Maps of metric spaces

Marius Buliga

Institute of Mathematics, Romanian Academy
P.O. BOX 1-764, RO 014700
Bucureşti, Romania
Marius.Buliga@imar.ro

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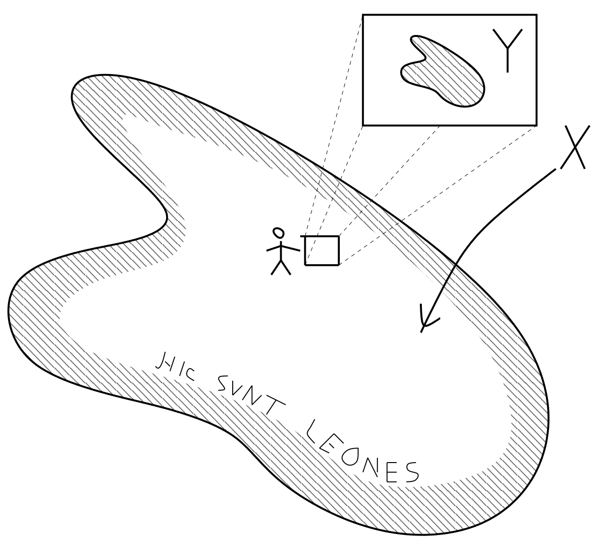
Abstract

This is a pedagogical introduction covering maps of metric spaces, Gromov-Hausdorff distance and its "physical" meaning, and dilation structures as a convenient simplification of an exhaustive database of maps of a metric space into another.

1 Exploring space

Suppose we send an explorer to make maps of an unknown territory X. The explorer wants to record her discoveries on maps, or charts, done in the metric space (Y, D).



I shall suppose that we can put a distance on the set X, that is a function $d : X \times X \to [0, +\infty)$ which satisfies the following requirement: for any three points $x, y, z \in X$ there is a bijective correspondence with a triple A,B,C in the plane such that the sizes (lengths) of AB, BC, AC are equal respectively with $d(x, y)$, $d(y, z)$, $d(z, x)$. Basically, we accept that we can represent in the plane any three points from the space X. An interpretation of the distance $d(x, y)$ is the following: the explorer has a ruler and $d(x, y)$ is the numerical value shown by the ruler when streched between points marked with "x" and "y". (Then the explorer has to use somehow these numbers in order to make a chart of X.)

How many maps are needed? "Understanding" the space X (with respect to the choice of the "gauge" function d) into the terms of the more familiar space Y means making a chart $f : X \to Y$ of X into Y which is not deforming distances too much. Ideally, a perfect chart has to be Lipschitz, that is the distances between points in X are transformed by the chart into distances between points in Y, with a precision independent of the scale: the chart f is (bi)Lipschitz if there are positive numbers $c < C$ such that for any two points $x, y \in X$

$$c\, d(x, y) \leq D(f(x), f(y)) \leq C\, d(x, y)$$

This would be a very good chart, because it would tell how X is at all scales. There are two difficulties related to this model. First, it is impossible to make such a chart in practice. What we can do instead, is to sample the space X (take a ε-dense subset of X with respect to the distance d) and try to represent as good as possible this subspace in Y. Mathematically this is like asking for the chart function f to have the following property: there are supplementary positive constants a, A such that for any two points $x, y \in X$

$$c\, d(x, y) - a \leq D(f(x), f(y)) \leq C\, d(x, y) + A$$

The second difficulty is that such a chart might not exist at all, from mathematical reasons (there is no quasi-isometry between the metric spaces (X, d) and (Y, D)). Such a chart exists of course if we want to make charts of regions with bounded distance, but remark that all details are erased at small scale. The remedy would be to make better and better charts, at smaller and smaller scales, eventually obtaining something resembling a road atlas, with charts of countries, regions, counties, cities, charts which have to be compatible one with another in a clear sense.

2 From maps to dilation structures

Imagine that the metric space (X, d) represents a territory. We want to make maps of (X, d) in the metric space (Y, D) (a piece of paper, or a scaled model).

In fact, in order to understand the territory (X, d), we need many maps, at many scales. For any point $x \in X$ and any scale $\varepsilon > 0$ we shall make a map of a neighbourhood of x, ideally. In practice, a good knowledge of a territory amounts to have, for each of several scales $\varepsilon_1 > \varepsilon_2 > ... > \varepsilon_n$ an atlas of maps of overlapping parts of X (which together form a cover of the territory X). All the maps from all the atlasses have to be compatible one with another.

The ideal model of such a body of knowledge is embodied into the notion of a manifold. To have X as a manifold over the model space Y means exactly this.

Examples from metric geometry (like sub-riemannian spaces) show that the manifold idea could be too rigid in some situations. We shall replace it with the idea of a dilation structure.

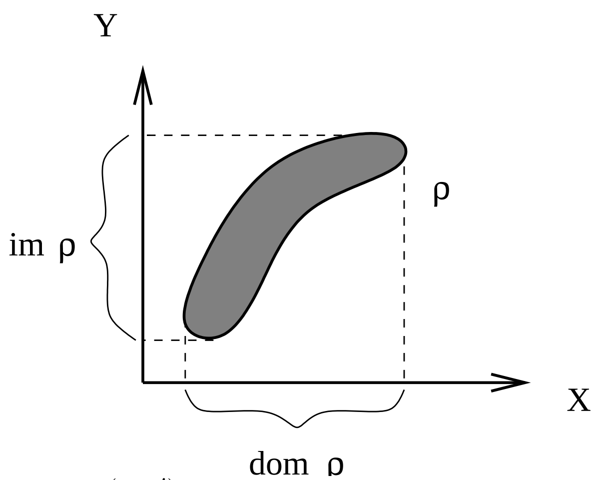
We shall see that a dilation structure (the right generalization of a smooth space, like a manifold), represents an idealization of the more realistic situation of having at our disposal many maps, at many scales, of the territory, with the property that the accuracy, precision and resolution of such maps, and of relative maps deduced from them, are controlled by the scale (as the scale goes to zero, to infinitesimal).

There are two facts which I need to stress. First is that such a generalization is necessary. Indeed, by looking at the large gallery of metric spaces which we now know, the metric spaces with a manifold structure form a tiny and very very particular class. Second is that we tend to take for granted the body of knowledge represented by a manifold structure (or by a dilation structure). Think as an example at the manifold structure of the Earth. It is an idealization of the collection of all cartographic maps of parts of the Earth. This is a huge data basis and it required a huge amount of time and energy in order to be constructed. To know, understand the territory is a huge task, largely neglected. We "have" a manifold, "let X be a manifold". And even if we do not doubt that the physical space (whatever that means) is a boring $\mathbb{R}^3$, it is nevertheless another task to determine with the best accuracy possible a certain point in that physical space, based on the knowledge of the coordinates. For example GPS costs money and time to build and use. Or, it is rather easy to collide protons, but to understand and keep the territory fixed (more or less) with respect to the map, that is where most of the effort goes.

A model of such a map of (X, d) in (Y, D) is a relation $\rho \subset X \times Y$, a subset of a cartesian product $X \times Y$ of two sets. A particular type of relation is the graph of a function $f : X \to Y$, defined as the relation

$$\rho = \{(x, f(x)) : x \in X\}$$

but there are many relations which cannot be described as graphs of functions.



