第 21 卷 第 4 期　宁 波 大 学 学 报（理 工 版）　Vol.21 No.4
2008 年 12 月　JOURNAL OF NINGBO UNIVERSITY (NSEE)　Dec. 2008

文章编号:1001-5132（2008）04-0520-04

Characterizations of the Minus Ordering in Fuzzy Matrix Set

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Abstract: The matrix minus ordering is introduced into fuzzy matrix set. The minus ordering is a partial ordering in $F^-_{m,n}$. Some characterizations of the minus ordering are given.

Key words: Fuzzy matrix; minus ordering; characterization

CLC number: O159　**Document code:** A

Let $F_{m,n}$ stand for the set of all $m\times n$ fuzzy matrices. Given $A\in F_{m,n}$, $A\{1\}$ and $A\{2\}$ will denote the sets of all inner and outer inverses of A, specified as

$$A\{1\}=\{X\in F_{n,m}\mid AXA=A\}, \tag{1}$$

and

$$A\{2\}=\{X\in F_{n,m}\mid XAX=X\}, \tag{2}$$

write $A\{1,2\}=A\{1\}\cap A\{2\}$. And, A^-, $A^=$ or $A^{\sim}$ will denote an element in $A\{1\}$ and $A^\wedge$, $A^\vee$ or $A^{(1,2)}$ an element in $A\{1,2\}$. Write $F^-_{m,n}=\{A\mid A\{1\}\neq\varnothing, A\in F_{m,n}\}$.

Now, we define the minus ordering $A\leq^- B$ and the preorder $A\preceq B$ in $F_{m,n}$.

Let $A\in F^-_{m,n}, B\in F_{m,n}$. The minus ordering $A\leq^- B$ in $F_{m,n}$ is defined as follow:

$$A\leq^- B\Longleftrightarrow A^-A=A^-B,\ AA^==BA^=, \tag{3}$$

where $A^-, A^=\in A\{1\}$.

It is clear that $A\leq^- A$, for each $A\in F^-_{m,n}$. In general, the minus ordering $A\leq^- B$ is not a partial ordering in $F_{m,n}$. In section 2, We will prove that the minus ordering $A\leq^- B$ is a partial ordering in $F^-_{m,n}$.

Let $A\in F_{m,n}$. Write

$$AF_{n,n}=\{AX\mid X\in F_{n,n}\},\ F_{m,m}A=\{YA\mid Y\in F_{m,m}\}.$$

Let $A, B\in F_{m,n}$. The preorder $A\preceq B$ in $F_{m,n}$ is defined as follow:

$$A\preceq B\Longleftrightarrow AF_{n,n}\subseteq BF_{n,n},\ F_{m,m}A\subseteq F_{m,m}B. \tag{4}$$

It is clear that $A\preceq A$, for each $A\in F_{m,n}$. The preorder $A\preceq B$ in $F_{m,n}$ is not a partial ordering in $F_{m,n}$. In section 3, by use of this preorder in $F_{m,n}$, we will discuss some characterizations of the minus partial ordering in $F^-_{m,n}$.

1 Minus partial ordering

In this section, we will prove that the minus ordering is a partial ordering in $F^-_{m,n}$. First, we have the following.

Theorem 1 Let $A\in F^-_{m,n}, B\in F_{m,n}$. The following statements are equivalent:

(i) $A\leq^- B$.

(ii) There exists $A^\wedge$ in $A\{1,2\}$ such that

$$AA^\wedge B=A=BA^\wedge A. \tag{5}$$

(iii) There exists $A^\wedge$ in $A\{1,2\}$ such that

$$A^\wedge A=A^\wedge B,\ AA^\wedge=BA^\wedge. \tag{6}$$

Proof (i) $\Rightarrow$ (ii): Set $A^\wedge=A^=AA^-$ where A^-, $A^=\in A\{1\}$. Then,

$$AA^\wedge A=AA^=AA^-A=AA^-A=A,$$

$$A^\wedge AA^\wedge=A^=AA^-AA^=AA^-=A^=AA^=AA^-=A^=AA^-=A^\wedge,$$

Thus, $A^\wedge\in A\{1,2\}$, and

Received date: 2007-10-18.　JOURNAL OF NINGBO UNIVERSITY (NSEE): http://3xb.nbu.edu.cn
Foundation item: Zhejiang Provincial Natural Science Foundation (Y604089).
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$AA^{\wedge}B = AA^{=}AA^{-}B = AA^{-}B = AA^{-}A = A$,

$BA^{\wedge}A = BA^{=}AA^{-}A = BA^{=}A = AA^{=}A = A$.

