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利用改进的(G'/G)函数法求解非线性发展方程的行波解

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摘要: 借助于Matlab软件, 利用改进的(G'/G)函数法获得了修正的非线性Degasperis-Procesi方程和非线性波动方程精确形式的行波解, 并且把用改进的(G'/G)函数法获得的结果与双曲正切函数法或(G'/G)函数法得到的结果进行比较. 结果表明, 该方法更有效, 且可得到更多的精确形式行波解.

关键词: 改进的(G'/G)函数法; 修正的非线性Degasperis-Procesi方程; 非线性波动方程; 行波解

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Traveling Wave Solutions of Nonlinear Evolution Equations by Improved (G'/G) Method

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Abstract: With the help of Matlab software, we employed the improved (G'/G) method to obtain exact traveling wave solutions of modified nonlinear Degasperis-Procesi equation and nonlinear wave equation. The improved (G'/G) method provided more general forms of solutions. This method is effective.

Key words: improved (G'/G) method; modified nonlinear Degasperis-Procesi equation; nonlinear wave equation; tranveling wave solution

求解非线性发展方程精确形式的行波解目前已有许多方法, 如逆散射法[1]、Hirota双线性技巧法[2]、截断的Painleve展开法[3]和基于双曲正余切法[4]等. Wang等[5]基于双曲正切函数法的思想[6]提出了(G'/G)函数法, 该方法可以快速有效地得到非线性发展方程精确形式的行波解[7-8]. 本文将改进的(G'/G)函数法应用于修正的非线性Degasperis-Procesi方程和非线性波动方程, 结果表明, 与双曲正切函数法或(G'/G)函数法得到的结果相比, 利用改进的(G'/G)函数法可以得到更多的精确形式行

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波解.

1 改进的(G'/G)函数法

考虑非线性发展方程

$$P(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)=0, \tag{1}$$

其中: $u(x,t)$是关于x,t的实函数; P是关于u及其各阶导数的多项式. 步骤如下.

1) 假设方程(1)的行波解为$u(x,t)=U(\tau)$, $\tau=kx+\omega t$, 其中k,ω为待定常数, 将其代入方程(1), 显然有

$$u_x=\frac{\mathrm{d}U}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}=k\frac{\mathrm{d}U}{\mathrm{d}\tau}=kU',\quad u_t=\omega U',\quad u_{xx}=k^2U'',\quad u_{xt}=k\omega U'',\quad u_{tt}=\omega^2U'',\quad\cdots$$

从而方程(1)可转化为含行波变量τ的常微分方程:

$$P(U,kU',\omega U',k^2U'',k\omega U'',\omega^2U'',\cdots)=0. \tag{2}$$

2) 假设式(2)的解可以表示为如下关于(G'/G)的多项式形式:

$$U=a_n\left(\frac{G'}{G}\right)^n+\cdots+b_n\left(\frac{G'}{G}\right)^{-n}+c_0, \tag{3}$$

其中: 最高次数n是待定常数, a_n,b_n不同时为0; $G(\tau)$满足一个二阶常微分方程

$$G''+\lambda G'+\mu G=0, \tag{4}$$

λ,μ为常数. 将式(3)代入方程(2), 再利用齐次平衡法, 可得次数n.

3) 将式(3)代入方程(2), 利用式(4)合并(G'/G)的相同幂次项, 并令(G'/G)各阶幂次项的系数为零, 即可得关于$c_0,a_1,b_1,\cdots,a_n,b_n,k,\omega$的代数方程组, 通过该方程组, 可得式(3)的各项系数.

下面只需确定G'/G的表达式. 注意到式(4)为二阶奇次常微分方程, 其特征根满足方程$\Delta=\lambda^2-4\mu$.

