riemann integral, $e^2 e^1$ is tensor product, represented area unit. We research in non-Euclid space, For examples, linear integral, that is inner product forming

$$\int \sum_i A_i(p)e^i(p)d(e_i(p)x^i(p)) \qquad \oint_c \sum_i A_i(p)e^i(p)d(e_i(p)x^i(p)). \tag{2}$$

In small scope, since differential and integral are mutually inverse to each other,

$$dA = d\sum_i A_i e^i = \sum_{ij}\partial_j A_i e^j dx^j e_j e^i + \sum_k A_k de^k = \sum_{ij}((\partial_j A_i - \sum_k \Gamma^k_{ji}A_k)e^j dx^j e_j)e^i =$$
$$\sum_{ij}(\nabla_j A_i)e^i e^j dx^j e_j, \tag{3}$$

(Here, for simplicity, $e^i e^j$ is a tensor product, by dropping mark $\otimes$. $e^j \mathrm{d}x^j e_i$ is the scalar product approximating $\mathrm{d}x^j$. We shall adopt this notation hereinafter).

$$A(x^i + \mathrm{d}x^i) - A(x^i) = \int \sum_j \left(\sum_i (\nabla_j A_i) e^i e^j \right) \mathrm{d}x^j e_j, \tag{4}$$

Where c is the segment of straight line from x^i to $x^i + \mathrm{d}x^i$.

Definition 2 (absolute circulation) The (absolute) circulation of a vector A along a closed curve c is the integral of the scalar product $A \cdot \mathrm{d}l$ along the contour c

$$I = \oint_c (A \cdot \mathrm{d}l) = \oint_c \sum_i (A \cdot e^i) e^i \mathrm{d}(e_i x^i) = \oint_c \sum_i A_i e^i \mathrm{d}(e_i x^i), \tag{5}$$

In small scope, may configure frame at every point, neglecting the infinitesimal of the second order and when x is small, we have

$$\mathrm{d}e_i x^i = e_i \mathrm{d}x^i + x^i e_i' \mathrm{d}x \approx e_i \mathrm{d}x^i \qquad (\mathrm{d}e_i x^i = e_i \mathrm{d}x^i \text{ when } x = 0).$$

Thus in small scope or in compact space, the common integral formulas may be applied. Notice that on manifolds (space), as in this paper, we shall emphasize that the tensor at every point can be expressed with respect to the frame at that point. We shall denote $A_i = A_i(p)$, $\mathrm{d}x^i = \mathrm{d}x^i(p)$, to mean that they are expressed with respect to the frame at point p, Hence

$$I = \oint_c \sum_i A_i e^i \mathrm{d}e_i x^i =, \oint_c \sum_i A_i \mathrm{d}x^i = \oint_c \sum_i A_i(p) \mathrm{d}x^i(p) \qquad p \in c. \tag{5$_1$}$$

For the expression, which is expressed with respect to the frame at the point o, where the point o is in a neighborhood of p, we shall denote by $(A_i(p))_o$. Suppose the relation of frames is

$$e^i(p) = \sum_j a_j^i(p) e^j(o), \tag{6}$$

we have

$$\mathrm{d}x^i(p) = \sum_j a_j^i(p) \mathrm{d}x^j(o), \qquad (A_j(p))_o = \sum_i a_j^i(p) A_i(p). \tag{6$_1$}$$

Usually, in the neighborhood of the point o, differential forms are expressed as

$$\omega_1 = \sum_i (\omega_i(p))_o \mathrm{d}x^i(o),$$

But in common literatures p and o are omitted in the differential forms[7]. For the sake of symmetry and better reflecting the Riemannian space, in this paper, we use the notation $\omega_2 = \sum_i \omega_i(p) \mathrm{d}x^i(p)$, They are equivalent because by (6_1), we have

$$\omega = \sum_{ij} \omega_i(p) a_j^i(p) \mathrm{d}x^j(o) = \omega_1 = \omega_2.$$