Imagine that pairs $(u, u') \in \rho$ are pairs

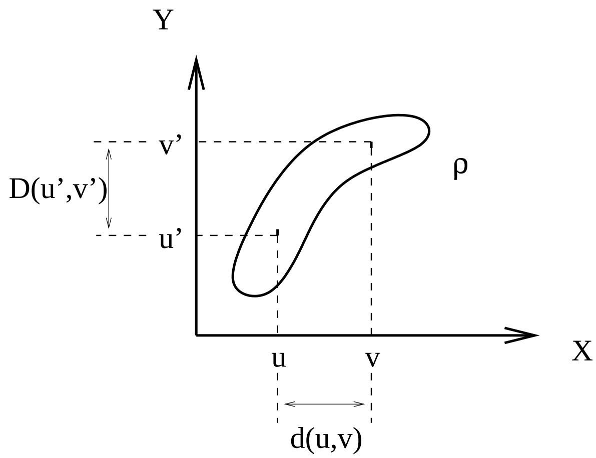
(point in the space X, pixel in the "map space" Y)

with the meaning that the point u in X is represented as the pixel u' in Y.

I don't suppose that there is a one-to-one correspondence between points in X and pixels in Y, for various reasons, for example: due to repeated measurements there is no unique way to associate pixel to a point, or a point to a pixel. The relation ρ represents the cloud of pairs point-pixel which are compatible with all measurements.

I shall use this model of a map for simplicity reasons. A better, more realistic model could be one using probability measures, but this model is sufficient for the needs of this paper.

For a given map ρ the point $x \in X$ in the space X is associated the set of points $\{y \in Y : (x, y) \in \rho\}$ in the "map space" Y. Similarly, to the "pixel" $y \in Y$ in the "map space" Y is associated the set of points $\{x \in X : (x, y) \in \rho\}$ in the space X.

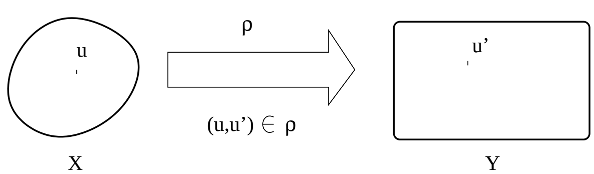


A good map is one which does not distort distances too much. Specifically, considering any two points $u, v \in X$ and any two pixels $u', v' \in Y$ which represent these points, i.e. $(u, u'), (v, v') \in \rho$, the distortion of distances between these points is measured by the number

$$| \ d(u, v) - D(u', v') \ |$$

3 Accuracy, precision, resolution, Gromov-Hausdorff distance

Notations concerning relations. Even if relations are more general than (graphs of) functions, there is no harm to use, if needed, a functional notation. For any relation $\rho \subset X \times Y$ we shall write $\rho(x) = y$ or $\rho^{-1}(y) = x$ if $(x, y) \in \rho$. Therefore we may have $\rho(x) = y$ and $\rho(x) = y'$ with $y \neq y'$, if $(x, y) \in f$ and $(x, y') \in f$. In some drawings, relations will be figured by a large arrow, as shown further.



The domain of the relation ρ is the set $dom \ \rho \ \subset \ X$ such that for any $x \in \ dom \ \rho$ there is $y \in Y$ with $\rho(x) = y$. The image of ρ is the set of $im \ \rho \ \subset Y$ such that for any $y \in \ im \ \rho$ there is $x \in X$ with $\rho(x) = y$. By convention, when we write that a statement $R(f(x), f(y), ...)$ is true, we mean that $R(x', y', ...)$ is true for any choice of $x', y', ...$, such that $(x, x'), (y, y'), ... \in f$.

The inverse of a relation $\rho \subset X \times Y$ is the relation

$$\rho^{-1} \subset Y \times X\,, \quad \rho^{-1} \,=\, \{(u', u) \,:\, (u,u') \in \rho\}$$

and if $\rho' \subset X \times Y$, $\rho" \subset Y \times Z$ are two relations, their composition is defined as

$$\rho = \rho" \circ \rho' \subset X \times Z$$

$$\rho \,=\, \{(u, u") \in X \times Z \,:\, \exists u' \in Y\, (u,u') \in \rho'\, (u', u") \in \rho"\}$$

I shall use the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function such that $\lim\limits_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

In metrology, by definition, accuracy is [13] 2.13 (3.5) "closeness of agreement between a measured quantity value and a true quantity value of a measurand". (Measurement) precision is [13] 2.15 "closeness of agreement between indications or measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions". Resolution is [13] 2.15 "smallest change in a quantity being measured that causes a perceptible change in the corresponding indication".

For our model of a map, if $(u, u') \in \rho$ then u' represent the measurement of u. Moreover, because we see a map as a relation, the definition of the resolution can be restated as the supremum of distances between points in X which are represented by the same pixel. Indeed, if the distance between two points in X is bigger than this supremum then they cannot be represented by the same pixel.

Definition 3.1 *Let $\rho \subset X \times Y$ be a relation which represents a map of dom $\rho \subset (X, d)$ into im $\rho \subset (Y, D)$. To this map are associated three quantities: accuracy, precision and resolution.*

The accuracy of ρ is defined by:

$$acc(\rho) \,=\, \sup\left\{\mid D(y_1, y_2) - d(x_1, x_2) \mid \,:\, (x_1, y_1) \in \rho\,,\, (x_2, y_2) \in \rho\right\} \quad (3.0.1)$$

The resolution of ρ at $y \in im\ \rho$ is

$$res(\rho)(y) \,=\, \sup\left\{d(x_1, x_2) \,:\, (x_1, y) \in \rho\,,\, (x_2, y) \in \rho\right\} \quad (3.0.2)$$

and the resolution of ρ is given by:

$$res(\rho) \,=\, \sup\left\{res(\rho)(y) \,:\, y \in\ im\ \rho\right\} \quad (3.0.3)$$

The precision of ρ at $x \in dom\ \rho$ is

$$prec(\rho)(x) \,=\, \sup\left\{D(y_1, y_2) \,:\, (x, y_1) \in \rho\,,\, (x, y_2) \in \rho\right\} \quad (3.0.4)$$

and the precision of ρ is given by:

$$prec(\rho) \,=\, \sup\left\{prec(\rho)(x) \,:\, x \in\ dom\ \rho\right\} \quad (3.0.5)$$

After measuring (or using other means to deduce) the distances $d(x', x")$ between all pairs of points in X (we may have several values for the distance $d(x', x")$), we try to represent the collection of these distances in (Y, D). When we make a map ρ we are not really measuring the distances between all points in X, then representing them as accurately as possible in Y.

What we do is that we consider a relation ρ, with domain $M = dom(\rho)$ which is ε-dense in (X, d), then we perform a " cartographic generalization"[1] of the relation ρ to a relation $\bar{\rho}$, a map of (X, d) in (Y, D), for example as in the following definition.

Definition 3.2 *A subset $M \subset X$ of a metric space (X, d) is ε-dense in X if for any $u \in X$ there is $x \in M$ such that $d(x, u) \leq \varepsilon$.*

Let $\rho \subset X \times Y$ be a relation such that dom ρ is ε-dense in (X, d) and im ρ is μ-dense in (Y, D). We define then $\bar{\rho} \subset X \times Y$ by: $(x, y) \in \bar{\rho}$ if there is $(x', y') \in \rho$ such that $d(x, x') \leq \varepsilon$ and $D(y, y') \leq \mu$.

If ρ is a relation as described in definition 3.2 then accuracy $acc(\rho)$, ε and μ control the precision $prec(\rho)$ and resolution $res(\rho)$. Moreover, the accuracy, precision and resolution of $\bar{\rho}$ are controlled by those of ρ and ε, μ, as well. This is explained in the next proposition.

Proposition 3.3 *Let ρ and $\bar{\rho}$ be as described in definition 3.2. Then:*

(a) $res(\rho) \leq acc(\rho)$,

(b) $prec(\rho) \leq acc(\rho)$,

(c) $res(\rho) + 2\varepsilon \leq res(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(d) $prec(\rho) + 2\mu \leq prec(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(e) $\mid acc(\bar{\rho}) - acc(\rho) \mid \leq 2(\varepsilon + \mu)$.

