ON KHOVANOV-SEIDEL QUIVER ALGEBRAS AND BORDERED FLOER HOMOLOGY

DENIS AUROUX, J. ELISENDA GRIGSBY, AND STEPHAN M. WEHRLI

Abstract. We discuss a relationship between Khovanov- and Heegaard Floer-type homology theories for braids. Explicitly, we define a filtration on the bordered Heegaard-Floer homology bimodule associated to the double-branched cover of a braid and show that its associated graded bimodule is equivalent to a similar bimodule defined by Khovanov and Seidel.

1. Introduction

The low-dimensional topology community has been energized in recent years by the introduction of a wealth of so-called *homology-type* invariants. These invariants are defined by associating to a topological object (for example, a link or a 3–manifold) an abstract chain complex whose quasi-isomorphism class–hence, homology–is an invariant of the object.

One obtains such invariants from two apparently unrelated points of view:

(1) algebraically, via the higher representation theory of quantum groups, and
(2) geometrically/analytically, via symplectic geometry and gauge theory.

Although the invariants themselves share a number of formal properties, finding explicit connections between the two viewpoints has proven challenging.

A striking success in this direction is a result of Ozsváth and Szabó relating the $\mathbb{Z}/2\mathbb{Z}$ versions of Khovanov homology and Heegaard Floer homology:

Theorem 1.1. [31] *Let $L \subset S^3$ be a link and $\overline{L} \subset S^3$ denote its mirror. There exists a spectral sequence whose E^2 term is $\widetilde{Kh}(\overline{L})$, the reduced Khovanov homology of the mirror of L, and whose E^∞ term is $\widehat{HF}(\mathbf{\Sigma}(L))$, the Heegaard-Floer homology of the double-branched cover of L.*

This result has generated applications in a number of directions (see, e.g., [32], [41], [7]). It also served as inspiration for Kronheimer and Mrowka's construction of an analogous spectral sequence from Khovanov homology to a version of instanton knot homology, yielding a proof that Khovanov homology detects the unknot [25].

The aim of the present paper is to move toward a more "atomic" understanding of the Ozsváth-Szabó spectral sequence and its sutured generalizations ([33, 13, 11, 14]). In particular, viewing a link in S^3 as the closure of a braid, we can ask whether there are appropriate Khovanov-type (algebraic) and Heegaard-Floer-type (geometric/analytic) invariants associated to braids such that the Ozsváth-Szabó spectral sequence emerges as an algebraic consequence of a relationship between these invariants.

DA was partially supported by NSF grant DMS-1007177.
JEG was partially supported by a Viterbi-endowed MSRI postdoctoral fellowship and NSF grant number DMS-0905848.
SMW was partially supported by an MSRI postdoctoral fellowship.

Such a description would not only be of theoretical interest. Ozsváth-Szabó's original description of the above spectral sequence involves holomorphic polygon counts in Heegaard multi-diagrams. Since these counts are tricky to carry out in practice, finding ways to perform them combinatorially should prove valuable, especially in light of subsequent work of Baldwin [6] (see also L. Roberts [34]) proving that the terms of the Ozsváth-Szabó spectral sequence are themselves link invariants.

We should at this point remark that recent work of Lipshitz-Ozsváth-Thurston, in [30] and its sequel, does precisely this. In addition, Szabó [40] has constructed a combinatorial filtration on the Khovanov cube of resolutions associated to a link diagram that he conjectures yields the original Ozsváth-Szabó spectral sequence.

In the present paper, we address a slightly different question from a substantially different direction. First, we focus not on the original Ozsváth-Szabó spectral sequence but rather on (a direct summand of) one of its sutured generalizations [33, 11]. Second, we take as our starting point a paper of Khovanov-Seidel [21], which explores a concrete instance of Kontsevich's homological mirror symmetry conjecture [24]. The constructions found there, when combined with work of the first author [3], lead naturally to a new view on the filtered complexes appearing in [33, 11].

Explicitly, given a braid $\sigma \subset D^2 \times I$, we consider the closure of the braid, not in the three-ball but in the solid torus (viewed as a product sutured annulus, $A \times I$). Associated to the resulting *annular link* are Khovanov-type and Heegaard-Floer-type invariants connected by a *sutured* spectral sequence [1, 33, 11] that splits along an extra grading measuring "wrapping" around the S^1 factor.[1] In [4], building on work in [28], we obtain a similar spectral sequence in the "next-to-top" graded piece as the Hochschild homology of a filtered A_∞ bimodule associated to the original braid, σ.

The purpose of the present paper is to give an explicit combinatorial construction of this filtered A_∞ bimodule. Informally, the resulting spectral sequence interpolates between the "open" Khovanov- and Heegaard-Floer-type invariants of a braid $\sigma \subset D^2 \times I$ just as the sutured spectral sequence interpolates between the analogous "closed" invariants of its closure, $\hat{\sigma} \subset A \times I$.

More precisely:

(1) On the algebraic side, we show how to use ideas of Khovanov-Seidel in [21] to construct an A_∞ bimodule, $\mathcal{M}_\sigma^{Kh}$, via Yoneda imbedding of a distinguished collection of objects in the derived category of a quiver algebra.
(2) On the geometric/analytic side, we use the bordered Floer homology package of Lipshitz-Ozsváth-Thurston in [27, 28] to construct an A_∞ bimodule, $\mathcal{M}_\sigma^{HF}$, the 1–strand CFDA bimodule associated to the mapping class $\hat{\sigma}$ obtained as the double-branched cover of $\sigma \subset D^2 \times I$.