Now let us discuss the exterior differential forms. by (1) we have,

$$(\partial_k)_p (a_j^i(p), j = 1, 2, \cdots, n) = -\sum_s \Gamma_{ks}^i (a_j^s(p), j = 1, 2, \cdots, n),$$

The $(\partial_k)_p$ means the partial differential with respect to the frame at the point p. Thus,

$$(\partial_k)_p (\mathrm{d}x^i(p)) = (\partial_k)_p \left(\sum_j a_j^i(p) \mathrm{d}x^j(o) \right) = \sum_{ij} - \Gamma_{ks}^i a_j^s(p) \mathrm{d}x^j(o) = \sum_s - \Gamma_{ks}^i \mathrm{d}x^s(p),$$

We have

$$\mathrm{d}(\mathrm{d}x^i(p)) = \sum_{sk} - \Gamma_{ks}^i \mathrm{d}x^s(p) \mathrm{d}x^k(p). \tag{7}$$

Usually that $\mathrm{d}(\mathrm{d}x^i) = 0$ really means is $\mathrm{d}(\mathrm{d}x^i(o)) = 0$[7,9], because the differential is taken with respect to a moving point p.

$$\mathrm{d}\omega = \mathrm{d}\left(\sum_i \omega_i(p) \mathrm{d}x^i(p) \right) = \sum_{ij} (\partial_j)_p \omega_i(p) \mathrm{d}x^j(p) \mathrm{d}x^i(p) + \omega_i(p) (\partial_j)_p (\mathrm{d}x^i(p)) \mathrm{d}x^j(p) =$$

$$\sum_{ij} (\partial_j)_p \omega_i(p) \mathrm{d}x^j(p) \mathrm{d}x^i(p) + \omega_i(p) \sum_s - \Gamma_{ks}^i \mathrm{d}x^s(p) \mathrm{d}x^j(p) =$$

$$\sum_{ij}\{(\partial_j)_p\omega_i(p) + \sum_s - \omega_s(p)\Gamma_{ij}^s\}\mathrm{d}x^i(p)\,dx^j(p).$$

So

$$\mathrm{d}\omega = \sum_{ij}\nabla_j\omega_i(p)\,\mathrm{d}x^i(p)\,\mathrm{d}x^j(p). \qquad (7_1)$$

It is easy to prove that the usually exterior differential forms

$$\mathrm{d}\omega = \sum_{ij}(\partial_j)_o(\omega_i(p))_o\mathrm{d}x^i(o)\,\mathrm{d}x^j(o),$$

is identified with above formula.

Stokes' formula (with respect to the frame at every point). Let us illustrate it by example.

Give tensor A_{ik}^{st}, we may make two forms: 1 - form W^1 and 2 - form W^2

$$W^1 = \sum_i\left(\sum_{kst}A_{ik}^{st}e_se_te^k\right)e^i\mathrm{d}x^ie^i = \sum_i\left(\sum_{kst}A_{ik}^{st}e_se_te^k\right)\mathrm{d}x^i(p),$$

$$W^2 = \sum_{ik}\left(\sum_{st}A_{ik}^{st}e_se_t\right)e^ke^i\mathrm{d}x^ie_i\mathrm{d}x^ke_k = \sum_{ik,\,i<k}\left(\sum_{st}A_{ik}^{st}e_se_t\right)\mathrm{d}x^i(p)\wedge\mathrm{d}x^k(p),$$

If anti-symmetrical in i,k, $\oint_{\partial s}W^1 = \iint_s\mathrm{d}W^1$, $\oiint_{\partial v}W^2 = \iiint_v\mathrm{d}W^2$.

In it d is common differential, but it made absolute differential and so on.

Definition 3 (absolute curl)　An absolute curl of a vector A is $\mathrm{rot}_{ij}A_i = \nabla_iA_j - \nabla_jA_i$. The curl of a vector has c_n^2 components.

