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Cauchy problem for a class of coupled hyperbolic systems of conservation laws*

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Abstract: The Cauchy problem for a class of coupled hyperbolic system of conservation laws is studied. Using convex hull of a potential function, the global solution including delta shock waves is constructed explicitly, and is then proved to be a measure solution directly.

Key words: Cauchy problem; coupled hyperbolic system; convex hull; delta-shock
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Consider the coupled hyperbolic system

$$\begin{cases} v_t + (vf(u))_x = 0, \\ (vu)_t + (vuf(u))_x = 0, \end{cases} \tag{1}$$

where $f(u)$ is a smooth monotone function, the sign of v is assumed to be unchangeable. The system (1) is coinciding with the one-dimensional transportation equations (i. e., zero-pressure gas dynamics) when $f(u) = u$, $v \geqslant 0$ and u are thought as the density and velocity, respectively.

The Riemann problem for (1) was solved completely in Ref. [1]. A distinctive feature is that delta shock waves develop in solutions. In earlier paper [2], the delta shock waves were found independently for a simplified mathematical model of Euler system. This kind of waves has been studied by many authors. The generalized Rankine-Hugoniot condition for a delta shock wave was proposed to describe the relationship among the location, propagation speed, weight and assignment of u on its discontinuity relative to the delta-shock [1—3]. In particular, the weight denotes the mass of concentrated particles. Thus the delta-shock may be interpreted as the galaxies in the universe, or the concentration of particles.

As for the Cauchy problem involving the delta-shock solutions, by introducing convex hull of a potential function, Chen et al (see [4]) obtained explicit construction of global measure solutions of Cauchy problem for the transportation equations. Making use of the method in [4], the present paper consider Cauchy problem of (1) with the initial data

$$(u, v)(0, x) = (u_0(x), v_0(x)), \tag{2}$$

where $v_0(x)$ expresses the mass distribution at $t = 0$. $u_0(x)$ is assumed to be boundedly measurable with respect to $v_0(x)$ and the total mass $\int_{R^1} v_0(dx) < +\infty$. If supp (v) is unbounded, we assume

$$\int_0^x \eta v_0(d\eta) \to +\infty \text{ as } |x| \to +\infty. \tag{3}$$

Definition 1 A pair $(u(x, t), v(\Delta, t))$ is called a measure solution of (1-2), if $(u(x, t), v(\Delta, t))$ satisfies

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$$\begin{cases}\int_{R_+^2}(\varphi_t + f(u)\varphi_x)v(\mathrm{d}x;t)\mathrm{d}t + \\ \qquad \int_{t=0}\varphi(x,0)v_0(\mathrm{d}x) = 0,\\ \int_{R_+^2} u(\phi_t + f(u)\phi_x)v(\mathrm{d}x;t)\mathrm{d}t + \\ \qquad \int_{t=0} u_0(x)\phi(x,0)(\mathrm{d}x) = 0,\end{cases} \tag{4}$$

for all test functions $\varphi, \phi \in C_0^\infty(R_+^2)(R_+^2 = R \times [0, \infty))$, where $v(\Delta; t) \geqslant 0$ denote the mass distribution on Borel measurable set Δ at t and $u(x, t)$ is boundedly measurable with respect to $v(\Delta; t)$.

1 Construction of solution and main result

The system (1) has a double eigenvalue $\lambda = f(u)$ and only one right eigenvector $\boldsymbol{r} = (1, 0)^T$, furthermore $\nabla\lambda \cdot \boldsymbol{r} \equiv 0$, which shows (1) is non-strictly hyperbolic. The characteristic equations of (1) are

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(u),\\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = 0,\\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -v\dfrac{\partial}{\partial x}(f(u)).\end{cases} \tag{5}$$

For simplicity, we suppose

$$f'(u) > 0, \quad v \geqslant 0. \tag{6}$$

The rest cases are considered similarly. If $u'_0(x) > 0$, then the characteristics do not intersect. If $u'_0(x) < 0$, the characteristics will overlap and the δ-shock will appear in the solutions. To solve (1—2), we need to construct a convex hull.