1) 当$\Delta>0$时, 方程(4)的解为

$$G=c_1\mathrm{e}^{(-\lambda+\sqrt{\Delta})\tau/2}+c_2\mathrm{e}^{(-\lambda-\sqrt{\Delta})\tau/2},$$

$$G'=-\frac{\lambda}{2}(c_1\mathrm{e}^{(-\lambda+\sqrt{\Delta})\tau/2}+c_2\mathrm{e}^{(-\lambda-\sqrt{\Delta})\tau/2})+\frac{\sqrt{\Delta}}{2}(c_1\mathrm{e}^{(-\lambda+\sqrt{\Delta})\tau/2}-c_2\mathrm{e}^{(-\lambda-\sqrt{\Delta})\tau/2}),$$

其中c_1,c_2为任意非零常数. 从而有

$$\frac{G'}{G}=-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{c_1\mathrm{e}^{\sqrt{\Delta}\tau/2}-c_2\mathrm{e}^{-\sqrt{\Delta}\tau/2}}{c_1\mathrm{e}^{\sqrt{\Delta}\tau/2}+c_2\mathrm{e}^{-\sqrt{\Delta}\tau/2}}. \tag{5}$$

令$c_1=C_1+C_2$, $c_2=C_1-C_2$, 则式(5)可化简为

$$\frac{G'}{G}=-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}. \tag{6}$$

2) 当$\Delta=0$时, 可得

$$\frac{G'}{G}=-\frac{\lambda}{2}+\frac{C_2\mathrm{e}^{-\lambda\tau/2}}{\mathrm{e}^{-\lambda\tau/2}(C_1+C_2\tau)}=-\frac{\lambda}{2}+\frac{C_2}{(C_1+C_2\tau)}. \tag{7}$$

3) 当$\Delta<0$时, 同理可得

$$\frac{G'}{G}=-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\times\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}. \tag{8}$$

4) 将3)中得到的系数和上述关于G'/G的表达式(6)~(8)代入式(4), 即可得到方程(1)的多个行波解.

2 修正的非线性 Degasperis-Procesi 方程

下面利用改进的(G'/G)函数法获得 Degasperis-Procesi 方程新的、更多的精确行波解. 考虑如下形式的 Degasperis-Procesi 方程[9]:

$$u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + u_{xxx} + u^3u_x. \tag{9}$$

Lundmark 等[10]利用逆散射法得到了具有 n 重孤波解的存在性. 将 $u(x,t)=U(\tau)$，$\tau = kx+\omega t$ 代入方程(9)，不妨设常数 $k=1$，从而可得

$$\omega U' - \omega U''' + 4U^2U' = 3U'U'' + U''' + U^3U'. \tag{10}$$

将式(10)两端同时积分，并取积分常数为零，则可得

$$\omega(U-U'') + \frac{3}{4}U^3 - UU'' - (U')^2 - \frac{1}{4}U^4 = 0. \tag{11}$$

将式(3)和(4)代入方程(11)，则可得关于 G'/G 的多项式，由齐次平衡法，则可得 $4n=2n+2$，即 $n=1$. 所以 $U=a_1\left(\frac{G'}{G}\right)+b_1\left(\frac{G'}{G}\right)^{-1}+c_0$，其中 a_1,b_1 不同时为零. 显然可得

$$\begin{aligned}
U' &= -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) + b_1\mu\left(\frac{G'}{G}\right)^{-2} + b_1\lambda\left(\frac{G'}{G}\right)^{-1} - a_1\mu + b_1,\\
U'' &= 2a_1\left(\frac{G'}{G}\right)^3 + 3\lambda a_1\left(\frac{G'}{G}\right)^2 + (\lambda^2a_1+2\mu a_1)\left(\frac{G'}{G}\right) + (\lambda^2b_1+2\mu b_1)\left(\frac{G'}{G}\right)^{-1} + \\
&\quad 3\lambda\mu b_1\left(\frac{G'}{G}\right)^{-2} + 2\mu^2b_1\left(\frac{G'}{G}\right)^{-3} + \lambda\mu a_1 + \lambda b_1.
\end{aligned} \tag{12}$$