Proof. Remark that (a), (b) are immediate consequences of definition 3.1 and that (c) and (d) must have identical proofs, just by switching ε with μ and X with Y respectively. I shall prove therefore (c) and (e).

For proving (c), consider $y \in Y$. By definition of $\bar{\rho}$ we write

$$\{x \in X \,:\, (x, y) \in \bar{\rho}\} \;=\; \bigcup_{(x',y') \in \rho, y' \in \bar{B}(y,\mu)} \bar{B}(x', \varepsilon)$$

Therefore we get

$$res(\bar{\rho})(y) \;\geq\; 2\varepsilon + \sup\left\{res(\rho)(y') \,:\, y' \in\, im(\rho) \cap \bar{B}(y, \mu)\right\}$$

By taking the supremum over all $y \in Y$ we obtain the inequality

$$res(\rho) + 2\varepsilon \leq\; res(\bar{\rho})$$

For the other inequality, let us consider $(x_1, y), (x_2, y) \in \bar{\rho}$ and $(x'_1, y'_1), (x'_2, y'_2) \in \rho$ such that $d(x_1, x'_1) \leq \varepsilon, d(x_2, x'_2) \leq \varepsilon, D(y'_1, y) \leq \mu, D(y'_2, y) \leq \mu$. Then:

$$d(x_1, x_2) \leq 2\varepsilon + d(x'_1, x'_2) \leq 2\varepsilon +\; acc(\rho) + d(y'_1, y'_2) \leq 2(\varepsilon + \mu) +\; acc(\rho)$$

Take now a supremum and arrive to the desired inequality.

For the proof of (e) let us consider for $i = 1, 2$ $(x_i, y_i) \in \bar{\rho}, (x'_i, y'_i) \in \rho$ such that $d(x_i, x'_i) \leq \varepsilon, D(y_i, y'_i) \leq \mu$. It is then enough to take absolute values and transform the following equality

$$d(x_1, x_2) - D(y_1, y_2) = d(x_1, x_2) - d(x'_1, x'_2) + d(x'_1, x'_2) - D(y'_1, y'_2) +$$

[1] `http://en.wikipedia.org/wiki/Cartographic_generalization`, "Cartographic generalization is the method whereby information is selected and represented on a map in a way that adapts to the scale of the display medium of the map, not necessarily preserving all intricate geographical or other cartographic details.

$$+D(y'_1, y'_2) - D(y_1, y_2)$$

into well chosen, but straightforward, inequalities. □

The following definition of the Gromov-Hausdorff distance for metric spaces is natural, owing to the fact that the accuracy (as defined in definition 3.1) controls the precision and resolution.

Definition 3.4 *Let (X, d), (Y, D), be a pair of metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X \times Y$ such that dom $\rho = X$, im $\rho = Y$, and $acc(\rho) \leq \mu$. The Gromov-Hausdorff distance between (X, d) and (Y, D) is the infimum of admissible numbers μ.*

As introduced in definition 3.4, the Gromov-Hausdorff (GH) distance is not a true distance, because the GH distance between two isometric metric spaces is equal to zero. In fact the GH distance induces a distance on isometry classes of compact metric spaces.

The GH distance thus represents a lower bound on the accuracy of making maps of (X, d) into (Y, D). Surprising as it might seem, there are many examples of pairs of metric spaces with the property that the GH distance between any pair of closed balls from these spaces, considered with the distances properly rescaled, is greater than a strictly positive number, independent of the choice of the balls. Simply put: *there are pairs of spaces X, Y such that is impossible to make maps of parts of X into Y with arbitrarily small accuracy.*

Any measurement is equivalent with making a map, say of X (the territory of the phenomenon) into Y (the map space of the laboratory). The possibility that there might a physical difference (manifested as a strictly positive GH distance) between these two spaces, even if they both might be topologically the same (and with trivial topology, say of a $\mathbb{R}^n$), is ignored in physics, apparently. On one side, there is no experimental way to confirm that a territory is the same at any scale (see the section dedicated to the notion of scale), but much of physical explanations are based on differential calculus, which has as the most basic assumption that locally and infinitesimally the territory is the same. On the other side the imposibility of making maps of the phase space of a quantum object into the macroscopic map space of the laboratory might be a manifestation of the fact that there is a difference (positive GH distance between maps of the territory realised with the help of physical phenomena) between "small" and "macroscopic" scale.

4 Scale

Let $\varepsilon > 0$. A map of (X, d) into (Y, D), at scale ε is a map of $(X, \frac{1}{\varepsilon} d)$ into (Y, D). Indeed, if this map would have accuracy equal to 0 then a value of a distance between points in X equal to L would correspond to a value of the distance between the corresponding points on the map in (Y, D) equal to εL.

In cartography, maps of the same territory done at smaller and smaller scales (smaller and smaller ε) must have the property that, at the same resolution, the accuracy and precision (as defined in definition 3.1) have to become smaller and smaller.

In mathematics, this could serve as the definition of the metric tangent space to a point in (X, d), as seen in (Y, D).

Definition 4.1 *We say that (Y, D, y) $(y \in Y)$ represents the (pointed unit ball in the) metric tangent space at $x \in X$ of (X, d) if there exist a pair formed by:*

- *a "zoom sequence", that is a sequence*

$$\varepsilon \in (0,1] \mapsto \rho_\varepsilon^x \subset (\bar{B}(x,\varepsilon), \frac{1}{\varepsilon} d) \times (Y, D)$$

such that dom $\rho_\varepsilon^x = \bar{B}(x,\varepsilon)$*, im* $\rho_\varepsilon^x = Y$*,* $(x,y) \in \rho_\varepsilon^x$ *for any* $\varepsilon \in (0,1]$ *and*

- *a "zoom modulus"* $F : (0,1) \to [0,+\infty)$ *such that* $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$*,*

such that for all $\varepsilon \in (0,1)$ *we have* $acc(\rho_\varepsilon^x) \leq F(\varepsilon)$*.*

Using the notation proposed previously, we can write $F(\varepsilon) = \mathcal{O}(\varepsilon)$, if there is no need to precisely specify a zoom modulus function.

Let us write again the definition of resolution, accuracy, precision, in the presence of scale. The accuracy of ρ_ε^x is defined by:

$$acc(\rho_\varepsilon^x) \ = \ \sup \left\{ \mid D(y_1, y_2) - \frac{1}{\varepsilon} d(x_1, x_2) \mid \ : \ (x_1, y_1), \, (x_2, y_2) \in \rho_\varepsilon^x \right\} \quad (4.0.1)$$

The resolution of ρ_ε^x at $z \in Y$ is

$$res(\rho_\varepsilon^x)(z) \ = \ \frac{1}{\varepsilon} \sup \left\{ d(x_1, x_2) \, : \, (x_1, z) \in \rho_\varepsilon^x \, , \, (x_2, z) \in \rho_\varepsilon^x \right\} \quad (4.0.2)$$

and the resolution of ρ_ε^x is given by:

$$res(\rho_\varepsilon^x) \ = \ \sup \left\{ res(\rho_\varepsilon^x)(y) \, : \, y \in Y \right\} \quad (4.0.3)$$

The precision of ρ_ε^x at $u \in \bar{B}(x,\varepsilon)$ is

$$prec(\rho_\varepsilon^x)(u) \ = \ \sup \left\{ D(y_1, y_2) \, : \, (u, y_1) \in \rho_\varepsilon^x \, , \, (u, y_2) \in \rho_\varepsilon^x \right\} \quad (4.0.4)$$

and the precision of ρ_ε^x is given by:

$$prec(\rho_\varepsilon^x) \ = \ \sup \left\{ prec(\rho_\varepsilon^x)(u) \, : \, u \in \bar{B}(x,\varepsilon) \right\} \quad (4.0.5)$$

If (Y, D, y) represents the (pointed unit ball in the) metric tangent space at $x \in X$ of (X, d) and ρ_ε^x is the sequence of maps at smaller and smaller scale, then we have:

