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Strong convergence theorems for common fixed points of a finite family of multi-valued Φ-pseudocontractive mappings*

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Abstract: By introducing a new modified Ishikawa iterative process with errors, it is identified that the iterative sequence converges strongly to unique common fixed point of a finite family of multi-valued Φ-pseudocontractive mappings in Hilbert space or uniformly smooth Banach space. The results presented in this paper improve and extend some corresponding results in this field.

Key words: modified Ishikawa iterative process; errors; common fixed points; finite family of multi-valued Φ-pseudocontractive mappings

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The main objective of this paper is to study the strong convergence theorems for common fixed points of a finite family of multi-valued Φ-pseudocontractive mappings by introducing a new modified Ishikawa iteration with errors. First of all, we describe the setting of our results.

Let E be a real Banach space, E^* be the dual space of E. $\langle .,. \rangle$ denote the pairing of E and E^*. The mapping $J: E \to 2^{E^*}$ defined by

$$J(x) = \{j \in E^* : \langle x, j \rangle = \|x\| \cdot \|j\|, \|j\| = \|x\|\} \tag{1}$$

is called the normalized duality mapping.

Definition 1 Let E be a real Banach space, K a nonempty subset of E. Suppose that $T: K \to 2^K$ is a multi-valued mapping.

(1) T is said to be multi-valued strongly pseudocontractive, if for any $x, y \in K$, there exists a $j(x-y) \in J(x-y)$ such that $\langle u-v, j(x-y) \rangle \leqslant k\|x-y\|^2$ for all $u \in Tx, v \in Ty$, where k is a constant in $(0, 1)$.

(2) T is said to be multi-valued ϕ-pseudocontractive, if for any $x, y \in K$, there exists a $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi: [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that $\langle u-v, j(x-y) \rangle \leqslant \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$ for all $u \in Tx, v \in Ty$.

(3) T is said to be multi-valued Φ-pseudocontractive, if for any $x, y \in K$, there exists a $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi: [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that $\langle u-v, j(x-y) \rangle \leqslant \|x-y\|^2 - \Phi(\|x-y\|)$ for all $u \in Tx, v \in Ty$.

(4) T is said to be multi-valued pseudocontractive, if for any $x, y \in K$, there exists a $j(x-y) \in$

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$J(x-y)$ such that $\langle u-v, j(x-y)\rangle \leqslant \|x-y\|^2$ for all $u \in Tx$, $v \in Ty$.

Obviously, Φ-pseudocontractive mapping is a more general mapping than ψ-pseudocontractive mapping or strongly pseudocontractive mapping.

The concept of accretive mapping was introduced independently by Browder[1] and Kato[2] in 1967. An early fundamental result in the theory of accretive operator, due to Browder, states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \qquad u(0) = u_0 \tag{2}$$

is solvable if T is locally Lipschitzian and accretive on E.

In general, Lipschitzian condition is necessary when the fixed point problems of mappings are discussed. For weakening this condition, we introduce a concept of multi-valued generalized Lipschitzian.

Definition 2 Let E be a real Banach space, T be a multi-valued mapping from E into 2^E. T is said to be multi-valued generalized Lipschitzian if there exists $C > 1$ such that

$$\|u-v\| \leqslant C(1+\|x-y\|) \tag{3}$$

for any given $x \in E$, $y \in E$ and all $u \in Tx$, $v \in Ty$, where C is said to be generalized Lipschitzian constant of T.

In fact, if T is Lipschitzian, uniformly continuous or the range of T is bounded, then T is generalized Lipschitzian. Conversely, it is not certainly true. So multi-valued generalized Lipschitzian mapping is a kind of more general mappings than Lipschitzian mappings.

For studying the fixed points of strongly pseudocontractive mapping, XU Y G[3] introduced the concept of modified Ishikawa iteration with random errors. The definition is as follows.