Letting $\mathbb{1}$ denote the identity braid of the same index as σ, we prove:

Theorem 6.1. *There exists a filtration on $\mathcal{M}_\sigma^{HF}$ whose associated graded bimodule is quasi-isomorphic, as an ungraded A_∞ bimodule over $\left\{gr(\mathcal{M}_{\mathbb{1}}^{HF}) = \mathcal{M}_{\mathbb{1}}^{Kh}\right\}$, to $\mathcal{M}_\sigma^{Kh}$.*

In particular, for each braid there exists a spectral sequence connecting the Khovanov-Seidel (algebraic) bimodule to the Lipshitz-Ozsváth-Thurston (geometric/analytic) one. Moreover, these "open" spectral sequences can be defined without reference to holomorphic curves. In fact, our construction is based on a remarkably simple toy model (Lemma

[1] This extra grading has a natural interpretation on the Khovanov side in terms of $U_q(\mathfrak{sl}_2)$ weight space decompositions and on the Heegaard-Floer side in terms of relative Spin^c structures. See [12] for more details.

5.3): a filtered complex interpolating between the cohomology of S^1 and the cohomology of S^0 (both over $\mathbb{Z}/2\mathbb{Z}$) coming from a $\mathbb{Z}/2\mathbb{Z}$–equivariant cochain complex for S^1. This toy model was, in turn, inspired by work of Seidel and Smith [38].

It is worth noting that the quiver algebras of Khovanov-Seidel are a special case (for $k = 1$) of certain algebras $A^{k,n-k}$ introduced by Chen-Khovanov [10] and independently by Stroppel [39]. We conjecture that Theorem 6.1 admits a generalization which, for every n-strand braid σ, provides a relationship between the k-strand part of the Lipshitz-Ozsváth-Thurston bimodule associated to $\widehat{\sigma}$ and a Khovanov-type bimodule defined over the Ext-algebra of the direct sum of all standard $A^{k,n-k}$-modules.

The paper is organized as follows:

In Section 2, we establish notation and collect a number of useful definitions and elementary algebraic results.

In Section 3, we describe the topological input needed for the algebraic constructions in the remainder of the paper. After reviewing the key points in [21], we proceed to the construction and description of

- an algebra, B^{Kh}, associated to a marked disk D_m equipped with a specific *basis of curves* and
- a module, $\mathcal{M}_\sigma^{Kh}$, associated to each braid σ, decomposed as a product of elementary Artin generators.

We conclude the section with a brief geometric interpretation of the Khovanov-Seidel algebra and bimodules in terms of the Fukaya category of a particular Lefschetz fibration.

In Section 4, we turn to the construction and description of the analogous bordered Floer algebra B^{HF} and bimodules $\mathcal{M}_\sigma^{HF}$, using the same topological input.

In Section 5, we describe a natural filtration on B^{HF} whose associated graded algebra is isomorphic to B^{Kh}. Our construction is based on a simple "toy model" (Lemma 5.3).

In Section 6, we describe a filtration on $\mathcal{M}_\sigma^{HF}$ whose associated graded homology bimodule is quasi-isomorphic to $\mathcal{M}_\sigma^{Kh}$. We proceed by choosing a decomposition

$$\sigma = \sigma_{k_1}^{\pm} \cdots \sigma_{k_n}^{\pm}$$

of σ as a product of elementary Artin generators, explicitly constructing a filtration on $\mathcal{M}_{\sigma_k^{\pm}}^{HF}$ for each elementary generator, then realizing $\mathcal{M}_\sigma^{HF}$ as the (filtered) A_∞ tensor product of the elementary bimodules $\mathcal{M}_{\sigma_{k_1}^{\pm}}^{HF}, \dots, \mathcal{M}_{\sigma_{k_n}^{\pm}}^{HF}$.

1.1. **Acknowledgements.** We are grateful to Tony Licata, Robert Lipshitz, Peter Ozsváth, Catharina Stroppel, and Dylan Thurston for a great number of interesting conversations, and to the MSRI semester-long program on Homology Theories of Knots and Links for making these conversations possible. We would also like to thank Joshua Sussan for bringing to our attention that some of the algebraic results of Section 3 (in particular, Lemma 3.12) were independently obtained by Angela Klamt and Catharina Stroppel in [22] and [23].

2. Algebraic preliminaries

In this section, we establish some basic facts about filtered A_∞ algebras and modules. We assume throughout that we are working over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. In addition, many of the spaces we discuss will be graded either by $\mathbb{Z}$, in which case we say it is *graded*, or by $\mathbb{Z}^2$, in which case we say it is *bigraded*.

Notation 2.1. If V is a bigraded vector space, i.e.

$$V = \bigoplus_{i,j \in \mathbb{Z}} V_{(i,j)},$$

and $k_1, k_2 \in \mathbb{Z}$, then $V[k_1]\{k_2\}$ will denote the vector space whose bigrading has been shifted by (k_1, k_2). Explicitly,

$$(V[k_1]\{k_2\})_{(i,j)} \cong V_{(i-k_1, j-k_2)}.$$

First, we recall (see [17, 18] for more details):

Definition 2.2. An A_∞ algebra, $\mathbf{A}$, over a field $\mathbb{F}$ is a graded $\mathbb{F}$–vector space endowed with grading-preserving linear maps

$$m_n : \mathbf{A}^{\otimes n} \to \mathbf{A}[2-n],$$

defined for $n \geq 1 \in \mathbb{Z}$, satisfying:

$$\sum_{i+j+\ell=n} m_{i+1+\ell} \circ (\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes \ell}) = 0.$$

If $\mathbf{A}$ is ungraded but otherwise satisfies all of the conditions above, we call $\mathbf{A}$ an *ungraded A_∞ algebra.*

A graded (resp., ungraded) A_∞ algebra satisfying $m_n = 0$ for all $n > 2$ is a *differential graded algebra (dga)* (resp., a *differential algebra*) with differential $\partial := m_1$ and multiplication m_2.

