Normed groupoids with dilations

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Abstract

We study normed groupoids with dilations and their induced deformations.

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1 Normed groupoids

1.1 Groupoids

A groupoid is a small category whose arrows are all invertible. More precisely we have the following definition.

Definition 1.1 *A* **groupoid** *over a set X is a set of* **arrows** *G along with a* **target map** *$\omega : G \to X$, a* **source map** *$\alpha : G \to X$, a* **identity section** *$e : X \to G$ which is a injective function, a* **partially defined operation** *(or product) m on G, which is a function:*

$$m : G^{(2)} = \left\{ (g,h) \in G^2 : \omega(h) = \alpha(g) \right\} \to G, \quad m(g,h) = gh$$

and a **inversion map** *$inv : G \to G$, $inv(g) = g^{-1}$. These are the structure maps of the groupoid. They satisfy several identities.*

(a) For any $(g,h) \in G^{(2)}$ we have

$$\omega(gh) = \omega(g), \quad \alpha(gh) = \alpha(h)$$

(b) Then for any $(g,h), (gh,k) \in G^{(2)}$ we have also $(g,hk) \in G^{(2)}$. This allows us to write the expression $g(hk)$ and to state that the operation m is associative:

$$(gh)k = g(hk)$$

(c) for any $g \in G$ the identity section satisfies $(e(\omega(g)),g), (g,e(\alpha(g))) \in G^{(2)})$ and

$$e(\omega(g))\,g = g\,e(\alpha(g)) = g$$

(c) The inversion map is an involution: $inv\,inv = id$. For any $g \in G$ we have $(g^{-1},g) \in G^{(2)}$ and $(g,g^{-1}) \in G^{(2)}$ and

$$g^{-1}\,g = e(\alpha(g)), \quad g\,g^{-1} = e(\omega(g))$$

An equivalent definition of a groupoid emphasizes the fact that a groupoid may be defined only in terms of its arrows.

Definition 1.2 *A groupoid is a set G with two operations $inv : G \to G$, $m : G^{(2)} \subset G \times G \to G$, which satisfy a number of properties. With the notations $inv(a) = a^{-1}$, $m(a,b) = ab$, these properties are: for any $a, b, c \in G$*

(i) *if $(a,b) \in G^{(2)}$ and $(b,c) \in G^{(2)}$ then $(a,bc) \in G^{(2)}$ and $(ab,c) \in G^{(2)}$ and we have $a(bc) = (ab)c$,*

(ii) *$(a, a^{-1}) \in G^{(2)}$ and $(a^{-1}, a) \in G^{(2)}$,*

(iii) *if $(a,b) \in G^{(2)}$ then $abb^{-1} = a$ and $a^{-1}ab = b$.*

Starting with the definition 1.2, we can reconstruct the objects from definition 1.1. The set $X = Ob(G)$ is formed by all products $a^{-1}a$, $a \in G$. For any $a \in G$ we let $\alpha(a) = a^{-1}a$ and $\omega(a) = aa^{-1}$. The identity section is just the identity function on X.

1.2 Notations.

A groupoid is denoted either by $\mathcal{G} = (X, G, \omega, \alpha, e, m, \ inv)$, or by (G, m, inv). In the second case we shall use the notation $X = Ob(G) = \{a^{-1}a : a \in G\}$. In most of this paper we shall simply denote a groupoid (G, m, inv) by G.

Definition 1.3 *The transformation $(f, F) : G \to G'$ is a **morphism of groupoids** defined from $\mathcal{G} = (X, G, \omega, \alpha, e, m, \ inv)$ to $\mathcal{G}' = (X', G', \omega', \alpha', e', m', \ inv')$ is a pair of maps: $f : X \to X'$ and $F : G \to G'$ which commutes with the structure, that is: $f\omega = \omega' F$, $f\alpha = \alpha' F$, $Fe = e'f$, $F\,inv = inv'\,F$ and F is a morphism of operations, from the operation m to operation m'.*

Definition 1.4 *A **Hausdorff topological groupoid** is a groupoid G which is also a Hausdorff topological space, such that inversion is continuous and the multiplication is continuous with respect to the topology on $G^{(2)}$ induced by the product topology on G^2.*

We denote by $dif : G \times_\alpha G \to G$ the **difference function**:

$$dif(g,h) = gh^{-1} \quad \forall (g,h) \in G \times G \quad \alpha(g) = \alpha(h)$$

1.3 Norms

We shall consider the convergence of nets (a_ε) of arrows, with $\varepsilon \in I$ a parameter in a directed set I. In this paper the most encountered directed set I will be $(0, +\infty)$.

Definition 1.5 *A **normed groupoid** (G, d) is a groupoid $\mathcal{G} = (X, G, \omega, \alpha, e, m, \ inv)$ with a **norm** function $d : G \to [0, +\infty)$, such that:*

(i) *$d(g) = 0$ if and only if there is a $x \in X$ with $g = e(x)$,*

(ii) *for any $(g,h) \in G^{(2)}$ we have $d(gh) \leq d(g) + d(h)$,*

(iii) *for any $g \in G$ we have $d(inv(g)) = d(g)$.*

*A **norm** d **is separable** if it satisfies the property:*

(iv) *if there is a net $(a_\varepsilon) \subset G$ such that for any $n \in \mathbb{N}$ $\alpha(a_\varepsilon) = x$, $\omega(a_\varepsilon) = y$ and $\lim\limits_{\varepsilon \in I} d(a_\varepsilon) = 0$ then $x = y$.*

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1.4 Other groupoids associated to a normed groupoid

Let (G, m, inv, d) be a normed groupoid and dif its difference function. The norm d composed with the function dif gives a new function $\tilde{d}$:

$$\tilde{d} : G \times_\alpha G \to [0, +\infty) \quad , \quad \tilde{d}(g, h) = d\, dif(g, h) = d(gh^{-1})$$

which induces a distance on the space $\alpha^{-1}(x)$, for any $x \in X = Ob(G)$:

$$d_x : \alpha^{-1}(x) \times \alpha^{-1}(x) \to [0, +\infty) \quad , \quad d_x(g, h) = \tilde{d}d(g, h) = d(gh^{-1})$$

Definition 1.6 *The* **metric groupoid** G_m *associated to the normed groupoid G is the following metric groupoid:*

- *the objects of G_m are the metric spaces $(\alpha^{-1}(x), d_x)$, with $x \in Ob(G)$;*

- *the arrows are right translations*

$$R_u : (\alpha^{-1}(\omega(u)), d_{\omega(u)}) \to (\alpha^{-1}(\alpha(u)), d_{\alpha(u)}) \quad , \quad R_u(g) = gu$$

- *the multiplication of arrows is the composition of functions;*

- *the norm is defined by: $d_m(R_u) = d_{\alpha(u)}(\alpha(u), u) = d(u)$.*

Remark that arrows in the metric groupoid G_m are isometries. It is also clear that G_m is isomorphic with G by the morphism $u \in G \mapsto R_u^{-1}$.

Definition 1.7 *The α-double groupoid $G \times_\alpha G$ associated to G is another way to assembly the metric spaces $\alpha^{-1}(x)$, $x \in Ob(G)$, into a groupoid. The definition of this groupoid is:*

- *the arrows are $G \times_\alpha G = \displaystyle\bigcup_{x \in Ob(G)} \alpha^{-1}(x) \times \alpha^{-1}(x)$;*

- *the composition of arrows is: $(g, h)(h, l) = (g, l)$, the inverse is $(g, h)^{-1} = (h, g)$, therefore as a groupoid $G \times_\alpha G$ is just the union of trivial groupoids over $\alpha^{-1}(x)$, $x \in Ob(G)$;*

- *it follows that $Ob(G \times_\alpha G) = \{(g, g) : g \in G\}$ and the induced α and ω maps are : $\tilde{\alpha}(g, h) = (h, h)$ and $\tilde{\omega}(g, h) = (g, g)$, for any $g, h \in G$ with $\alpha(g) = \alpha(h)$;*

- *the norm is the function $\tilde{d}$.*

This groupoid has the property that dif is a morphism of normed groupoids.

Finally, suppose that for any $x, y \in Ob(G)$ there is $g \in G$ such that $\alpha(g) = x$ and $\omega(g) = y$. Then any separable norm d on G induces a distance on $X = Ob(G)$, by the formula:

$$d_{ob}(x, y) = \inf \{d(g) : \alpha(g) = x, \omega(g) = y\}$$

If the groupoid is not connected by arrows then d_{ob} may take the value $+\infty$ and the space X decomposes into a disjoint union of metric spaces.

1.5   Notions of convergence

Any norm d on a groupoid G induces three notions of convergence on the set of arrows G.

Definition 1.8 *A net of arrows (a_ε) **simply converges** to the arrow $a \in G$ (we write $a_\varepsilon \to a$) if:*

(i) *for any $\varepsilon \in I$ there are elements $g_\varepsilon, h_\varepsilon \in G$ such that $h_\varepsilon a_\varepsilon g_\varepsilon = a$,*

(ii) *we have $\lim\limits_{\varepsilon \in I} d(g_\varepsilon) = 0$ and $\lim\limits_{\varepsilon \in I} d(h_\varepsilon) = 0$.*

*A net of arrows (a_ε) **left-converges** to the arrow $a \in G$ (we write $a_\varepsilon \overset{L}{\to} a$) if for all $i \in I$ we have $(a_\varepsilon^{-1}, a) \in G^{(2)}$ and moreover $\lim\limits_{\varepsilon \in I} d(a_\varepsilon^{-1} a) = 0$.*

*A net of arrows (a_ε) **right-converges** to the arrow $a \in G$ (we write $a_\varepsilon \overset{R}{\to} a$) if for all $i \in I$ we have $(a_\varepsilon, a^{-1}) \in G^{(2)}$ and moreover $\lim\limits_{\varepsilon \in I} d(a_\varepsilon a^{-1}) = 0$.*

It is clear that if $a_\varepsilon \overset{L}{\to} a$ or $a_\varepsilon \overset{R}{\to} a$ then $a_\varepsilon \to a$.