Definition 3 Let E is a real Banach space, K be a nonempty bounded closed convex subset of E and $T: K \to K$ be a mapping. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0, 1)$ and $\{e_n\}$, $\{f_n\}$ are arbitrary sequences in K. The sequence $\{x_n\} \subset K$ defined by

$$x_0 \in K,\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n e_n, \qquad n \geqslant 0, \tag{4}$$
$$y_n = (1-\beta_n-\delta_n)x_n + \beta_n T x_n + \delta_n f_n, \qquad n \geqslant 0$$

is called the modified Ishikawa iteration with errors.

In real Banach space or uniformly smooth Banach space, Xu's concept has been widely used to study the approximation problems of fixed points for strongly pseudocontractive mapping, ψ-pseudocontractive mapping and Φ-pseudocontractive mapping by various authors and some convergence results[4-14] are obtained. But all their work is just for one mapping. For studying the common fixed points of a finite family of multi-valued mappings, we introduce the following modified Ishikawa iterative process with errors. Meanwhile, Theorem 2 and Theorem 3 presented in this paper improve and extend the corresponding results in ref. [4-14].

Definition 4 Let E be a real Banach space, K be a nonempty closed convex subset of E. Suppose that $T_1, T_2, \cdots, T_N: K \to 2^K$ are N multi-valued mappings. For any given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$x_{n+1} \in (1-\alpha_n-\gamma_n)x_n + \alpha_n T_{n+1} y_n + \gamma_n e_n,$$
$$y_n \in (1-\beta_n-\delta_n)x_n + \beta_n T_n x_n + \delta_n f_n, \qquad n \geqslant 0 \tag{5}$$

is called the Modified Ishikawa iterative sequence with errors, where $T_n = T_{n \bmod N}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are four real sequences in $[0, 1)$, $\{e_n\}$, $\{f_n\}$ are two bounded sequences in K.

Note In Definition 4, when $T_1 = T_2 = \cdots = T_N$ and T_i are single valued mappings($i = 1, 2, \cdots, N$), then the modified Ishikawa iterative sequence $\{x_n\}$ defined in (5) is same as Xu's.

Putting $F = \bigcap_{i=1}^{N} F(T_i)$ and $I = \{1, 2, \cdots, N\}$, where $F(T_i)$ is the set of fixed points of T_i, i.e.,

$F(T_i) = \{x \in K: x \in T_i x\}, i \in I$.

1 Preliminaries

In the proof of our main results, we shall need the following proposition and lemmas.

The following proposition can be found in ref. [15, Chap. 1].

Proposition 1 Let E be a real Banach space and J be a normalized duality mapping from E into 2^{E^*}. Then,

(1) J is bounded, i. e., for any bounded set $A \subset E$, $J(A)$ is bounded;

(2) E^* is strictly convex $\Leftrightarrow E$ is refletive and smooth;

(3) E is smooth $\Leftrightarrow J$ is single valued;

(4) E^* is uniformly convex $\Leftrightarrow E$ is uniformly smooth $\Leftrightarrow J$ is uniformly continuous on any bounded subset of E.

Lemma 1[4] Let E be a Banach space and J be a normality duality mapping. Then for any given $x, y \in E$, the following inequality holds

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \forall j \in J(x + y). \tag{6}$$

Lemma 2[16] Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leqslant (1 + \delta_n) a_n + b_n. \qquad n \geqslant 1$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In addition, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

2 Main results

Now, many results about iterative approximation for fixed points of multi-valued mappings need K is bounded or the ranges of mappings are bounded. The following result abolishes the boundness of K when E is a Hilbert space and mappings are multi-valued generalized Lipschitzian.

Theorem 1 Let E be a Hilbert space, K be a nonempty closed convex subset of E. Suppose that $\{T_i, i \in I\}$ are N multi-valued generalized Lipschitzian Φ-pseudocontractive mappings from K into 2^K, $\{x_n\}$ is defined by (5), where $\{e_n\}$, $\{f_n\}$ are bounded and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \beta_n = 0$, $\gamma_n = o(\alpha_n^2)$;

(2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty} \alpha_n(\beta_n + \delta_n) < +\infty$.

If $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$, then for arbitrary $x_0 \in E$, $\{x_n\}$ converges strongly to unique common fixed point of the mappings $\{T_i, i \in I\}$ in K.

