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Original Article

Premium and reinsurance control of an ordinary insurance system with liabilities driven by a fractional Brownian motion

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This paper investigates the problem of premium and reinsurance control of an ordinary insurance system when liabilities are driven by a fractional Brownian motion. The reserve equation is considered using two alternative routes: the first with no reinsurance option, and the second with some controllable proportional reinsurance coverage. Recent results from the theory of fractional linear-quadratic control (fractional calculus) are discussed, partially extended and utilized to derive compact analytical formulae for the optimal functionals of the safety loading (consequently for the respective premium rate), and the volume of the retained risk (or equivalently, for the proportion of the reinsurance coverage).

Keywords: Insurance reserve process; Fractional Brownian motion; Stochastic linear-quadratic (L-Q) control; Ito Integral; Riccati equation; Malliavin derivative

1. Introduction

The Cramer-Lundberg model sets the insurance systems and their problems within a dynamic framework. The vital mechanism of a typical insurance system is described by the following simple mathematical dynamic equation:

$$F(t) = F(0) + \Pi \cdot t - S(t) \quad t \geq 0, \tag{1}$$

where $F(t)$ is the reserve value at time t, $F(0)$ is the initial capital of the insurance system, Π is the constant rate of premium which inflows in the system per unit time, and $S(t)$ is the aggregate claims (liabilities) up to (and including) time t.

The approach above has been fully reviewed by De Finetti (1957), who proposed a modified random walk for the reserve process with a reflecting barrier at a predefined level, and Borch (1967) who provided further extension assuming that the premium rate Π may not be constant, but a smoothly controlled item. Then, the premium rate is composed of the net premium rate p, which equals the constant expected claim rate plus a variable safety loading $\theta(t)$ (expressed as a percentage of p). Hence, the basic Eq. (1) becomes

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$$F(t) = F(0) + \int_0^t (1 + \theta(s))\, p ds - S(t). \tag{2}$$

The ideal mathematical framework for the solution of the problem above has been proved to be the control theory, and especially the stochastic approach. Vandebroeck & Dhaene (1990), Martin & Löf (1994) and Zimbidis & Haberman (2001) investigated the problem of insurance pricing in a discrete control setting, while Ruohonen (1980), Garrido (1989) and Norberg (1999) employed a continuous mathematical framework.

In this paper, we adapt the continuous formulation described by Eq. (2), and investigate two different management decisions with respect to the level of the safety loading, $\theta(s)$, $s \in [0,t]$ and the level of the reinsurance coverage, $1-u(s)$, $s \in [0,t]$, $0 \le u(s) \le 1$, assuming that the liabilities of the system $S(t)$ are driven by a fractional Brownian motion.

Kolmogorov (1940) first introduced such a process, while Mandelbrot & Van Ness (1968) proposed and investigated potential applications of fBm in financial models forty years ago. More recently, and motivated also from further applications in other fields (hydrology, telecommunications, queuing theory) fractional Brownian motion and the respective fractional noise have gained much popularity. The fBm is considered the ideal tool for modeling the long-range dependence or long-memory which is suspected to be present in the financial or insurance market. However, although fBm is suspicious of any presence behind different financial or insurance data, this is not fully justified yet, especially for the insurance part.

Of course there are certain empirical studies, such as Cheung (1993) and references cited therein, that shows the existence of 'long memory' in financial time series (e.g., for the exchange rates). Shiryaev (1999) provides an adequate investigation with respect to the modeling of financial quantities and fractional Brownian motion. Beran (1994) presents different statistical methods and tools that can identify the 'long-range dependency' within time series data. One of the main difficulties for this test is: how to distinguish whether there is 'dependency' or 'non-stationarity' in the available time series. This difficulty is further enlarged when we have a very small set of data.

Finally, we should state that fractional noise is a process that possesses both the 'Joseph and Noah effect'. This characterization is due to Mandelbrot (1968) who was inspired from the biblical Joseph (with the long sequence of seven good and bad harvests) and Noah (who survived with his family an enormous flood), providing the names for the cases for strong dependency and extremal events (see the discussion in Embrechts *et al.* (1999)).

2. Stochastic calculus and linear control for fractional Brownian motion

In the remainder of the paper all random variables and processes are defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^{\mathcal{H}})$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra generated as the $\mathbb{P}^{\mathcal{H}}$-completion of the natural filtration of the $\mathcal{W}^{\mathcal{H}}$ process, and $\mathbb{P}^{\mathcal{H}}$ is the respective probability measure.

The $\mathcal{W}^{\mathcal{H}} = \{\mathcal{W}^{\mathcal{H}}(t);\ t \geq 0\}$ process is assumed to be a fractional Brownian motion with a Hurst exponent $\mathcal{H}$, where $0.5 \leq \mathcal{H} \leq 1$, and may be regarded as the fractional time derivative (refer to Jumarie (2005)) of the Gaussian white noise. The basic properties of such a process are:

$\Pr[\mathcal{W}^{\mathcal{H}}(0) = 0] = 1$;
$\mathcal{W}^{\mathcal{H}}(t)$ is a $\mathcal{F}$ measurable function (random variable) for each $t \in \Re_+$, with $\mathbb{E}[\mathcal{W}^{\mathcal{H}}(t)] = 0$;
$\mathbb{E}[\mathcal{W}^{\mathcal{H}}(s)\mathcal{W}^{\mathcal{H}}(t)] = \frac{1}{2}\,[s^{2\mathcal{H}} + t^{2\mathcal{H}} - |s-t|^{2\mathcal{H}}]$ for any $s, t \in \Re_+$.

The expectation operator $\mathbb{E}[\ ]$ is applicable using the $\mathbb{P}^{\mathcal{H}}$ probability measure.

Using the properties above and Kolmogorov's continuity criterion, we know that fractional Brownian motion has a version with continuous sample paths with probability one, but these paths are nowhere differentiable. The physical derivation of such a process may be found in Mandelbrot & Van Ness (1968). The Hurst exponent is indicated with $\mathcal{H}$ due to the climatologist Hurst (1951) who first observed that $n^{\mathcal{H}}$ with $\mathcal{H} \neq \frac{1}{2}$ is required for weak convergence of the centered cumulative sums in his statistical analysis describing the yearly water flows of the Nile river (refer to Embrechts *et al.* (1999)).