Right-convergence of a_ε to a is just convergence of a_ε to a in the distance $d_{\alpha(a)}$, that is $\lim\limits_{\varepsilon \in I} d_{\alpha(a)}(a_\varepsilon, a) = 0$.

Left-convergence of a_ε to a is just convergence of a_ε^{-1} to a^{-1} in the distance $d_{\omega(a)}$, that is $\lim\limits_{\varepsilon \in I} d_{\omega(a)}(a_\varepsilon^{-1}, a^{-1}) = 0$.

Proposition 1.9 *Let G be a groupoid with a norm d.*

(i) *If $a_\varepsilon \overset{L}{\to} a$ and $a_\varepsilon \overset{L}{\to} b$ then $a = b$. If $a_\varepsilon \overset{R}{\to} a$ and $a_\varepsilon \overset{R}{\to} b$ then $a = b$.*

(ii) *The following are equivalent:*

    1. G is a Hausdorff topological groupoid with respect to the topology induced by the simple convergence,

    2. d is a separable norm,

    3. for any net (a_ε), if $a_\varepsilon \to a$ and $a_\varepsilon \to b$ then $a = b$.

    4. for any net (a_ε), if $a_\varepsilon \overset{R}{\to} a$ and $a_\varepsilon \overset{L}{\to} b$ then $a = b$.

Proof.   (i) We prove only the first part of the conclusion. We can write $b^{-1}a = b^{-1}a_\varepsilon a_\varepsilon^{-1}a$, therefore
$$d(b^{-1}a) \leq d(b^{-1}a_\varepsilon) + d(a_\varepsilon^{-1}a)$$
The right hand side of this inequality is arbitrarily small, so $d(b^{-1}a) = 0$, which implies $a = b$.

(ii) Remark that the structure maps are continuous with respect to the topology induced by the simple convergence. We need only to prove the uniqueness of limits.

3. $\Rightarrow$ 4. is trivial. In order to prove that 4.$\Rightarrow$ 3., consider an arbitrary net (a_ε) such that $a_\varepsilon \to a$ and $a_\varepsilon \to b$. This means that there exist nets $(g_\varepsilon), (g_\varepsilon'), (h_\varepsilon), (h_\varepsilon')$ such that $h_\varepsilon a_\varepsilon g_\varepsilon = a$, $h_\varepsilon' a_\varepsilon g_\varepsilon' = b$ and $\lim\limits_{i \in I}(d(g_\varepsilon) + d(g_\varepsilon') + d(h_\varepsilon) + d(h_\varepsilon')) = 0$. Let $g"_\varepsilon = g_\varepsilon^{-1} g_\varepsilon'$ and $h"_\varepsilon = h_\varepsilon' h_\varepsilon^{-1}$. We have then $b = h"_\varepsilon a g"_\varepsilon$ and $\lim\limits_{i \in I}(d(g"_\varepsilon) + d(h"_\varepsilon)) = 0$. Then $h"_\varepsilon a \overset{L}{\to} b$ and $h"_\varepsilon a \overset{R}{\to} a$. We deduce that $a = b$.

1.$\Leftrightarrow$ 3. is trivial. So is 3. $\Rightarrow$ 2. We finish the proof by showing that 2. $\Rightarrow$ 3. By a reasoning made previously, it is enough to prove that: if $b = h_\varepsilon a g_\varepsilon$ and $\lim_{i \in I} (d(g_\varepsilon) + d(h_\varepsilon)) = 0$ then $a = b$. Because d is separable it follows that $\alpha(a) = \alpha(b)$ and $\omega(a) = \omega(b)$. We have then $a^{-1} b = a^{-1} h_\varepsilon a g_\varepsilon$, therefore

$$d(a^{-1} b) \le d(a^{-1} h_\varepsilon a) + d(g_\varepsilon)$$

The norm d induces a left invariant distance on the vertex group of all arrows g such that $\alpha(g) = \omega(g) = \alpha(a)$. This distance is obviously continuous with respect to the simple convergence in the group. The net $a^{-1} h_\varepsilon a$ simply converges to $\alpha(a)$ by the continuity of the multiplication (indeed, h_ε simply converges to $\alpha(a)$). Therefore $\lim_{i \in I} d(a^{-1} h_\varepsilon a) = 0$. It follows that $d(a^{-1} b)$ is arbitrarily small, therefore $a = b$. $\qquad\square$

1.6 Families of seminorms

Instead of a norm we may use families of seminorms.

Definition 1.10 *A* **family of seminorms** *on a groupoid G is a family S of functions $\rho : G \to [0, +\infty)$ with the properties:*

(i) *for any $x \in X$ and $\rho \in S$ we have $\rho(e(x)) = 0$; if $\rho(g) = 0$ for any $\rho \in S$ then there is $x \in X$ such that $g = e(x)$,*

(ii) *for any $\rho \in S$ and $(g, h) \in G^{(2)}$ we have $\rho(gh) \le \rho(g) + \rho(h)$,*

(iii) *for any $\rho \in S$ and $g \in G$ we have $\rho(inv(g)) = \rho(g)$.*

A groupoid G endowed with a family of seminorms S is called a **seminormed groupoid***. A family of seminorms S is separable if it satisfies the property:*

(iv) *if there is a net $(a_\varepsilon) \subset G$ such that for any $n \in \mathbb{N}$ $\alpha(a_\varepsilon) = x$, $\omega(a_\varepsilon) = y$ and for any $\rho \in S$ we have $\lim_{i \in I} \rho(a_\varepsilon) = 0$ then $x = y$.*

Families of morphisms induce families of seminorms.

Definition 1.11 *Let G be a groupoid and (H, d) be a normed groupoid. A (H, d)* **family of morphisms** *is a set L of morphisms from G to H such that for any $g \in G$ there is $A \in L$ with $A(g) \notin Ob(H)$.*

The following proposition has a straightforward proof which we omit.

Proposition 1.12 *Let G be a groupoid, (H, d) be a normed groupoid and L a (H, d) family of morphisms. Then the set*

$$d(L) = \{d\, A \; : \; A \in L\}$$

is a family of seminorms.

Definition 1.8 can be modified for the case of families of seminorms.

Definition 1.13 *Let (G, S) be a semi-normed groupoid. A net of arrows (a_ε)* **simply converges** *to the arrow $a \in G$ (we write $a_\varepsilon \to a$) if:*

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(i) for any $i \in I$ there are elements $g_\varepsilon, h_\varepsilon \in G$ such that $h_\varepsilon a_\varepsilon g_\varepsilon = a$,

(ii) for any $\rho \in S$ we have $\lim_{i \in I} (\rho(g_\varepsilon) + \rho(h_\varepsilon)) = 0$.

*A net of arrows (a_ε) **left-converges** to the arrow $a \in G$ (we write $a_\varepsilon \xrightarrow{L} a$) if for all $i \in I$ we have $(a_\varepsilon^{-1}, a) \in G^{(2)}$ and moreover for any $\rho \in S$ we have $\lim_{i \in I} \rho(a_\varepsilon^{-1} a) = 0$.*

*A net of arrows (a_ε) **right-converges** to the arrow $a \in G$ (we write $a_\varepsilon \xrightarrow{R} a$) if for all $i \in I$ we have $(a_\varepsilon, a^{-1}) \in G^{(2)}$ and moreover for any $\rho \in S$ we have $\lim_{i \in I} \rho(a_\varepsilon a^{-1}) = 0$.*

With these slight modifications, the proposition 1.9 still holds true. This is visible from the examination of its proof.

Let us finally remark that if (G, dL) is a seminormed groupoid, where L is a (H, d) family of morphisms, then a net $(a_\varepsilon) \in G$ converges (simply, left or right) to $a \in G$ if and only if for any $A \in L$ the net $(A(a_\varepsilon))$ respectively converges in (H, d).

1.7 Uniform convergence on bounded sets

We shall use right-convergence, according to definition 1.8, but left-convergence or simple convergence could also be used. In relation to this see for example the remark 6.12.

Definition 1.14 *Let G be a normed groupoid with a separable norm. A net (f_ε) of functions $f_\varepsilon : G \times_\alpha G \to G$ **uniformly converges on bounded sets** to the function $f : G \times_\alpha G \to G$ (in the sense of the left convergence) if:*

(i) for any $\varepsilon > 0$ and $(h, g) \in G^{(2)}$ we have $\alpha(f_\varepsilon(h, g)) = \alpha(f(h, g))$,

(ii) for any $\lambda, \mu > 0$ there is $\varepsilon(\lambda, \mu) > 0$ such that for any $\varepsilon \in \Gamma$, $\mid \varepsilon \mid \leq \varepsilon(\lambda, \mu)$ and any $(h, g) \in G^{(2)}$ with $d(h) \leq \lambda$, $d(g) \leq \lambda$, we have:

$$d\left(f_\varepsilon(h, g)inv(f(h, g))\right) \leq \mu$$

In the case of a groupoid G with a separable family of seminorms S, the definition of uniform convergence is the same, excepting the modification of (ii) above into: for any $\lambda, \mu > 0$ and any seminorm $\rho \in S$ there is $\varepsilon(\lambda, \mu, \rho) > 0$ such that for any $\varepsilon \in \Gamma$, $\mid \varepsilon \mid \leq \varepsilon(\lambda, \mu)$ and any $(h, g) \in G^{(2)}$ with $\rho(h) \leq \lambda$, $\rho(g) \leq \lambda$, we have:

$$\rho\left(f_\varepsilon(h, g)inv(f(h, g))\right) \leq \mu$$

Similarly, in a normed groupoid with a separable norm d, the uniform convergence on bounded sets of a net of functions $f_\varepsilon : G \to \mathbb{R}$ to $f : G \to \mathbb{R}$ means that for any $\lambda, \mu > 0$ there is $\varepsilon(\lambda, \mu) > 0$ such that for any $\varepsilon \in \Gamma$, $\mid \varepsilon \mid \leq \varepsilon(\lambda, \mu)$ and any $g \in G$ with $d(g) \leq \lambda$ we have: $\mid f_\varepsilon(g) - f(g) \mid \leq \mu$.