Introduce two projections $F: R^1 \to R^1$ and $B: R^1 \to R^1$

$$\begin{cases}F(x_0, t) = \int_A (\eta + tf(u_0(\eta)))v_0(\mathrm{d}\eta),\\ z = B(x_0) = \int_{\infty}^{x_0} v_0(\mathrm{d}\eta),\end{cases} \tag{7}$$

in which $A = [0, x_0)$ if $x_0 > 0$, and $A = [x_0, 0)$ if $x_0 < 0$. It is obvious that $B(x_0)$ is nondecreasing, then there exists inverse $B^{-1}(z) = x_0 = x_0(z)$. Let

$$G(z) = F \circ x_0(z) = F \circ B^{-1}(z), \tag{8}$$

and when $x_0 = B^{-1}(z) = \bar{x}$ (constant) $(z \in [z_1, z_2])$, we define

$$G(z) = F(x_0(z_1)) + \frac{F(x_0(z_2)) - F(x_0(z_1))}{z_2 - z_1}(z - z_1), \quad z \in (z_1, z_2), \tag{9}$$

where $z_1 = B(x_0 - 0)$, $z_2 = B(x_0 + 0)$. Noting (6), we get $G(z)$ is convex if only $u_0(x)$ is nondecreasing and

$$x = \left.\frac{\mathrm{d}G}{\mathrm{d}z}\right|_{z = B(x_0)} = x_0 + tf(u_0(x_0)), \text{ a. e.}, \tag{10}$$

where x can be explained as the particle location of x_0 at t. If $u_0(x)$ is decreasing, particles will interact and stick together at time t, i. e., δ-shock will appear. To get the global solution, we now construct the convex hull $H(z)$ of $G(z)$

$$H(z) = \inf\{(G(z), \tilde{G}(z)); \tilde{G}(z) = G(z_1) + \frac{G(z_1) - G(z_2)}{z_1 - z_2}(z - z_1), \quad \forall z_1, z_2 \in \int_{-\infty}^{+\infty} v_0(\mathrm{d}\eta)\}, \tag{11}$$

which is tuitively a rubber band covering $G(z)$ tightly, and has the property

$$H'(z - 0) \leqslant H'(z + 0). \tag{12}$$

Then R^1 can be divided into two kinds intervals L_1 and L_2, where $L_1 = \{x_0;\ G(z(x_0)) = H(z(x_0))\}$. For any $x_0 \in L_1$, the convex hull satisfies

$$G'(z - 0) \leqslant H'(z - 0) \leqslant H'(z + 0) \leqslant G'(z + 0). \tag{13}$$

If $G'(z - 0) = G'(z + 0)$, then

$$\left.\frac{\mathrm{d}H}{\mathrm{d}z}\right|_{z = B(x_0)} = \left.\frac{\mathrm{d}G}{\mathrm{d}z}\right|_{z = B(x_0)} = x_0 + tf(u(x_0)) \text{ hold a. e.} \tag{14}$$

$L_2 = \{x_0 \in R^1; H(z(x_0)) \neq G(z(x_0))\}$ is constructed by non-intersecting intervals Δ, which are open sets and $\Delta_i \cap \Delta_j = \Phi(i \neq j)$. It is obvious that Δ are countable. On every Δ_j, the tangent of $H(z)$ is a constant

$$\frac{\mathrm{d}H}{\mathrm{d}z} = \frac{G(z_2) - G(z_1)}{z_2 - z_1} =$$

$$\frac{\int_{(x_1,x_2)}(\eta+tf(u_0(\eta)))v_0(\mathrm{d}\eta)}{\int_{(x_1,x_2)}v_0(\mathrm{d}\eta)}, \tag{15}$$

where $z_1 = B(x_1), z_2 = B(x_2), (x_1, x_2) = \Delta_t$. Then we can define reflection $h(\cdot; t): R^1 \to R^1$

$$x = h(x_0; t) = \left.\frac{\mathrm{d}H(z)}{\mathrm{d}z}\right|_{z=B(x_0)}, \tag{16}$$

which reflects a point or a set to a point. Thus $h^{-1}(\cdot; t)$ maps a point to a point or a set. We now give some properties about $h(x_0; t)$.