The above definition is justified. When the limiting position of a curve is in i,j $(i<j)$ plane. We denote the area enclosed by it by S^{ij}. By Stokes formula,

$$\frac{1}{S^{ij}}\oint_c\sum_iA_ie^i\mathrm{d}e_ix^i = \frac{1}{S^{ij}}\oint_c\sum_iA_i(p)\,\mathrm{d}x^i(p) = \frac{1}{S^{ij}}\iint_{S^{ij}}\sum_{ij,\,i<j}(\partial_i(A_j\mathrm{d}x^j)\,\mathrm{d}x^i - \partial_j(A_i\mathrm{d}x^i)\,\mathrm{d}x^j) =$$

$$\frac{1}{S^{ij}}\iint_{S^{ij}}\sum_{ij,\,i<j}(\nabla_iA_j - \nabla_jA_i)\,\mathrm{d}x^i\mathrm{d}x^j \to \mathrm{rot}_{ij}A_i\,(\text{as } A^{ij}\to 0), \qquad (8)$$

$$\mathrm{rot}_{ij}A_ie^ie^j = (\nabla_iA_j - \nabla_jA_i)\,e^ie^j. \qquad (8_1)$$

For mixed tensor A_i^{st} of higher order, the curl with respect to covariant index, use $\sum_{st}A_i^{st}e_te_s$ instead of A_i, we have

$$\mathrm{rot}_{ij}A_t^{st} = \nabla_iA_j^{st} - \nabla_jA_i^{st}.$$

In generally, a divergence is the sum of outward flow of unit volume through its boundary surface (in process of limit). Therefore the quantity expressing flow and the quantity expressing area should be matched. There are two ways to express the quantity of area, namely $\mathrm{n}\,\mathrm{d}S$ and $\mathrm{d}e_ix^i\mathrm{d}e_jx^j$ (i.e. vector mode, form of anti-symmetric tensor mode). Correspondingly, use vector and anti-symmetric tensor to express flow. Combines them, we get the outflow through a small area is $\mathrm{A\,n\,d}S$ or $A_{ij}e^je^i\mathrm{d}e_ix^i\mathrm{d}e_jx^j$. Both of them are may be used in 3 - dimensional space. But, in n - dimensional case only the latter is convenience, We discuss it latter.

$$(V_{ijk})^{-1}\oiint_{s_{ij},\,i<j}\sum A_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad (\text{because } A_{ij}\mathrm{d}x^i\mathrm{d}x^j = A_{ij}e^fe^i\mathrm{d}e_ix^i\mathrm{d}e_jx^j,\ A_{ij} = -A_{ji}) =$$

$$(V_{ijk})^{-1}\iiint_{V_{ijk},\,i<j}\sum\partial_k(A_{ij}\mathrm{d}x^j)\,e^k\mathrm{d}e_kx^k = (V_{ijk})^{-1}\iiint_{V_{ijk},\,i<j}\sum\nabla_kA_{ij}\mathrm{d}x^k\mathrm{d}x^i\mathrm{d}x^j \to$$

$$(\nabla_kA_{ij} + \nabla_iA_{jk} + \nabla_jA_{ki}) \qquad \text{when } V\to V_{ijk}\to 0,$$

(V_{ijk} is the volume in 3 - dimensional space with base vectors i,j,k).

Definition 4 (absolute divergence)　In n - dimensional space, the divergence of 2nd - order anti-symmetrical, covariant tensor (A_{ij}) is

$$\mathrm{div}_{ijk}A_{ij} = (\nabla_kA_{ij} + \nabla_iA_{jk} + \nabla_jA_{ki}) \qquad (\text{with } c_n^3 \text{ ccomponents}). \qquad (9)$$

For mixed tensor (A_{ijk}^{st}) of higher covariant order, they are similar.