2 Normed categories

Sometimes it is interesting to work with normed categories, instead of normed groupoids. Briefly said, a normed category is a small category endowed with an involutionary antimorphism (an "inverse") and with a norm function defined on arrows.

Definition 2.1 *A **normed category** (G, d) is a small category $\mathcal{C}$ with inverses: the set of objects is $X = Obj(\mathcal{C})$, the set of arrows is identified with $\mathcal{C}$; there are two functions $s, t : \mathcal{C} \to X$, named **source** and **target**, a **multiplication** $m : \mathcal{C}^{(2)} \to \mathcal{C}$ (notation: $m(g, h) = gh$ or $m(g, h) = g \circ h$), where*

$$\mathcal{C}^{(2)} = \left\{ (g, h) \in \mathcal{C}^2 : s(g) = t(h) \right\}$$

*a **inversion map** $inv : \mathcal{C} \to \mathcal{C}$ (notation $inv(g) = g^{-1}$) and a **norm** $d : \mathcal{C} \to [0, +\infty)$, which satisfy the following conditions:*

(i) $s(gh) = s(h)$, $t(gh) = t(g)$, for any $(g, h) \in \mathcal{C}^{(2)}$,

(ii) the multiplication is associative $(gh)k = g(hk)$

(iii) the inverse is an involution and a antimorphism: $s(g^{-1}) = t(g)$, $(gh)^{-1} = h^{-1}g^{-1}$.

(iv) $d(g) = 0$ if and only if there is $h \in \mathcal{C}$ such that $g = h^{-1}h$,

(v) for any $(g, h) \in \mathcal{C}^{(2)}$ we have $d(gh) \leq d(g) + d(h)$,

(vi) for any $g \in \mathcal{C}$ we have $d(inv(g)) = d(g)$.

*Let $\alpha, \omega : \mathcal{C} \to \mathcal{C}$ be defined by $\alpha(g) = g^{-1}g$, $\omega(g) = gg^{-1}$. The **norm** d **is separable** if it satisfies the property:*

(vii) if there is a net $(a_\varepsilon) \subset \mathcal{C}$ such that for any $n \in \mathbb{N}$ $\alpha(a_\varepsilon) = x$, $\omega(a_\varepsilon) = y$ and $\lim_{\varepsilon \in I} d(a_\varepsilon) = 0$ then $x = y$.

Seminormed categories are defined further, by making a slight modification of definition 1.10.

Definition 2.2 *Let $\mathcal{C}$ be a category with inverses (which satisfies (i)–(iv) definition 2.1). Let $X = \alpha(\mathcal{C}$.*

*A **family of seminorms** on a category with inverses $\mathcal{C}$ is a family S of functions $\rho : \mathcal{C} \to [0, +\infty)$ with the properties:*

(iv)' for any $x \in X$ and $\rho \in S$ we have $\rho(x) = 0$; if $\rho(g) = 0$ for any $\rho \in S$ then $g \in X$,

(v)' for any $\rho \in S$ and $(g, h) \in \mathcal{C}^{(2)}$ we have $\rho(gh) \leq \rho(g) + \rho(h)$,

(vi)' for any $\rho \in S$ and $g \in \mathcal{C}$ we have $\rho(inv(g)) = \rho(g)$.

*A category $\mathcal{C}$ with inverses endowed with a family of seminorms S is called a **seminormed category**.*

A family of seminorms S is separable if it satisfies the property:

(vii)' if there is a net $(a_\varepsilon) \subset \mathcal{C}$ such that for any $i \in I$ $\alpha(a_\varepsilon) = x$, $\omega(a_\varepsilon) = y$ and for any $\rho \in S$ we have $\lim_{i \in I} \rho(a_\varepsilon) = 0$ then $x = y$.

All considerations made before concerning convergence for normed or seminormed separable groupoids, extend without effort to normed categories, or to seminormed separable categories. A little bit of care is needed though: everywhere we should replace source and target maps (of groupoids) by the (algebraically defined) $\alpha(g) = g^{-1}g$ and $\omega(g) = gg^{-1}$.

3   Examples of normed groupoids and normed categories

We give several examples of normed groupoids which will be of interest later in this paper.

3.1   Metric spaces

Let (X, d) be a metric space. We form the **normed trivial groupoid** (G, d) over X:

   - the set of arrows is $G = X \times X$ and the multiplication is

$$(x, y)(y, z) = (x, z)$$

  Therefore we have $\alpha(x, y) = y$, $\omega(x, y) = x$, $e(x) = (x, x)$, $(x, y)^{-1} = (y, x)$.

   - the norm is just the distance function $d : G \to [0, +\infty)$.

It is easy to see that if $(X \times X, d)$ is a normed trivial groupoid over X then (X, d) is a metric space.

3.1.1   Associated groupoids.

The metric groupoid $(X \times X)_m$ can be described as the groupoid with objects pointed metric spaces (X, d, x), $x \in X$, arrows $R_{(x,y)} : (X, d, x) \to (X, d, y)$, $R_{(x,y)}(z) = z$, and norm $d_m\left(R_{(x,y)}\right) = d(x, y)$. The α-double groupoid $(X \times X) \times_\alpha (X \times X)$ can be described as the groupoid with arrows $X \times X \times X$, composition $(x, y, z)(y, v, z) = (x, v, z)$, inverse $(x, y, z)^{-1} = (y, x, z)$ and norm $\tilde{d}(x, y, z) = d(x, y)$.

3.1.2   Convergence.

Remark first that d is a separable norm, according to definition 1.5 (iv). Indeed, for any $x, y \in X$ there is only one arrow $a \in X \times X$ such that $\alpha(a) = x$, $\omega(a) = y$, namely the arrow $a = (y, x)$. Any net (a_ε) with $\alpha(a_\varepsilon) = x$, $\omega(a_\varepsilon) = y$ is the constant net $a_\varepsilon = (y, x)$. If $\lim_{n \to \infty} d(a_\varepsilon) = 0$ then $d(y, x) = 0$, therefore $x = y$. We deduce from proposition 1.9 that we have only one interesting notion of convergence, which is simple convergence.

In the particular case of normed trivial groupoids the definition 1.8 of simple convergence becomes: a net $(x_\varepsilon, y_\varepsilon) \subset (X \times X$ simply converges to (x, y) if we have

$$\lim_{n \to \infty} (d(x, x_\varepsilon) + d(y_\varepsilon, y)) = 0$$

that is if the nets $x_\varepsilon, y_\varepsilon$ converge respectively to x, y. Indeed this is coming from the fact that for any $n \in \mathbb{N}$ there are unique $h_\varepsilon, g_\varepsilon \in X \times X$ such that $h_\varepsilon(x_\varepsilon, y_\varepsilon)g_\varepsilon = (x, y)$. These are $h_\varepsilon = (x, x_\varepsilon)$ and $g_\varepsilon = (y_\varepsilon, y)$.

3.1.3   Nice families of seminorms on metric spaces.

Let X be a non empty set, let (Y, d) be a metric spaces and (Y^2, d) its associated normed trivial groupoid. Any function $f : X \to Y$ induces a morphism $\bar{f}$ from the trivial groupoid X^2 to Y^2 by $\bar{f}(x, y) = (f(x), f(y))$. Any family $\mathcal{L}$ of functions from X to Y with the separation property: for any $x, y \in X$ $x \neq y$ there is $f \in \mathcal{L}$ with $f(x) \neq f(y)$, gives us a (Y^2, d) family of morphisms, which in turn induces a family of seminorms on X^2.

3.2 Normed groupoids from α-double groupoids

We can construct normed groupoids starting from definition 1.7 of α-double groupoids.

Proposition 3.1 *Let (G, d) be a groupoid and $(G \times_\alpha G, \tilde{d})$ the associated α-double groupoid. Then for any $(g, h) \in G \times_\alpha G$ and for any $u \in G$ with $\omega(u) = \alpha(g) = \alpha(h)$ we have*

$$d_{\omega(u)}(g, h) = d_{\alpha(u)}(gu, hu) \tag{3.2.1}$$

Conversely, suppose that G is a groupoid and that for any $x \in Ob(G)$ we have a distance $d_x : \alpha^{-1}(x) \times \alpha^{-1}(x) \to [0, +\infty)$. If (3.2.1) is true for any $(g, h) \in G \times_\alpha G$ and for any $u \in G$ with $\omega(u) = \alpha(g) = \alpha(h)$ then

$$d(g) = d_{\alpha(g)}(g, \alpha(g)) \quad and \quad \tilde{d}(g, h) = d_{\alpha(g)}(g, h)$$

define a norm on G such that $(G \times_\alpha G, \tilde{d})$ is the associated α-double groupoid.

