研究快报

2n 阶时滞微分方程周期解的存在性

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摘要: 利用上下解方法研究 2n 阶时滞微分方程周期边值问题, 建立了 $2n(n\geqslant 1)$ 阶时滞微分方程周期边值问题解存在的充分条件.

关键词: 2n 阶边值问题; 先验边界; 周期解
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Existence of Periodic Solutions of 2n-Order Delay Differential Equation

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Abstract: We studied problems about 2n-order delay differential equations with periodic boundary value using the upper and lower solution method, and established some suffcient conditions for the existence of the solutions of $2n(n\geqslant 1)$-order periodic boundary value problems.

Key words: 2n-order boundary value problem; priori boundary; periodic solutions

时滞微分方程在动力系统、宇航技术、生态系统和金融等领域应用广泛. 近年来, 时滞微分方程的研究已取得许多结果[1-8].

本文考虑如下 2n 阶时滞微分方程周期边值问题:

$$\begin{aligned}&u^{(2n)}(t)+D(t,u(t-r),u(t),\cdots,u^{(2n-2)}(t))u^{(2n-1)}(t)+\\&\qquad g(t,u(t-r),u(t),\cdots,u^{(2n-2)}(t))=h(t),\quad t\in[0,T],\\&\begin{cases}u(\theta)=0, & \theta\in[-r,0],\\ u^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ u^{(2n-2)}(0)=u^{(2n-2)}(T), \\ u^{(2n-1)}(0)=u^{(2n-1)}(T),\end{cases}\end{aligned}\tag{1}$$

其中: $h\in L^1(0,T)$; $D\in C([0,T]\times\mathbb{R}^{2n},\mathbb{R})$; g: $[0,T]\times\mathbb{R}^{2n}\to\mathbb{R}$ 是关于第一个变量以 T 为周期的 Carathéodory 函数.

定理 1　设 $h\in L^1(0,T)$; $D\in C([0,T]\times\mathbb{R}^{2n},\mathbb{R})$; g: $[0,T]\times\mathbb{R}^{2n}\to\mathbb{R}$, 是关于第一个变量以 T 为周期的 Carathéodory 函数. 如果存在两个实数 a_1,a_2, 满足 $a_1<0$, $a_1+a_2>0$, 并使得: 当 $x_{2n-2}<a_1$ 且 $t\in[0,T]$时, $g(t,x_{-1},x_0,\cdots,x_{2n-2})>h(t)$; 当 $a_1\leqslant x_{2n-2}\leqslant a_2$ 且 $t\in[0,T]$时, $g(t,x_{-1},x_0,\cdots,x_{2n-2})\leqslant h(t)$, 则边值问题(1)至少有一个解.

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为了证明边值问题(1)有周期解，先考虑如下辅助边值问题：

$$x^{(2n)}(t)=f(t,x(t-r),x(t),\cdots,x^{(2n-1)}(t)),$$
$$\begin{cases}x(\theta)=0, & \theta\in[-r,0],\\ x^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ x^{(2n-2)}(0)=x^{(2n-2)}(T), & n\geqslant 1,\quad t\in[0,T],\\ x^{(2n-1)}(0)=x^{(2n-1)}(T);\end{cases}\tag{2}$$

$$x^{(2n)}(t)=F(t,x(t-r),x(t),\cdots,x^{(2n-1)}(t)),$$
$$\begin{cases}x(\theta)=0, & \theta\in[-r,0],\\ x^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ x^{(2n-2)}(0)=x^{(2n-2)}(T),\\ x^{(2n-1)}(0)=x^{(2n-1)}(T),\end{cases}\tag{3}$$

其中 $F(t,x(t-r),x(t),\cdots,x^{(2n-1)}(t))$ 定义如下：

$$F:=\begin{cases}f(t,\alpha(t-r),\alpha(t),\cdots,\alpha^{(2n-2)}(t),x^{(2n-1)}(t))+m(x^{(2n-2)}(t)-\alpha^{(2n-2)}(t)),\\ \qquad \alpha^{(i)}(t)>x^{(i)}(t),\quad i=0,1,\cdots,2n-2,\\ f(t,x(t-r),x(t),\cdots,x^{(2n-2)}(t),x^{(2n-1)}(t)),\quad \alpha^{(i)}(t)\leqslant x^{(i)}(t)\leqslant\beta^{(i)}(t),\ i=0,1,\cdots,2n-2,\\ f(t,\beta(t-r),\beta(t),\cdots,\beta^{(2n-2)}(t),x^{(2n-1)}(t))+m(x^{(2n-2)}(t)-\beta^{(2n-2)}(t)),\\ \qquad x^{(i)}(t)>\beta^{(i)}(t),\ i=0,1,\cdots,2n-2,\end{cases}$$

记

$$E:=\{(t,x_{-1},x_0,x_1,\cdots,x_{2n-1})\,|\,t\in[0,T],\ \alpha(t-r)\leqslant x_{-1}\leqslant\beta(t-r),$$
$$\alpha^{(i)}(t)\leqslant x_i\leqslant\beta^{(i)}(t),\ i=0,1,\cdots,2n-2,\ x_{2n-1}\in\mathbb{R}\},$$