Furthermore, we should state that fBm is a Gaussian process, self-similar with stationary increments exhibiting long-range dependence. Below, we provide a short mathematical description for those properties (for further details, refer to Nualart (2006)).

$\mathcal{W}^{\mathcal{H}}(t)$ is a Gaussian process if for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the distribution of the random vector $(\mathcal{W}^{\mathcal{H}}(t_1), \mathcal{W}^{\mathcal{H}}(t_2), \ldots, \mathcal{W}^{\mathcal{H}}(t_n))$ on $\Re^n$ is normal or Gaussian.

$\mathcal{W}^{\mathcal{H}}(t)$ is a self-similar process if for any constant $s \in \Re_+$, the processes $\{s^{\mathcal{H}}\mathcal{W}^{\mathcal{H}}(t);\ t \geq 0\}$ and $\{\mathcal{W}^{\mathcal{H}}(st);\ t \geq 0\}$ have the same probability distribution.

$\mathcal{W}^{\mathcal{H}}(t)$ has stationary increments when for any $s, t \in \Re_+$ the increment of the process in the interval $[s,t]$ follows the same probability distribution (normal with zero mean and variance equal to $|s-t|^{2\mathcal{H}}$).

$\mathcal{W}^{\mathcal{H}}(t)$ exhibits long-range dependence when the following summation of the co-variances diverges $\sum_{n=1}^{\infty}\mathbb{E}[\mathcal{W}^{\mathcal{H}}(1)(\mathcal{W}^{\mathcal{H}}(n+1) - \mathcal{W}^{\mathcal{H}}(n))] = \infty$, although each term of the summation tends (slowly) to zero.

Obviously, for the special value of the Hurst exponent $\mathcal{H} = 0.5$, the process is reduced to the standard Brownian motion. For $\mathcal{H} \neq 0.5$, the respective process $\mathcal{W}^{\mathcal{H}}$ is outside the 'wonderful world' of Markovian, Martingales or even semi-Martingales processes. So, the classical stochastic calculus, the respective theory of integration, and the other powerful tools of stochastic analysis are not, as yet, available, although there are certain simple integral transformations connecting the fractional Brownian motion with standard Brownian motion (refer to Norros *et al.* (1999) and Embrechts *et al.* (1999) for the formula below).

$$\mathcal{W}^{\mathcal{H}}(t) = c(\mathcal{H}) \int_{\Re} [((t-x)^+)^{\mathcal{H}-1/2} - ((-x)^+)^{\mathcal{H}-1/2}]\, d\mathcal{W}(t). \tag{3}$$

Significant research efforts have been made in two directions.

a. Stochastic integration, i.e., to establish an analogous theory of stochastic integration retaining the good properties of the Ito integral as the zero mean property (refer to Lin (1995), Dai & Heyde (1996), Decreusefond & Üstünel (1998), Alos *et al.* (1999), Duncan *et al.* (2000), Carmona & Coutin (2003) and Hu (2005)).
b. Stochastic control, i.e., to launch a concrete general theory for the control and optimization of systems driven by fractional Brownian motion (refer to Kleptsyna *et al.* (2003) and Hu & Zhou (2005)).

2.1. *Stochastic integration for the fBm and the Malliavin derivative*

The first attempts by Lin (1995) and Dai & Heyde (1996) to define an integral with respect to the fractional Brownian motion with a Hurst exponent $\mathcal{H} > 0.5$ resulted in a stochastic integral which does not possess the zero-mean property. Their approach was rather standard, defining the stochastic integral as the limit of Riemann sums in $\boldsymbol{L}^2\,(\Omega; \Re)$ – the space of real-valued square integrable functions. As the absence of zero-mean property is not convenient for either the theoretical development or practical (usually financial) applications, new proposals have been raised (refer to Decreusefond & Ustunel (1998), Alos *et al.* (1999), Duncan *et al.* (2000) and Carmona & Coutin (2003)). These alternative proposals have been developed using the techniques of Malliavin calculus and Wick product.

Hence, given a time interval $[0,T]$, a function F continuously differentiable that also satisfies an exponential growth condition and $\mathcal{H} > 0.5$, we can define the Wick stochastic integral with respect to fractional Brownian motion as follows

$$\int_0^T F(\mathcal{W}^{\mathcal{H}}(t)) \diamondsuit d\mathcal{W}^{\mathcal{H}}(t) = \int_0^T F(\mathcal{W}^{\mathcal{H}}(t)) d\mathcal{W}^{\mathcal{H}}(t) - \mathcal{H} \int_0^T F'(\mathcal{W}^{\mathcal{H}}(t)) t^{2\mathcal{H}-1} dt. \tag{4}$$

The $\diamondsuit$ symbol corresponds to the Wick product, while $\int_0^T F(\mathcal{W}^{\mathcal{H}}(t)) d\mathcal{W}^{\mathcal{H}}(t)$ is the path-wise Riemann-Stielges integral. That is well defined using the results of Young (1936) for $\mathcal{H} > 0.5$ given that F is continuously differentiable.

Moreover, the following result holds if $\Phi' = F$

$$\int_0^T F(\mathcal{W}^{\mathcal{H}}(t)) d\mathcal{W}^{\mathcal{H}}(t) = \Phi(\mathcal{W}^{\mathcal{H}}(T)) - \Phi(0) \tag{5}$$

It is proved that the Wick stochastic integral described above has the zero-mean property (refer to Duncan *et al.* (2000)).