Definition 2.3. Let $\mathbf{A}, \mathbf{B}$ be two A_∞ algebras. Then an A_∞ *morphism* $f : \mathbf{A} \to \mathbf{B}$ is a family $f_n : \mathbf{A}^{\otimes n} \to \mathbf{B}[1-n]$ of $\mathbb{F}$–multilinear maps for $n \geq 1 \in \mathbb{Z}$, homogeneous of degree 0, respecting the A_∞ relations in the following sense:

$$\sum_{i+j+\ell=n} f_{i+1+\ell} \circ \left(\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes \ell}\right) = \sum_{i_1+\ldots+i_s=n} m_s \circ (f_{i_1} \otimes \ldots \otimes f_{i_s}).$$

If $f_n = 0$ for all $n \geq 2$, then we say that $f = f_1 : \mathbf{A} \to \mathbf{B}$ is a *strict morphism* of A_∞ algebras. In particular, a strict morphism $f : \mathbf{A} \to \mathbf{B}$ of differential (graded) algebras is a chain map intertwining the multiplication, m_2.

Definition 2.4. An A_∞ morphism f is said to be a *quasi-isomorphism* if f_1 induces an isomorphism on homology.

The homology of an A_∞ algebra is itself an A_∞ algebra. The following proposition explains how to understand this A_∞ structure.

Proposition 2.5. *(*[16]*, cf.* [18, Thm. 2.3]*) Let $\mathbf{A}$ be an A_∞ algebra with multiplication maps*

$$m_n^A : \mathbf{A}^{\otimes n} \to \mathbf{A}[2-n].$$

Then $H_*(\mathbf{A})$ admits an A_∞ algebra structure such that

(1) *$m_1 = 0$ and m_2 is induced from m_2^A,*
(2) *there is an A_∞ quasi-isomorphism $\mathbf{A} \to H_*(\mathbf{A})$ inducing the identity in homology.*

Moreover, this structure is unique up to (non unique) A_∞ isomorphism, and can be described explicitly as follows.

Choose chain maps $p : \mathbf{A} \to H_*(\mathbf{A})$, $\iota : H_*(\mathbf{A}) \to \mathbf{A}$, and a homotopy $h : \mathbf{A} \to \mathbf{A}[-1]$ satisfying

$$p\iota = Id, \qquad \iota p = Id + m_1^A h + h m_1^A, \qquad h^2 = 0. \tag{1}$$

Then the nth A_∞ multiplication

$$m_n : (H_*(\mathbf{A}))^{\otimes n} \to H_*(\mathbf{A})[2-n]$$

is given by

$$m_n := \sum_T m_n^T$$

where the sum ranges over all planar rooted trees T with n leaves and m_n^T is defined by applying the T–shaped diagram with

(1) *leaves labeled with ι,*
(2) *interior edges labeled with h,*
(3) *vertices labeled with the multiplication maps m_i in the algebra $\mathbf{A}$, and*
(4) *root labeled with p*

to an element of $(H_*(\mathbf{A}))^{\otimes n}$.

See Figure 1 for an enumeration of all such rooted trees T specifying the multiplication m_n when $n = 4$.

Definition 2.6. A *minimal model* of an A_∞ algebra $\mathbf{A}$ is an A_∞ algebra $H_*(\mathbf{A})$ endowed with the structure provided by Proposition 2.5. An A_∞ algebra is said to be *formal* if a minimal model can be chosen so that $m_n = 0$ for all $n > 2$.

Henceforth, whenever we refer to *the* minimal model, $H_*(\mathbf{A})$, for $\mathbf{A}$ an A_∞ algebra, we shall always assume it has been endowed with the structure provided by Proposition 2.5 for suitable maps ι, p, h.

Remark 2.7. The "A_∞ Transfer Theorem," [8, Thm. 2.1] gives an explicit recursive construction of (homotopy inverse) A_∞ quasi-isomorphisms $\mathbf{A} \leftrightarrow H_*(\mathbf{A})$. In particular, ι and p admit extensions to A_∞ morphisms ι' and p'; hence, $\iota' : H_*(\mathbf{A}) \to \mathbf{A}$ and $p' : \mathbf{A} \to H_*(\mathbf{A})$ give A_∞ quasi-isomorphisms between $\mathbf{A}$ and its minimal model, $H_*(\mathbf{A})$.

Definition 2.8. A *strict unit* for an A_∞ algebra $\mathbf{A}$ is an element $\mathbb{1}$ in the 0–th graded component of $\mathbf{A}$ satisfying

- $m_2(a \otimes \mathbb{1}) = m_2(\mathbb{1} \otimes a) = a$ for all $a \in \mathbf{A}$, and
- $m_n(a_1 \otimes \ldots \otimes \mathbb{1} \otimes \ldots a_{n-1}) = 0$ for all $n \neq 2$ and all $a_1, \ldots, a_{n-1} \in \mathbf{A}$.

A *homological unit* for $\mathbf{A}$ is a strict unit for $H_*(\mathbf{A})$. An A_∞ algebra $\mathbf{A}$ is called *strictly unital* (resp., *homologically unital*) if it contains a strict (resp., homological) unit.