Proof Let $q_1, q \in F$, then $q_1 \in T_i q_1$, $q \in T_i q$ for all $i \in I$. Since T_i is a Φ-pseudocontractive mapping, we have

$$\|q_1 - q\|^2 = \langle q_1 - q, j(q_1 - q)\rangle \leqslant \|q_1 - q\|^2 - \Phi_i(\|q_1 - q\|).$$

So $\Phi_i(\|q_1 - q\|) \leqslant 0$. It follows from Definition 1 that $q_1 = q$. It implies that $T_1, T_2, \cdots, T_N$ have only one common fixed point when $F \neq \phi$. Setting $q \in F = \bigcap_{i=1}^{N} F(T_i)$, C_i is the generalized Lipschitzian constant of T_i and $C = \max_{i\in I} C_i$.

From (5), we know that there exists $u_n \in T_{n+1} y_n$ and $v_n \in T_n x_n$ such that

$$x_{n+1} = (1 - \alpha_n - \gamma_n) x_n + \alpha_n u_n + \gamma_n e_n,$$

$$y_n = (1-\beta_n-\delta_n)x_n+\beta_n v_n+\delta_n f_n. \tag{7}$$

Putting $M = \sup\{\|e_n - q\|, \|f_n - q\|, n \geqslant 0\}$. By (7), we have

$$\|y_n - q\| = \|(1-\beta_n-\delta_n)(x_n-q)+\beta_n(v_n-q)+\delta_n(f_n-q)\| \leqslant (1+C\beta_n)\|x_n-q\|+C\beta_n+\delta_n M, \tag{8}$$

$$\left\|\frac{y_n-q}{1+\|x_n-q\|}\right\| \leqslant 1+2C\beta_n+M\delta_n \leqslant 1+2C+M, \tag{9}$$

$$\|x_{n+1}-q\| = \|(1-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(u_n-q)+\gamma_n(e_n-q)\| \leqslant (1+C\alpha_n+C^2\alpha_n\beta_n)\|x_n-q\|+C\alpha_n+C^2\alpha_n\beta_n+C\alpha_n\delta_n M+\gamma_n M, \tag{10}$$

$$\left\|\frac{x_{n+1}-q}{1+\|x_n-q\|}\right\| \leqslant 1+4C^2+CM+M, \tag{11}$$

$$\left\|\frac{u_n-q}{1+\|x_n-q\|}\right\| \leqslant C(2+2C+M), \tag{12}$$

$$\begin{aligned}\|x_{n+1}-y_n\| = {}& \|(1-\alpha_n-\gamma_n)x_n+\alpha_n u_n+\gamma_n e_n-(1-\beta_n-\delta_n)x_n-\beta_n v_n-\delta_n f_n\| = \\ & \|(\beta_n+\delta_n-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(u_n-q)+\gamma_n(e_n-q)- \\ & \beta_n(v_n-q)-\delta_n(f_n-q)\| \leqslant (\beta_n+\delta_n)\|x_n-q\|+C\alpha_n(1+\|y_n-q\|)+ \\ & C\beta_n(1+\|x_n-q\|)+\gamma_n\|e_n-q\|+\delta_n\|f_n-q\| \leqslant \\ & (2C^2\alpha_n+2C\beta_n+\delta_n)\|x_n-q\|+2C^2\alpha_n+C\beta_n+\gamma_n M+2C\delta_n M,\end{aligned} \tag{13}$$

$$\left\|\frac{x_{n+1}-y_n}{1+\|x_n-q\|}\right\| \leqslant 3C^2\alpha_n+3C\beta_n+\delta_n+\gamma_n M+2C\delta_n M \to 0 (\text{as } n \to \infty). \tag{14}$$