2.1.1. The Malliavin derivative. Malliavin (1978) launched the Malliavin calculus in order to provide a probabilistic proof to the Hormander hypoellipticity theorem. It is actually an infinite dimensional differential calculus with two basic operators – the Malliavin derivative D, and its adjoint the divergence operator δ. Actually, the Malliavin derivative is a linear map from the space of random variables to a space of processes indexed by a Hilbert space. Below, we formally provide the definition of the Malliavin derivative for a special category of random variables F that may be written as

$$F = f(\mathcal{W}^{\mathcal{H}}(h_1),\ \mathcal{W}^{\mathcal{H}}(h_2), \ldots, \mathcal{W}^{\mathcal{H}}(h_n))$$

where f is infinitely differentiable with all partial derivatives polynomially bounded, $\mathcal{W}^{\mathcal{H}}(h_1),\ \mathcal{W}^{\mathcal{H}}(h_2), \ldots, \mathcal{W}^{\mathcal{H}}(h_n)$ are random variables jointly zero-mean Gaussian, and $\mathbb{E}[\mathcal{W}^{\mathcal{H}}(h_i) \cdot \mathcal{W}^{\mathcal{H}}(h_j)] = \frac{1}{2}[h_i^{2\mathcal{H}} + h_j^{2\mathcal{H}} - |h_i - h_j|^{2\mathcal{H}}] = \langle h_i, h_j \rangle$ for any i,j, while $h_1,\ h_2,\ \ldots,\ h_n$ are elements of the respective index Hilbert space.

Then, the Malliavin derivative of the random variable F is defined as

$$D^{\mathcal{H}} F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F\left(\mathcal{W}^{\mathcal{H}}(h_1), \mathcal{W}^{\mathcal{H}}(h_2), \ldots, \mathcal{W}^{\mathcal{H}}(h_n)\right) h_i.$$

If $\mathcal{W}^{\mathcal{H}}(h_i) = \int_0^T h_i(t) \diamond d\mathcal{W}^{\mathcal{H}}(t)\ i = 1,2, \ldots n$, i.e., stochastic integrals with respect to fBm, then we may assume the derivative $D^{\mathcal{H}} F$ is a stochastic process indexed by the interval $[0,T]$ and write

$$D_t^{\mathcal{H}} F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F\left(\mathcal{W}^{\mathcal{H}}(h_1), \mathcal{W}^{\mathcal{H}}(h_2), \ldots, \mathcal{W}^{\mathcal{H}}(h_n)\right) h_i(t).$$

The existence of the Malliavin derivative may be extended to a wider domain of random variables. It may also be shown some kind of a chain-rule. Finally, we should stress that a random variable may be (Malliavin) differentiable but not continuous (refer to Nualart (2006) for further details).

2.2. Stochastic controls for linear systems driven by fractional noises

We assume the general format of a linear stochastic controlled differential equation

$$dx(t) = (A(t)x(t) + B(t)u(t))dt + (C(t)x(t) + D(t)u(t) + \zeta(t))dW^{\mathcal{H}}(t) \tag{6}$$

where $x(t)$ is the state variable, $u(t)$ is the control variable, $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are given essentially bounded deterministic (matrix-valued) functions of t, $\zeta(t)$ a given real-valued essentially bounded function of t, and $\mathcal{W}^{\mathcal{H}} = \{\mathcal{W}^{\mathcal{H}}(t);\ t \geq 0\}$ is a fractional Brownian motion with a Hurst exponent $\mathcal{H}$.

Then, we denote with $\mathcal{U}^{\mathcal{H}}$ the class of $(\mathcal{F}_t^{\mathcal{H}})$-adapted processes u, where $u = \{u_t,\ t \geq 0\}$, for which the system admits a unique strong solution x_u. Of course, then x_u is an $(\mathcal{F}_t^{\mathcal{H}})$-adapted process. Actually, for control purposes, we are interested only in closed-loop policies. Therefore, we initiate a sub-class of admissible controls as the class $\mathcal{U}^{\mathcal{H}}{}_{\text{ad}}$ of those us in $\mathcal{U}^{\mathcal{H}}$ which are $(\mathcal{F}_{t,u}^{\mathcal{H}})$-adapted processes, where $(\mathcal{F}_{t,u}^{\mathcal{H}})$ is the natural filtration of the corresponding state process x_u. Then, the pair $(u,\ x_u)$ is called an admissible pair.

Now, we introduce the functional

$$J(u) := J(x_0, u) = \mathbb{E}\left[\int_0^T (x^*(t)Q(t)x(t) + u^*(t)R(t)u(t))dt + x^*(T)Gx(T)\right] \tag{7}$$

where, $Q(t)$, $R(t)$ and G are positive definite matrices.

Hence, the pair (u^{opt}, $x_u{}^{\text{opt}}$) is defined as the optimal pair if

$$J(u^{\text{opt}}) = \inf\{J(x_0, u); \quad u \in U_{\text{ad}}\} \tag{8}$$

while the $J(u^{\text{opt}})$ is established as the optimal cost for the system.

As regards the theoretical results with respect to the derivation of the optimal solution u^{opt} for the system described by expressions (6) and (7), we have the following two basic theorems.

Theorem 2.1 *Assume that for a.e. $t \in [0,T]$, $\zeta(t) = 0$, the optimal solution for the system of expressions (6) and (7) is determined via the following feedback formula*

$$u^{\text{opt}}(t) = K(t) \cdot x(t) \tag{9}$$

where $K(t) = (K_1(t), \ldots, K_m(t))^*$ satisfies the equation below

$$\begin{aligned} &K^*(t) \cdot R(t) \cdot \Theta(t) + B(t) \cdot \int_t^T \Theta(s)(Q(s) + K^*(s) \cdot R(s) \cdot K(s))ds \\ &+D(t) \cdot \int_t^T \int_0^s \Theta(s) \cdot \phi(s', t) \cdot (C(s') + D(s') \cdot K(s'))^* (Q(s) + K^*(s) \cdot R(s) \cdot K(s)) ds' ds \\ &+G \cdot \Theta(T) \left[B(t) + D(t) \int_0^T \phi(s', t)(C(s') + D(s') \cdot K(s'))^* ds' \right] = 0, \quad a.e. \;\; s \in [0, T] \end{aligned} \tag{10}$$

while

$$\phi(s, t) = \mathcal{H}(2\mathcal{H} - 1)|s - t|^{2\mathcal{H}-2} \tag{11}$$

$$\begin{aligned} \Theta(t) = x_0^2 \cdot \mathrm{E}\Bigg\{ \exp\Bigg[2 \int_0^t (A(s) + B(s) \cdot K(s)) ds \\ + \int_0^t \int_0^t \phi(s, s')(C(s) + D(s) \cdot K(s))(C(s') + D(s') \cdot K(s'))^* ds ds' \Bigg] \Bigg\}. \end{aligned} \tag{12}$$

Proof. Refer to the second special case of Theorem 4.1 of Hu & Zhou (2005).