Remark 3.2 *Therefore any normed groupoid (G, d) can be seen as the bundle of metric spaces $\alpha : G \to Ob(G)$, such that (a) each fiber $\alpha^{-1}(x)$ has a distance d_x, and (b) the distances d_x are right invariant with respect to the groupoid composition, in the sense of relation (3.2.1).*

Proof. For the first implication remark that $(gu, hu) \in G \times_\alpha G$. Moreover let $g' = gu$, $h' = hu$. Then $g'(h')^{-1} = gh^{-1}$, therefore

$$d_{\omega(u)}(g, h) = d(gh^{-1}) = d_{\alpha(u)}(gu, hu)$$

For the converse implication, we have to prove that if $g'(h')^{-1} = gh^{-1}$ then $d(gh^{-1}) = d(g'(h')^{-1})$, with d defined as in the formulation of the proposition. This is easy: Let $u = (h')^{-1})h$, then $g = g'u$, $h = h'u$ and (3.1) implies the desired equality. The verification that d is indeed a norm on G is straightforward, as well as the fact that $\tilde{d}$ is the induced norm on $G \times_\alpha G$. $\square$

3.3 Group actions

Let G be a group with neutral element e, which acts from the left on the space X. Associated with this is the **action groupoid** $G \ltimes X$ over X. The action groupoid is defined as: the set of arrows is $X \times G$ and the multiplication is

$$(g(x), h)(x, g) = (x, hg)$$

Therefore $\alpha(x, g) = x$, $\omega(x, g) = g(x)$, $e(x) = (x, e)$, $(x, g)^{-1} = (g(x), g^{-1})$,

As a particular case of definition 1.5, a **normed action groupoid** is an action groupoid $G \ltimes X$ endowed with a norm function $d : X \times G \to [0, +\infty)$ with the properties:

(i) $d(x, g) = 0$ if and only if $g = e$,

(ii) $d(g(x), g^{-1}) = d(x, g)$,

(iii) $d(x, hg) \leq d(x, g) + d(g(x), h)$.

Remark that the norm function is no longer a distance function. In the case of a free action (if $g(x) = x$ for some $x \in X$ then $g = e$) we may obtain a norm function from a distance function on X. Indeed, let $d' : X \times X \to [0, +\infty)$ be a distance. Define then $d : X \times G \to [0, +\infty)$ by

$$\bar{d}(x, g) = d'(g(x), x)$$

Then $(G \ltimes X, \bar{d})$ is a normed action groupoid.

3.3.1 Associated groupoids.

The associated α-double groupoid can be seen as $X \times G \times G$, with composition $(x, g, h)(x, h, l) = (x, g, l)$ and inverse $(x, g, h)^{-1} = (x, h, g)$. For any $x \in X$ we have a distance $d_x : G \times G \to [0, +\infty)$, defined by

$$d_x(g, h) = d(h(x), gh^{-1})$$

Conversely, according to proposition 3.1 and relation (3.2.1), a norm on a action groupoid can be constructed from a function $x \in X \mapsto d_x$ which associates to any $x \in X$ a distance d_x on G, such that for any $x \in X$ and $u, g, h \in G$ we have

$$d_{u(x)}(g, h) = d_x(gu, hu)$$

In this case we can define the norm on the action groupoid by $d(x, g) = d_x(g, e)$.

A particular case is $X = \{x\}$, when a normed action groupoid is just a group endowed with a right invariant distance.

3.3.2 Convergence.

The norm d is separable if the following condition is satisfied: for any $x, y \in X$ and any net $g_\varepsilon \in G$ with the property $g_\varepsilon(x) = y$ for all ε, if $\lim_{\varepsilon \in I} d_x(g_\varepsilon, e) = 0$ then $x = y$.

3.4 Groupoids actions

Let G and M be two groupoids. We denote by $Aut(M)$ the groupoid which has as objects sub-groupoids of M and invertible morphisms between sub-groupoids of M as arrows. A groupoid action of G on M is just a morphism F of groupoids from G to $Aut(M)$. In fewer words, for any $g \in G$ let $F(g)$ be the associated morphism of sub-groupoids, defined from the sub-groupoid denoted by $dom\, g$ to the sub-groupoid denoted by $im\, g$. For any $x \in dom\, g$ we use the notation $g.x = F(g)(x)$. Compositions in G and in M are denoted multiplicatively. Let $G \ltimes M$ be the set

$$G \ltimes M = \{(x, g) : x \in dom\, g\}$$

The action of G on M satisfies the following conditions:

- for any $u, v \in G$ such that $\alpha(v) = \omega(u)$ we have $dom\, u = dom\, vu$, $im\, u = dom\, v$ and for any $x \in dom\, u$ we have $v.(u.x) = (vu).x$;

- for any $u \in G$ and $x, y \in dom\, u$ we have $u.(xy) = (u.x)(u.y)$.

Any groupoid action induces a groupoid structure on $G \ltimes M$, by the composition law $(g.x, h)(x, g) = (x, hg)$.

At a closer look we may notice an example of a groupoid action in proposition 3.1. Indeed, let G be a groupoid and $G \times_\alpha G$ the associated α-double groupoid. Then G acts on $G \times_\alpha G$ by $u.(g, h) = (gu^{-1}, hu^{-1})$, for any $u \in G$ and any $(g, h) \in G \times_\alpha G$ such that $\alpha(u) = \alpha(g) = \alpha(h)$. Therefore $dom\, u = \left(\alpha^{-1}(\alpha(u))\right)^2$ and the associated action groupoid is

$$G \ltimes (G \times_\alpha G) = \{(g, h, u) : \alpha(g) = \alpha(h) = \alpha(u)\}$$

with multiplication defined by

$$(gu^{-1}, hu^{-1}, v)(g, h, u) = (g, h, vu)$$

Relation (3.2.1) in proposition 3.1 tells that G acts on the normed groupoid $(G \times_\alpha G, \tilde{d})$ by isometries. In general, the action groupoid induced by the action of a groupoid G on a normed groupoid M *by isometries* may be an object as interesting as a normed groupoid.

3.5 Riemannian groupoids (not normed)

The references used are Glickenstein [7] and Lott [9] section 5.

Definition 3.3 *A smooth étale groupoid G is Riemannian if there is a riemannian metric on $Ob(G)$ such that the germs of the arrows (arrows are seen as local diffeomeorphisms on $Ob(G)$) are isometries.*

A Riemannian groupoid is not a normed groupoid.

3.6 Transport category

A good reference for this section is [1] chapters 5 and 6. Much more details could be found, for instance, in [13].

Let (X, d) be a a complete, separable metric space (Polish space). The class of Borel probabilities on X is denoted by $\mathcal{P}(X)$. If (X_1, d_1), (X_2, d_2) are Polish spaces then $(X_1 \times X_2, d)$ is also a Polish space, with the distance d defined by $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$.

If $f : X_1 \to X_2$ is a Borel map and $\mu \in \mathcal{P}(X_1)$ then the push-forward of μ through f is the measure $f \sharp \mu \in \mathcal{P}(X_2)$ defined by: for any $B \in \mathcal{B}(X_2)$

$$(f \sharp \mu)(B) = \mu(f^{-1}(B))$$

In particular, the projections $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, $\pi_i(x_1, x_2) = x_i$, define push-forwards from $\mathcal{P}(X_1 \times X_2)$ to $\mathcal{P}(X_i)$.

Definition 3.4 *Let X, Y be Polish spaces. A measure-valued map $x \in X \mapsto \mu_x \in \mathcal{P}(Y)$ is a Borel map if for any Borel set $B \in \mathcal{B}(Y)$ the function $x \mapsto \mu_x(B)$ is Borel.*

The following is the disintegration theorem [1] theorem 5.3.1 (see also references therein). (In [1] the theorem is formulated for Radon spaces, here we stay in the frame of Polish spaces).

Theorem 3.5 *Let X, Y be two Polish spaces, $\pi : X \to Y$ a Borel map and $mu \in \mathcal{P}(X)$, $\nu = \pi \sharp \mu \in \mathcal{P}(Y)$. Then there exists a Borel measure-valued function $y \in Y \mapsto \mu_y \in \mathcal{P}(X)$, which is ν-a.e. uniquely determined, such that:*

(a) for ν-a.e. $y \in Y$ we have $\mu_y(X \setminus \pi^{-1}(x)) = 0$,

(b) for every Borel map $f : X \to [0, +\infty]$

$$\int_X f(x) \, d\mu(x) = \int_Y \left(\int_{\pi^{-1}(y)} f(x) \, d\mu_y(x) \right\} \, d\nu(y)$$

In the particular case $X = X_1 \times X_2$, $\pi = \pi_i$ and $Y = X_i$, $i = 1, 2$, the disintegration theorem implies that for any $\gamma \in \mathcal{P}(X_1 \times X_2)$ with marginals

$$\phi_1 \sharp \gamma = \mu \, , \, \phi_2 \sharp \gamma = \nu$$

there exist Borel measure-valued maps $x_1 \in X_1 \mapsto \gamma_{x_1} \in \mathcal{P}(X_2)$, μ-a.e. uniquely determined, and $x_2 \in X_2 \mapsto \gamma_{x_2} \in \mathcal{P}(X_1)$, ν-a.e. uniquely determined, such that for any Borel function $f : X_1 \times X_2 \to [0, +\infty]$ we have:

$$\int_{X_1 \times X_2} f(x_1, x_2) \, \mathrm{d}\gamma(x_1, x_2) = \int_{X_1} \left(\int_{X_2} f(x_1, x_2) \, \mathrm{d}\gamma_{x_1}(x_2) \right) \, \mathrm{d}\mu(x_1) =$$

$$= \int_{X_2} \left(\int_{X_1} f(x_1, x_2) \, \mathrm{d}\gamma_{x_2}(x_1) \right) \, \mathrm{d}\nu(x_2)$$

For any two probability measures μ, ν on X the set of all transport plans between μ and ν is defined as:

$$\Pi(\mu, \nu) = \{ \gamma \in \mathcal{P}(X \times X) : \phi_1 \sharp \gamma = \mu \,, \, \phi_2 \sharp \gamma = \nu \}$$