Now we shall point out the common characters and the difference between the absolute curl and usual curl.

a) Their relations with circulation are the same ,according (8) ,

b) The corresponding expression of ∇_j and ∂_j are similar ,

c) The divergence of the curl equals zero ,latter see (11) ,

$\mathrm{div}(\mathrm{rot}\, A) = \nabla_k(\mathrm{rot}_{ij}A_i) + \nabla_i(\mathrm{rot}_{jk}A_j) + \nabla_j(\mathrm{rot}_{ki}A_k) = 0.$

This shows the curvature (flow) is stationary ,forming tube shape field , (regarding the n - dimension space is decomposed into several 3 - dimension spaces) ,this is beneficial for studying motion of the particle.

d) Anti-symmetrical ,$\mathrm{rot}_{ij}A_i = -\mathrm{rot}_{ij}A_j$,

e) The curl of a vector is the ‘sum’ of its projection on tangential direction of the boundary of volume per unit. This is extension of the article a.[6] Notice that this ‘sum’ depends on the fashion of rotation. Considers small rectangular parallelepiped of dimensions $\mathrm{d}x^1 e_1, \mathrm{d}x^2 e_2, \mathrm{d}x^3 e_3$. In ‘summing’ $A_3 e^3 \mathrm{d}x^3 e_3 \mathrm{d}x^1 e_1$ takes positive sign ,while $A_2 e^2 \mathrm{d}x^1 e_1 \mathrm{d}x^2 e_2$ takes minus. The general formula[6]

$$\iiint_V (\partial_2 A_3 - \partial_3 A_2)\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \oiint_s A_3\mathrm{d}x^3\mathrm{d}x^1 - A_2\mathrm{d}x^1\mathrm{d}x^2$$

still holds. Because $\mathrm{rot}_{ij}A_i = \nabla_i A_j - \nabla_j A_i = \partial_i A_j - \partial_j A_i$[3] ,we have

$$\iiint_V \mathrm{rot}_{ij}A_i e^i e^j \mathrm{d}e_k x^k \mathrm{d}e_i x^i \mathrm{d}e_j x^j = \iiint_V (\nabla_i A_j - \nabla_j A_i)\, e^i e^j \mathrm{d}e_k x^k \mathrm{d}e_i x^i \mathrm{d}e_j x^j =$$

$$\oiint_s A_j e^j \mathrm{d}e_j x^j \mathrm{d}e_k x^k - A_i e^t \mathrm{d}e_k x^k \mathrm{d}e_i x^i.$$

i. e. $$\iiint_V (\mathrm{rot}_{ij}A_i)\, e_k \mathrm{d}x^k \mathrm{d}x^i \mathrm{d}x^j = \oiint_s (A_j \mathrm{d}x^j \mathrm{d}e_k x^k - A_i \mathrm{d}e_k x^k \mathrm{d}x^t).$$

f) In generalrot rotgrad $\neq 0$,in non-Euclidean-spaces. This situation leads us to introduce the curvature tensor R^i_{kjh}. Hereinafter we denote $e^i \mathrm{d}e_i x^i$ as $\mathrm{d}x^i$ for simplicity

$$\oint_c \sum_{ji} \nabla_j A_i e^i \mathrm{d}x^j = \frac{1}{2}\iint_{S^{kj}} \sum_{ijk} (\partial_k(\nabla_j A_i e^i \mathrm{d}x^j)\,\mathrm{d}x^k + \partial_j(\nabla_k A_i e^i \mathrm{d}x^k)\,\mathrm{d}x^j) =$$

$$\frac{1}{2}\iint_{S^{kj}} \sum_{ijk} (\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i))\, e^i \mathrm{d}x^j \mathrm{d}x^k =$$

$$\iint_{S^{kj}} \sum_{kj,\,k<j} \sum_i \sum_h R^h_{ij\,k} A_h e^i \mathrm{d}x^j \mathrm{d}x^k.$$