Setting $h_n = \left\|\frac{x_{n+1}-y_n}{1+\|x_n-q\|}\right\|$. It follows from (14) that $\lim_{n\to\infty} h_n = 0$. Since E is a Hilbert space, normalized duality mapping j is a identity mapping. By Lemma 1, we have

$$\begin{aligned}\left\|\frac{y_n-q}{1+\|x_n-q\|}\right\|^2 \leqslant {}& (1-\beta_n-\delta_n)^2\frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+ \\ & 2\left\langle\frac{\beta_n(v_n-q)+\delta_n(f_n-q)}{1+\|x_n-q\|}, \frac{y_n-q}{1+\|x_n-q\|}\right\rangle \leqslant \\ & \frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\beta_n\left\|\frac{v_n-q}{1+\|x_n-q\|}\right\|\cdot\left\|\frac{y_n-q}{1+\|x_n-q\|}\right\|+ \\ & 2\delta_n\left\|\frac{e_n-q}{1+\|x_n-q\|}\right\|\cdot\left\|\frac{y_n-q}{1+\|x_n-q\|}\right\| \leqslant \\ & \frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2(1+2C+M)(C\beta_n+M\delta_n).\end{aligned} \tag{15}$$

By (15) and Lemma 1, we have

$$\begin{aligned}\left\|\frac{x_{n+1}-q}{1+\|x_n-q\|}\right\|^2 = {}& \left\|\frac{(1-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(u_n-q)+\gamma_n(e_n-q)}{1+\|x_n-q\|}\right\|^2 \leqslant \\ & \frac{(1-\alpha_n-\gamma_n)^2\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\left\langle\frac{\alpha_n(u_n-q)+\gamma_n(e_n-q)}{1+\|x_n-q\|}, \frac{x_{n+1}-q}{1+\|x_n-q\|}\right\rangle \leqslant \\ & \frac{(1-\alpha_n-\gamma_n)^2\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\alpha_n\left\langle\frac{u_n-q}{1+\|x_n-q\|}, \frac{x_{n+1}-q}{1+\|x_n-q\|}-\right. \\ & \left.\frac{y_n-q}{1+\|x_n-q\|}\right\rangle+2\alpha_n\left\langle\frac{u_n-q}{1+\|x_n-q\|}, \frac{y_n-q}{1+\|x_n-q\|}\right\rangle+ \\ & 2\gamma_n\left\langle\frac{e_n-q}{1+\|x_n-q\|}, \frac{x_{n+1}-q}{1+\|x_n-q\|}\right\rangle \leqslant \\ & \frac{(1-\alpha_n)^2\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\alpha_n\frac{\|y_n-q\|^2}{(1+\|x_n-q\|)^2}-2\alpha_n\frac{\Phi_{n+1}(\|y_n-q\|)}{(1+\|x_n-q\|)^2}+\end{aligned}$$

$$2M\gamma_n\left\|\frac{x_{n+1}-q}{1+\|x_n-q\|}\right\|+2\alpha_n h_n\frac{\|u_n-q\|}{1+\|x_n-q\|}\leqslant$$
$$\frac{(1+\alpha_n^2)\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+4(1+2C+M)\alpha_n(C\beta_n+M\delta_n)-$$
$$2\alpha_n\frac{\Phi_{n+1}(\|y_n-q\|)}{(1+\|x_n-q\|)^2}+2M\gamma_n(1+4C^2+CM+M)+2\alpha_n h_n C(2+2C+M). \tag{16}$$

It follows from (16) that

$$\|x_{n+1}-q\|^2\leqslant(1+\alpha_n^2)\|x_n-q\|^2+L_n(1+\|x_n-q\|)^2-2\alpha_n\Phi_{n+1}(\|y_n-q\|)\leqslant$$
$$(1+\alpha_n^2)\|x_n-q\|^2+2L_n(1+\|x_n-q\|^2)-2\alpha_n\Phi_{n+1}(\|y_n-q\|)\leqslant$$
$$(1+2L_n+\alpha_n^2)\|x_n-q\|^2+2L_n-2\alpha_n\Phi_{n+1}(\|y_n-q\|). \tag{17}$$

Where $L_n=4(1+2C+M)\alpha_n(C\beta_n+M\delta_n)+2M\gamma_n(1+4C^2+CM+M)+2\alpha_n h_n C(2+2C+M)$. Because $\gamma_n=o(\alpha_n^2)$, $\sum_{n=1}^{\infty}\alpha_n^2<\infty$, there exists a sequence $\{d_n\}$ such that $\gamma_n=d_n\alpha_n^2$ where $\lim_{n\to\infty}d_n=0$. So, $\sum_{n=1}^{\infty}\gamma_n<\infty$ and (17) can be written as follows

$$\|x_{n+1}-q\|^2\leqslant(1+2\alpha_n D_n+\alpha_n^2)\|x_n-q\|^2+2\alpha_n D_n-2\alpha_n\Phi_{n+1}(\|y_n-q\|). \tag{18}$$