Lemma 1 For fixed $t \geqslant 0$ and $x_1, x_2 \in R^1$, if $h(x_1; t) \leqslant h(x_2; t)$, then $x_1 \leqslant x_2$.

Lemma 2 For $0 < t^* < t$, then $h^{-1}(\cdot; t^*) \subset h^{-1}(\cdot; t)$.

Lemma 3 If $x_0 \in h^{-1}(x; t)$, then

$$|x_0 - x| \leqslant \max_{x\in R^1}|f(u_0(x))|\, t. \tag{17}$$

Based on the above analysis, the following main result can be obtained.

Theorem 1 If the initial data (2) satisfies the condition imposed, then Cauchy problem (1—2) possesses a measure solution $(v(\Delta, t), u(x, t))$ constructed as follows

$$\begin{cases} u(x;t) = \begin{cases} u_0(x_0), \text{ hold on } R^1 \text{ a.e.}, & h^{-1}(x;t) = x_0, \\ \dfrac{\int_{(x_1,x_2)}u_0(\eta)v_0(\mathrm{d}\eta)}{\int_{(x_1,x_2)}v_0(\mathrm{d}\eta)}, & h^{-1}(x;t) = (x_1, x_2), \end{cases} \\ v(\Delta; t) = v_0(h^{-1}(\Delta; t)), \end{cases} \tag{18}$$

where v denotes the mass distribution on Borel measurable set Δ at t.

2 Proof of theorem

For convenience, denote $h(h^{-1}(x; t), t+\delta_t) = h(x; t, t+\delta_t)$, which presents the location of particle x from time t to $t+\delta_t$. We first introduce some lemmas before we proof the theorem.

Lemma 4 For $\forall t^*, 0 \leqslant t^* \leqslant t$, if $h^{-1}(x; t)$ is a open set (x_1, x_2), then

$$x = \frac{\int_{h((x_1,x_2);t^*)}(\eta+(t-t^*)f(u(\eta,t^*)))v(\mathrm{d}\eta, t^*)}{\int_{h((x_1,x_2);t^*)}v(\mathrm{d}\eta; t^*)}. \tag{19}$$

It can be verified easily by (15).

Lemma 5 Suppose $x^* \in h(h^{-1}(x;t); t^*)$, $(0 < t^* < t)$, then

$$|x - x^*| \leqslant \max_{x\in R^1}|f(u(x, t^*))|(t-t^*). \tag{20}$$

Lemma 6 Suppose $\tilde{u}(t) = \max_{x\in R^1}|u(x,t)|$, then $\tilde{u}(t)$ is monotone and non-increasing.

Proof of Theorem Let′s first verify the solution (18) satisfying the first equation of (4). For any $\varphi \in C_0^\infty(R^2)$, $\exists\, T$ and (a, b) having

$$\int_{(a,b)}(\varphi(x,T)v(\mathrm{d}x;T) - \varphi(x,0)v_0(\mathrm{d}x)) + \int_{(a,b)}\varphi(x)v_0(\mathrm{d}x) = 0. \tag{21}$$

So we just need to prove

$$\int_{(a,b)}(\varphi(x,T)v(\mathrm{d}x;T) - \varphi(x,0)v_0(\mathrm{d}x)) = \int_0^T\int_{(a,b)}(\varphi_t + f(u)\varphi_x)v(\mathrm{d}x;t)\mathrm{d}t. \tag{22}$$

We now separate $[0, T]$ into n parts $[t_i, t_{i+1}]$ $(i = 0, 1, \cdots n-1)$, where $t_0 = 0$, $t_n = T$, and denote $t_{i+1} - t_i$ by Δt_i, then the right side of (22) is transformed into

$$\sum_{i=0}^{n-1}\int_{(a,b)}(\varphi(x,t_{i+1})v(\mathrm{d}x;t_{i+1}) - \varphi(x,t_i)v(\mathrm{d}x;t_i)) =$$

$$\sum_{i=0}^{n-1}\Big[\int_{(a,b)}(\varphi(x,t_{i+1}) - \varphi(x,t_i)v(\mathrm{d}x;t_{i+1})) + \int_{(a,b)}(\varphi(x,t_i)v(\mathrm{d}x;t_{i+1}) - \varphi(x,t_i)v(\mathrm{d}x;t_i))\Big] =$$