Definition 3.6 *The category $Trans(X)$ of transport plans on X is the category with inverses which has as objects the elements of $\mathcal{P}(X)$ and the class of arrows between $\mu, \nu \in \mathcal{P}(X)$ is $\Pi(\mu, \nu)$.*

The composition of arrows is given by composition of transport plans, defined further. Let $\gamma \in \Pi(\mu, \nu)$ and $\gamma' \in \Pi(\nu, \mu')$. By the disintegration theorem 3.5, there are ν-a.e. uniquely defined measure-valued maps $x \in X \mapsto \gamma'_x, \gamma_x \in \mathcal{P}(X)$ such that for any Borel function $f : X_1 \times X_2 \to [0, +\infty]$ we have:

$$\int_{X \times X} f(x_1, x_2) \, d\gamma(x_1, x_2) = \int_X \left(\int_X f(x_1, x_2) \, d\gamma_{x_2}(x_1) \right) \, d\nu(x_2)$$

$$\int_{X \times X} f(x_1, x_2) \, d\gamma'(x_1, x_2) = \int_X \left(\int_X f(x_1, x_2) \, d\gamma'_{x_1}(x_2) \right) \, d\nu(x_1)$$

Then the composition $\gamma' \circ \gamma \in \Pi(\mu, \mu')$ is defined by: for any Borel function $f : X_1 \times X_2 \to [0, +\infty]$

$$\int_{X \times X} f(x_1, x_3) \, d\left(\gamma' \circ \gamma \right)(x_1, x_3) = \int_X \left(\int_{X \times X} f(x_1, x_3) \, d\gamma'_{x_2}(x_3) \, d\gamma_{x_2}(x_1) \right) \, d\nu(x_2)$$

Finally, this category has a contravariant "inverse" functor, which associates to each arrow $\gamma \in \Pi(\mu, \nu)$ the arrow $\gamma^{-1} \in \Pi(\nu, \mu)$ defined by: for any Borel function $f : X_1 \times X_2 \to [0, +\infty]$

$$\int_{X \times X} f(x_1, x_2) \, d\gamma^{-1}(x_1, x_2) = \int_{X \times X} f(x_1, x_2) \, d\gamma(x_2, x_1)$$

We denote by $Lip_1(X)$ the class of 1-Lipschitz maps from X to $\mathbb{R}$.

Proposition 3.7 *The category $Trans(X)$ is a separable seminormed category, with the family S of seminorms: for any $u \in Lip_1(X)$ and any $\gamma \in Trans(X)$*

$$\rho_u(\gamma) = \left| \int_{X \times X} (u(x) - u(y)) \, d\gamma(x, y) \right|$$

$Trans(X)$ it is also a normed category, with the norm:

$$d(\gamma) = \int_{X \times X} d(x, y) \, d\gamma(x, y)$$

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The following relation between the norm d in the transport category and the seminorms ρ_u, is related to Kantorovich formulation of the transport problem and the dual of this problem: for any $u \in Lip_1(X)$ and any $\gamma \in Trans(X)$ we have

$$d(\gamma) \geq \rho_u(\gamma)$$

The topology of $Trans(X)$ with the norm d is called "strong topology", while the topology of $Trans(X)$ as a seminormed groupoid is called the "weak topology".

For any pair formed by a Borel function $f : X \to X$ and by a measure $\mu \in \mathcal{P}(X)$, there is an associated transport plan $(f, \mu) = (id \times f)\sharp\mu \in \Pi(\mu, f\sharp\mu)$.

Definition 3.8 *If $\gamma \in \mathcal{P}(X \times X)$ is representable as a measure (f, μ) then we say that γ is induced by a transport map. The category $Tmap(X)$ is the subcategory of $Trans(X)$ with objects elements of $\mathcal{P}(X)$ and arrows transport plans induced by transport maps.*

Several facts deserve to be mentioned. Notice that if $(f, \mu) = (g, \mu)$ then $f(x) = g(x)$ μ-a.e. in X. Also, in the category $Tmap(X)$ the composition of arrows is the following:

$$(g, f\sharp\mu) \circ (f, \mu) = (g \circ f, \mu)$$

Finally, the sub-category $Invtrans(X)$ of $Tmap(X)$ with objects elements of $\mathcal{P}(X)$ and arrows (f, μ) such that there exists a Borel function g with $(f, \mu)^{-1} = (g, f\sharp\mu)$, is a groupoid.

4   Dilations induced deformations of normed groupoids

Let (G, d) be a normed groupoid with a separable norm. A deformation of (G, d) is basically a "local action" of a commutative group Γ on G which satisfies several properties.

$(\Gamma, | \cdot |)$ is a commutative group endowed with a group morphism $| \cdot |: \Gamma \to (0, +\infty)$ to the multiplicative group of positive real numbers. This morphism induces a invariant topological filter over Γ (a end of Γ). Further we shall write $\varepsilon \to 0$ for ε converging to this end, and meaning that $| \varepsilon | \to 0$. The neutral element of Γ is denoted by e.

To any $\varepsilon \in \Gamma$ is associated a transformation $\delta_\varepsilon : dom(\varepsilon) \to im(\varepsilon)$, which may be called a dilatation, dilation, homothety or contraction.

For the precise properties of the domains and codomains of δ_ε for $\varepsilon \in \Gamma$ see the subsection 4.2. For the moment is sufficient to know that for any $\varepsilon \in \Gamma$ we have $Ob(G) = X \subset dom(\varepsilon)$ and $Ob(G) = X \subset im(\varepsilon)$. Basically the domain and codomain of δ_ε are neighbourhoods of X. Moreover, these sets are chosen so that various compositions of transformations δ_ε are well defined.

In the formulation of properties of deformations we shall use a uniform convergence on bounded sets. We explain further what uniform convergence on bounded sets means in the case of nets of functions indexed with the directed net the group Γ (ordered such that limits are taken in the sense $| \varepsilon | \to 0$.

4.1   Introducing dilations

Definition 4.1 *A dilation of a separated normed groupoid (G, d) is a map assigning to any $\varepsilon \in \Gamma$ a transformation $\delta_\varepsilon : dom(\varepsilon) \to im(\varepsilon)$ which satisfies the following:*

A1. *For any $\varepsilon \in \Gamma$ $\alpha\delta_\varepsilon = \alpha$. Moreover $\varepsilon \in \Gamma \mapsto \delta_\varepsilon$ is an action of Γ on G, that is for any $\varepsilon, \mu \in \Gamma$ we have $\delta_\varepsilon\delta_\mu = \delta_{\varepsilon\mu}$, $(\delta_\varepsilon)^{-1} = \delta_{\varepsilon^{-1}}$ and $\delta_e = id$.*

A2. For any $x \in Ob(G)$ and any $\varepsilon \in \Gamma$ we have $\delta_\varepsilon(x) = x$. Moreover the transformation δ_ε contracts $dom(\varepsilon)$ to $X = Ob(G)$ uniformly on bounded sets, which means that the net $d\,\delta_\varepsilon$ converges to the constant function 0, uniformly on bounded sets, in the sense of definition 1.14.

Moreover the domains and codomains $dom(\varepsilon)$, $im(\varepsilon)$ satisfy the conditions from definition 4.3, section 4.2.

A **dilation of a separated normed or seminormed category** *is defined in the same way as a dilation of a normed groupoid, only that α and ω functions are no longer the source and arrow functions, but $\alpha(g) = g^{-1}g$ and $\omega(g) = gg^{-1}$.*

4.1.1 Deformation of the α-double groupoid.

The dilation δ of (G, d) induces a right-invariant dilation of the normed groupoid $(G \times_\alpha D, \tilde{d})$. The proof of the following proposition is straightforward and we do not write it.

Proposition 4.2 *For any $\varepsilon \in \Gamma$ we define $\tilde{\delta}$ on $G \times_\alpha G$, given by:*

$$\tilde{\delta}_\varepsilon(g, h) = (\delta_\varepsilon(gh^{-1})h, h) \tag{4.1.1}$$

This is a deformation of the normed groupoid $(G \times_\alpha G, \tilde{d})$ is a normed groupoid and moreover dif is a morphism of normed groupoids (that is a norm preserving morphism of groupoids), which commutes with dilations in the sense: for any $\varepsilon \in \Gamma$ $dif\,\tilde{\delta}_\varepsilon = \delta_\varepsilon\,dif$.

4.1.2 Morphisms of dilations.

Let (G_1, d_1, δ^1) and (G_2, d_2, δ^2) be two dilations of the normed groupoids (G_1, d_1), (G_2, d_2) respectively. Then $F : (G_1, d_1, \delta^1) \to (G_2, d_2, \delta^2)$ is a morphism of dilations if: F is a morphism of groupoids, it preserves the norms (it is a isometry) and it commutes with dilations (it is "**linear**").

4.2 Domains and codomains of dilations

Dilations are locally defined. This is explained in the following definition, which should be seen as the axiom A0 of dilations.