Where $D_n=4(1+2C+M)(C\beta_n+M\delta_n)+2Md_n\alpha_n(1+4C^2+CM+M)+2h_nC(2+2C+M)$. Since E is a Hilbert space, therefore

$$h_n=\left\|\frac{x_{n+1}-y_n}{1+\|x_n-q\|}\right\|\leqslant 3C^2\alpha_n+3C\beta_n+\delta_n+\gamma_n M+2C\delta_n M.$$

It is easy to see that $\sum_{n=1}^{\infty}\alpha_n h_n$ is convergent from the conditions in this theorem. It follows that $\sum_{n=1}^{\infty}\alpha_n D_n<\infty$. From Lemma 2 and (18), we know that $\{x_n\}$ converges strongly to q if $\inf\{\|x_n-q\|, n\geqslant 0\}=0$.

Now, we prove that $\inf\{\|y_n-q\|, n\geqslant 0\}=0$. If $\inf\{\|y_n-q\|, n\geqslant 0\}=\delta>0$, then $\|y_n-q\|\geqslant\delta$ for any $n\geqslant 0$. Let $\Phi(\delta)=\min_{i\in I}\Phi_i(\delta)$. Since $\lim_{n\to\infty}2D_n=0$, there exists positive integer K such that $2D_n\leqslant\Phi(\delta)$ for all $n\geqslant K$. From (18), as $n\geqslant K$, we have

$$\|x_{n+1}-q\|^2\leqslant(1+2\alpha_n D_n+\alpha_n^2)\|x_n-q\|^2-\alpha_n\Phi(\delta)\leqslant$$
$$\|x_K-q\|^2\prod_{i=K}^{n}(1+2\alpha_i D_i+\alpha_i^2)-\Phi(\delta)\sum_{i=K}^{n}\alpha_i\leqslant$$
$$\|x_K-q\|^2e^{\sum_{i=K}^{n}(2\alpha_i D_i+\alpha_i^2)}-\Phi(\delta)\sum_{i=K}^{n}\alpha_i.$$

Since $\sum_{n=1}^{\infty}(2\alpha_n D_n+\alpha_n^2)<\infty$, setting $M=\sum_{n=1}^{\infty}(2\alpha_n D_n+\alpha_n^2)$, then, as $n\geqslant K$,

$$\|x_{n+1}-q\|^2\leqslant\|x_K-q\|^2e^M-\Phi(\delta)\sum_{i=K}^{n}\alpha_i. \tag{19}$$

In addition, since $\sum_{n=1}^{\infty}(4C\alpha_n h_n+\alpha_n^2)<\infty$ and $\sum_{n=1}^{\infty}4C\alpha_n h_n<\infty$, by Lemma 2 and (18), we know that $\lim_{n\to\infty}\|x_n-q\|$ exists. It is easy to see that $\sum_{n=1}^{\infty}\alpha_n<\infty$ from (19), which is a contradiction. By virtue of this contradiction, we know that $\inf\{\|y_n-q\|, n\geqslant 0\}=0$. Therefore, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\|y_{n_j}-q\|\to 0$ as $j\to\infty$. In addition,

$$y_{n_j}-q=(1-\beta_{n_j}-\delta_{n_j})(x_{n_j}-q)+\beta_{n_j}(v_{n_j}-q)+\delta_{n_j}(f_{n_j}-q).$$

Therefore

$$\|y_{n_j}-q\| \geqslant (1-\beta_{n_j}-\delta_{n_j})\|x_{n_j}-q\|-\beta_{n_j}\|v_{n_j}-q\|-\delta_{n_j}\|x_{n_j}-q\|.$$

So $\lim_{j\to\infty}\|x_{n_j}-q\|=0$, i.e., the subsequence $\{x_{n_j}-q\}$ of $\{x_n-q\}$ converges strongly to zero. Now we can conclude that $\inf\{\|x_n-q\|, n\geqslant 0\}=0$. Therefore, $\{x_n\}$ converges strongly to q. This completes the proof.