$$\sum_{i=0}^{n-1}\Big[\int_{(a,b)}\varphi_t(x,\bar{t}_i)\,v(\mathrm{d}x;t_{i+1})\cdot\Delta_i+$$
$$\int_{(a,b)}(\varphi(h(x;t_i,t_{i+1}),t_i)-\varphi(x,t_i))\,v(\mathrm{d}x;t_i)\Big]=$$
$$J_1+J_2, \tag{23}$$

where $\bar{t}_i\in(t_i,t_{i+1})$. Because

$$\int_{(a,b)}\varphi_t(x,t+\Delta t)\,v(\mathrm{d}x;t+\Delta t)-$$
$$\varphi_t(x,t)\,v(\mathrm{d}x;t)=$$
$$\int_{(a,b)}(\varphi_t(h(x;t,t+\Delta t),t)-$$
$$\varphi_t(x,t))\,v(\mathrm{d}x;t)=$$
$$\int_{(a,b)}\varphi_{tx}(x+\theta(h(x;t,t+\Delta t)-x),t)\cdot$$
$$(h(x;t,t+\Delta t)-x)\,v(\mathrm{d}x;t), \tag{24}$$

where $\theta\in(0,1)$, and

$$h(x;t,t+\Delta t)-x\to 0(\Delta t\to 0), \tag{25}$$

then $\int_{(a,b)}\varphi_t(x,t)\,v(\mathrm{d}x;t)$ is continuous, and

$$J_1\to\int_0^T\int_{(a,b)}\varphi_t(x,t)\,v(\mathrm{d}x;t),\ n\to\infty,$$
$$\max(\Delta t_i)\to 0. \tag{26}$$

For J_2, we have

$$J_2=\sum_{i=0}^{n-1}\int_{(a,b)}\varphi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-$$
$$x)\,v(\mathrm{d}x;t_i), \tag{27}$$

where (a,b) can be separated into $\Sigma_1\cup\Sigma_2$, $\Sigma_1=\{\cup h^{-1}(x;t_i,t_{i+1});h^{-1}(x;t_i,t_{i+1})=\{x_i\}\}$, and $\Sigma_2=\{\cup\Delta_j;\ \Delta_i\cap\Delta_j=\Phi(i\neq j),$
$h(\Delta_j;t_i,t_{i+1})=x\}$. Then

$$J_2=\sum_{i=0}^{n-1}\Big[\int_{\Sigma_1}\varphi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-x)\cdot$$
$$v(\mathrm{d}x;t_i)+$$
$$\int_{\Sigma_2}\varphi_x(\bar{x},t_i)(h(x;t_i,t_{i+1})-x)\cdot$$
$$v(\mathrm{d}x;t_i)\Big]=J_{21}+J_{22}. \tag{28}$$

From (10), we have

$$J_{21}=\sum_{i=0}^{n-1}\int_{\Sigma_1}\varphi_x(\bar{x},t_i)f(u(x,t_i))\,v(\mathrm{d}x;t_i)\,\Delta t_i. \tag{29}$$

For J_{22}, we have

$$J_{22}=\sum_{i=0}^{n-1}\sum_j\int_{\Delta_j}\varphi_x(\bar{x},t_i)\cdot$$
$$\frac{\int_{\Delta_j}(\eta-x)\,v(\mathrm{d}\eta;t_i)}{\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}v(\mathrm{d}x;t_i)+$$
$$\sum_{i=0}^{n-1}\sum_j\int_{\Delta_j}\varphi_x(\bar{x},t_i)\cdot$$
$$\Big[\frac{\int_{\Delta_j}f(u(\eta,t_i))\,v(\mathrm{d}\eta;t_i)}{\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}-$$
$$f(u(x,t_i))\Big]v(\mathrm{d}x;t_i)\,\Delta t_i+$$
$$\sum_{i=0}^{n-1}\sum_j\int_{\Delta_j}\varphi_x(\bar{x},t_i)f(u(x,t_i))\cdot$$
$$v(\mathrm{d}x;t_i)\,\Delta t_i=J_{22}^{(1)}+J_{22}^{(2)}+J_{22}^{(3)}. \tag{30}$$