将式(12)代入式(11)，合并 G'/G 的相同幂次系数，并令这些系数为零，则可得如下非线性方程组：

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^4: a_1^2 - \frac{a_1^4}{4} = 0,\\
&\left(\frac{G'}{G}\right)^3: -2\omega a_1 + \lambda a_1^2 + \frac{3a_1^3}{4} + 2a_1c_0 - a_1^3c_0 = 0,\\
&\left(\frac{G'}{G}\right)^2: -3\lambda\omega a_1 + 4a_1b_1 - a_1^3b_1 + 3\lambda a_1c_0 + \frac{9}{4}a_1^2c_0 - \frac{3}{2}a_1^2c_0^2 = 0,\\
&\frac{G'}{G}: (\omega - \lambda^2\omega + 2\mu\omega)a_1 - \lambda\mu a_1^2 + 8\lambda a_1b_1 + \frac{9}{4}a_1^2b_1 + \lambda^2a_1c_0 + \\
&\qquad 2\mu a_1c_0 - 3a_1^2b_1c_0 + \frac{9}{4}a_1c_0^2 - a_1c_0^3 = 0,\\
&\left(\frac{G'}{G}\right)^0: -\lambda\mu\omega a_1 - \mu^2a_1^2 - \lambda\omega b_1 + 4\lambda^2a_1b_1 + 8\mu a_1b_1 - b_1^2 - \frac{3}{2}a_1^2b_1^2 + \omega c_0 + \\
&\qquad \lambda\mu a_1c_0 + \lambda b_1c_0 + \frac{9}{2}a_1b_1c_0 - 3a_1b_1c_0^2 + \frac{3}{4}c_0^3 - \frac{1}{4}c_0^4 = 0,\\
&\left(\frac{G'}{G}\right)^{-1}: \omega b_1(1-\lambda^2-2\mu) + 8\lambda\mu a_1b_1 - \lambda b_1^2 + \frac{9}{4}a_1b_1^2 + \lambda^2b_1c_0 + \\
&\qquad 2\mu b_1c_0 - 3a_1b_1^2c_0 + \frac{9}{4}b_1c_0^2 - b_1c_0^3 = 0,\\
&\left(\frac{G'}{G}\right)^{-2}: -3\lambda\mu\omega b_1 + 4\mu^2a_1b_1 - a_1b_1^3 + 3\lambda\mu b_1c_0 + \frac{9}{4}b_1^2c_0 - \frac{3}{2}b_1^2c_0^2 = 0,\\
&\left(\frac{G'}{G}\right)^{-3}: -2\mu^2\omega b_1 + \lambda\mu b_1^2 + \frac{3b_1^3}{4} + 2\mu^2b_1c_0 - b_1^3c_0 = 0,\\
&\left(\frac{G'}{G}\right)^{-4}: \mu^2b_1^2 - \frac{b_1^4}{4} = 0.
\end{aligned} \tag{13}$$

利用 Mathematica 软件可得方程组(13)的解：

$$\omega = 0,\quad \mu = \frac{9}{64},\quad \lambda = 0,\quad a_1 = -2,\quad b_1 = -\frac{9}{32},\quad c_0 = \frac{3}{2};$$

$$\omega = 0,\quad \mu = -\frac{9}{64},\quad \lambda = 0,\quad a_1 = 2,\quad b_1 = \frac{9}{32},\quad c_0 = \frac{3}{2};$$

$$\omega=0,\quad \mu=0,\quad \lambda=\pm\frac{3}{2},\quad a_1=2,\quad b_1=0,\quad c_0=\frac{1}{2}(3+2\lambda);$$

$$\omega=0,\quad \mu=\frac{9}{16},\quad \lambda=\frac{3}{2},\quad a_1=2,\quad b_1=\frac{9}{8},\quad c_0=3;$$

$$\omega=0,\quad \lambda=\pm\frac{1}{2}\sqrt{9+16\mu},\quad a_1=-2,\quad b_1=0,\quad c_0=\frac{1}{2}(3-2\lambda),\quad \mu\neq 0;$$

$$\omega=0,\quad \lambda=\pm\frac{1}{2}\sqrt{9+16\mu},\quad a_1=0,\quad b_1=2\mu,\quad c_0=\frac{1}{2}(3+2\lambda),\quad \mu\neq 0;$$

$$\omega=0,\quad \mu=0,\quad \mu=\frac{9}{16},\quad \lambda=-\frac{3}{2},\quad \lambda=\frac{1}{6}(9-32\mu),$$
$$a_1=-2,\quad b_1=-2\mu,\quad c_0=\frac{1}{2}(3-2\lambda);$$