Recall the symbol c and S^{kj} ,

$$\frac{1}{S^{kj}} \oint_c \sum_{ji} \nabla_A{}_i e^i \mathrm{d}x^j \to \sum_i \mathrm{rot}_{kj,\,i} \nabla_j A_i e^i \;(\text{when } S^{kj} \to 0) ,$$

$\mathrm{rot}_{kj,\,i} \nabla_j A_i$ is defined by this identity above

$$\frac{1}{S^{kj}} \iint_{S^{kj}} \sum_{ih} R^h_{ij\,k} A_h e^i \mathrm{d}x^j \mathrm{d}x^k \to \sum_{ih} R^h_{ij\,k} A_h e^i \qquad (\text{when } S^{kj} \to 0) ,$$

I. e. , $$\sum_i \mathrm{rot}_{kj,\,i} \nabla_j A_i e^i = \sum_i (\nabla_k(\nabla_j A_i) - \nabla_j(\nabla_k A_i))\,{}^i e = \sum_{ih} R^h_{ij\,k} A_h e^i.$$

In case only one of the components of A say A_i is unity and the others are zero ,i. e. A is e^i ,we have

$$\mathrm{rot}_{kj,\,i} \nabla_j A_i = R^t_{ij\,k} , \qquad (\text{or ,simply ,}\mathrm{rot}_{kj} , \nabla_j e^t = R^t_{ij\,k}) . \tag{10}$$

Now the meaning of curvature $R^t_{ij\,k}$ **is very clearly :in** k , j - **plane ,it is the** i - **component of the curl of the gradient of base** e^t (t - **component of the curl of the gradient of base** e_i**)** . Thus it is can be seen that the curvature is a curl or its factor. It is a geometric representation of forming total difference of field on boundary (along tangent line or tangent plane) . Of course ,it represents the rotation force or its coefficient.

Now let us to prove c.) . By Stokes theorem ,the integral of the curl of a vector field taken over any closed surface (2 - dimension) equal zeros, then by Gauss theorem. It is may be calculated immediately also

$$\nabla_k(\mathrm{rot}_{ij}A_i) + \nabla_i(\mathrm{rot}_{jk}A_j) + \nabla_i(\mathrm{rot}_{ki}A_k) = 0. \tag{11}$$

Because $\nabla_k(\nabla_j A_i - \nabla_i A_j) + \nabla_j(\nabla_i A_k - \nabla_k A_i) + \nabla_i(\nabla_k A_j - \nabla_j A_k) =$

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)A_i + (\nabla_i\nabla_k - \nabla_k\nabla_i)A_j + (\nabla_j\nabla_i - \nabla_i\nabla_j)A_k =$$

$$\sum_h (R^h_{kji} + R^h_{ikj} + R^h_{jik})A_h = 0 \qquad (\text{According to } R^h_{kji} + R^h_{ikj} + R^h_{jik} = 0^{[5]}).$$

Conversely, from div rot = 0 and the arbitrariness of A_h, we conclude that $R^h_{kji} + R^h_{ikj} + R^h_{jik} = 0$. Its essence now is clear. Yields also

$$\nabla_s(\mathrm{rot}^i_{kj}\nabla_j A^i) + \nabla_k(\mathrm{rot}^i_{js}\nabla_s A^i) + \nabla_j(\mathrm{rot}^i_{sk}\nabla_k A^i) = 0.$$

$\mathrm{rot}^i_{kj}\nabla_j A^i$ is the i - componet of the curl of tensor $\nabla_j A^i$ in k, j - plane, when in the component of A, one and only one A_i is unity and the others are zero, i. e. It is base e^i. Give

$$\nabla_s(R^i_{kjt}) + \nabla_k(R^i_{jst}) + \nabla_j(R^i_{skt}) = 0. \tag{11$_1$}$$

This is famous Bianchi identity, its essence is div rot grad $e_t = 0$.