We now prove $J_{22}^{(1)}\to 0(\max_i\Delta_i\to 0)$. For $\forall\xi$, $\eta\in\Delta_j$,

$$|\eta-\xi|\leqslant|\Delta_j|\leqslant 2\max_{x\in\Delta_j}|f(u(x;t_i))|\,\Delta t_i\leqslant$$
$$2\max_{x\in R^1}|f(u_0(x,t_i))|\,\Delta t_i=C_1\Delta t_i, \tag{31}$$

in which $|\Delta_j|$ is the Radon measurement of Δ_j, $C_1=2\max_x|f(u_0(x,t_i))|$. Then

$$J_{22}^{(1)}=\sum_{i=0}^{n-1}\sum_j\frac{1}{2\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}\cdot$$
$$\int_{\Delta_j}\int_{\Delta_j}(\varphi_x(\bar{x},t_i)(\eta-x)+$$
$$\varphi_x(\bar{\eta},t_i)(x-\eta))\,v(\mathrm{d}x;t)=$$
$$\sum_{i=0}^{n-1}\sum_j\frac{1}{2\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}\cdot$$
$$\int_{\Delta_j}\int_{\Delta_j}\varphi_{xx}(\tilde{x},t_i)(\bar{\eta}-\tilde{x})(\eta-x)\cdot$$
$$v(\mathrm{d}\eta;t_i)\,v(\mathrm{d}x;t_i), \tag{32}$$

where $\bar{\eta}=\eta+\theta(h(\eta;t_i)-\eta)(\theta\in(0,1))$, $\tilde{x}\in(\bar{x},\bar{\eta})$, then

$$|J_{22}^{(1)}|\leqslant\sum_{i=0}^{n-1}\sum_j\frac{1}{2\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}\cdot$$
$$\int_{\Delta_j}\int_{\Delta_j}|\varphi_{xx}(\tilde{x},t_i)|\,|\bar{\eta}-\bar{x}|\cdot$$
$$|(\eta-x)|\,v(\mathrm{d}\eta;t_i)\,v(\mathrm{d}x;t_i)\leqslant$$

$$\sum_{i=0}^{n-1}\sum_{j}\frac{C_1^2C_2(\Delta t_i)^2}{2\int_{\Delta_j}v(\mathrm{d}\eta;t_i)}\cdot$$

$$\int_{\Delta_j}\int_{\Delta_j}v(\mathrm{d}\eta;t_i)v(\mathrm{d}x;t_i)=$$

$$\sum_{i=0}^{n-1}\frac{C_1^2C_2}{2}\int_{\sum_2}v(\mathrm{d}\eta;t_i)(\Delta t_i)^2\leqslant$$

$$\frac{C_1^2C_2TC_3}{2}\max_i(\Delta t_i),\qquad(33)$$

in which $C_2=\max_{x\in(a,b)}|\varphi_{xx}(x,t)|$, $C_3=\int_{(a,b)}v(\mathrm{d}\eta)$. Then $J_{22}^{(1)}\to 0(\max_i(\Delta t_i)\to 0)$. Similarly we can prove $J_{22}^{(2)}\to 0(\max_i(\Delta t_i)\to 0)$. Then we get

$$J_2\to\int_0^T\int_{(a,b)}\varphi_x(x,t)f(u,t)v(\mathrm{d}x;t)\mathrm{d}t,$$
$$\max(\Delta t_i)\to 0.\qquad(34)$$

Combining with (26), the proof of (21) is completed. By introducing momentum

$$I(\Delta;t)=\int_{\Delta}u(\eta,t)v(\mathrm{d}\eta;t),\qquad(35)$$

the second equation of (4) can be verified in the same way. The proof is omitted.

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一类偶合双曲守恒律系统的柯西问题*

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摘要: 借助于一个势函数的凸包, 研究了一类偶合双曲守恒律系统的柯西问题. 构造了包含狄拉克激波的整体显示解, 并直接证明了所构造的解是一个测度解.

关键词: 柯西问题; 偶合双曲系统; 凸包; 狄拉克激波

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