We also discuss A_∞ modules over A_∞ algebras.

Definition 2.9. An **A**-**B** A_∞ bimodule, $\mathbf{M}$, over homologically unital A_∞ algebras $\mathbf{A}$ and $\mathbf{B}$, is a graded vector space over $\mathbb{F}$ endowed with grading-preserving linear maps

$$m_{(n_1|1|n_2)} : \mathbf{A}^{\otimes n_1} \otimes \mathbf{M} \otimes \mathbf{B}^{\otimes n_2} \to \mathbf{M}[2 - (n_1 + 1 + n_2)],$$

defined for $n_1, n_2 \geq 0 \in \mathbb{Z}$, satisfying:

$$\sum_{\substack{0 \leq i_1 < n_1, \\ 1 \leq j_1 \leq n_1, \\ 1 \leq i_1 + j_1 \leq n_1}} m_{(n_1 - j_1 + 1|1|n_2)} \circ \left(\mathrm{Id}^{\otimes i_1} \otimes m_{j_1} \otimes \mathrm{Id}^{\otimes n_1 - (i_1 + j_1) + 1 + n_2}\right) +$$

$$\sum_{\substack{0 \leq i_1 \leq n_1 \\ 0 \leq i_2 \leq n_2 \\ 1 \leq i_1 + i_2 < n_1 + n_2}} m_{(i_1|1|i_2)} \circ \left(\mathrm{Id}^{\otimes i_1} \otimes m_{(n_1 - i_1|1|n_2 - i_2)} \otimes \mathrm{Id}^{\otimes i_2}\right) +$$

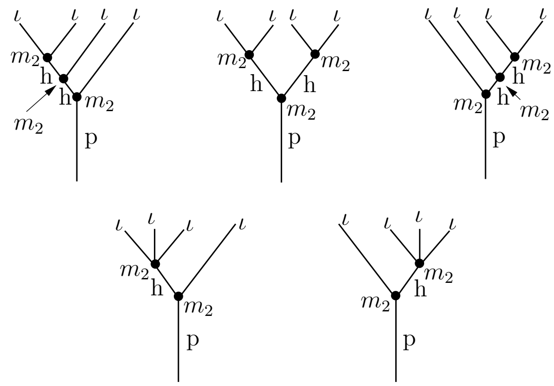


FIGURE 1. The full collection of rooted trees with 4 inputs specifying the multiplication m_4 described by Proposition 2.5.

$$\sum_{\substack{0\leq i_2<n_2\\ 1\leq j_1\leq n_1\\ 1\leq i_1+j_1\leq n_1}} m_{(n_1|1|n_2-j_2+1)}\circ\left(\mathrm{Id}^{\otimes n_1+1+n_2-(i_2+j_2)}\otimes m_{j_2}\otimes\mathrm{Id}^{\otimes i_2}\right)$$

and such that the induced actions

$$H_*(\mathbf{A})\otimes H_*(\mathbf{M})\to H_*(\mathbf{M}),\qquad H_*(\mathbf{M})\otimes H_*(\mathbf{B})\to H_*(\mathbf{M})$$

are unital.

By an A_∞ *bimodule over* $\mathbf{A}$ we shall always mean an $\mathbf{A}$-$\mathbf{A}$ A_∞ bimodule.

A module $\mathbf{M}$ endowed only with a left A_∞ action:

$$m_{(n_1|1|0)}:\mathbf{A}^{\otimes n_1}\otimes\mathbf{M}\to\mathbf{M}[2-(n_1+1)]$$

will be called a *left* A_∞ *module over* $\mathbf{A}$, and a module $\mathbf{M}$ endowed only with a right A_∞ action:

$$m_{(0|1|n_2)}:\mathbf{M}\otimes\mathbf{B}^{\otimes n_2}\to\mathbf{M}$$

will be called a *right* A_∞ *module over* $\mathbf{B}$.

By an A_∞ *module over* $\mathbf{A}$ we shall always mean an A_∞ left, right, or bi- module over $\mathbf{A}$, as appropriate from the context.

If $\mathbf{M}$ is an ungraded module over ungraded A_∞ algebras $\mathbf{A}$ and/or $\mathbf{B}$ but otherwise satisfies all of the conditions above, we call $\mathbf{M}$ an *ungraded* A_∞ *module.*

A graded (resp., ungraded) A_∞ module that satisfies $m_{(n_1|1|n_2)}=0$ whenever $n_1+1+n_2>2$ is a *differential graded module* (resp., a *differential module*) with differential $\partial:=m_{(0|1|0)}$ and left (resp., right) multiplication $m_{(1|1|0)}$ (resp., $m_{(0|1|1)}$).

Remark 2.10. The definitions of morphism and quasi-isomorphism are analogous for A_∞ modules over A_∞ algebras. In particular, a morphism $f:\mathbf{M}\to\mathbf{N}$ between $\mathbf{A}$-$\mathbf{B}$ A_∞ bimodules $\mathbf{M},\mathbf{N}$ is a collection of maps

$$f_{(n_1|1|n_2)}:\mathbf{A}^{\otimes n_1}\otimes\mathbf{M}\otimes\mathbf{B}^{\otimes n_2}\to\mathbf{N}$$

for all $n_1,n_2\geq 0\in\mathbb{Z}$ satisfying the appropriate analogues of the A_∞ relations for morphisms described in Definition 2.3.