(ii) holds.

(ii) $\Longrightarrow$ (iii): Since $A^{\wedge} \in A\{1,2\}$, $A^{\wedge}A = A^{\wedge}AA^{\wedge}B = A^{\wedge}B, AA^{\wedge} = BA^{\wedge}AA^{\wedge} = BA^{\wedge}$. Then, (iii) holds.

(iii) $\Longrightarrow$ (i): It is clear.

Lemma 1 Let $A \in F_{m,n}^{-}, B \in F_{m,n}$. If $A \leq^{-} B$, then

(i) $A \preceq B$.

(ii) There exists $A^{\wedge}$ in $A\{1,2\}$ such that $A = BA^{\wedge}B, A^{\wedge} = A^{\wedge}BA^{\wedge}$.

Proof (i) holds clearly by Theorem 1(ii). And, by (6) and (5) in Theorem 1,

$BA^{\wedge}B = BA^{\wedge}A = A, A^{\wedge}BA^{\wedge} = A^{\wedge}AA^{\wedge} = A^{\wedge}$.

(ii) holds.

Lemma 2 Let $A, B \in F_{m,n}^{-}$. If $A \leq^{-} B$, then

(i) For each $B^{-} \in B\{1\}$, $AB^{-}A = A, AB^{-}B = A = BB^{-}A$.

(ii) For each $A^{(1,2)} \in A\{1,2\}, B^{-} \in B\{1\}$, $B^{-}BA^{(1,2)} \cdot BB^{-} \in A\{1,2\}$.

(iii) There exists $A^{\vee}$ in $A\{1,2\}$ such that $AA^{\vee} = BA^{\vee} = AB^{-}, A^{\vee}A = A^{\vee}B = B^{-}A, \forall B^{-} \in B\{1\}$.

Proof (i) By Lemma 1(ii), there exists $A^{\wedge}$ in $A\{1,2\}$ such that $A = BA^{\wedge}B, A^{\wedge} = A^{\wedge}BA^{\wedge}$. Thus, for each $B^{-} \in B\{1\}$, $AB^{-}A = BA^{\wedge}BB^{-}BA^{\wedge}B = BA^{\wedge}BA^{\wedge}B = BA^{\wedge}B = A$, $A = BA^{\wedge}B = BB^{-}BA^{\wedge}B = BB^{-}A$. Similarly, we have that $A = AB^{-}B$. (i) holds.

(ii) By (i),

$AB^{-}BA^{(1,2)}BB^{-}A = AA^{(1,2)}A = A$,

$B^{-}BA^{(1,2)}BB^{-}AB^{-}BA^{(1,2)}BB^{-} =$

$B^{-}BA^{(1,2)}BB^{-}AA^{(1,2)}BB^{-} =$

$B^{-}BA^{(1,2)}AA^{(1,2)}BB^{-} = B^{-}BA^{(1,2)}BB^{-}$.

That is $B^{-}BA^{(1,2)}BB^{-} \in A\{1,2\}, \forall B^{-} \in B\{1\}$. (ii) holds.

(iii) Set $A^{\vee} = B^{-}BA^{\wedge}BB^{-}$ where $A^{\wedge}$ in Lemma 1(ii). Then, $A^{\vee} \in A\{1,2\}$ by (ii). And, by Theorem 1 (ii) and Lemma 1(ii),

$AA^{\vee} = AB^{-}BA^{\wedge}BB^{-} = AA^{\wedge}BB^{-} = AB^{-} =$

$BA^{\wedge}BB^{-} = BB^{-}BA^{\wedge}BB^{-} = BA^{\vee}$.

Similarly, we can obtain that $A^{\vee}A = B^{-}A = A^{\vee}B$. Thus, (iii) holds.