Theorem 2.1 (fully proved in Hu & Zhou (2005)) may solve the advanced model with reinsurance, as described in Section 4. As regards the simple model with no reinsurance, we use another Theorem 3.1 from Hu & Zhou (2005). The additional problem that must be dealt with is the existence of the non-zero function $\zeta(t)$. This complication is eliminated by the proof of the following lemma.

Lemma 2.1 *Let $x(t)$ be the solution of the uncontrolled version of system described in Eq. (6), i.e., $u(t) = 0$. If $A(t)$ is a measurable and essentially bounded deterministic function in t, while $C(t) = 0$ a.e. $t \in [0,T]$, then the Malliavin derivative of $x(t)$, represented by $D_t^{\mathcal{H}} x(t)$ can be represented as*

$$D_t^{\mathcal{H}} x(t) = \exp\left(\int A(t)dt\right) \cdot \left[\psi + \int \exp\left(\int A(t)dt\right) \cdot \zeta(t) \cdot \phi(r,t)dt\right] \tag{13}$$

where ψ is a constant (see expression (19)) and consequently, the Malliavin derivative of $x(t)$ is bounded.

Proof. The solution of Eq. (6), $x(t)$, is understood as

$$x(t) = \int_0^t A(s)x(s)ds + \int_0^t \zeta(s)dW^{\mathcal{H}}(s). \tag{14}$$

We may compute the Malliavin derivative of $x(t)$ using the results of Duncan *et al.* (2000) (see especially Theorem 4.2 of their paper). We have

$$D_r^{\mathcal{H}} x(t) = \int_0^t A(s) \cdot D_r^{\mathcal{H}} x(s)ds + \int_0^t \zeta(s) \cdot \phi(r,s)ds \quad \forall r, t \in [0,T] \text{ a.s.} \tag{15}$$

We fix r and denote $z(r,t) := D_r^{\mathcal{H}} x(t)$. Then, we may rewrite Eq. (15) in its differential format

$$dz(r,t) = A(t) \cdot z(r,t) \cdot dt + \zeta(t)\phi(r,t)dt$$

or equivalently,

$$\frac{d}{dt} z(r,t) - A(t) \cdot z(r,t) = \zeta(t)\phi(r,t) \tag{16}$$

with

$$z(r,0) = D_r^{\mathcal{H}} x(0) = 0. \tag{17}$$

This is an ordinary linear differential equation, so the solution can be represented as

$$z(r,t) = \exp\left(\int A(t)dt\right) \cdot \left[\psi + \int \exp\left(\int A(t)dt\right) \cdot \zeta(t) \cdot \phi(r,t)dt\right] \tag{18}$$

where ψ is determined by the initial condition (17).

$$\psi = -\int \exp\left(\int A(t)dt\right) \cdot \zeta(t) \cdot \phi(r,t)dt\Big|_{t=0}. \tag{19}$$

The proof is completed.

Theorem 2.2 *Assume that for a.e. $t \in [0,T]$, $D(t) = 0$, $Q(t) \geq 0$, and $R(t) > \delta I$ for some given $\delta > 0$ and $G \geq 0$. Then, the optimal solution for the system of expressions (6) and (7), is determined again via the following feedback formula*

$$u^{\text{opt}}(t) = K(t) \cdot x(t) \tag{20}$$

where

$$K(t) = -R^{-1}(t) \cdot B^*(t) \cdot p(t) \tag{21}$$

and $p(t)$ is determined by the solution of the following Riccati equation

$$\frac{d}{dt}p(t) + 2p(t)\left[A(t) + C(t)\cdot\int_0^t \phi(t,s)C(s)ds\right] + Q(t) - B(t)R^{-1}(t)B^*(t)p^2(t) = 0 \quad (22)$$

with the terminal condition

$$p(T) = G. \quad (23)$$

Proof. Since the Malliavin derivative of $x(t)$ is bounded (see Lemma 2.1), we follow the exact methodology and arguments of the proof of Theorem 3.1 in Hu & Zhou (2005). That completes the proof.

3. The basic version of the framework model with no reinsurance

We consider a typical insurance system with an initial reserve. Variable premiums inflow while claims (liabilities) outflow at a stochastic pattern that may be modeled with $S(t)$ a drifted fractional Brownian motion under the respective measure $\mathbb{P}^{\mathcal{H}}$. So, the system is well described by the integral Eq. (2) or its equivalent differential format

$$dF(t) = [1 + \theta(t)]\cdot m(t)\cdot dt - dS(t) \quad (24)$$

where

$$dS(t) = m(t)\cdot dt + \sigma(t)\cdot dW^{\mathcal{H}}(t). \quad (25)$$

A similar approach has been proposed by several authors, such as Højgaard & Taksar (1998) and Schmidli (2001), assuming that the aggregate claims $S(t)$ may be approximated by a drifted Brownian motion given that the insurance portfolio 'is made up of a large number of independent individual risks, none of which is large enough to affect the total result significantly' (Norberg (1999)).

In our model, we extend the specific approach by relaxing the assumption of independence of the individual risks and incorporating some kind of dependence as far as the Hurst exponent deviates from the critical value of $H = 0.5$, which corresponds to the traditional Brownian motion. Furthermore, we can also relax the assumption for the small size of the claims. As we have seen in the introductory section, fractional noise is a process that exhibits both the 'Joseph effect' and 'Noah effect'. So, we may also model insurance risks that require heavy-tailed distributions either for the inter-arrival times (e.g., earthquake insurance) or for the amounts of individual risks (e.g., huge industrial risks).