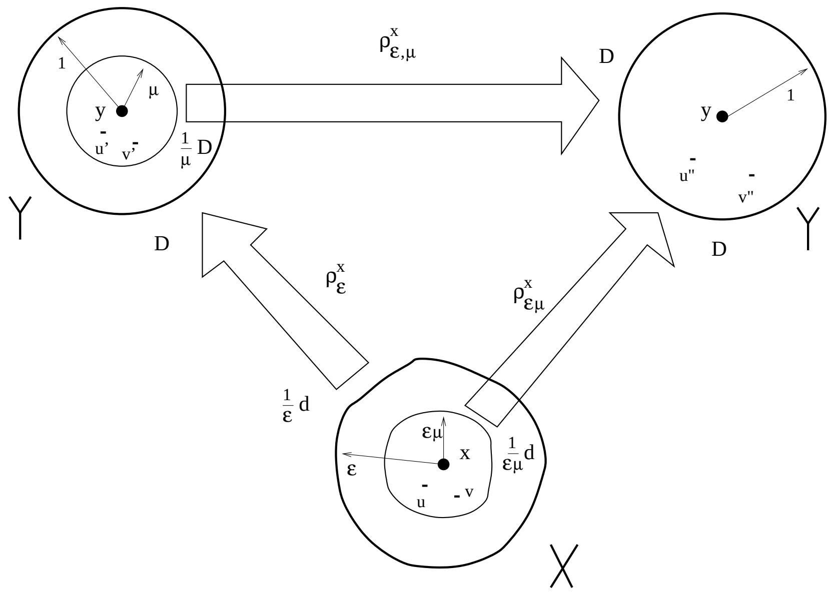
$$\sup \left\{ \mid D(y_1, y_2) - \frac{1}{\varepsilon} d(x_1, x_2) \mid \ : \ (x_1, y_1), \, (x_2, y_2) \in \rho_\varepsilon^x \right\} \ = \ \mathcal{O}(\varepsilon) \quad (4.0.6)$$

$$\sup \left\{ D(y_1, y_2) \, : \, (u, y_1) \in \rho_\varepsilon^x \, , \, (u, y_2) \in \rho_\varepsilon^x \, , \, u \in \bar{B}(x,\varepsilon) \right\} \ = \ \mathcal{O}(\varepsilon) \quad (4.0.7)$$

$$\sup \left\{ d(x_1, x_2) \, : \, (x_1, z) \in \rho_\varepsilon^x \, , \, (x_2, z) \in \rho_\varepsilon^x \, , \, z \in Y \right\} \ = \ \varepsilon \, \mathcal{O}(\varepsilon) \quad (4.0.8)$$

Of course, relation (4.0.6) implies the other two, but it is interesting to notice the mechanism of rescaling.

4.1 Scale stability. Viewpoint stability



I shall suppose further that there is a metric tangent space at $x \in X$ and I shall work with a zoom sequence of maps described in definition 4.1.

Let $\varepsilon, \mu \in (0,1)$ be two scales. Suppose we have the maps of the territory X, around $x \in X$, at scales ε and $\varepsilon\mu$,

$$\rho^x_\varepsilon \subset \bar{B}(x,\varepsilon) \times \bar{B}(y,1)$$

$$\rho^x_{\varepsilon\mu} \subset \bar{B}(x,\varepsilon\mu) \times \bar{B}(y,1)$$

made into the tangent space at x, $(\bar{B}(y,1), D)$. The ball $\bar{B}(x,\varepsilon\mu) \subset X$ has then two maps. These maps are at different scales: the first is done at scale ε, the second is done at scale $\varepsilon\mu$.

What are the differences between these two maps? We could find out by defining a new map

$$\rho^x_{\varepsilon,\mu} = \left\{ (u', u") \in \bar{B}(y,\mu) \times \bar{B}(y,1) : \right. \tag{4.1.9}$$

$$\left. \exists u \in \bar{B}(x,\varepsilon\mu)\, (u,u') \in \rho^x_\varepsilon \ , \ (u,u") \in \rho^x_{\varepsilon\mu} \right\}$$

and measuring its accuracy, with respect to the distances $\frac{1}{\mu} D$ (on the domain) and D (on the image).

Let us consider $(u,u'), (v,v') \in \rho^x_\varepsilon$ and $(u,u"), (v,v") \in \rho^x_{\varepsilon\mu}$ such that $(u',u"), (v',v") \in \rho^x_{\varepsilon,\mu}$. Then:

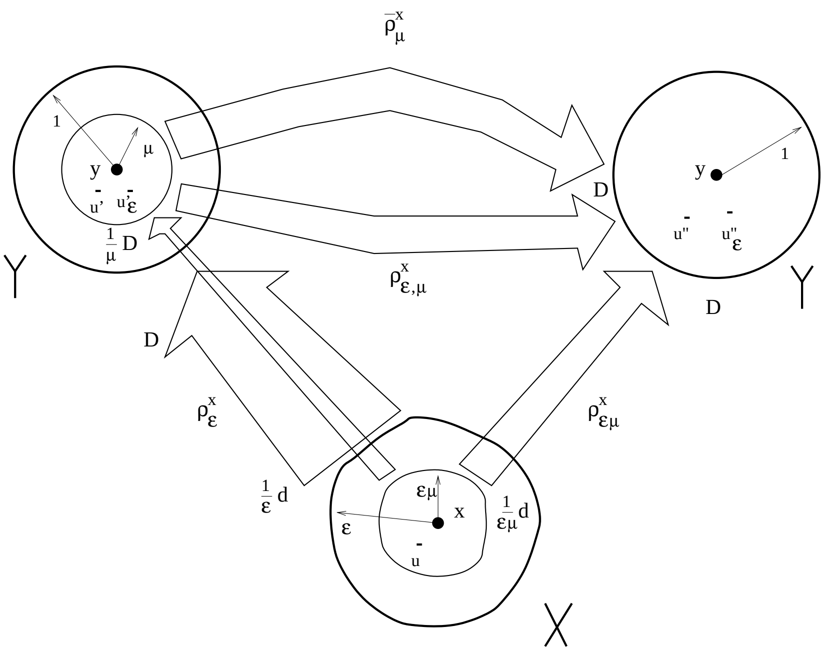
$$\mid D(u",v") - \frac{1}{\mu} D(u',v') \mid \leq \mid \frac{1}{\mu} D(u',v') - \frac{1}{\varepsilon\mu} d(u,v) \mid + \mid \frac{1}{\varepsilon\mu} d(u,v) - D(u",v") \mid$$

We have therefore an estimate for the accuracy of the map $\rho^x_{\varepsilon,\mu}$, coming from estimate (4.0.6) applied for ρ^x_ε and $\rho^x_{\varepsilon\mu}$:

$$acc(\rho^x_{\varepsilon,\mu}) \leq \frac{1}{\mu} \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\mu) \tag{4.1.10}$$

This explains the cascading of errors phenomenon, namely, for fixed μ, as ε goes to 0 the accuracy of the map $\rho^x_{\varepsilon,\mu}$ becomes smaller and smaller, meaning that the maps of the ball $\bar{B}(x, \varepsilon\mu) \subset X$ at the scales $\varepsilon, \varepsilon\mu$ (properly rescaled) are more and more alike. On the contrary, for fixed ε, as μ goes to 0, the bound on the accuracy becomes bigger and bigger, meaning that by using only the map at scale ε, magnifications of a smaller scale region of this map may be less accurate than the map of this smaller region done at the smaller scale.

I shall add a supplementary hypothesis to the one concerning the existence of the metric tangent space. It is somehow natural to suppose that as ε converges to 0 the map $\rho^x_{\varepsilon,\mu}$ converges to a map $\bar{\rho}^x_\mu$. This is described further.