Theorem 2 Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E. Suppose that $\{T_i, i\in I\}$ are N multi-valued Φ-pseudocontractive mappings from K into 2^K. $\{x_n\}$ is defined by (5), where $\{e_n\}$, $\{f_n\}$ are bounded in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(1) $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\delta_n=0$ and $\sum_{k=1}^{\infty}\alpha_n=+\infty$; (2) $\gamma_n=0(\alpha_n)$.

If $F=\bigcap_{i=1}^{N}F(T_i)\neq\phi$, then for arbitrary $x_0\in K$, $\{x_n\}$ converges strongly to unique common fixed point of the mappings $\{T_i, i\in I\}$ in K.

Proof Similar to the proof of Theorem 1, we see that $T_1, T_2, \cdots, T_N$ have only one common fixed point when $F\neq\phi$. From (5), we know that there exists $u_n\in T_{n+1}y_n$ and $v_n\in T_nx_n$ such that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\gamma_n)x_n+\alpha_nu_n+\gamma_ne_n,\\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_nv_n+\delta_nf_n. \end{aligned} \tag{20}$$

Using (20) and Lemma 1, we have

$$\begin{aligned} \|x_{n+1}-q\|^2 \leqslant{} & \|(1-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(u_n-q)+\gamma_n(e_n-q)\|^2 \leqslant\\ & (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\|y_n-q\|^2-2\alpha_n\Phi_{n+1}(\|y_n-q\|)+\\ & 2\gamma_n\|e_n-q\|\cdot\|x_{n+1}-q\|+2\alpha_nh_n\|u_n-q\|, \end{aligned} \tag{21}$$

where $h_n=\|j(x_{n+1}-q)-j(y_n-q)\|$ and $\Phi_{n+1}=\Phi_{n+1\bmod N}$. And

$$\begin{aligned} \|y_n-q\|^2 ={} & \|(1-\beta_n-\delta_n)(x_n-q)+\beta_n(v_n-q)+\delta_n(f_n-q)\|^2 \leqslant\\ & (1+\beta_n^2)\|x_n-q\|^2+2\beta_nl_n\|v_n-q\|+2\delta_n\|f_n-q\|\cdot\|y_n-q\|, \end{aligned} \tag{22}$$

where $l_n=\|j(y_n-q)-j(x_n-q)\|$. Since K is bounded, there exists $M>0$ such that $M=\sup\{\|x\|, x\in K\}$. We have

$$\begin{aligned} & \|u_n-q\|\leqslant 2M,\ \|v_n-q\|\leqslant 2M,\ \|x_n-q\|\leqslant 2M,\ \|y_n-q\|\leqslant 2M,\ \|e_n-q\|\leqslant\\ & 2M,\ \|f_n-q\|\leqslant 2M, \end{aligned} \tag{23}$$

$$(x_{n+1}-q)-(y_n-q)=(\beta_n+\delta_n-\alpha_n-\gamma_n)x_n+\alpha_nu_n+\gamma_ne_n-\beta_nv_n-\delta_nf_n\to 0, \text{ as } n\to\infty,$$

$$(y_n-q)-(x_n-q)=(-\beta_n-\delta_n)x_n+\beta_nv_n+\delta_nf_n\to 0 \text{ as } n\to\infty.$$