We will often refer to the map $f_{(n_1|1|n_2)}$ associated to the A_∞ morphism f as the "$(n_1|1|n_2)$ *term of* f." In addition, we will use the terminology "$(n_1|1|n_2)$ A_∞ *relation*" to refer to the A_∞ relation corresponding to n_1 left inputs and n_2 right inputs. For example, the $(1|1|0)$ A_∞ relation for a morphism $f : \mathbf{M} \to \mathbf{N}$ is given by:

$$f_{(1|1|0)}(m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_{(0|1|0)}) + f_{(0|1|0)}m_{(1|1|0)} = m_{(1|1|0)}(\mathbb{1} \otimes f_{(0|1|0)}) + m_{(0|1|0)}f_{(1|1|0)}.$$

In addition, the induced A_∞ structure on $H_*(\mathbf{M})$ is defined exactly as described in Proposition 2.5, where the leaves and root of each rooted tree have been labeled with $H_*(\mathbf{M})$ or $H_*(\mathbf{B})$ rather than $H_*(\mathbf{A})$, as appropriate. As before, whenever we write $H_*(\mathbf{M})$, for $\mathbf{M}$ an A_∞ module, we shall always assume it has been endowed with the A_∞ structure provided by Proposition 2.5 (for some admissible choice of maps ι, p, h).

Definition 2.11. Let $\mathbf{A}$ be a homologically unital A_∞–algebra. The *derived category* $\mathrm{D}_\infty(\mathbf{A})$ is the category with objects A_∞–modules (left, right, or bi–, depending on the context) and morphisms A_∞–homotopy classes of A_∞–morphisms.

Remark 2.12. Since every A_∞ quasi-isomorphism has a homotopy inverse (see [9, Lemma 10.12.2.2]), passing to the derived category has the effect of making A_∞ quasi-isomorphisms invertible.

Definition 2.13. Let $\mathbf{A}$ be an A_∞ algebra, $\mathbf{M}$ a right A_∞ module over $\mathbf{A}$ and $\mathbf{N}$ a left A_∞ module over $\mathbf{A}$. Then their A_∞ *tensor product* is the complex

$$\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N} := \bigoplus_{i=0}^{n} \mathbf{M} \otimes \mathbf{A}^{\otimes n}[n] \otimes \mathbf{N}$$

with differential given by

$$\begin{aligned}\partial(\mathbf{x} \otimes a_1 \otimes \ldots \otimes a_n \otimes \mathbf{y}) := &\\ &\sum_{i=0}^{n} m_{(0|1|i)}(\mathbf{x} \otimes a_1 \otimes \ldots \otimes a_i) \otimes \ldots \otimes a_n \otimes \mathbf{y} \\ &+ \sum_{i=1}^{n} \sum_{\ell=1}^{n-i+1} \mathbf{x} \otimes a_1 \otimes \ldots \otimes m_i(a_\ell \otimes \ldots \otimes a_{\ell+i-1}) \otimes \ldots \otimes a_n \otimes \mathbf{y} \\ &+ \sum_{i=0}^{n} \mathbf{x} \otimes a_1 \otimes \ldots \otimes m_{(i|1|0)}(a_{n-i+1} \otimes \ldots \otimes a_n \otimes \mathbf{y}).\end{aligned}$$

Definition 2.14. Two A_∞–algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *derived equivalent* if there exists a $\mathbf{B}$-$\mathbf{A}$ bimodule $\mathbf{X}$ and an $\mathbf{A}$-$\mathbf{B}$ bimodule $\mathbf{Y}$ such that

$$\mathbf{X}\widetilde{\otimes}_{\mathbf{A}}(-)\widetilde{\otimes}_{\mathbf{A}}\mathbf{Y} : \mathrm{D}_\infty(\mathbf{A}) \to \mathrm{D}_\infty(\mathbf{B})$$

is an equivalence of categories.

Definition 2.15. A (graded or ungraded) *filtered A_∞ algebra* $\mathbf{A}$ is a (graded or ungraded) A_∞ algebra equipped with a sequence of subsets, for $i \in \mathbb{Z}$:

$$0 \subseteq \ldots \subseteq \mathcal{F}_i \subseteq \mathcal{F}_{i+1} \subseteq \ldots \subseteq \mathbf{A}$$

that are compatible with the A_∞ structure in the following sense:

$$m_n\left(\mathcal{F}_{i_1} \otimes \ldots \otimes \mathcal{F}_{i_n}\right) \subseteq \mathcal{F}_{i_1+\ldots+i_n}.$$

If $m_n = 0$ for all $n > 2$, $\mathbf{A}$ is a (graded or ungraded) *filtered differential algebra.* (Graded or ungraded) *filtered A_∞ modules* and *filtered differential modules* are defined analogously.

Note that the compatibility of the filtration with the multiplicative structure ensures that if $\mathbf{A}$ is a filtered A_∞ algebra, the associated graded algebra $\bigoplus_i \mathcal{F}_i/\mathcal{F}_{i-1}$ is a well-defined (graded or ungraded) A_∞ algebra, and if $\mathbf{M}$ is a filtered A_∞ module over a filtered A_∞ algebra $\mathbf{A}$, then the associated graded module $\bigoplus_i \mathcal{F}_i/\mathcal{F}_{i-1}$ is a well-defined A_∞ module over the associated graded algebra of $\mathbf{A}$.

Definition 2.16. A filtered A_∞ algebra $\mathbf{A}$ (resp., module $\mathbf{M}$) is said to be *bounded* if there exist $n, N \in \mathbb{Z}$ such that $0 = \mathcal{F}_n$ and $\mathbf{A} = \mathcal{F}_N(\mathbf{A})$ (resp., $\mathbf{M} = \mathcal{F}_N(\mathbf{M})$).