Definition 4.2 *Let the zoom sequence ρ^x_ε be as in definition 4.1 and for given $\mu \in (0,1)$, the map $\rho^x_{\varepsilon,\mu}$ be defined as in (4.1.9). We say that the zoom sequence ρ^x_ε is scale stable at scale μ if there is a relation $\bar{\rho}^x_\mu \subset \bar{B}(y,\mu) \times \bar{B}(y,1)$ such that the Haussorff distance between $\rho^x_{\varepsilon,\mu}$ and $\bar{\rho}^x_\mu$, in the metric space $\bar{B}(y,\mu) \times \bar{B}(y,1)$ with the distance*

$$D_\mu\left((u', u"), (v', v")\right) \;=\; \frac{1}{\mu} D(u', v') + D(u", v")$$

can be estimated as:

$$D^{Hausdorff}_\mu\left(\rho^x_{\varepsilon,\mu}, \bar{\rho}^x_\mu\right) \;\leq\; F_\mu(\varepsilon)$$

with $F_\mu(\varepsilon) = \mathcal{O}_\mu(\varepsilon)$. Such a function $F_\mu(\cdot)$ is called a scale stability modulus of the zoom sequence ρ^x_ε.

This means that for any $(u', u") \in \bar{\rho}^x_\mu$ there is a sequence $(u'_\varepsilon, u"_\varepsilon) \in \rho^x_{\varepsilon,\mu}$ such that

$$\lim_{\varepsilon \to 0} u'_\varepsilon \;=\; u' \quad \lim_{\varepsilon \to 0} u"_\varepsilon \;=\; u"$$

Proposition 4.3 *If there is a scale stable zoom sequence ρ^x_ε as in definitions 4.1 and 4.2 then the space (Y, D) is self-similar in a neighbourhood of point $y \in Y$, namely for any $(u', u"), (v', v") \in \bar{\rho}^x_\mu$ we have:*

$$D(u", v") = \frac{1}{\mu} D(u', v')$$

In particular $\bar{\rho}^x_\mu$ is the graph of a function (the precision and resolution are respectively equal to 0).

Proof. Indeed, for any $\varepsilon \in (0, 1)$ let us consider $(u'_\varepsilon, u"_\varepsilon), (v'_\varepsilon, v"_\varepsilon) \in \rho^x_{\varepsilon,\mu}$ such that

$$\frac{1}{\mu} D(u', u'_\varepsilon) + D(u", u"_\varepsilon) \leq \mathcal{O}_\mu(\varepsilon)$$

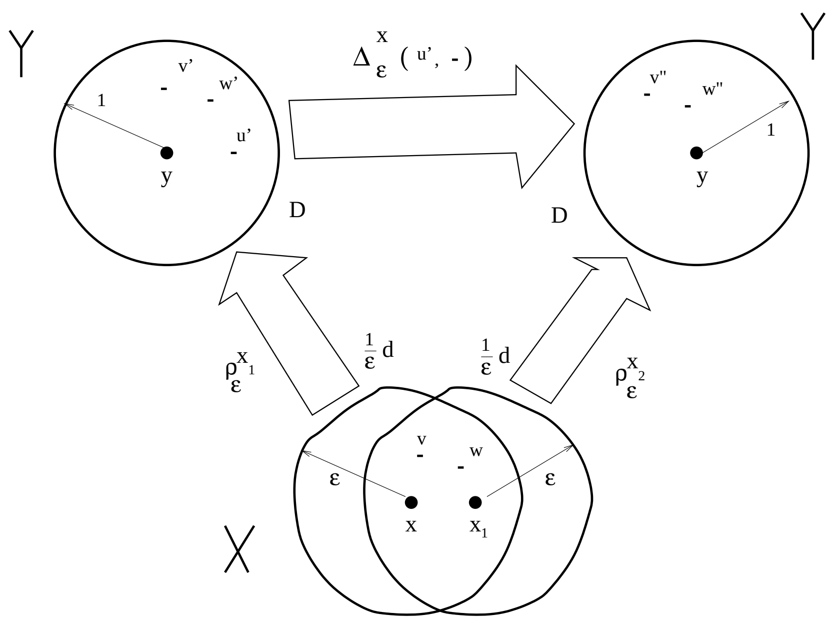
$$\frac{1}{\mu} D(v', v'_\varepsilon) + D(v", v"_\varepsilon) \leq \mathcal{O}_\mu(\varepsilon)$$

Then we get the following inequality, using also the cascading of errors inequality (4.1.10),

$$\mid D(u", v") - \frac{1}{\mu} D(u', v') \mid \leq 2\mathcal{O}_\mu(\varepsilon) + \frac{1}{\mu} \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\mu)$$

We pass with ε to 0 in order to obtain the conclusion. □

Instead of changing the scale (i.e. understanding the scale stability of the zoom sequence), we could explore what happens when we change the point of view.



This time we have a zoom sequence, a scale $\varepsilon \in (0, 1)$ and two points: $x \in X$ and $u' \in \bar{B}(y, 1)$. To the point u' from the map space Y corresponds a point $x_1 \in \bar{B}(x, \varepsilon)$ such that

$$(x_1, u') \in \rho^x_\varepsilon$$

The points x, x_1 are neighbours, in the sense that $d(x, x_1) < \varepsilon$. The points of X which are in the intersection

$$\bar{B}(x, \varepsilon) \cap \bar{B}(x_1, \varepsilon)$$

are represented by both maps, ρ^x_ε and $\rho^{x_1}_\varepsilon$. These maps are different; the relative map between them is defined as:

$$\Delta^x_\varepsilon(u', \cdot) \,=\, \big\{(v', v") \in \bar{B}(y, 1) : \exists v \in \bar{B}(x, \varepsilon) \cap \bar{B}(x_1, \varepsilon) \qquad (4.1.11)$$

$$(v, v') \in \rho^x_\varepsilon \,,\, (v, v") \in \rho^{x_1}_\varepsilon\}$$

and it is called "difference at scale ε, from x to x_1, as seen from u'".

The viewpoint stability of the zoom sequence is expressed as the scale stability: the zoom sequence is stable if the difference at scale ε converges in the sense of Hausdorff distance, as ε goes to 0.

Definition 4.4 *Let the zoom sequence ρ^x_ε be as in definition 4.1 and for any $u' \in \bar{B}(y, 1)$, the map $\Delta^x_\varepsilon(u', \cdot)$ be defined as in (4.1.11). The zoom sequence ρ^x_ε is viewpoint stable if there is a relation $\Delta^x(u', \cdot) \subset \bar{B}(y, 1) \times \bar{B}(y, 1)$ such that the Haussorff distance can be estimated as:*

$$D^{Hausdorff}_\mu \left(\Delta^x_\varepsilon(u', \cdot), \Delta^x(u', \cdot)\right) \,\leq\, F_{diff}(\varepsilon)$$

with $F_{diff}(\varepsilon) = \mathcal{O}(\varepsilon)$. Such a function $F_{diff}(\cdot)$ is called a viewpoint stability modulus of the zoom sequence ρ^x_ε.

There is a proposition analoguous with proposition 4.3, stating that the difference relation $\Delta^x(u', \cdot)$ is the graph of an isometry of (Y, D).

4.2 Foveal maps

The following proposition shows that if we have a scale stable zoom sequence of maps ρ^x_ε as in definitions 4.1 and 4.2 then we can improve every member of the sequence such that all maps from the new zoom sequence have better accuracy near the "center" of the map $x \in X$, which justifies the name "foveal maps".