$$\omega=0,\quad \mu(-9+16\mu)\neq 0,\quad \lambda=-\frac{3}{2},\quad a_1=-2,\quad b_1=-2\mu,\quad c_0=3;$$

$$\omega=0,\quad -9+16\mu\neq 0,\quad \lambda=\frac{3}{2},\quad a_1=2,\quad b_1=2\mu,\quad c_0=3,\quad \mu\neq 0;$$

$$\omega=0,\quad \mu\neq 0,\quad \lambda=\pm\frac{1}{2}\sqrt{9+16\mu},\quad a_1=0,\quad b_1=-2\mu,\quad -9\lambda^4+4\lambda^6-81\mu\neq 0,$$
$$c_0=-\frac{2(-9\lambda^3\mu+4\lambda^5\mu-216\mu^2+108\lambda\mu^2-192\mu^3+64\lambda\mu^3)}{-9\lambda^4+4\lambda^6-81\mu}.$$

将方程组的解及式(6)~(8)代入 U, 则易得 U 的表达式, 从而可得对应于方程(9)的行波解:

$$u(x,t)=-2\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)-$$
$$\frac{9}{32}\left(\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\right)\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)^{-1}+\frac{3}{2},\quad \Delta<0; \tag{14}$$

$$u(x,t)=2\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)-$$
$$\frac{9}{32}\left(\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\right)\frac{-C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}+\frac{3}{2},\quad \Delta>0; \tag{15}$$

$$u(t,x)=\sqrt{\Delta}\frac{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}+\frac{3}{2},\quad \Delta>0; \tag{16}$$

$$u(x,t)=2\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\tau}\right)+\frac{9}{8}\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\tau}\right)^{-1}+3,\quad \Delta=0; \tag{17}$$

$$u(x,t)=-\sqrt{\Delta}\frac{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}+\frac{3}{2},\quad \Delta>0; \tag{18}$$

$$u(x,t)=2u\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}+\frac{3}{2}+\lambda,\quad \Delta>0; \tag{19}$$

$$u(x,t)=-2\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)-$$
$$2u\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}+\frac{3}{2}-\lambda,\quad \Delta>0; \tag{20}$$

$$u(x,t)=-2\left(-\frac{\lambda}{2}+\frac{C_1}{C_1+C_2\tau}\right)-2u\left(-\frac{\lambda}{2}+\frac{C_1}{C_1+C_2\tau}\right)^{-1}+\frac{3}{2}-\lambda,\quad \Delta>0; \tag{21}$$

$$u(x,t)=-2\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\times\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)-2u\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\times\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}+3,\quad \Delta>0; \tag{22}$$

$$u(x,t)=-2\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\times\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)-2u\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\times\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)^{-1}+3,\quad \Delta<0; \tag{23}$$

$$u(x,t)=2\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\times\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)+2u\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\times\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}+3,\quad \Delta>0; \tag{24}$$

$$u(x,t)=2\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\times\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)+2u\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\times\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)^{-1}+3,\quad \Delta<0; \tag{25}$$

$$u(x,t)=-2u\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\times\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1}-\frac{2(-9\lambda^3\mu+4\lambda^5\mu-216\mu^2+108\lambda\mu^2-192\mu^3+64\lambda\mu^3)}{-9\lambda^4+4\lambda^6-81\mu},\quad \Delta>0. \tag{26}$$

将扩展的G'/G函数法最终得到的行波解与其他方法得到行波解进行比较表明，行波解(19)和(26)只能由扩展的G'/G函数法得到.