Additionally, if we consider that the reserve fund is also invested within a non-defaulting bank account bearing a variable force of interest, say $\delta(t)$, then the differential equation governing the reserve process becomes

$$dF(t) = [\delta(t)\cdot F(t) + m(t)\cdot\theta(t)]\cdot dt - \sigma(t)\cdot dW^{\mathcal{H}}(t). \quad (26)$$

Obviously, Eq. (26) represents the mechanisms of a potentially controlled process, where $\theta(t)$ is the control variable based on the decisions of the insurance manager about the level of the respective safety loading of the premium rate at time t; $W^H(t)$ is the input variable reflecting the fluctuation of the claim experience; and $F(t)$ is the state variable which represents the reserve value at time t.

Having described the vital mechanism of the insurance system with Eq. (26), we can proceed with the exploration of the critical problem of the premium determination $\Pi(t)$ at time t or equivalently the determination of the safety loading $\theta(t)$ at time t, as

$$\Pi(t) = (1 + \theta(t))\cdot m(t) \tag{27}$$

We solve this problem assuming a finite control horizon up to time T with no reinsurance.

The problem of premium determination requires a criterion that will be optimized under the constraint of the regulating Eq. (26). Such a typical and well-used criterion is described by the minimization of the expectation of a quadratic functional:

$$\min_{\theta(t)} E\left\{ \int_0^T \beta(t)\ \theta^2(t)dt + [F(T) - F_\pi]^2 \right\} \tag{28}$$

where $\beta(t)$ is a weighting factor.

The last expression (28) may have a good verbal interpretation in the insurance context. Normally, the clients of an insurance company require small and smooth (as possible) premium rates, so the insurance manager should keep a constant zero (as possible) level for the safety loading $\theta(t)$ for any $t \in [0,T]$, i.e., the manager aims to minimize the following integral

$$\int_0^T \beta(t)\ \theta^2(t)dt. \tag{29}$$

Ideally, if $\theta(t)$ equals zero for any time t, then we obtain the smoothest pattern for premiums and the smallest value for expression (29). This simple strategy cannot be fully adapted as a constant zero safety loading increases dramatically the probability of ruin. An alternative route for smooth premium rates with increased security is the determination of a high constant positive safety loading. But this approach will ultimately explode the level of reserve to undesirable values. So, the manager should balance his decisions, and, consequently, his expectations between a small value for expression (29) and some bounded value for the final reserve $F(T)$ near the desired final value F_π at the end of a certain period, T.

The balancing effort is formulated and obtained by the introduction of the weighting factor $\beta(t)$. The stochastic differential system, described by expressions (26) and (28), may be solved using Theorem 2.2 analyzed in Section 2 and substituting

$$A(t) = \delta(t),\ B(t) = m(t),\ C(t) = 0,\ D(t) = 0,\ \zeta(t) = -\sigma(t),\ Q(t) = 0,\ R(t) = \beta(t),\ G = 1 \tag{30}$$

in the basic Eq. (22) and the final condition (23). So, we derive the following differential equation

$$\frac{d}{dt}p(t) + \delta(t)p(t) - \frac{1}{\beta(t)}m^2(t)p^2(t) = 0 \tag{31}$$

with

$$p(T) = 1. \tag{32}$$

The general solution of the ordinary differential Eq. (31) under the terminal condition (32) may be easily verified as the following one (see Polyanin & Zaitsev (2002), Eq. (31) is characterized as a Bernoulli type)

$$\frac{1}{p(t)} = \exp\left(2\int \delta(s)ds\right)\cdot\left[\psi_1 - \exp\left(2\int \delta(s)ds\right)\cdot\int_0^t \frac{1}{\beta(s)}m^2(s)\cdot\exp\left(-2\int \delta(s)ds\right)ds\right] \tag{33}$$

where ψ_1 is a constant determined by the terminal condition (32). For the special case where $\delta(t) = 0$ we obtain,

$$p(t) = \frac{1}{1 + \int_t^T \frac{m^2(s)}{\beta(s)}ds}. \tag{34}$$

Hence, the optimal control for the safety loading is

$$\theta^{\text{opt}}(t) = K(t)\cdot(F(t) - F_\pi) \tag{35}$$

or equivalently,

$$\theta^{\text{opt}}(t) = -\frac{m(t)}{\beta(t)}\frac{1}{1 + \int_t^T \frac{m^2(s)}{\beta(s)}ds}(F(t) - F_\pi). \tag{36}$$

4. The advanced version of the framework model with reinsurance

We extend the basic structure of the typical insurance system incorporating the standard concept of proportional reinsurance coverage. Then, the development of the surplus reserve follows the general equation below:

$$[\text{Reserve}] = \begin{bmatrix}\text{Initial}\\ \text{Reserve}\end{bmatrix} + \begin{bmatrix}\text{Premiums}\\ \text{Received}\end{bmatrix} - \begin{bmatrix}\text{Claims}\\ \text{Paid}\end{bmatrix} - \begin{bmatrix}\text{Reinsurance}\\ \text{Premiums}\end{bmatrix} + \begin{bmatrix}\text{Claims recovered}\\ \text{by the Reinsurer}\end{bmatrix}.$$

The mathematical formulation of the process described in the expression above is provided by the following integral equation

$$F(t) = F(0) + \int_0^t (1 + \theta_r(s))m(s)ds - S(t) - \int_0^t (1 + \xi(s))m(s)\,(1 - v(s))ds + \int_0^t v(s)dS(s) \tag{37}$$

or using the equivalent stochastic differential format and adding the investment parameter $\delta(t)$ (the force of interest), we obtain

$$dF(t) = \delta(t)F(t)dt + (1 + \theta_r(t))m(t)dt - (1 + \xi(t))m(t)(1 - v(t))dt - v(t)dS(t) \tag{38}$$

where $F(t)$, $F(0)$, are defined in the same way as at the beginning of Section 3, while $\theta_r(t)$ is the safety loading of the insurer at time t, $t>0$; $\xi(t)$ is the safety loading of the reinsurer at time t, $t>0$; $1-v(t)$ is the proportion of the reinsurance coverage at time t, $v(t)\in[0,1]$ for any $t>0$ or equivalently, $v(t)$ is the proportion retained by the insurer at time t.