Definition 4.3 *The domains and codomains of a dilation of (G, d) satisfy the following Axiom A0:*

(i) *for any $\varepsilon \in \Gamma$ $Ob(G) = X \subset dom(\varepsilon)$ and $dom(\varepsilon) = dom(\varepsilon)^{-1}$,*

(ii) *for any bounded set $K \subset Ob(G)$ there are $1 < A < B$ such that for any $\varepsilon \in \Gamma$, $|\varepsilon| \le 1$:*

$$d^{-1}(|\varepsilon|) \cap \alpha^{-1}(K) \subset \delta_\varepsilon\left(d^{-1}(A) \cap \alpha^{-1}(K)\right) \subset dom(\varepsilon^{-1}) \cap \alpha^{-1}(K) \subset$$

$$\subset \delta_\varepsilon\left(d^{-1}(B) \cap \alpha^{-1}(K)\right) \subset \delta_\varepsilon\left(dom(\varepsilon) \cap \alpha^{-1}(K)\right) \tag{4.2.2}$$

(iii) *for any bounded set $K \subset Ob(G)$ there are $R > 0$ and $\varepsilon_0 \in (0, 1]$ such that for any $\varepsilon \in \Gamma$, $|\varepsilon| \le \varepsilon_0$ and any $g, h \in|\ d^{-1}(R) \cap \alpha^{-1}(K)$ we have:*

$$dif(\delta_\varepsilon g, \delta_\varepsilon h) \in dom(\varepsilon^{-1}) \tag{4.2.3}$$

Remark 4.4 *Concerning (iii) definition 4.3, the first part of A1 definition 4.1 implies that $dif(\delta_\varepsilon g, \delta_\varepsilon h)$ is well defined for any $(g, h) \in G \times_\alpha G$ such that $g, h \in dom(\varepsilon)$.*

4.3   Induced deformations

The purpose of this section is to define several deformations of normed groupoids, such that the diagram from figure 4.1 becomes a commutative diagram of morphisms of dilations.

Let us consider a triple (G, d, δ) with (G, d) a normed groupoid and δ a dilation. For any $\mu \in \Gamma$ there are two normed induced groupoids, such that the arrows in the diagram (4.1) are morphisms.

Remark 4.5 *As dilatations are not globally defined and they are used to transport groupoid operations, it follows that the transported objects (operation, norms, ...) are not globally defined. Therefore the induced groupoids are not groupoids, but "local" groupoids, in a sense which is clear in the context.*

Figure 4.1: Commutative diagram of induced normed groupoids

Definition 4.6 *The deformation (G_μ, d_μ, δ) is equal to G as a set and its operations, norm and dilation are transported by the map $\delta_\mu : G_\mu \to G$ (with the precautions concerning the domains of definition of the transported objects mentioned in remark 4.5).*

The deformation $\left((G \times_\alpha G)_\mu, \tilde{d}_\mu, \tilde{\delta}_\mu \right)$ is equal to $G \times_\alpha G$ as a groupoid and its norm and dilation are transported by the map $\delta_\mu \times \delta_\mu : (G \times_\alpha G)_\mu \to G \times_\alpha G$.

More precisely, the deformation (G_μ, d_μ, δ) is described by:

- $G_\mu = G$ as a set, $\alpha_\mu = \alpha$ and $\omega_\mu = \omega \, \delta_\mu$, which follow from the computations using A1, A2 definition 4.1:

$$\alpha_\mu = \delta_\mu^{-1} \, \alpha \, \delta_\mu = \delta_\mu^{-1} \, \alpha = \alpha \quad , \quad \omega_\mu = \delta_\mu^{-1} \, \omega \, \delta_\mu = \omega \, \delta_\mu$$

  Also $Ob(G_\mu) = Ob(G)$.

- the composition operation and inverse are

$$m_\mu(g, h) \; = \; \delta_\mu^{-1} \left(\delta_\mu(g) \, \delta_\mu(h) \right) \quad , \quad inv_\mu \, g \; = \; \delta_\mu^{-1} \, inv \, \delta_\mu \, (g)$$

  These are well defined (at least locally) because of the axiom A0 definition 4.3. Notice that dif_μ from the diagram 4.1 appears as the difference function associated with the operation m_μ, defined as

$$dif_\mu : G_\mu \times_{\alpha_\mu} G_\mu \to G_\mu \quad , \quad dif_\mu(g, h) \; = \; \delta_\mu^{-1} \left(\delta_\mu(g) \, (\delta_\mu(h))^{-1} \right) \qquad (4.3.4)$$

- the norm d_μ is defined as:

$$d_\mu(g) \;=\; \frac{1}{\mid \mu \mid} \, d(\delta_\mu g) \tag{4.3.5}$$

- we may transport the dilation δ of (G, d) into a dilation $\delta_{\mu,.}$ of (G_μ, d_μ), but from the commutativity of Γ we get that

$$\delta_{\mu,\varepsilon} \;=\; \delta_\mu^{-1} \, \delta_\varepsilon \, \delta_\mu \;=\; \delta_\varepsilon$$

therefore it is the same dilation.

The deformation $\left((G \times_\alpha G)_\mu, \tilde{d}_\mu, \tilde{\delta}_\mu \right)$ is described by:

- $(G \times_\alpha G)_\mu = G \times_\alpha G$ as a groupoid; remark that this is compatible with the transport of operations using the map $\delta_\mu \times \delta_\mu$ (because this map is an endomorphism of the groupoid $G \times_\alpha G$),

- with respect to the relation (4.3.4), notice that $G_\mu \times_{\alpha_\mu} G_\mu = G \times_\alpha G = (G \times_\alpha G)_\mu$ and dif_μ as represented in figure 4.1 is a morphism of groupoids,

- the norm $\tilde{d}_\mu$ is defined as:

$$\tilde{d}_\mu(g, h) \;=\; \frac{1}{\mid \mu \mid} \, \tilde{d}(\delta_\mu g, \delta_\mu h) \tag{4.3.6}$$

and it is easy to check that dif_μ is also a isometry.

- we transport the dilation $\tilde{\delta}$ of $\left(G \times_\alpha G, \tilde{d} \right))$ into a dilation $\tilde{\delta}_{\mu,.}$

$$\tilde{\delta}_{\mu,\varepsilon}(g, h) \;=\; \left(\delta_{\mu^{-1}} \left(\delta_\varepsilon \left(\delta_\mu(g) \, (\delta_\mu(h))^{-1} \right) \delta_\mu(h) \right) , \, h \right) \tag{4.3.7}$$

The commutativity of the diagram 4.1 is clear now.

5   Algebraic operations from dilations

At the core of the introduction of dilations lies the fact that we can construct group operations from them. More precisely we are able to construct, by using compositions of dilations and the groupoid operation, approximately associative operations which shall lead us eventually to group operations in the tangent groupoid of a dilation.

5.1   A general construction

Let (G, d, δ) be a dilation and $\left(G \times_\alpha G, \tilde{d}, \tilde{\delta} \right)$ the associated dilation of the α-double groupoid. Further we shall be interested only in the properties of the following map.

Definition 5.1 *For any $x \in Ob(G)$ and any $\varepsilon \in \Gamma$ we define the* **dilatation***:*

$$\delta_\varepsilon^{(\cdot)}(\cdot) : \alpha^{-1}(x) \times \alpha^{-1}(x) \to \alpha^{-1}(x) \quad , \quad \delta_\varepsilon^h g \;=\; \delta_\varepsilon \left(g \, h^{-1} \right) h \tag{5.1.1}$$

Remark 5.2 *The domain of definition of $\delta_\varepsilon^{(\cdot)}(\cdot)$ is in fact only a subset of $\alpha^{-1}(x) \times \alpha^{-1}(x)$, according to the Axiom 0 explained in definition 4.3 section 4.2.*

This map comes from the definition (4.1.1) of the dilation $\tilde{\delta}$, namely

$$\tilde{\delta}_\varepsilon(g, h) = \left(\delta_\varepsilon^h g, h\right) \tag{5.1.2}$$

For any $\varepsilon \in \Gamma$ with $|\varepsilon|$ sufficiently small we can define dif_ε (as in figure 4.1) from (a subset of) $G \times_\alpha G$ to G. Remark that $\alpha(dif_\varepsilon(g, h)) = \omega(\delta_\varepsilon h)$, therefore the following composition is well defined:

$$\Delta_\varepsilon(g, h) = dif_\varepsilon(g, h)\, \delta_\varepsilon h \tag{5.1.3}$$

Then $\alpha\,\Delta_\varepsilon(g, h) = \alpha(g) = \alpha(h)$.

Related to the function Δ_ε is the following

$$inv_\varepsilon(g) = \Delta_\varepsilon(\alpha(g), g) \tag{5.1.4}$$

The following expression makes sense too, for any pair of elements (g, h) from (a subset of) $G \times_\alpha G$:

$$\Sigma_\varepsilon(g, h) = \delta_{\varepsilon^{-1}} \left[\delta_\varepsilon \left(g\, (\delta_\varepsilon h)^{-1} \right) \delta_\varepsilon h \right] \tag{5.1.5}$$

It is also true that $\alpha\,\Sigma_\varepsilon(g, h) = \alpha(g) = \alpha(h)$.

These three functions are interesting operations. The function Δ_ε is an approximate difference operation, inv_ε is an approximate inverse and Σ_ε is an approximate sum operation.

5.1.1   A graphic construction of approximate difference operation

A look at the figure 5.1 will help. There is graphically explained how $\Delta_\varepsilon(g, h)$ is constructed.
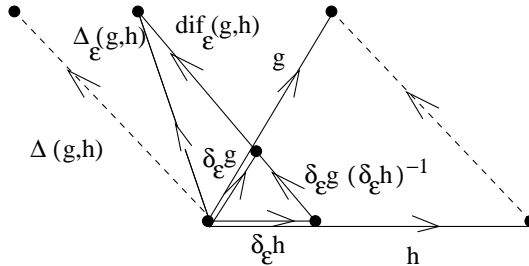

Figure 5.1: The meaning of the "approximate difference" operation $\Delta_\varepsilon(g, h)$.