Definition 4.5 *Let ρ^x_ε be a scale stable zoom sequence. We define for any $\varepsilon \in (0, 1)$ the μ-foveal map ϕ^x_ε made of all pairs $(u, u') \in \bar{B}(x, \varepsilon) \times \bar{B}(y, 1)$ such that*

- *if $u \in \bar{B}(x, \varepsilon\mu)$ then $(u, \bar{\rho}^x_\mu(u')) \in \rho^x_{\varepsilon\mu}$,*
- *or else $(u, u') \in \rho^x_\varepsilon$.*

Proposition 4.6 *Let ρ^x_ε be a scale stable zoom sequence with associated zoom modulus $F(\cdot)$ and scale stability modulus $F_\mu(\cdot)$. The sequence of μ-foveal maps ϕ^x_ε is then a scale stable zoom sequence with zoom modulus $F(\cdot) + \mu F_\mu(\cdot)$. Moreover, the accuracy of the restricted foveal map $\phi^x_\varepsilon \cap \left(\bar{B}(x, \varepsilon\mu) \times \bar{B}(y, \mu)\right)$ is bounded by $\mu F(\varepsilon\mu)$, therefore the right hand side term in the cascading of errors inequality (4.1.10), applied for the restricted foveal map, can be improved to $2F(\varepsilon\mu)$.*

Proof. Let $u \in \bar{B}(x, \varepsilon\mu)$. Then there are $u', u'_\varepsilon \in \bar{B}(y, \mu)$ and $u", u"_\varepsilon \in \bar{B}(y, 1)$ such that $(u, u') \in \phi^x_\varepsilon$, $(u, u") \in \rho^x_\varepsilon \mu)$, $(u', u") \in \bar{\rho}^x_\mu$, $(u'_\varepsilon, u"_\varepsilon) \in \rho^x_{\varepsilon,\mu}$ and

$$\frac{1}{\mu} D(u', u'_\varepsilon) + D(u", u"_\varepsilon) \ \leq \ F_\mu(\varepsilon)$$

Let $u, v \in \bar{B}(x, \varepsilon\mu)$ and $u', v' \in \bar{B}(y, \mu)$ such that $(u, u'), (v, v') \in \phi^x_\varepsilon$. According to the definition of ϕ^x_ε, it follows that there are uniquely defined $u", v" \in \bar{B}(y, 1)$ such that $(u, u"), (v, v") \in \rho^x_{\varepsilon\mu}$ and $(u', u"), (v', v") \in \bar{\rho}^x_\mu$. We then have:

$$\mid \frac{1}{\varepsilon} d(u, v) - D(u', v') \mid =$$

$$= \mid \frac{1}{\varepsilon} d(u, v) - \mu D(u", v") \mid =$$

$$= \ \mu \mid \frac{1}{\varepsilon\mu} d(u, v) - D(u", v") \mid \leq \ \mu F(\varepsilon\mu)$$

Thus we proved that the accuracy of the restricted foveal map

$$\phi^x_\varepsilon \cap \left(\bar{B}(x, \varepsilon\mu) \times \bar{B}(y, \mu) \right)$$

is bounded by $\mu F(\varepsilon\mu)$:

$$\mid \frac{1}{\varepsilon} d(u, v) - D(u', v') \mid \leq \ \mu F(\varepsilon\mu) \qquad (4.2.12)$$

If $u, v \in \bar{B}(x, \varepsilon) \setminus \bar{B}(x, \mu)$ and $(u, u'), (v, v') \in \phi^x_\varepsilon$ then $(u, u'), (v, v') \in \rho^x_\varepsilon$, therefore

$$\mid \frac{1}{\varepsilon} d(u, v) - D(u', v') \mid \leq \ F(\varepsilon)$$

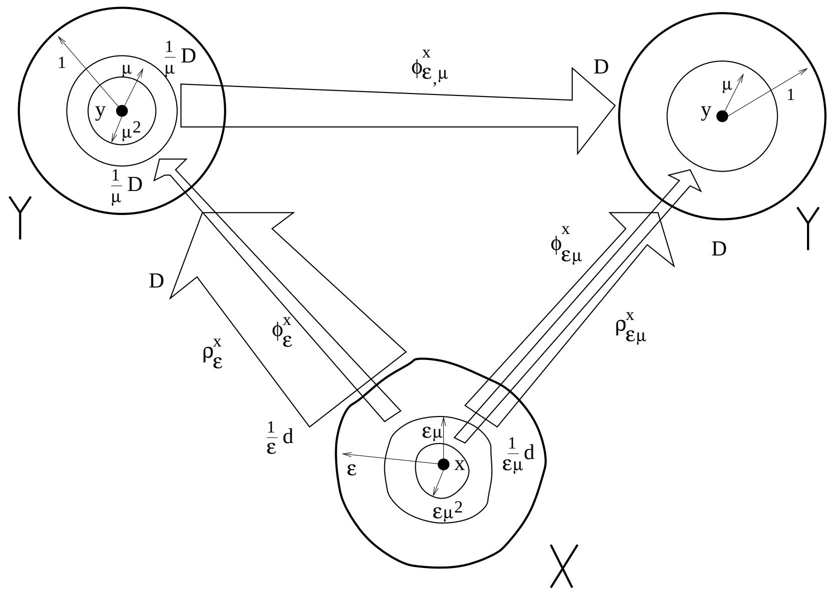
Suppose now that $(u, u'), (v, v') \in \phi^x_\varepsilon$ and $u \in \bar{B}(x, \varepsilon\mu)$ but $v \in \bar{B}(x, \varepsilon) \setminus \bar{B}(x, \mu)$. We have then:

$$\mid \frac{1}{\varepsilon} d(u, v) - D(u', v') \mid \leq$$

$$\leq \mid \frac{1}{\varepsilon} d(u, v) - D(u'_\varepsilon, v') \mid + D(u', u'_\varepsilon) \ \leq \ F(\varepsilon) + \mu F_\mu(\varepsilon)$$

We proved that the sequence of μ-foveal maps ϕ^x_ε is a zoom sequence with zoom modulus $F(\cdot) + \mu F_\mu(\cdot)$.

In order to prove that the sequence is scale stable, we have to compute $\phi^x_{\varepsilon,\mu}$, graphically shown in the next figure.



We see that $(u', u") \in \phi^x_{\varepsilon,\mu}$ implies that $(u', u") \in \rho^x_{\varepsilon,\mu}$ or $(u', u") \in \rho^x_{\varepsilon\mu,\mu}$. From here we deduce that the sequence of foveal maps is scale stable and that

$$\varepsilon \mapsto \max\left\{F_\mu(\varepsilon), \mu F_\mu(\varepsilon\mu)\right\}$$

is a scale stability modulus for the foveal sequence.

The improvement of the right hand side for the cascading of errors inequality (4.1.10), applied for the restricted foveal map is then straightforward if we use (4.2.12). □

5 Dilation structures

From definition 4.5 we see that

$$\bar{\rho}^x_\mu \circ \phi^x_\varepsilon \ = \ \rho^x_{\varepsilon\mu} \tag{5.0.1}$$

Remark that if the μ-foveal map ϕ^x_ε coincides with the chart ρ^x_ε for every ε (that is, if the zoom sequence ρ^x_ε is already so good that it cannot be improved by the construction of foveal maps), then relation (5.0.1) becomes

$$\bar{\rho}^x_\mu \circ \phi^x_\varepsilon \ = \ \phi^x_{\varepsilon\mu} \tag{5.0.2}$$

By proposition 4.3, it follows that μ-foveal map at scale $\varepsilon\mu$ is just a $1/\mu$ dilation of a part of the μ-foveal map at scale ε.

An idealization of these "perfect", stable zoom sequences which cannot be improved by the μ-foveal map construction for any $\mu \in (0,1)$, are dilation structures.

There are several further assumptions, which clearly amount to yet other idealizations. These are the following:

- the "map is the territory assumption", namely $Y = U(x)$, the "map space" is included in X, the "territory", and $y = x$.