Similarly, with the arguments and procedures adapted in Section 3, we reformulate the differential Eq. (38) as

$$dF(t) = [\delta(t)F(t) + [\theta_r(t) - (1 - v(t))\xi(t)]m(t)]dt - \sigma(t)v(t)dW^{\mathcal{H}}(t) \tag{39}$$

where $W^{\mathcal{H}}(t)$ is a fractional Brownian motion under the respective measure $\mathbb{P}^{\mathcal{H}}$.

We must also notice that normally, the following inequality does truly hold

$$\theta_r(t) \leq \xi(t) \quad \text{for any } t. \tag{40}$$

The supporting argument for the validity of the expression above is the 'no arbitrage' opportunity. If there exist t_0 such that $\theta_r(t_0) > \xi(t_0)$, then the insurer has the opportunity to make risk-less profit by ceding the total risk to the reinsurer paying a reinsurance premium smaller than the insurance premium.

Having described the mechanisms of the advanced version of the framework model which includes a reinsurance option, we proceed with the exploration of the basic question, as in Section 3, i.e., we determine the two potentially controlled variables $\theta_r(t)$ and $v(t)$.

We define a new quantity $\omega(t)$ such that

$$\theta_r(t) = \xi(t) - \omega(t). \tag{41}$$

This new variable $\omega(t)$ represents the difference between the insurer's and reinsurer's loading, and according to inequality (40), should be normally positive. Then, Eq. (39) becomes

$$dF(t) = [\delta(t)F(t) + [\omega(t) + v(t)\xi(t)]m(t)]dt - \sigma(t)v(t)dW^{\mathcal{H}}(t) \tag{42}$$

or in matrix format

$$dF(t) = \left\{ \delta(t)F(t) + [m(t) \quad m(t)\xi(t)] \begin{bmatrix} \omega(t) \\ v(t) \end{bmatrix} \right\} dt + [0 \quad -\sigma(t)] \begin{bmatrix} \omega(t) \\ v(t) \end{bmatrix} dW^{\mathcal{H}}(t). \tag{43}$$

As objective function, we use a similar criterion as described before in expression (28)

$$\min_{\omega(t),\ v(t)} E\left\{ \int_0^T \beta(t)\ (\omega(t) - \xi(t))^2 dt + [F(T) - F_\pi]^2 \right\} \tag{44}$$

(expressions (28) and (44) are equivalent since $(\omega(t) - \xi(t))^2 = (\xi(t) - \omega(t))^2 = \theta_r{}^2(t)$).

The actual problem is described by the objective function (44) under the constraint of the stochastic differential equation (43). We solve this optimization problem by applying Theorem 2.1 described in Section 2, and substituting the general matrices with the specific values

$$\begin{aligned} &A(t) = \delta(t), \quad B(t) = [m(t)\ m(t)\xi(t)], \quad C(t) = 0, \quad D(t) = [0 \quad -\sigma(t)], \\ &Q(t) = 0, \quad R(t) = \begin{bmatrix} \beta(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad G = 1 \end{aligned} \quad . \tag{45}$$

Hence, the optimal control for $\omega(t)$ and $v(t)$ is established as a feedback mechanism of the reserve value

$$u(t) = \Xi(t) + K(t)\cdot(F(t) - F_\pi) \tag{46}$$

where

$$u(t) = \begin{bmatrix} \omega(t) \\ v(t) \end{bmatrix}, \ \Xi = \begin{bmatrix} \xi(t) \\ 0 \end{bmatrix} \ \text{and} \ K(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \end{bmatrix} \tag{47}$$

while $K(t)$ is determined by the general Eq. (10) substituting the values determined by relationships (45). So, we derive the following matrix integral equation

$$\begin{aligned}
&\Theta(t)[\beta(t)K_1(t) \quad 0] + [m(t) \quad m(t)\xi(t)] \int_t^T \Theta(s)\cdot\beta(s)\cdot K_1(s)ds \\
&+ [0 \ -\sigma(t)]\cdot \int_t^T\int_0^s \Theta(s)\cdot\phi(s',t)\cdot(-\sigma(s')\cdot K_2(s'))(\beta(s)\cdot {K_1}^2(s))ds'ds, \quad a.e.\ s\in[0,T] \\
&+ \Theta(T)\left[[m(t) \quad m(t)\xi(t)] + [0 \ -\sigma(t)] \int_0^T \phi(s',t)(-\sigma(s')\cdot K_2(s'))ds'\right] = 0
\end{aligned} \tag{48}$$

where $\Theta(t)$ is determined via expression (12) as follows

$$\begin{aligned}
\Theta(t) = F^2(0)\cdot \mathrm{E}\Bigg\{\exp\Bigg[2\int_0^t (\delta(s) + m(s)\cdot K_1(s) + m(s)\cdot\xi(s)\cdot K_1(s))ds \\
+ \int_0^t\int_0^t \phi(s,s')\sigma(s)\cdot K_2(s)\cdot\sigma(s')\cdot K_2(s')dsds'\Bigg]\Bigg\}
\end{aligned} \tag{49}$$

or equivalently we obtain a system of two integral equations as below

$$\beta(t)\Theta(t)K_1(t) + m(t)\int_t^T \Theta(s)\cdot\beta(s)\cdot K_1(s)ds + m(t)\cdot\Theta(T) = 0 \tag{50}$$

$$\begin{aligned}
&m(t)\xi(t)\int_t^T \Theta(s)\cdot\beta(s)\cdot K_1(s)ds + \sigma(t)\cdot\int_t^T\int_0^s \Theta(s)\cdot\phi(s',t)\cdot\sigma(s')\cdot K_2(s')\cdot\beta(s)\cdot {K_1}^2(s)ds'ds \\
&+m(t)\xi(t)\Theta(T) + \sigma(t)\int_0^T \phi(s',t)\cdot\sigma(s')\cdot K_2(s')ds' = 0
\end{aligned} \tag{51}$$

The system above, in the most general format, cannot be solved analytically. So we examine a special case which appears quite interesting, where the insurer does not actually control its loading, but follows exactly the reinsurer's policy. In that case, we find an elegant result as regards the control of the retained risk or the proportion of reinsurance coverage.