Let us imagine that we are looking at a figure in the Euclidean plane. Then δ_ε is just a homothety, g, h are vectors with the same origin $\alpha(g) = \alpha(h)$, $\Delta(g, h)$ is the difference of vectors $-g + h$ (or $h - g$, it's the same as long as we are in a commutative world). In the Euclidean plane, as $|\varepsilon|$ goes to 0, the "vector" $dif_\varepsilon(g, h)$ slides towards $\Delta(g, h)$ and $\Delta_\varepsilon(g, h)$ is obtained from $dif_\varepsilon(g, h)$ by composition with the vector $\delta_\varepsilon h$. Thus $\Delta_\varepsilon(g, h)$ has the meaning of a approximate difference of vectors g, h.

5.2   Idempotent right quasigroup and induced operations

5.2.1   Approximate operations from dilatations

The functions Δ_ε, inv_ε and Σ_ε can be expressed in terms of dilatations introduced in definition 5.1. Indeed, let us define, for any triple $u, g, h \in G$ with $\alpha(u) = \alpha(g) = \alpha(h)$, and

such that $d(u), d(g), d(h)$ are sufficiently small, the following approximate difference function with three arguments:

$$\Delta_\varepsilon^u(g, h) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^u g} \delta_\varepsilon^u h \tag{5.2.6}$$

the approximate inverse function with two arguments:

$$inv_\varepsilon^u(g) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^u g} u = \Delta_\varepsilon^u(g, u) \tag{5.2.7}$$

and the following approximate sum function with three arguments:

$$\Sigma_\varepsilon^u(g, h) = \delta_{\varepsilon^{-1}}^u \delta_\varepsilon^{\delta_\varepsilon^u g} h \tag{5.2.8}$$

We have then:

$$\Delta_\varepsilon(hu^{-1}, gu^{-1}) = \Delta_\varepsilon^u(g, h)u^{-1}) \quad , \quad \Sigma_\varepsilon(hu^{-1}, gu^{-1}) = \Delta_\varepsilon^u(g, h)u^{-1}) \tag{5.2.9}$$

5.2.2   Γ-idempotent right quasigroups

We are in the framework of emergent algebras and idempotent right quasigroups, as introduced in [6]. We recall here the definition of a idempotent right quasigroup and induced operations.

Definition 5.3 *An idempotent right quasigroup (irq) is a set X endowed with two operations $\circ$ and $\bullet$, which satisfy the following axioms: for any $x, y \in X$*

(P1) $\qquad\qquad x \circ (x \bullet y) = x \bullet (x \circ y) = y$

(P2) $\qquad\qquad x \circ x = x \bullet x = x$

We use these operations to define the sum, difference and inverse operations of the irq: for any $x, u, v \in X$

(a) the difference operation is $(xuv) = (x \circ u) \bullet (x \circ v)$. By fixing the first variable x we obtain the difference operation based at x: $v -^x u = dif^x(u, v) = (xuv)$.

(b) the sum operation is $)xuv(= x \bullet ((x \circ u) \circ v)$. By fixing the first variable x we obtain the sum operation based at x: $u +^x v = sum^x(u, v) =)xuv($.

(a) the inverse operation is $inv(x, u) = (x \circ u) \bullet x$. By fixing the first variable x we obtain the inverse operator based at x: $-^x u = inv^x u = inv(x, u)$.

For any $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ we define also the following operations:

  - $x \circ_1 u = x \circ u$, $x \bullet_1 u = x \bullet u$,

  - for any $k > 0$ let $x \circ_{k+1} u = x \circ (x \circ_k u)$ and $x \bullet_{k+1} u = x \bullet (x \bullet_k u)$,

  - for any $k < 0$ let $x \circ_k u = x \bullet_{-k} u$ and $x \bullet_k u = x \circ_{-k} u$.

For any $k \in \mathbb{Z}^*$ the triple $(X, \circ_k, \bullet_k)$ is a irq. We denote the difference, sum and inverse operations of $(X, \circ_k, \bullet_k)$ by the same symbols as the ones used for $(X, \circ, \bullet)$, with a subscript "k".

For any $\varepsilon \in \Gamma$ and for any $x \in X$ we can define a irq operation on $\alpha^{-1}(x)$ by $g \circ_\varepsilon h = \delta_\varepsilon^g h$. We have then:

$$u +^g h = \Sigma_\varepsilon^g(u, h) \quad , \quad u -^g h = \Sigma_\varepsilon^g(h, u)$$

By computation it follows that $(\circ_\varepsilon)_k = \circ_{\varepsilon^k}$. The approximate difference, sum and inverse operations are exactly the ones introduced in the preceding section. In [6] we introduced idempotent right quasigroups and then iterates of the operations indexed by a parameter $k \in \mathbb{N}$. This was done in order to simplify the notations mostly. Here, in the presence of the group Γ, we might define a Γ-irq.

Definition 5.4 *Let Γ be a commutative group. A Γ-idempotent right quasigroup is a set X with a function $\varepsilon \in \Gamma \mapsto \circ_\varepsilon$ such that $(X, \circ_\varepsilon)$ is a irq and moreover for any $\varepsilon, \mu \in \Gamma$ and any $x, y \in X$ we have*

$$x \circ_\varepsilon (x \circ_\mu y) = x\circ_{\varepsilon\mu}$$

It is then obvious that if $(X, \circ)$ is a irq then $(X, k \in \mathbb{Z} \mapsto \circ_k)$ is a $\mathbb{Z}$-irq (we define $x \circ_0 y = y$).

The following is a slight modification of proposition 3.4 and point (k) proposition 3.5 [6], for the case of Γ-irqs (the proof of this proposition is almost identical, with obvious modifications, with the proof of the original proposition).

Proposition 5.5 *In any irq $(X, \circ_\varepsilon)_{\varepsilon \in \Gamma}$ be a Γ-irq. Then we have the relations:*

(a) $(u +^x_\varepsilon v) -^x_\varepsilon u = v$

(b) $u +^x_\varepsilon (v -^x_\varepsilon u) = v$

(c) $v -^x_\varepsilon u = (-^x_\varepsilon u) +^{x\circ u}_\varepsilon v$

(d) $-^{x\circ u}_\varepsilon (-^x_\varepsilon u) = u$

(e) $u +^x_\varepsilon (v +^{x\circ u}_\varepsilon w) = (u +^x_\varepsilon v) +^x_\varepsilon w$

(f) $-^x_\varepsilon u = x -^x_\varepsilon u$

(g) $x +^x_\varepsilon u = u$

(k) for any $\varepsilon, \mu \in \mathbb{Z}^*$ and any $x, u, v \in X$ we have the distributivity property:

$$(x \circ_\mu v) -^x_\varepsilon (x \circ_\mu u) = (x \circ_{\varepsilon\mu} u) \circ_\mu (v -^x_{\varepsilon\mu} u)$$

Later we shall apply this proposition for the irq $\alpha^{-1}(x)$ with the operations induced by dilatations δ_ε.

6 Limits of induced deformations

As $\mid \mu \mid \to 0$ the components of the deformations indexed by μ from the diagram 4.1 (namely the operation, norm and respective dilation maps) may converge in the sense of section 1.7 to the components of another normed groupoid with dilations.

6.1 The weak case: dilatation structures on metric spaces

This is the case when only $\left((G \times_\alpha G)_\mu, \tilde{d}_\mu, \tilde{\delta}_\mu \right)$ converges. There is no condition of convergence upon (G_μ, d_μ, δ), nor upon the difference function dif_μ.

Definition 6.1 *A dilation (G, d, δ) is a **groupoid weak δ-structure** (gw δ-structure) if it satisfies the following two axioms:*

A3. *There is a function $\bar{d} : G \times_\alpha G \cap U^2 \to \mathbb{R}$ which is the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\mid \varepsilon \mid} d \, dif(\delta_\varepsilon g, \delta_\varepsilon h) \ = \ \tilde{d}_0(g, h)$$

uniformly on bounded sets in the sense of definition 1.14. Moreover the convergence with respect to $\bar{d}$ is the same as the convergence with respect to $\tilde{d}$ and in particular $\tilde{d}_0(g, h) = 0$ implies $g = h$.

A4weak. *There is a dilation $\bar{\delta}$ of the normed groupoid $(G \times_\alpha G, \tilde{d}_0)$ such that for any $\varepsilon \in \Gamma$ the transformation $\tilde{\delta}_{\mu, \varepsilon}$ converges uniformly on bounded sets to $\bar{\delta}_\varepsilon$.*

Remark 6.2 *For a gw δ-structure the function $\bar{d}$ has the following properties of a distance: for any $(g, h) \in G \times_\alpha G \cap U^2$*

(a) $\bar{d}(g, h) = 0$ *if and only if $g = h$,*

(b) $\bar{d}(g, h) \le \bar{d}(g) + \bar{d}(h)$,

(c) $\bar{d}(g, h) = \bar{d}(h, g)$.

This means that for any $x \in Ob(G)$ the function $\bar{d}$ gives a distance on the set $\alpha^{-1}(x) \cap U$.

Indeed, these properties of the function $\bar{d}$ come from the following observation. Let us define on $G \times_\alpha G$ the function:

$$d(g, h) = d(gh^{-1}) = d \, dif(g, h)$$

Then for any $x \in Ob(G)$ the function d (with two arguments) gives a distance on the set $\alpha^{-1}(x)$. In the case of a δ-structure the axiom A3 can be written as:

$$\lim_{\varepsilon \to 0} \frac{1}{\mid \varepsilon \mid} d(\delta_\varepsilon g, \delta_\varepsilon h) \ = \ \bar{d}(g, h) \tag{6.1.1}$$

uniformly on bounded sets. This gives properties (b), (c) above from a passage to the limit of the properties of the distance d.