- "functions instead relations", that is the perfect stable zoom sequences $\rho_\varepsilon^x = \phi_\varepsilon^x$ are graphs of functions, called dilations. That means:

$$\rho_\varepsilon^x \subset \{(\delta_\varepsilon^x u', u') \, : \, u' \in Y = V_\varepsilon(x)\}$$

- "hidden uniformity", that is: in order to pass to the limit in various situations, we could choose the zoom modulus and stability modulus to not depend on $x \in X$. This innocuous assumption is the least obvious, but necessary one.

With these idealizations in force, remember that we want our dilations to form a stable zoom sequence and we want also the subtler viewpoint stability, which consists in being able to change the point of view in a coherent way, as the scale goes to zero. These are the axioms of a dilation structure.

We shall use here a slightly particular version of dilation structures. For the general definition of a dilation structure see [1]. More about this, as well as about length dilation structures, see [4].

Definition 5.1 *Let (X, d) be a complete metric space such that for any $x \in X$ the closed ball $\bar{B}(x, 3)$ is compact. A dilation structure (X, d, δ) over (X, d) is the assignment to any $x \in X$ and $\varepsilon \in (0, +\infty)$ of a homeomorphism, defined as: if $\varepsilon \in (0, 1]$ then $\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$, else $\delta_\varepsilon^x : W_\varepsilon(x) \to U(x)$, with the following properties.*

A0. *For any $x \in X$ the sets $U(x), V_\varepsilon(x), W_\varepsilon(x)$ are open neighbourhoods of x. There are $1 < A < B$ such that for any $x \in X$ and any $\varepsilon \in (0, 1)$ we have:*

$$B_d(x, \varepsilon) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset$$

$$\subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B)$$

Moreover for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in (0, \varepsilon_0)$, we have $\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u)$.

A1. *For any $x \in X$ $\delta_\varepsilon^x x = x$ and $\delta_1^x = id$. Consider the closure $Cl(dom\,\delta)$ of the set*

$$dom\,\delta = \{(\varepsilon, x, y) \in (0, +\infty) \times X \times X \, :$$

$$if\ \varepsilon \leq 1\ then\ y \in U(x)\ ,\ else\ y \in W_\varepsilon(x)\}$$

seen in $[0, +\infty) \times X \times X$ endowed with the product topology. The function $\delta : dom\,\delta \to X$, $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ is continuous, admits a continuous extension over $Cl(dom\,\delta)$ and we have $\lim_{\varepsilon \to 0} \delta_\varepsilon^x y = x$.

A2. *For any $x, \in X$, $\varepsilon, \mu \in (0, +\infty)$ and $u \in U(x)$, whenever one of the sides are well defined we have the equality $\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u$.*

A3. *For any x there is a distance function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that uniformly with respect to x in compact set we have the limit:*

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u, v) \mid \ : u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

A4. *Let us define $\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v$. Then we have the limit, uniformly with respect to x, u, v in compact set,*

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

It is algebraically straightforward to transport a dilation structure: given (X, d, δ) a dilation structure and $f : X \to Z$ a uniformly continuous homeomorphism from X (as a topological space) to another topological space Z (actually more than a topological space, it should be a space endowed with an uniformity), we can define the transport of (X, d, δ) by f as the dilation structure $(Z, f * d, f * \delta)$. The distance $f * d$ is defined as

$$(f * d)\,(u, v) = d(f(u), f(v))$$

which is a true distance, because we supposed f to be a homeomorphism. For any $u, v \in X$ and $\varepsilon > 0$, we define the new dilation based at $f(u) \in Z$, of coefficient ε, applied to $f(v) \in Z$ as

$$(f * \delta)_{\varepsilon}^{f(u)}\, f(v) \; = \; f\,(\delta_{\varepsilon}^{u} v)$$

It is easy to check that this is indeed a dilation structure.

In particular we may consider to transport a dilation structure by one of its dilations. Visually, this corresponds to transporting the atlas representing a dilation structure on X to a neighbourhood of one of its points. It is like a scale reduction of the whole territory (X, d) to a smaller set.

Inversely, we may transport the (restriction of the) dilation structure (X, d, δ) from $V_{\varepsilon}(x)$ to $U(x)$, by using $\delta_{\varepsilon^{-1}}^{x}$ as the transport function f. This is like a magnification of the "infinitesimal neighbourhood" $V_{\varepsilon}(x)$. (This neighbourhood is infinitesimal in the sense that we may consider ε as a variable, going to 0 when needed. Thus, instead of one neighbourhood $V_{\varepsilon}(x)$, there is a sequence of them, smaller and smaller).

This is useful, because it allows us to make "infinitesimal statements", i.e. statements concerning this sequence of magnifications, as $\varepsilon \to 0$.

Let us compute then the magnified dilation structure. We should also rescale the distance on $V_{\varepsilon}(x)$ by a factor $1/\varepsilon$. Let us compute this magnified dilation structure:

- the space is $U(x)$
- for any $u, v \in U(x)$ the (transported) distance between them is

$$d_{\varepsilon}^{x}(u, v) \; = \; \frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v)$$

- for any $u, v \in U(x)$ and scale parameter $\mu \in (0, 1)$ (we could take $\mu > 0$ but then we have to be careful with the domains and codomains of these new dilations), the transported dilation based at u, of coefficient μ, applied to v, is

$$\delta_{\varepsilon^{-1}}^{x}\, \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u}\, \delta_{\varepsilon}^{x} v \qquad (5.0.3)$$

It is visible that working with such combinations of dilations becomes quickly difficult. This is one of the reasons of looking for more graphical notations.

Here is the definition of "linearity" and "selfsimilarity" for dilation structures.

Definition 5.2 *Let X, Y be metric spaces endowed with dilation structures. A function $f : X \to Y$ is linear if and only if it is a morphism of dilation structures: for any $u, v \in X$ and any $\varepsilon \in \Gamma$*

$$f(\delta_{\varepsilon}^{u} v) \; = \; \delta_{\varepsilon}^{f(u)} f(v)$$

which is also a Lipschitz map from X to Y as metric spaces.

A dilation structure (X, d, δ) is (x, μ) self-similar (for a $x \in X$ and $\mu \in \Gamma$, different from 1, the neutral element of $\Gamma = (0, +\infty)$) if the dilation $f = \delta^x_\mu$ is linear from (X, d, δ) to itself and moreover for any $u, v \in X$ we have

$$d(\delta^x_\mu u, \delta^x_\mu v) \;=\; \mu \, d(u, v)$$

A dilation structure is linear if it is self-similar with respect to any $x \in X$ and $\mu \in \Gamma$.

Definition 5.3 *Let (X, d, δ) be a dilation structure. A property*

$$\mathcal{P}(x_1, x_2, x_3, ...)$$

is true for $x_1, x_2, x_3, ... \in X$ sufficiently close if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.

For a dilation structure the metric tangent spaces have the algebraic structure of a normed group with dilations.

We shall work further with local groups, which are spaces endowed with a locally defined operation which satisfies the conditions of a uniform group. See section 3.3 [1] for details about the definition of local groups.

5.1 Normed conical groups

This name has been introduced in section 8.2 [1], but these objects appear more or less in the same form under the name "contractible group" or "homogeneous group". Essentially these are groups endowed with a family of "dilations". They were also studied in section 4 [2].

In the following general definition appear a topological commutative group Γ endowed with a continuous morphism $\nu : \Gamma \to (0, +\infty)$ from Γ to the group $(0, +\infty)$ with multiplication. The morphism ν induces an invariant topological filter on Γ (other names for such an invariant filter are "absolute" or "end"). The convergence of a variable $\varepsilon \in \Gamma$ to this filter is denoted by $\varepsilon \to 0$ and it means simply $\nu(\varepsilon) \to 0$ in $\mathbb{R}$.