Lemma 3 Let $A \in F_{m,n}, B \in F_{m,n}^{-}$. Then,

$$A \preceq B \Longleftrightarrow AB^{-}B = A = BB^{-}A, \forall B^{-} \in B\{1\}. \quad (7)$$

Proof "$\Rightarrow$": Since $A \preceq B$, there exist X in $F_{n,n}$ such that

$A = BX = BB^{-}BX = BB^{-}A, \forall B^{-} \in B\{1\}$. Similarly, it is proved that $A = AB^{-}B$.

"$\Leftarrow$": Since $A = AB^{-}B$, for $YA \in F_{m,m}A$, $YA = YAB^{-}B \in F_{m,m}B$. Thus, $F_{m,m}A \subseteq F_{m,m}B$. Similarly, it is proved that $AF_{n,n} \subseteq BF_{n,n}$. Thus, $A \preceq B$.

Theorem 2 "$\leq^{-}$" is a partial ordering in $F_{m,n}^{-}$.

Proof Let $A \leq^{-} B, B \leq^{-} A$ where $A, B \in F_{m,n}^{-}$. If $A \leq^{-} B$, by Lemma 2(i), $A = BB^{-}A$ for each $B^{-} \in B\{1\}$. If $B \leq^{-} A$, by Theorem 1, there exists $B^{\wedge} \in B\{1,2\}$ such that $B = BB^{\wedge}A$. Then, $A = BB^{\wedge}A = B$. And, let $A \leq^{-} B, B \leq^{-} C$ where $A, B, C \in F_{m,n}^{-}$. If $A \leq^{-} B$, by Lemma 2(iii), there exists $A^{\vee} \in A\{1,2\}$ such that

$AA^{\vee} = AB^{-}, A^{\vee}A = B^{-}A, \forall B^{-} \in B\{1\}$.

If $B \leq^{-} C$, by Theorem 1(iii), there exists $B^{\wedge} \in B\{1,2\}$ such that

$B^{\wedge}C = B^{\wedge}B, CB^{\wedge} = BB^{\wedge}$.

By Lemma 2(i),

$(AA^{\vee})C = (AB^{\wedge})C = A(B^{\wedge}C) = AB^{\wedge}B =$

$A = BB^{\wedge}A = CB^{\wedge}A = CA^{\vee}A$,

and Lemma 2(i). Thus, $A \leq^{-} C$ by Theorem 1. Therefore, "$\leq^{-}$" is a partial ordering in $F_{m,n}^{-}$.

2 Characterizations of the minus ordering

In this section, we discuss only fuzzy matrices in $F_{m,n}^{-}$.

Theorem 3 Let $A, B \in F_{m,n}^{-}$. The following statements are equivalent:

(i) $A \leq^{-} B$.

(iv) There exists $A^{\vee}$ in $A\{1,2\}$ such that $AA^{\vee} = BA^{\vee} = AB^{-}, A^{\vee}A = A^{\vee}B = B^{-}A, \forall B^{-} \in B\{1\}$.

(v) There exists $A^{\vee}$ in $A\{1,2\}$ such that

$AA^{\vee} \leq^{-} BB^{(1,2)}$, $A^{\vee}A \leq^{-} B^{(1,2)}B$ and $BA^{\vee}B =$

$A = AB^{(1,2)}A, \ \forall B^{(1,2)} \in B\{1,2\}$.

(vi) There exists $A^{\vee}$ in $A\{1,2\}$ such that $AA^{\vee} \preceq^{-} BB^{(1,2)}$, $A^{\vee}A \leq^{-} B^{(1,2)}B$ and $A = AB^{-}A, \forall B^{-} \in B\{1\}$.

(vii) There exists $X \in F_{n,m}$ such that $A = BXB$, $B\{1\} \subseteq A\{1\}$.

(viii) $A \preceq B$ and $B\{1\} \subseteq A\{1\}$.

(ix) $A \preceq B$ and $A\{1\} \cap B\{1\} \neq \varnothing$.

(x) For all $B^{-}, B^{=}, B^{(1)} \in B\{1\}$, $AB^{-}B = BB^{=}A = A = AB^{(1)}A$.