In view of uniform continuity of J on any bounded of E, we have $h_n\to 0$ and $l_n\to 0$ as $n\to\infty$. Substituting (22) into (21), and using (23), we have

$$\|x_{n+1}-q\|^2\leqslant(1+\gamma_n^2-2\gamma_n)\|x_n-q\|^2+\alpha_nL_n+8M^2\gamma_n-2\alpha_n\Phi_{n+1}(\|y_n-q\|), \tag{24}$$

where $L_n=(4\alpha_n+8\gamma_n+8\beta_n^2+16\delta_n)M^2+8M\beta_nl_n+4Mh_n$. Obviously that $\lim_{n\to\infty}L_n=0$. Because $\gamma_n=o(\alpha_n)$, there exists a positive integer K_1 such that $\gamma_n^2\leqslant\gamma_n$ as $n\geqslant K_1$ and exists $\{d_n\}$ such that $\gamma_n=d_n\alpha_n$ where $\lim_{n\to\infty}d_n=0$. Furthermore, when $n\geqslant K_1$, we have

$$\|x_{n+1}-q\|^2\leqslant\|x_n-q\|^2+\alpha_n(L_n+8M^2d_n)-2\alpha_n\Phi_{n+1}(\|y_n-q\|). \tag{25}$$

Now, we prove that $\inf\{\|y_n-q\|, n\geqslant 0\}=0$. If $\inf\{\|y_n-q\|, n\geqslant 0\}=\delta>0$, then $\|y_n-q\|\geqslant\delta$ for any $n\geqslant 0$. Let $\Phi(\delta)=\min_{i\in I}\Phi_i(\delta)$. Since $\lim_{n\to\infty}(L_n+8M^2d_n)=0$, there exists positive integer K_2 such that $L_n+8M^2d_n\leqslant\Phi(\delta)$ for any $n\geqslant K_2$. Let $K_3=\max\{K_1, K_2\}$. From (25), as $n\geqslant K_3$, we have

$$\|x_{n+1}-q\|^2\leqslant\|x_n-q\|^2-\alpha_n\Phi(\delta),$$

$$\Phi(\delta)\sum_{j=K_3}^{n}\alpha_j \leqslant \sum_{j=K_3}^{n}(\|x_j-q\|^2-\|x_{j+1}-q\|^2) \leqslant \|x_{K_3}-q\|^2<+\infty.$$

This means that $\sum_{n=1}^{\infty}\alpha_n<\infty$, which is a contradiction. Using this contradiction, we know that $\delta=0$. Therefore, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\|y_{n_j}-q\|\to 0$ as $j\to\infty$. In addition, because $y_{n_j}-q=(1-\beta_{n_j}-\delta_{n_j})(x_{n_j}-q)+\beta_{n_j}(v_{n_j}-q)+\delta_{n_j}(f_{n_j}-q)$, we have $x_{n_j}\to q$ as $j\to\infty$. Therefore for any given $\varepsilon>0$, there exists j_0 such that $\|x_{n_j}-q\|<\varepsilon$ as $j>j_0$. For above ε and certain $j>j_0$, by induction, we can conclude that $\|x_{n_j+k}-q\|<\varepsilon$ for any positive integer k. This implies that $x_n\to q$. The proof is completed.

Theorem 3　Let E be a real uniformly smooth Banach space. Suppose that $T_1, T_2, \cdots, T_N$ are N multi-valued Φ-pseudocontractive mappings from E into 2^E and their ranges are bounded. If $\{x_n\}$ is defined by (5), where $\{e_n\}, \{f_n\}$ are bounded, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ satisfy the conditions as in Theorem 2 and $F=\bigcap_{i=1}^{N}F(T_i)\neq\phi$, then for arbitrary $x_0\in E$, $\{x_n\}$ converges strongly to unique common fixed point of the mappings $\{T_i, i\in I\}$ in E.

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(下转第 11 页)

代数正规类的上根*

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摘要: 利用 Puczylowski 建立的一般代数正规类的根理论, 对任一个代数类 **K** 构造由 **K** 确定的上根 **UK** 及讨论 **UK** 的一些性质.

关键词: 代数正规类; 上根; 半单性

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有限族多值 Φ- 伪压缩映象公共不动点的强收敛定理*

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摘要: 通过引入一个新的具误差的修正的 Ishikawa 迭代过程, 在 Hilbert 空间和一致光滑的 Banach 空间中, 证明了此迭代系列强收敛于有限族多值 Φ- 伪压缩映象的公共不动点, 所得结果改进和扩展了本领域中近期的一些相关结果.

关键词: 修正的 Ishikawa 迭代过程; 误差; 公共不动点; 有限族多值 Φ- 伪压缩映象

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