Particular, interesting examples of pairs (Γ, ν) are: $(0, +\infty)$ with identity, which is the case interesting for this paper, $\mathbb{C}^*$ with the modulus of complex numbers, or $\mathbb{N}$ (with addition) with the exponential, which is relevant for the case of normed contractible groups, section 4.3 [2].

Definition 5.4 *A normed group with dilations $(G, \delta, \| \cdot \|)$ is a local group G with a local action of Γ (denoted by δ), on G such that*

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit $\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$ is well defined in a compact neighbourhood of e and the limit is uniform with respect to x, y.

H2. the following relation holds: $\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$, where the limit from the left hand side exists in a neighbourhood $U \subset G$ of e and is uniform with respect to $x \in U$.

Moreover the group is endowed with a continuous norm function $\|\cdot\| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the properties:

(a) *for any x we have $\|x\| \geq 0$; if $\|x\| = 0$ then $x = e$,*

(b) *for any x, y we have $\|xy\| \leq \|x\| + \|y\|$,*

(c) *for any x we have $\|x^{-1}\| = \|x\|$,*

(d) *the limit $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ exists, is uniform with respect to x in compact set,*

(e) *if $\|x\|^N = 0$ then $x = e$.*

Theorem 5.5 *(Thm. 15 [1]) Let $(G, \delta, \| \cdot \|)$ be a locally compact normed local group with dilations. Then (G, d, δ) is a dilation structure, where the dilations δ and the distance d are defined by: $\delta^x_\varepsilon u = x \delta_\varepsilon (x^{-1} u)$, $d(x, y) = \|x^{-1} y\|$.*

Moreover (G, d, δ) is linear, in the sense of definition 5.2.

Definition 5.6 *A normed conical group N is a normed group with dilations such that for any $\varepsilon \in \Gamma$ the dilation δ_ε is a group morphism and such that for any $\varepsilon > 0$ $\|\delta_\varepsilon x\| = \nu(\varepsilon) \|x\|$.*

A normed conical group is the infinitesimal version of a normed group with dilations ([1] proposition 2).

Proposition 5.7 *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed local group with dilations. Then $(G, \beta, \delta, \| \cdot \|^N)$ is a locally compact, local normed conical group, with operation β, dilations δ and homogeneous norm $\| \cdot \|^N$.*

5.2 Tangent bundle of a dilation structure

The most important metric and algebraic first order properties of a dilation structure are presented here as condensed statements, available in full length as theorems 7, 8, 10 in [1].

Theorem 5.8 *Let (X, d, δ) be a dilation structure. Then the metric space (X, d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \{ \mid d(u, v) - d^x(u, v) \mid \ : d(x, u) \leq \varepsilon \ , \ d(x, v) \leq \varepsilon \} \ = \ 0 \ .$$

Theorem 5.9 *If (X, d, δ) is a dilation structure then for any $x \in X$ the triple (U^x, δ^x, d^x) is a locally compact normed conical group, with operation $\Sigma^x(\cdot, \cdot)$, neutral element x and inverse $inv^x(y) = \Delta^x(y, x)$.*

The conical group $(U(x), \Sigma^x, \delta^x)$ can be seen as the tangent space of (X, d, δ) at x. We shall denote it by $T_x(X, d, \delta) = (U(x), \Sigma^x, \delta^x)$, or by $T_x X$ if (d, δ) are clear from the context.

The following proposition is corollary 6.3 from [2], which gives a more precise description of the conical group $(U(x), \Sigma^x, \delta^x)$. In the proof of that corollary there is a gap pointed by S. Vodopyanov, namely that Siebert' proposition 5.4 [15], which is true for conical groups (in our language), is used for local conical groups. Fortunately, this gap was filled by the theorem 1.1 [12], which states that a locally compact, locally connected, contractible (with Siebert' wording) group is locally isomorphic to a contractive Lie group.

Proposition 5.10 *Let (X, d, δ) be a dilation structure. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a homogeneous group), given by the eigenspaces of δ^x_ε for an arbitrary $\varepsilon \in (0,1)$.*

There is a bijection between linear (in the sense of definition 5.2) dilation structures and normed conical groups. Any normed conical group induces a linear dilation structure, by theorem 5.5. Conversely, we have the following result (see theorem 6.1 [5] for a more general statement).

Theorem 5.11 *Let (G, d, δ) be a linear dilation structure. Then, with the notations from theorem 5.9, for any $x \in G$, the dilation structure $(U(x), d, \delta)$ coincides with the dilation structure of the conical group $(U(x), \Sigma^x, \delta^x)$.*

5.3 Differentiability with respect to dilation structures

For any dilation structure or there is an associated notion of differentiability (section 7.2 [1]). For defining differentiability with respect to dilation structures we need first the definition of a morphism of conical groups.

Definition 5.12 *Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A function $f : N \rightarrow M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.*

The definition of the derivative, or differential, with respect to dilations structures is a straightforward generalization of the definition of the Pansu derivative [14].

Definition 5.13 *Let (X, d, δ) and $(Y, \overline{d}, \overline{\delta})$ be two dilation structures and $f : X \rightarrow Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $D\, f(x) : T_x X \rightarrow T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta^x_\varepsilon u\right), \overline{\delta}^{f(x)}_\varepsilon D\, f(x)(u) \right) \;:\; d(x,u) \leq \varepsilon \right\} = 0, \qquad (5.3.4)$$

The morphism $D\, f(x)$ is called the derivative, or differential, of f at x.

The definition also makes sense if the function f is defined on a open subset of (X, d).

References

[1] M. Buliga, Dilatation structures I. Fundamentals, *J. Gen. Lie Theory Appl.*, **1** (2007), 2, 65-95.

[2] M. Buliga, Infinitesimal affine geometry of metric spaces endowed with a dilatation structure , *Houston Journal of Math.* 36, 1 (2010), 91-136, `http://arxiv.org/abs/0804.0135`

[3] M. Buliga, Dilatation structures in sub-riemannian geometry, in: Contemporary Geometry and Topology and Related Topics. Cluj-Napoca, Cluj-Napoca, Cluj University Press (2008), 89-105

[4] M. Buliga, A characterization of sub-riemannian spaces as length dilation structures constructed via coherent projections, *Commun. Math. Anal.* **11** (2011), No. 2, pp. 70-111, `http://arxiv.org/abs/0810.5042`

[5] M. Buliga, Emergent algebras as generalizations of differentiable algebras, with applications, (2009), http://arxiv.org/abs/0907.1520

[6] M. Buliga, Braided spaces with dilations and sub-riemannian symmetric spaces, (2010), http://arxiv.org/abs/1005.5031

[7] M. Buliga, What is a space? Computations in emergent algebras and the front end visual system (2010), http://arxiv.org/abs/1009.5028

[8] M. Buliga, More than discrete or continuous: a bird's view (2010), http://arxiv.org/abs/arXiv:1011.4485

[9] M, Buliga, Introduction to metric spaces with dilations (2010), http://arxiv.org/abs/1007.2362

[10] M. Buliga, Deformations of normed groupoids and differential calculus. First part, (2009), http://arxiv.org/abs/0911.1300

[11] M. Buliga, Tangent bundles to sub-Riemannian groups, (2003), http://xxx.arxiv.org/abs/math.MG/0307342

[12] L. van den Dries, I. Goldbring, Locally compact contractive local groups, (2009), http://arxiv.org/abs/0909.4565

[13] JCGM 200:2008 International vocabulary of metrology - Basic and general concepts and associated terms (VIM), http://www.bipm.org/utils/common/documents/jcgm/JCGM_200_2008.pdf

[14] P. Pansu, Métriques de Carnot-Carathéodory et quasi-isométries des espaces symétriques de rang un, Ann. of Math., (2) **129**, (1989), 1-60

[15] E. Siebert, Contractive automorphisms on locally compact groups, *Math. Z.*, 191, 73-90, (1986)