Notation 2.17. If $\mathbf{M}$ is a filtered A_∞ module and $k \in \mathbb{Z}$, $\mathbf{M}\{k\}$ will denote the filtered A_∞ module whose filtration has been shifted by k. Explicitly,

$$\mathcal{F}_n\left(\mathbf{M}\{k\}\right) := \mathcal{F}_{n-k}\left(\mathbf{M}\right).$$

A filtration on an A_∞ algebra (resp., module) induces a spectral sequence in the standard way, and if the filtered complex is bounded this spectral sequence converges in a finite number of steps. Furthermore, each page of the corresponding spectral sequence has the structure of an A_∞ algebra (resp., module), by Proposition 2.5. We will call the homology of the associated graded complex, $\bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i/\mathcal{F}_{i-1}$, the *associated graded homology algebra* (resp., the *associated graded homology module*) and the homology of the total complex (i.e., the E^∞ page of this spectral sequence) the *total homology algebra* (resp., the *total homology module*).

If $\mathbf{M}$ is a filtered left A_∞ $\mathbf{A}$-module, and $\mathbf{N}$ is a filtered right A_∞ $\mathbf{B}$-bimodule, then $\mathbf{M} \otimes \mathbf{N}$ inherits a filtration (and, hence, the structure of a filtered A_∞ $\mathbf{A}$-$\mathbf{B}$ bimodule in the sense of Definition 2.15) via: $a \otimes b \in \mathcal{F}_{m+n}\left(\mathbf{M}\otimes\mathbf{N}\right)$ if $a \in \mathcal{F}_m\left(\mathbf{M}\right)$ and $b \in \mathcal{F}_n\left(\mathbf{N}\right)$.

Similarly, the A_∞ tensor product of filtered A_∞ bimodules naturally inherits the structure of a filtered A_∞ bimodule:

Lemma 2.18. *Let $\mathbf{M}, \mathbf{N}$ be two filtered A_∞ bimodules over a filtered A_∞ algebra $\mathbf{A}$. Then the A_∞ tensor product, with underlying vector space:*

$$\mathbf{M}\widetilde{\otimes}\mathbf{N} := \bigoplus_{n=0}^{\infty} \mathbf{M} \otimes \mathbf{A}^{\otimes n} \otimes \mathbf{N}$$

inherits the structure of a filtered A_∞ bimodule as follows:

$$\mathcal{F}_\ell(\mathbf{M}\widetilde{\otimes}\mathbf{N}) := \bigoplus_{n=0}^{\infty}\left[\bigoplus_{i+j_1+\ldots+j_n+k=\ell} \mathcal{F}_i(\mathbf{M}) \otimes \mathcal{F}_{j_1}(\mathbf{A}) \otimes \ldots \otimes \mathcal{F}_{j_n}(\mathbf{A}) \otimes \mathcal{F}_k(\mathbf{N})\right]$$

Proof. Since $\mathbf{M}, \mathbf{N}$ are filtered A_∞ bimodules, the multiplications

$$\begin{aligned} m_{(0|1|i)} : \mathbf{M} \otimes \mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_i &\to \mathbf{M} \\ m_{(i|1|0)} : \mathbf{A}_{n-i+1} \otimes \ldots \otimes \mathbf{A}_n \otimes \mathbf{N} &\to \mathbf{N} \\ m_i : \mathbf{A}_\ell \otimes \ldots \otimes \mathbf{A}_{\ell+i-1} &\to \mathbf{A} \end{aligned}$$

contributing to the differential on the complex all respect the filtration in the sense of Definition 2.15. The same is true of the higher multiplications on the complex, for the same reason. □

Definition 2.19. An A_∞ morphism $f : \mathbf{M} \to \mathbf{N}$ between two filtered A_∞ modules is said to be *filtered* if

$$f_{(n_1|1|n_2)}\left(\mathcal{F}_{i_1} \otimes \ldots \otimes \mathcal{F}_{i_{n_1+n_2+1}}\right) \subseteq \mathcal{F}_{i_1+\ldots+i_{n_1+n_2+1}}.$$

Definition 2.20. Let $\mathbf{A}$ be a filtered A_∞ algebra, and $f : \mathbf{M} \to \mathbf{N}$ a filtered A_∞ morphism between filtered $\mathbf{A}$–modules $\mathbf{M}$ and $\mathbf{N}$. Let $m^M_{(n_1|1|n_2)}$ (resp., $m^N_{(n_1|1|n_2)}$) denote the A_∞ multiplication maps for $\mathbf{M}$ (resp., for $\mathbf{N}$).

Then the *mapping cone of* f, denoted $MC(f)$, is the filtered A_∞ $\mathbf{A}$–module with underlying $\mathbb{F}$–vector space $\mathbf{M} \oplus (\mathbf{N}[1])$, A_∞ multiplication maps:

$$m_{(n_1|1|n_2)} := \begin{pmatrix} m^M_{(n_1|1|n_2)} & 0 \\ f_{(n_1|1|n_2)} & m^N_{(n_1|1|n_2)} \end{pmatrix}.$$

and filtration given by:

$$\mathcal{F}_n(MC(f)) := \{(a,b) \in MC(f) \mid a \in \mathcal{F}_n(\mathbf{M}) \text{ and } b \in \mathcal{F}_n(\mathbf{N})\}.$$

The following lemma will be useful in the proof of Theorem 6.1.