3　非线性波方程

考虑非线性波方程

$$u_{tt}=-u_{xx}+u(u^2-1). \tag{27}$$

将$u(x,t)=U(\tau)$，$\tau=kx+\omega t$代入方程(27)，可得

$$(\omega^2+k^2)U''-U^3+U=0.$$

同理，设$U=a_n\left(\frac{G'}{G}\right)^n+\cdots+b_n\left(\frac{G'}{G}\right)^{-n}+c_0$，其中，$G=G(\tau)$满足二阶常微分方程$G''+\lambda G'+\mu G=0$，且$a_n,b_n$不同时为零，$n,\lambda,\mu$为待定常数. 利用齐次平衡法，则可得$3n=n+2$，即$n=1$，从而有

$$U=a_1\left(\frac{G'}{G}\right)+b_1\left(\frac{G'}{G}\right)^{-1}+c_0, \tag{28}$$

其中，a_1,b_1不同时为零，则可得

$$U'=-a_1\left(\frac{G'}{G}\right)^2-a_1\lambda\left(\frac{G'}{G}\right)+b_1\mu\left(\frac{G'}{G}\right)^{-2}+b_1\lambda\left(\frac{G'}{G}\right)^{-1}-a\mu+b_1+(\lambda^2b_1+2\mu b_1)\left(\frac{G'}{G}\right)^{-1}, \tag{29}$$

$$U'' = a_1\left(\frac{G'}{G}\right)^3 + 3\lambda a_1\left(\frac{G'}{G}\right)^2 + (\lambda^2 a_1 + 2\mu a_1)\left(\frac{G'}{G}\right) + 3\lambda\mu b_1\left(\frac{G'}{G}\right)^{-2} + 2\mu^2 b_1\left(\frac{G'}{G}\right)^{-3} + \lambda\mu a_1 + \lambda b_1. \tag{30}$$

将式(20),(30)代入式(11), 比较 G'/G 的同次幂系数并令其为零, 则可得如下代数方程组:

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^3: 2ka_1 + 2\omega a_1 - a_1^3 = 0,\\
&\left(\frac{G'}{G}\right)^2: 3k\lambda a_1 + 3\lambda\omega a_1 - 3a_1^2 c_0 = 0,\\
&\left(\frac{G'}{G}\right)^1: a_1 + k\lambda^2 a_1 + 2k\mu a_1 + \lambda^2\omega a_1 + 2\mu\omega a_1 - 3a_1^2 b_1 - 3a_1 c_0^2 = 0,\\
&\left(\frac{G'}{G}\right)^0: k\lambda\mu a_1 + \lambda\mu\omega a_1 + k\lambda b_1 + \lambda\omega b_1 + c_0 - 6a_1 b_1 c_0 - c_0^3 = 0,\\
&\left(\frac{G'}{G}\right)^{-1}: b_1 + k\lambda^2 b_1 + 2k\mu b_1 + \lambda^2\omega b_1 + 2\mu\omega b_1 - 3a_1 b_1^2 - 3b_1 c_0^2,\\
&\left(\frac{G'}{G}\right)^{-2}: 3k\lambda\mu b_1 + 3\lambda\mu\omega b_1 - 3b_1^2 c_0 = 0,\\
&\left(\frac{G'}{G}\right)^{-3}: 2k\mu^2 b_1 + 2\mu^2\omega b_1 - b_1^3.
\end{aligned} \tag{31}$$

求得方程组(31)的解为:

$\lambda^2 - 4\mu \neq 0$, $k = \dfrac{2 - \lambda^2\omega + 4\mu\omega}{\lambda^2 - 4\mu}$, $a_1 = \pm\sqrt{2}\sqrt{k+\omega}$, $b_1 = 0$, $c_0 = \dfrac{\lambda a_1}{2}$;

$\mu(-\lambda^2 + 4\mu) \neq 0$, $k = \dfrac{2 - \lambda^2\omega + 4\mu\omega}{\lambda^2 - 4\mu}$, $a_1 = 0$, $b_1 = \pm\sqrt{2}\sqrt{k\mu^2 + \mu^2\omega}$,

$$c_0 = \frac{\lambda(1 + k\lambda^2 - 4k\mu + \lambda^2\omega - 4\mu\omega)b_1}{6\mu};$$

$\lambda = 0$, $\mu \neq 0$, $k = \dfrac{-1 - 8\mu\omega}{8\mu}$, $a_1 = \pm\sqrt{2}\sqrt{k+\omega}$, $b_1 = \dfrac{1}{3}\mu(-1 + 16k\mu + 16\mu\omega)a_1$, $c_0 = 0$;