(xi) There exist an idempotent fuzzy matrix $E_m \in F_{m,m}$ and an idempotent fuzzy matrix $E_n \in F_{n,n}$ such that $E_m B = A = BE_n$.

(xii) There exist an idempotent fuzzy matrix $E_m \in F_{m,m}$ and $D \in F_{n,n}$ such that $E_m B = A = BD$.

(xiii) There exist $C \in F_{m,m}$ and $D \in F_{n,n}$ such that $CA = CB = A = BD$.

(xiv) There exist $C \in F_{m,m}$ and an idempotent fuzzy matrix $E_n \in F_{n,n}$ such that $CB = A = BE_n$.

(xv) There exist $C \in F_{m,m}$ and $D \in F_{n,n}$ such that $CB = A = AD = BD$.

(xvi) There exist $C \in F_{m,m}$ and $D \in F_{n,n}$ such that $CB = CA = A = AD = BD$.

Proof (i) $\Rightarrow$ (iv): It is clear by Lemma 2(iii).

(iv) $\Rightarrow$ (v): There exists $A^{\vee}$ in $A\{1,2\}$ such that $A = AA^{\vee}A = AB^{(1,2)}A, \forall B^{(1,2)} \in B\{1,2\}$. And

$$BA^{\vee}B = BB^{(1,2)}A = BA^{\vee}A = AA^{\vee}A = A.$$

Also, we have that

$$AA^{\vee}BB^{(1,2)} = AB^{(1,2)}BB^{(1,2)} = AB^{(1,2)} = AA^{\vee} = BA^{\vee} = BB^{(1,2)}BA^{\vee} = BB^{(1,2)}AA^{\vee}.$$

Since $AA^{\vee}$ is idempotent, $AA^{\vee} \in (AA^{\vee})\{1\}$. Write $(AA^{\vee})^{-} = AA^{\vee}$. Then,

$$(AA^{\vee})(AA^{\vee})^{-} = AA^{\vee} = BB^{(1,2)}AA^{\vee} = (BB^{(1,2)})(AA^{\vee})^{-},$$

$$(AA^{\vee})^{-}(AA^{\vee}) = AA^{\vee} = AA^{\vee}BB^{(1,2)} = (AA^{\vee})^{-}BB^{(1,2)}.$$

That is, $AA^{\vee} \leq^{-} BB^{(1,2)}$. Similarly, we have that $AA^{\vee} \preceq^{-} B^{(1,2)}B$. Thus, (v) holds.

(v) $\Rightarrow$ (vi): $\forall B^{-} \in B\{1\}$, $A = AB^{(1,2)}A = BA^{\vee}B \cdot B^{(1,2)}BA^{\vee}B = BA^{\vee}BB^{-}BA^{\vee}B = AB^{-}A$. Thus, (vi) holds.

(vi) $\Rightarrow$ (vii): Since $BB^{(1,2)}$ is idempotent and $AA^{\vee} \leq^{-} BB^{(1,2)}$ by Lemma 2(i),

$$AA^{\vee} = BB^{(1,2)}(BB^{(1,2)})^{-}AA^{\vee} = BB^{(1,2)}AA^{\vee}.$$

Thus, $A = AA^{\vee}A = BB^{(1,2)}AA^{\vee}A = BB^{(1,2)}A$. Similarly, we can prove that $A = AB^{(1,2)}B$. Therefore,

$$A = AA^{\vee}A = BB^{(1,2)}AA^{\vee}AB^{(1,2)}B = BB^{(1,2)}AB^{(1,2)}B = BXB.$$

where $X = B^{(1,2)}AB^{(1,2)} \in F_{n,m}$. And, $A = AB^{-}A$, $\forall B^{-} \in B\{1\}$. Thus, $B\{1\} \subseteq A\{1\}$. Then, (vii) holds.

(vii) $\Rightarrow$ (viii): Since $A = BXB$, it is clear that $A \preceq B$ by (6). (viii) holds.

(viii) $\Rightarrow$ (ix): It is clear.