Theorem: A Riemannian-space to be Euclidean if and only if

$$\mathrm{rot\ grad} = 0. \tag{12}$$

Proof Because of $\mathrm{rot}^i_{kj}\nabla_j e_t = R^i_{kjt}$, when $R^i_{kjt} = 0$, Riemannian-space to be Euclidean-space.

Add a word: 1 When i is fixed, Γ^i_{ki} is 2 - order tensor (any one of the three indexes fixed is the same), the calculation of the curvature is easier.

$$R^i_{kjt} = \mathrm{rot}^i_{kj}\nabla_j e_i = \mathrm{rot}^i_{kj}\Gamma^i_{jt} = \nabla_k\Gamma^i_{jt} - \nabla_i\Gamma^i_{kt} \qquad (t: \text{fixed})$$

$$\nabla_k\Gamma^i_{jt} = \partial_k\Gamma^i_{jt} + \sum_s(\Gamma^i_{ks}\Gamma^s_{jt} - \Gamma^s_{kj}\Gamma^i_{st})$$

It is easy to know: $R^i_{kjt} = \partial_k\Gamma^i_{jt} - \partial_j\Gamma^i_{kt} + \sum_s(\Gamma^i_{ks}\Gamma^s_{jt} - \Gamma^i_{js}\Gamma^s_{kt})$.

2: It is easily seen for torsion tensor T^i_{kl} (in non Riemanian space)

$$T^i_{kl} = \mathrm{rot}_{lk}e^i.$$

Fibre bundle space (Gauge field): (E, B, π), surjective mapping $\pi: E \to B$, $\pi^{-1}(x)$ is the fibre at x, $x \in B^{[7]}$. By simply regarding them as: a set of points, there is a frame of $\pi^{-1}(x)$ at every point such that every point has a neighborhood homeomorphic to Euclidean-space. To discuss some quantities as $\sum_i A_i e^i$, e^i represent the base of frame of fibre $\pi^{-1}(x)$, $e_i = \sum_j g_{ij}e^j$, $(g_{ij}) = (g^{ig})^{-1}$, $g^{ij} = (e^i, e^j)$ is scalar product[6]. Except those formulas special in Riemannian space, the connection satisfy

$$\Gamma^k_{\mu\lambda} = \Gamma^k_{\lambda\mu}, \quad \sum_k A^k_{k'}\Gamma^{k'}_{\mu'\lambda'} = \sum_{\mu\lambda} A^\mu_{\mu'}A^\lambda_{\lambda'}\Gamma^k_{\mu\lambda} + \partial_{\mu'}A^k_{\lambda'} \quad (e_{\lambda'} = \sum_\lambda A^\lambda_{\lambda'}e_\lambda),$$

All the above discussion is still hold.

In physics, from general theory of relativity, to the particle physics (Gauge field theory), the curvature tensor plays very important role, in cosmology regarding as one of three most important numbers[8]. According this work, no other than the field of the curl of gradients play very important role. This is extremely accorded with the philosophy thought: difference (in total) is motive force. Regarding the curvature as basic dynamic notion is very reasonable.

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曲率张量 ,规范场的实质 :梯度的旋度场*

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摘要 :在黎曼空间、纤维丛空间(规范场)中 ,坚持使用逐点标架的基础上 ,普遍地引入绝对积分的概念(绝对微分逆运算) ,并对通常外微分 $d(dx^i) \neq 0$ 的意义和条件加以讨论 ,改写微分形为对称形式 ,使外微分和绝对微分联系起来. 在此基础上 ,改进 Stokes '公式 ,引入环量、旋度、散度(通常借助或类比欧氏空间的概念来 ,不精确 ,不能很好应用) . 证实 :曲率它正是非欧氏空间不为零的梯度的旋度. 并发现 :Bianchi 等式实质是 div(rot(grad)) = 0 ,曲率形成管形场 ,沿管不变. 附带 ,得到挠率也是旋度.

关键词 : Riemann - 空间 ;规范场 ;绝对积分 ;旋度 ;曲率

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