Lemma 2.21. *Let $\mathbf{M}\widetilde{\otimes}\mathbf{N}$ be the filtered A_∞ bimodule (over the filtered algebra $\mathbf{A}$) obtained as the A_∞ tensor product of the two filtered A_∞ bimodules $\mathbf{M}$ and $\mathbf{N}$ as in Lemma 2.18. Let $gr(-)$ denote the associated graded A_∞ module of $-$.*

Then $gr(\mathbf{M})\widetilde{\otimes}_{gr(\mathbf{A})} gr(\mathbf{N}) = gr(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N})$ as A_∞ bimodules over $gr(\mathbf{A})$.

Proof. We construct chain maps

$$\mathrm{gr}(\mathbf{M})\widetilde{\otimes}_{\mathrm{gr}(\mathbf{A})}\mathrm{gr}(\mathbf{N}) \xrightarrow{\Phi} \mathrm{gr}(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N})$$

$$\mathrm{gr}(\mathbf{M})\widetilde{\otimes}_{\mathrm{gr}(\mathbf{A})}\mathrm{gr}(\mathbf{N}) \xleftarrow[\Psi]{} \mathrm{gr}(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N})$$

and show that Φ and Ψ are mutually inverse.

Suppose $x \otimes a_1 \otimes \ldots \otimes a_n \otimes y \in \mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N}$ represents an element

$$\begin{aligned} [x] \otimes [a_1] \otimes \ldots \otimes [a_n] \otimes [y] \quad &\in \quad \frac{\mathcal{F}_i}{\mathcal{F}_{i-1}}(\mathbf{M}) \otimes \frac{\mathcal{F}_{j_1}}{\mathcal{F}_{j_1-1}}(\mathbf{A}) \otimes \ldots \otimes \frac{\mathcal{F}_{j_n}}{\mathcal{F}_{j_n-1}}(\mathbf{A}) \otimes \frac{\mathcal{F}_k}{\mathcal{F}_{k-1}}(\mathbf{N}) \\ &\subseteq \quad \mathrm{gr}(\mathbf{M})\widetilde{\otimes}_{\mathrm{gr}(\mathbf{A})}\mathrm{gr}(\mathbf{N}). \end{aligned}$$

Letting $I := i + j_1 \ldots + j_n + k$, then we define

$$\begin{aligned} \Phi([x] \otimes [a_1] \otimes \ldots \otimes [a_n] \otimes [y]) \quad &:= \quad [x \otimes a_1 \otimes \ldots \otimes a_n \otimes y] \\ &\in \quad \frac{\mathcal{F}_I}{\mathcal{F}_{I-1}}(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N}). \end{aligned}$$

This map is well-defined, since any other representative, $x' \otimes a'_1 \otimes \ldots \otimes a'_n \otimes y' \in \mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N}$, of $[x] \otimes [a_1] \otimes \ldots \otimes [a_n] \otimes [y]$ will differ from $x \otimes a_1 \otimes \ldots \otimes a_n \otimes y$ by an element in $\mathcal{F}_{I-1}$, by the definition of the filtration on $\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N}$.

Similarly, we send an equivalence class $[x \otimes a_1 \otimes \ldots \otimes a_n \otimes y] \in \mathrm{gr}(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N})$ to the uniquely-specified equivalence class

$$\begin{aligned} \Psi([x \otimes a_1 \otimes \ldots \otimes a_n \otimes y]) \quad &:= \quad [x] \otimes [a_1] \otimes \ldots \otimes [a_n] \otimes [y] \\ &\in \quad \mathrm{gr}(\mathbf{M})\widetilde{\otimes}_{\mathrm{gr}(\mathbf{A})}\mathrm{gr}(\mathbf{N}). \end{aligned}$$

Furthermore, the differentials on $\mathrm{gr}(\mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N})$ and $\mathrm{gr}(\mathbf{M})\widetilde{\otimes}_{\mathrm{gr}(\mathbf{A})}\mathrm{gr}(\mathbf{N})$ agree, by the same argument above applied to the image of the differential of a representative $x \otimes a_1 \otimes \ldots \otimes a_n \otimes y \in \mathbf{M}\widetilde{\otimes}_{\mathbf{A}}\mathbf{N}$. □

2.1. **Formality and derived equivalence.** The following results will be useful throughout the paper.

Lemma 2.22. *Let* $\mathbf{A}$ *be a formal dg algebra and* $H_*(\mathbf{A})$ *its homology algebra. Then* $D_\infty(\mathbf{A})$ *and* $D_\infty(H_*(\mathbf{A}))$ *are equivalent triangulated categories.*

Proof. Since $\mathbf{A}$ is formal, there is an A_∞ quasi-isomorphism $\phi\colon \mathbf{A} \to H_*(\mathbf{A})$, and by Proposition 2.4.10 of [28], this A_∞ quasi-isomorphism induces two mutually quasi-inverse functors $\mathrm{Induct}_\phi\colon D_\infty(\mathbf{A}) \to D_\infty(H_*(\mathbf{A}))$ and $\mathrm{Rest}_\phi\colon D_\infty(H_*(\mathbf{A})) \to D_\infty(\mathbf{A})$. (Note that although Proposition 2.4.10 of [28] is formulated for categories of A_∞ right modules, similar statements also hold for categories of A_∞ left modules and A_∞ bimodules; see [28] for details). □

Remark 2.23. Lemma 2.22 can also be obtained as a consequence of the following facts:

- If two dg algebras are related by an A_∞ quasi-isomorphim, then there is also a zig-zag of honest quasi-isomorphisms connecting the two dg algebras (this follows from [26, Corollaire 1.3.1.3c]). In particular, a dg algebra is formal (in the sense of Definition 2.6) if and only if it is connected to its homology algebra by a zig-zag of honest quasi-isomorphisms.
- An honest quasi-isomorphism between two dg algebras induces an equivalence between the ordinary derived categories of the two dg algebras. Explicitly, this equivalence is given by scalar restriction and derived scalar extension along the given quasi-isomorphism (see [5, 3.6.2]).
- The ordinary derived category of a dg algebra is equivalent to the A_∞ derived category of the given dg algebra (see [28, Proposition 2.4.1]).