(ix) $\Rightarrow$ (x): By Lemma 3, since $A \preceq B$, $AB^{-}B = A = BB^{=}A, \forall B^{-}, B^{=} \in B\{1\}$. Since $A\{1\} \cap B\{1\} \neq \varnothing$, there exist $B^{\sim} \in A\{1\} \cap B\{1\}$ such that $A = AB^{\sim}A = AB^{-}BB^{\sim}BB^{=}A = AB^{-}BB^{(1)}BB^{=}A = AB^{(1)}A, \forall B^{(1)} \in B\{1\}$. Thus, (x) holds.

(x) $\Rightarrow$ (xi): In $AB^{-}B = BB^{-}A = A$, set $AB^{-} = E_m$, $B^{-}A = E_n$. Since $AB^{-}A = A$, E_m and E_n are idempotent. Thus, (xi) holds.

(xi) $\Rightarrow$ (xii): It is clear.

(xii) $\Rightarrow$ (xiii): Set $C = E_m$, then $CA = E_m E_m B = E_m B = CB = A = BD$. Thus, (xiii) holds.

(xiii) $\Rightarrow$ (xiv): $A = AA^{-}A = BDA^{-}A, A^{-} \in A\{1\}$. Set $X = DA^{-}A \in F_{n,n}$, Then,

$$X^2 = DA^{-}ADA^{-}A = DA^{-}CBDA^{-}A = DA^{-}CAA^{-}A = DA^{-}A = X.$$

Thus, $CB = A = BE_n$ where $E_n = X$. Therefore, (xiv) holds.

(xiv) $\Rightarrow$ (xv): Similar to the proof of "(xii)$\Rightarrow$(xiii)".

(xv) $\Rightarrow$ (xvi): $CA = CAD = CBD = AD = A$. (xvi) holds.

(xvi) $\Rightarrow$ (i): Let $A^{(1,2)} \in A\{1,2\}$, Write $A^{\vee} = A^{(1,2)}C$. Then,

$$AA^{\vee}A = AA^{(1,2)}CA = AA^{(1,2)}CBD = AA^{(1,2)}AD = AD = A,$$

$$A^{\vee}AA^{\vee} = A^{(1,2)}CAA^{(1,2)}C = A^{(1,2)}CBDA^{(1,2)}C = A^{(1,2)}ADA^{(1,2)}C = A^{(1,2)}AA^{(1,2)}C = A^{(1,2)}C = A^{\vee}$$

That is, $A^{\vee} \in A\{1,2\}$. And, $AA^{\vee}B = AA^{(1,2)}CB = AA^{(1,2)}A = A$. Then, $A^{\vee}A = A^{\vee}AA^{\vee}B = A^{\vee}B$.

Set $A^{\wedge} = DA^{(1,2)}$. Similarly, we have $A^{\wedge} \in A\{1,2\}$ and $AA^{\wedge} = BA^{\wedge}$. Thus, $A \leq^{-} B$. Therefore, (i) holds.

Corollary 1 Let $A, B \in F_{m,n}^{-}$.

(i) If $BB^{-} = I_m, B^{-} \in B\{1\}$, $A \le^{-} B \Longleftrightarrow AB^{-} \cdot A = A = AB^{-}B$.

(ii) If $B^{-}B = I_n, B^{-} \in B\{1\}$, $A \le^{-} B \Longleftrightarrow AB^{-} \cdot A = A = BB^{-}A$.

(iii) If B^{-1} exists, $A \le^{-} B \Longleftrightarrow AB^{-1}A = A$.

Corollary 2 Let $A, B \in F_{m,n}^{+}$. Then, the following statements are equivalent:

(i) $A \le^{-} B$.

(ii) $AB^{+}A = AB^{+}B = A = BB^{+}A$.

(iii) $AB^{+}B = A = BB^{+}A$, and $B^{+}A$ and AB^{+} are idempotent.

(iv) $BAB^{+} = A = B^{+}AB$, and $B^{+}A$ and AB^{+} are idempotent.

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Fuzzy 矩阵集中减序的特征刻划

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摘要：在 Fuzzy 矩阵集中引进 Fuzzy 矩阵减序，减序是 $F_{m,n}^{-}$ 中的偏序. 给出了 Fuzzy 矩阵减序的一些特征刻划.

关键词：Fuzzy 矩阵；减序；特征刻划

中图分类号：O159 文献标识码：A

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