其中 α 和 β 在 $[0,T]$ 上连续，且满足 $\alpha^{(i)}(t)\leqslant\beta^{(i)}(t)$，$i=0,1,\cdots,2n-2$，$t\in[0,T]$，$\alpha(t)\leqslant0\leqslant\beta(t)$，$t\in[-r,0]$；

f 满足如下条件：

(H_1) f: $E\to\mathbb{R}$ 是一个定义在 E 上的 Carathéodory 函数，并且对 $(t,x_{-1},x_0,\cdots,x_{2n-2},y_1)$，$(t,x_{-1},x_0,\cdots,x_{2n-2},y_2)\in E$，存在 $L\in L^1(0,T)$，使得

$$|f(t,x_{-1},x_0,\cdots,x_{2n-2},y_1)-f(t,x_{-1},x_0,\cdots,x_{2n-2},y_2)|\leqslant L(t)\,|y_1-y_2|;$$

(H_2) 存在 $\alpha,\beta\in W^{n,n-1}(0,T)$，在 $[0,T]$ 上几乎处处有

$$\alpha^{(2n)}(t)\geqslant f(t,\alpha(t-r),\alpha(t),\alpha^{(1)}(t),\cdots,\alpha^{(2n-1)}(t)),$$
$$\begin{cases}\alpha(\theta)\leqslant0, & \theta\in[-r,0],\\ \alpha^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ \alpha^{(2n-1)}(0)=\alpha^{(2n-2)}(T),\\ \alpha^{(2n-1)}(0)\geqslant\alpha^{(2n-2)}(T);\end{cases}$$

$$\beta^{(2n)}(t)\leqslant f(t,\beta(t-r),\beta(t),\beta^{(1)}(t),\cdots,\beta^{(2n-1)}(t)),$$
$$\begin{cases}\beta(\theta)\leqslant0, & \theta\in[-r,0],\\ \beta^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ \beta^{(2n-1)}(0)=\beta^{(2n-2)}(T),\\ \beta^{(2n-1)}(0)\geqslant\beta^{(2n-2)}(T);\end{cases}$$

(H_3) 存在 $m\in L^1(0,T)$，对任意的 $|y|\leqslant1$，有

$$\max\{|f(t,\alpha(t-r),\alpha(t),\cdots,\alpha^{(2n-2)}(t),y)|,|f(t,\beta(t-r),\beta(t),\cdots,\beta^{(2n-2)}(t),y)|,\ L(t)<m(t)\}\leqslant m(t).$$

在假设(H_1)~(H_3)下，易证问题(2)至少存在一个周期解. 下面利用上下解方法证明边值问题(1)至少有一个周期解.

令 $\alpha^{(2n-2)}(t)=\alpha_1/2$，$t\in[0,T]$，则有 $\alpha^{(2n)}(t)=\alpha^{(2n-1)}(t)=0$. 容易验证 $\alpha(t)$ 是边值问题(1)的

下解. 令 $c=(\alpha_1+\alpha_2)/2$，$\varphi=(\alpha_2-\alpha_1)/(2T^2)>0$，则边值问题

$$\beta_0^{(2n)}(t)+D\Big(t,\beta_0(t-r)+\frac{c}{(2n-2)!}(t-r)^{2n-2},\beta_0(t)+\frac{c}{(2n-2)!}t^{2n-2},\cdots,\beta_0^{(2n-2)}(t)+c\Big)\times \beta_0^{(2n-1)}(t)+\varphi=0,$$

$$\begin{cases}\beta_0(\theta)=0,\\ \beta_0^{(i)}(\theta)=0, & \theta\in[-r,0],\\ \beta_0^{(2n-2)}(\theta)=\beta_0^{(2n-2)}(T), & i=0,1,\cdots,2n-3,\\ \beta_0^{(2n-1)}(\theta)=\beta_0^{(2n-1)}(T),\\ \overline{\beta_0^{(2n-2)}}=0\end{cases} \tag{4}$$

在 $[0,T]$ 上有一个解为 $\beta_0(t)$.

当 $\lambda\in[0,1]$ 时，边值问题

$$\beta^{(2n)}(t)+\lambda D\Big(t,\beta(t-r)+\frac{c}{(2n-2)!}(t-r)^{2n-2},\beta(t)+\frac{c}{(2n-2)!}t^{2n-2},\cdots,\beta^{(2n-2)}(t)+c\Big)\times \beta^{(2n-1)}(t)+\varphi=0,$$

$$\begin{cases}\beta(\theta)=0, & \theta\in[-r,0],\\ \beta^{(i)}(0)=0, & i=0,1,\cdots,2n-3,\\ \beta^{(2n-2)}(0)=\beta^{(2n-2)}(T),\\ \beta^{(2n-1)}(0)=\beta^{(2n-1)}(T),\\ \overline{\beta^{(2n-2)}}=0\end{cases} \tag{5}$$

的解有先验边界. 由于 D 在 $[0,T]\times\mathbb{R}^{2n}$ 上连续，故存在一个 $M>0$，使得

$$\left|D\Big(t,\beta(t-r)+\frac{c}{(2n-2)!}(t-r)^{2n-2},\beta(t)+\frac{c}{(2n-2)!}t^{2n-2},\cdots,\beta^{(2n-2)}(t)+c\Big)\right|\leqslant M,\quad t\in[0,T].$$

利用辅助边值问题容易证明 $\beta(t)$ 是边值问题(1)的上解，所以由边值问题(2)解的存在性可知，边值问题(1)至少有一个周期解.

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