$\lambda = 0$, $\mu \neq 0$, $k = \dfrac{1 - 4\mu\omega}{4\mu}$, $a_1 = \pm\sqrt{2}\sqrt{k+\omega}$, $b_1 = \dfrac{1}{3}\mu(-1 + 16k\mu + 16\mu\omega)a_1$, $c_0 = 0$;

$\mu = 0$, $\lambda \neq 0$, $k = \dfrac{2 - \lambda^2\omega}{\lambda^2}$, $a_1 = \pm\sqrt{2}\sqrt{k+\omega}$, $b_1 = 0$, $c_0 = \dfrac{\lambda a_1}{2}$.

将方程组的解及式(6)~(8)代入式(29), 可求出 U 的表达式, 进而求出方程(27)有如下形式的行波解:

$$u(x,t) = \pm\sqrt{2k+2\omega}\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau + C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau + C_2\cosh(\sqrt{\Delta}/2)\tau}\right) \pm \frac{\lambda}{2}\sqrt{2k+2\omega},\ \Delta > 0; \tag{32}$$

$$u(x,t) = \pm\sqrt{2k+2\omega}\left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2}\frac{-C_1\cos(\sqrt{-\Delta}/2)\tau + C_2\sin(\sqrt{-\Delta}/2)\tau}{C_1\sin(\sqrt{-\Delta}/2)\tau + C_2\cos(\sqrt{-\Delta}/2)\tau}\right) + \frac{\lambda}{2}\sqrt{2k+2\omega},\quad \Delta < 0; \tag{33}$$

$$u(x,t) = \pm\sqrt{2k\mu^2 + 2\omega\mu^2}\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau + C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau + C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1} \pm \frac{\lambda\sqrt{2k\mu^2 + 2\omega\mu^2}(1 + k\lambda^2 + \omega\lambda^2 - 4k\mu - 4\omega\mu)}{6\mu},\quad \Delta > 0; \tag{34}$$

$$u(x,t)=\pm\sqrt{2k\mu^2+2\omega\mu^2}\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)^{-1}\pm \frac{\lambda\sqrt{2k\mu^2+2\omega\mu^2}(1+k\lambda^2+\omega\lambda^2-4k\mu-4\omega\mu)}{6\mu},\quad \Delta<0; \tag{35}$$

$$u(x,t)=\pm\sqrt{2k+2\omega}\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)\pm\sqrt{2k+2\omega}\frac{-\mu+16k\mu+16\omega\mu}{3}\times \left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)^{-1},\quad \Delta>0; \tag{36}$$

$$u(x,t)=\pm\sqrt{2k+2\omega}\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)\pm\sqrt{2k+2\omega}\times \frac{-\mu+16k\mu+16\omega\mu}{3}\left(-\frac{\lambda}{2}+\frac{\sqrt{-\Delta}}{2}\frac{-C_1\sin(\sqrt{-\Delta}/2)\tau+C_2\cos(\sqrt{-\Delta}/2)\tau}{C_1\cos(\sqrt{-\Delta}/2)\tau+C_2\sin(\sqrt{-\Delta}/2)\tau}\right)^{-1},\quad \Delta<0; \tag{37}$$

$$u(x,t)=\pm\sqrt{2k+2\omega}\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\frac{C_1\cosh(\sqrt{\Delta}/2)\tau+C_2\sinh(\sqrt{\Delta}/2)\tau}{C_1\sinh(\sqrt{\Delta}/2)\tau+C_2\cosh(\sqrt{\Delta}/2)\tau}\right)\pm\frac{\lambda\sqrt{2k+2\omega}}{2},\quad \Delta>0. \tag{38}$$

对比结果可见，应用 G'/G 函数扩展法求解方程(27)得到的行波解在结果准确性及行波解个数上都有较大提高.

本文研究结果表明，利用 G'/G 函数扩展法求解非线性发展方程行波解非常有效，该方法不仅克服了其他方法在不能求得部分非线性发展方程行波解的难题，而且还提供了一种更易于理解和求解的模式，通过对两个非线性方程的研究也表明，利用该方法可得到比其他方法更多的行波解.

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