Special case: $\theta_r(t) = \xi(t)$

We consider the case where the insurer's loading is fully equated with the level of the reinsurer's loading. This action offsets any undesirable and unaffordable losses for the insurer, which may occur when there is a great distance between the two loadings: $\theta_r(t)$ and $\xi(t)$.

So, we put $\omega(t) = 0$ for any t and consequently

$$\theta_r(t) = \xi(t). \tag{52}$$

Substituting the relationship (52) in the differential equation (39) we obtain

$$dF(t) = [\delta(t)F(t) + m(t)\xi(t)v(t)]dt - \sigma(t)v(t)dW^{\mathcal{H}}(t). \tag{53}$$

We solve the problem assuming a finite control horizon up to time T and using a reduced version of the objective function compared with the basic version of the control problem, i.e.

$$\min_{v(t)} E\{[F(T) - F_\pi]^2\}. \tag{54}$$

Actually, the structure of expression (54) supports that the decision-maker of the system penalize the large deviations between the desired final value F_π and the actual final value $F(T)$ of the reserve process. Alternatively, the decision-maker is trying to secure that the system will finally arrive closely to the desired profitable (assuming that $F_\pi > F(0)$) financial position.

Hence, the actual problem is described by the objective function (54) under the constraint of the stochastic differential equation (53). We solve this optimization problem by applying Theorem (2.1) described in Section 2, and substituting the general matrices with the specific values:

$$A(t) = \delta(t)\ ,\ B(t) = m(t)\xi(t)\ ,\ C(t) = 0\ ,\ D(t) = -\sigma(t)\ ,\ Q(t) = 0,\ R(t) = 0\ ,\ G = 1. \tag{55}$$

Finally, the optimal control is established as a feedback mechanism of the reserve value

$$v(t) = K(t)\cdot(F(t) - F_\pi) \tag{56}$$

where $K(t)$ is determined by the general Eq. (10) substituting the values determined by relationships (55). So, we derive a special integral equation described as Carleman type.

$$\int_0^T \phi(s,t)\cdot K(s)ds = -\frac{m(t)}{\sigma^2(t)}\xi(t). \tag{57}$$

The general solution of Eq. (57) is:

$$K(t) = -a_{\mathcal{H}}\cdot t^{0.5-\mathcal{H}}\cdot\frac{d}{dt}\int_t^T\Bigg[w^{2\mathcal{H}-1}\cdot(w-t)^{0.5-\mathcal{H}}\cdot \frac{d}{dw}\times\int_0^w z^{0.5-\mathcal{H}}\cdot(w-z)^{0.5-\mathcal{H}}\cdot\left(-\frac{m(t)}{\sigma^2(t)}\xi(z)\right)dz\Bigg]dw \tag{58}$$

where

$$a_{\mathcal{H}} = \frac{\Gamma(2-2\mathcal{H})}{2\mathcal{H}\cdot\Gamma(0.5+\mathcal{H})\cdot\Gamma(1.5-\mathcal{H})^3}. \tag{59}$$

The relationship (58) may be further simplified for the special case where

$$\frac{m(t)}{\sigma^2(t)}\xi(z) = \frac{m}{\sigma^2}\xi, \quad \forall t \in [0, T]. \tag{60}$$

By removing the functional structure, the second integral in expression (58) may be easily calculated using the Beta function, i.e.

$$\int_0^w z^{0.5-\mathcal{H}}\cdot(w-z)^{0.5-\mathcal{H}}dz = w^{2(1-\mathcal{H})}\cdot B(1.5-\mathcal{H}, 1.5-\mathcal{H}). \tag{61}$$

Consequently,

$$\frac{d}{dw}\int_0^w z^{0.5-\mathcal{H}}\cdot(w-z)^{0.5-\mathcal{H}}dz = 2(1-\mathcal{H})\cdot w^{1-2\mathcal{H}}\cdot B(1.5-\mathcal{H}, 1.5-\mathcal{H}). \tag{62}$$

Substituting the last result (62) in the complex expression (58), we finally obtain a short compact analytic expression for the feedback factor

$$K(t) = -a_{\mathcal{H}}\cdot b_{\mathcal{H}}\cdot\frac{m}{\sigma^2}\xi\cdot t^{0.5-\mathcal{H}}\cdot(T-t)^{0.5-\mathcal{H}} \tag{63}$$

where

$$b_{\mathcal{H}} = 2\cdot(1-\mathcal{H})\cdot B(1.5-\mathcal{H},\ 1.5-\mathcal{H}). \tag{64}$$

Using the general relationship between the Beta and Gamma functions, i.e.,

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}, \tag{65}$$

the recursive relationship for the values of the Gamma function, i.e.

$$\Gamma(a+1) = a\cdot\Gamma(a) \tag{66}$$

and substituting expression (63) in relationship (56), we finally obtain the rule for the optimal functional for the retained percentage of risk via a typical feedback mechanism

$$v(t) = -a_{\mathcal{H}}\cdot b_{\mathcal{H}}\cdot\frac{m}{\sigma^2}\xi\cdot t^{0.5-\mathcal{H}}\cdot(T-t)^{0.5-\mathcal{H}}\cdot(F(t)-F_\pi) \tag{67}$$

or equivalently

$$v(t) = -\frac{1}{\Gamma(0.5+\mathcal{H})\cdot\Gamma(1.5-\mathcal{H})}\cdot\frac{m\xi}{2\mathcal{H}\sigma^2}\cdot t^{0.5-\mathcal{H}}\cdot(T-t)^{0.5-\mathcal{H}}\cdot(F(t)-F_\pi). \tag{68}$$

Keeping in mind that $v(t)$ is a percentage in the interval [0, 1], we may easily conclude that when the reserve exceeds the predefined desired value of the profitable level ($F(t) > F_\pi$) then $v(t) = 0$. That means the insurer will be fully reinsured passing the total risk to the reinsurer.

Table 1.