For any $x \in Ob(G)$ the restriction of the norm $\tilde{d}$ on the trivial groupoid $\alpha^{-1}(x) \times \alpha^{-1}(x)$ gives a distance on the space $\alpha^{-1}(x)$. The dilatation $\tilde{\delta}$ has the property: for any $\varepsilon \in \Gamma$ and $x \in Ob(G)$

$$\tilde{\delta}_\varepsilon \, \alpha^{-1}(x) \subset \alpha^{-1}(x)$$

therefore we can define δ_ε^h from (a subset of) $\alpha^{-1}(\alpha(h))$ to $\alpha^{-1}(\alpha(h))$ by:

$$\delta_\varepsilon^h g = \delta_\varepsilon(gh^{-1})h \tag{6.1.2}$$

Theorem 6.3 *Suppose that (G, d, δ) is a gw δ-structure. Then for any $x \in Ob(G)$ the triple $(\alpha^{-1}(x), \tilde{d}, \delta)$ is a dilatation structure, with δ defined by (6.1.2) and $\tilde{d}$ restrictioned to $\alpha^{-1}(x)$.*

The proof is just a translation of the definition 6.1 in terms of metric spaces, using the equivalence between metric spaces and normed trivial groupoids. At the end we obtain definition 6.4 of dilatation structures on metric spaces, given further.

6.1.1 Dilatation structures on metric spaces.

For simplicity we shall list the axioms of a dilatation structure (X, d, δ) without concerning about domains and codomains of dilatations. For the full definition of dilatation structure, as well as for their main properties and examples, see [3], [4], [5]. The notion appeared from my efforts to understand the last section of the paper [2] (see also [12], [8], [10], [11]).

However, notice several differences with respect to the original definition of dilatation structures:

(a) in the following definition 6.4 we are no longer asking the metric space (X, d) to be locally compact. Also, uniform convergence in compact sets is replaced by uniform convergence in bounded sets.

(b) because of the modifications explained at (a), we have to ask explicitly that the uniformities induced by d^x and d are the same.

(c) finally, dilatation structures in the sense of the following definition 6.4 are a bit stronger than dilatation structures in the sense introduced and studied in [3], [4], namely we ask for the existence of a "limit dilatation", see the last axiom. This limit exists for strong dilatation structures, but not for dilatation structures in the sense introduced in [3], [4].

Definition 6.4 *A triple (X, d, δ) is a **dilatation structure** if (X, d) is a metric space and*

$$\delta : \Gamma \times \{(x, y) \in X \times X \ : \ y \in dom(\varepsilon, x)\} \to X \quad , \quad \delta(\varepsilon, x, y) = \delta_\varepsilon^x y$$

is a function with the following properties:

A1. *For any point $x \in X$ the function δ induces an action $\delta^x : \Gamma \to End(X, d, x)$, where $End(X, d, x)$ is the collection of all continuous, with continuous inverse transformations $\phi : (X, d) \to (X, d)$ such that $\phi(x) = x$.*

A2. *The function δ is continuous. Moreover, it can be continuously extended to $\bar{\Gamma} \times X \times X$ by $\delta(0, x, y) = x$ and the limit*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^x y = x$$

is uniform with respect to x, y in bounded set.

A3. *There is $A > 1$ such that for any x there exists a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that*

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ |\ \frac{1}{|\varepsilon|} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u, v) \ | \ : u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

uniformly with respect to x in bounded set. Moreover the uniformity induced by d^x is the same as the uniformity induced by d, in particular $d^x(u, v) = 0$ implies $u = v$.

A4weak. *(for metric spaces) The following limit exists:*

$$\lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}^x \delta_\mu^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ \bar{\delta}_\mu^{x, u} v$$

for any $\mu \in \Gamma$, uniformly with respect to x, u, v in bounded sets.

Remark 6.5 *In particular the axiom A1 tells us that $\delta_\varepsilon^x x = x$ for any $x \in X$, $\varepsilon \in \Gamma$, also $\delta_1^x y = y$ for any $x, y \in X$, and $\delta_\varepsilon^x \delta_\mu^x y = \delta_{\varepsilon\mu}^x y$ for any $x, y \in X$ and $\varepsilon, \mu \in \Gamma$.*

Remark 6.6 *In axiom A2 we may alternatively put that the limit is uniform with respect to $d(x, y)$. Similarly, we may ask in axiom A4weak (for metric spaces) that the limit is uniform with respect to $d(x, u)$, $d(x, v)$.*

Remark 6.7 *It is easy to see that:*

(a) *If (X, d) is locally compact then the function d^x is continuous as an uniform limit of continuous functions on a compact set. If (X, d) is also separable then from the existence of the limit d^x and from axiom A1 we obtain the fact that d^x and d induce the same uniformities.*

(b) *By definition d^x is symmetric and satisfies the triangle inequality, but it can be a degenerated distance function: there might exist v, w such that $d^x(v, w) = 0$. But the end of axiom A2 eliminates this possibility.*

Proposition 6.8 *Let (X, d, δ) be a dilatation structure, $x \in X$, and let*

$$\delta_\varepsilon^x \, d(u, v) \; = \; \frac{1}{\mid \varepsilon \mid} \, d(\delta_\varepsilon^x u, \delta_\varepsilon^x v)$$

Then the net of metric spaces $(\bar{B}_d(x, A), \delta_\varepsilon^x d)$ converges in the Gromov-Hausdorff sense to the metric space $(\bar{B}_d(x, A), d^x)$. Moreover this metric space is a metric cone, in the following sense: for any $\mu \in \Gamma$ such that $\mid \mu \mid < 1$ we have $\bar{\delta}_\mu^{x,x} = \delta_\mu^x$ and

$$d^x(\delta_\mu^x u, \delta_\mu^x v) \; = \mid \mu \mid \, d^x(u, v)$$

Proof. The first part of the proposition is just a reformulation of axiom A3, without the condition of uniform convergence. For the second part remark that

$$\delta_{\varepsilon^{-1}}^x \delta_\mu^{\delta_\varepsilon^x x} \delta_\varepsilon^x v \; = \; \delta_\mu^x v$$

and also that

$$\frac{1}{\mid \varepsilon \mid} d(\delta_\varepsilon^x \delta_\mu^x u, \delta_\varepsilon^x \delta_\mu^x v) \; = \mid \mu \mid \, \delta_{\varepsilon\mu}^x d(u, v)$$

Therefore if we pass to the limit with $\varepsilon \to 0$ in these two relations we get the desired conclusion. $\square$

6.1.2   The translation groupoid associated to a dilatation structure

We may associate to a dilatation structure a groupoid which is not normed, which look similar to a Riemannian groupoid (section 3.5).

Definition 6.9 *Let (X, d, δ) be a dilatation structure. The translation groupoid $Tr(X, d, \delta)$ has as objects the distances*

$$d_\varepsilon^x(u, v) \, \frac{1}{\mid \varepsilon \mid} d\left(\delta_\varepsilon^x u, \delta_\varepsilon^x v\right)$$

for all $x \in X$ and all $\varepsilon \in \Gamma$. The arrows are of the form $\Sigma_\varepsilon^x(u, \cdot)$, with

$$\alpha\left(\Sigma_\varepsilon^x(u, \cdot)\right) \; = \; d_\varepsilon^{\delta_\varepsilon^x u} \quad , \quad \omega\left(\Sigma_\varepsilon^x(u, \cdot)\right) \; = \; d_\varepsilon^x$$

Composition of arrows is composition of functions.

Proposition 6.10 $Tr(X, d, \delta)$ *is a groupoid. Moreover, arrows are isometries, in the sense that for any $\varepsilon \in \Gamma$ and $x, u \in X$ the arrow $\Sigma_\varepsilon^x(u, \cdot)$ is an isometry from the source to the range: for ant $v, w \in X$ we have:*

$$d_\varepsilon^x \left(\Sigma_\varepsilon^x(u, v), \Sigma_\varepsilon^x(u, w) \right) = d_\varepsilon^{\delta_\varepsilon^x u}(v, w)$$

Proof. Use proposition 5.5. For example, the fact that composition of arrows is well defined is equivalent with (e) from the mentioned proposition. Invertibility of arrows is (a), and so on. The isometry claim follows from a straight computation. □

This is not a normed groupoid but it can be made into one by adding a random walk which embedd the translation groupoid into the transport normed category.

6.2 The strong case

Definition 6.11 *A **groupoid strong δ-structure** (or a gs δ-structure) is a triple (G, d, δ) such that δ is a map assigning to any $\varepsilon \in \Gamma$ a transformation $\delta_\varepsilon : dom(\varepsilon) \to im(\varepsilon)$ which satisfies the axioms A1, A2 from definition 6.1 and the following axioms A3mod and A4:*

A3mod. There is a function $\bar{d} : U \to \mathbb{R}$ which is the limit

$$\lim_{\varepsilon \to 0} \frac{1}{|\varepsilon|} d \, \delta_\varepsilon(g) = \bar{d}(g)$$

uniformly on bounded sets in the sense of definition 1.14. Moreover, if $\bar{d}(g) = 0$ then $g \in Ob(G)$.

A4. the net Δ_ε converges uniformly on bounded sets to a function Δ.

Remark 6.12 *In the case of a gs δ-structure, notice that A2 and A4 imply that the net dif_ε simply converges to Δ, uniformly on bounded sets.*

Proposition 6.13 *A gs δ-structure is a gw δ-structure. More precisely A1, A2, A3mod and A4 imply A3 with*

$$\bar{d}(g, h) = \bar{d}\Delta(g, h)$$

Proof. Indeed, we have:

$$\frac{1}{|\varepsilon|} d(dif(\delta_\varepsilon(g), \delta_\varepsilon(h)) = \frac{1}{|\varepsilon|} d \, \delta_\varepsilon \, dif_\varepsilon(g, h)$$

We reach to the conclusion by using the remark 6.12 and A3mod. □

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