	MFF(t)					
t	$\mathcal{H}=0.5$	$\mathcal{H}=0.6$	$\mathcal{H}=0.7$	$\mathcal{H}=0.8$	$\mathcal{H}=0.9$	$\mathcal{H}=1.0$
0	1,000	1,530	1,429	1,334	1,245	1,162
1	1,000	1,221	0,910	0,678	0,505	0,376
2	1,000	1,146	0,801	0,559	0,391	0,273
3	1,000	1,106	0,747	0,504	0,340	0,230
4	1,000	1,082	0,714	0,471	0,311	0,205
5	1,000	1,065	0,691	0,449	0,292	0,189
6	1,000	1,053	0,676	0,434	0,279	0,179
7	1,000	1,044	0,665	0,424	0,270	0,172
8	1,000	1,039	0,658	0,417	0,264	0,167
9	1,000	1,035	0,654	0,413	0,261	0,165
10	1,000	1,034	0,653	0,412	0,260	0,164
11	1,000	1,035	0,654	0,413	0,261	0,165
12	1,000	1,039	0,658	0,417	0,264	0,167
13	1,000	1,044	0,665	0,424	0,270	0,172
14	1,000	1,053	0,676	0,434	0,279	0,179
15	1,000	1,065	0,691	0,449	0,292	0,189
16	1,000	1,082	0,714	0,471	0,311	0,205
17	1,000	1,106	0,747	0,504	0,340	0,230
18	1,000	1,146	0,801	0,559	0,391	0,273
19	1,000	1,221	0,910	0,678	0,505	0,376
20	1,000	1.530	1,429	1,334	1,245	1,162

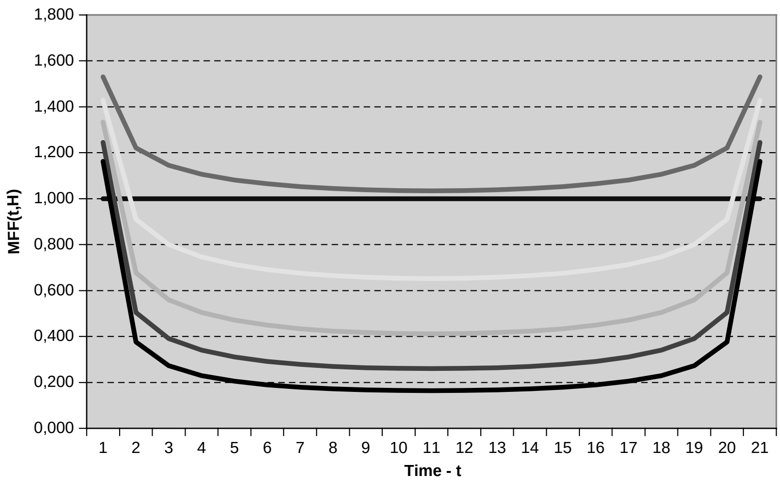

Figure 1. The pattern of the MFF quantity is quite interesting. Here are some basic notes.

- The shape of the MFF graph is symmetric and similar for all values of $\mathcal{H}$, apart from the extreme value $\mathcal{H}=0.5$ where we obtain a straight line.
- Again apart from the extreme value $\mathcal{H}=0.5$, as $\mathcal{H}$ increases the MFF also increases for all values of t and the specific rate of increase is also increasing.
- The MFF is greater at the beginning or at the end of the control period approaching infinity for $t=0.20$. It achieves its minimum (for all values of $\mathcal{H}$) exactly in the middle of the control period, $t=10$. That is quite a peculiar result. It suggests that the decision-maker should retain greater volumes of risk at the beginning or at the end of the control period as long as the actual reserve value differs from the desired target value.

5. Conclusions and further research

The management of both premium and reinsurance policies is a critical aspect for all insurance organizations. The responsible decision-makers should always balance the demand of policyholders for smooth development or constant pattern for premium rates and the requirement of shareholders for increased protection of the system, and ultimately achievement of a profitable position. In this paper, we deal with both these two problems and obtain compact analytical formulae, which reveal very interesting results.

As regards the safety loading which actually determines the level of insurance premium, we derive Eq. (36). It is obvious that whenever the reserve $F(t)$ exceeds the desired predefined value F_π, then the safety loading is negative while in the opposite case it is positive. That is expected, as whenever the reserve exceeds the predefined value, then there is no need to receive any additional loading to further enhance that difference, but receive a reduced premium in order to cut down the specific undesirable surplus and attract the clients to remain with the company. In the opposite case, where $F(t)$ remains below F_π, we increase the loading to recover the undesirable deficit. Additionally, the structure of Eq. (36) supports that the level of the loading is proportional to the drift $m(t)$ of the fractional Brownian motion (representing the liabilities of the system) and the future sequence of weighting factors $\beta(s)$, $t<s<T$, while reversal analogous to the weighting factor $\beta(t)$ and the future sequence of the drifts $m(s)$, $t<s<T$.

As regards the retained proportion of risk $v(t)$ (or equivalently, the proportion of reinsurance coverage $1\text{-}v(t)$), we conclude that whenever $F(t)$ exceeds the desired predefined value F_π, then the decision-maker retains no risk ($v(t)=0$) as they have no incentive to make any insurance business since the profitability of the system is well above expectations. In the opposite situation, where $F(t)$ remains below F_π, the decision-maker (greatly concerned to reach the profitability target) reduces the proportion of reinsurance in order to save money and upgrade the reserve level near to the desired value. The retained proportion of risk is also analogous to the modified feedback factor MFF(t) that is defined explicitly below as

$$\mathrm{MFF}(t,\mathcal{H}) = \frac{1}{2\cdot\mathcal{H}\cdot\Gamma(0.5+\mathcal{H})\cdot\Gamma(1.5-\mathcal{H})}\cdot t^{0.5-\mathcal{H}}\cdot(T-t)^{0.5-\mathcal{H}}. \tag{69}$$

We have plotted the MFF quantity using a certain control period $T=20$ and six different values for the Hurst exponent $\mathcal{H}=0.5$, 0.6, 0.7, 0.8, 0.9 and 1.0 (see Table 1 and Figure 1).

In conclusion, we should point out some further directions for future research. We can incorporate additional control variables for the investment management of the reserve fund or consider other reinsurance treaties as the excess of loss or stop loss cover.

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