The following lemmas provide sufficient (but not necessary) conditions for formality of an A_∞ module.

Lemma 2.24. *Let* $\mathbf{A}$ *be a differential (graded) algebra (resp., let* $\mathbf{M}$ *be a differential (graded) module over* $\mathbf{A}$*), and let* ι, p, h *be maps satisfying the conditions in Proposition 2.5. If, in addition,*

(1) $h\iota = 0$, *and*
(2) $m_2^A(\iota\otimes\iota)(\mathbf{A}^{\otimes 2}) \subseteq \iota(\mathbf{A})$ *(resp.,* $m^M_{(n_1|1|n_2)}(\iota\otimes\iota)(\mathbf{A}^{\otimes n_1}\otimes\mathbf{M}\otimes\mathbf{A}^{\otimes n_2}) \subseteq \iota(\mathbf{M})$ *whenever* $n_1 + 1 + n_2 = 2$*),*

then $\mathbf{A}$ *is formal (resp.,* $\mathbf{M}$ *is formal).*

Furthermore, $\iota : \mathbf{A} \to H_*(\mathbf{A})$ *(resp.,* $\iota : \mathbf{M} \to H_*(\mathbf{M})$*) is a strict* A_∞ *quasi-isomorphism.*

Proof. In the interest of brevity, we give the argument for the case of $\mathbf{A}$ a differential (graded) algebra, leaving the completely analogous proof in the case of $\mathbf{M}$ a differential (graded) module to the reader.

Each tree T contributing to the definition of

$$m_n : (H_*(\mathbf{A}))^{\otimes n} \to H_*(\mathbf{A})$$

for $n > 2$ yields the 0 map, since each such tree T involves a product of terms in $\mathbf{A}$, at least one of which is either:

- of the form $h \circ m_2^A \circ (\iota \otimes \iota)$ (if T is trivalent) or
- of the form $m_n^A(\iota \otimes \ldots \otimes \iota)$, for $n > 2$ (if T is not trivalent).

In both cases, such a term is 0 in $\mathbf{A}$ by assumption, hence the corresponding map is 0, implying formality of $\mathbf{A}$.

To see that $\iota : \mathbf{A} \to H_*(\mathbf{A})$ is a strict quasi-isomorphism, we note that by definition ι is a chain map inducing an isomorphism on homology. We therefore need only show that there are no higher terms in the A_∞ morphism generated by ι.

The "A_∞ Transfer Theorem" ([8, Thm. 2.1]) tells us that ι_n can be defined recursively as

$$\iota_n := \sum_{\substack{i_1+\ldots+i_r=n \\ r>1}} hm_r^A\left(\iota_{i_1}\otimes\ldots\otimes\iota_{i_r}\right).$$

Assumptions (1) and (2), combined with the assumption that $m_r^A = 0$ for $r > 2$, now allow us to conclude inductively that $\iota_n = 0$ for $n \geq 2$, as desired.

□

Lemma 2.25. *Let* $\mathbf{M}$ *be a differential (graded) module over an algebra* $\mathbf{A}$*, and let*

$$\iota_M : H_*(\mathbf{M}) \to \mathbf{M}, \quad p_M : \mathbf{M} \to H_*(\mathbf{M}), \quad h_M : \mathbf{M} \to \mathbf{M}$$

satisfy the conditions in Proposition 2.5. Suppose in addition that

1. $p_M h_M = 0$, *and*
2. *Im*(h_M) *and Im*$(m^M_{(0|1|0)})$ *are both submodules of* $\mathbf{M}$ *over* $\mathbf{A}$ *(i.e., left or/and right multiplication by an element of* $\mathbf{A}$ *preserves Im*(h_M) *and Im*$(m^M_{(0|1|0)})$*).*

Then $\mathbf{M}$ *is formal, and the projection map* $p_M : \mathbf{M} \to H_*(\mathbf{M})$ *is a strict quasi-isomorphism.*

Proof. We give the proof in the case that $\mathbf{M}$ is a differential (graded) bimodule over $\mathbf{A}$. If Assumption (2) holds only under left (resp., right) multiplication, then p_M will be a strict quasi-isomorphism of left (resp., right) $\mathbf{A}$–modules.

Since $\mathbf{A}$ is an algebra, $m_n^A = 0$ unless $n = 2$, and $\mathbf{A}$ is trivially A_∞ isomorphic to its homology. Choosing $\iota_A : H_*(\mathbf{A}) \to \mathbf{A}$ and $p_A : \mathbf{A} \to H_*(\mathbf{A})$ to be the identity morphism, and $h_A : \mathbf{A} \to \mathbf{A}$ to be the zero morphism, we now claim that any tree T contributing to the definition of

$$m_{(n_1|1|n_2)} : \mathbf{A}^{\otimes n_1} \otimes (H_*(\mathbf{M})) \otimes \mathbf{A}^{\otimes n_2} \to H_*(\mathbf{M})$$

is zero if $n_1 + n_2 + 1 > 2$. This follows because:

- If T is trivalent then it corresponds to a summand of the form $p_M \circ h_M(m)$, since Im(h_M) is an $\mathbf{A}$–bimodule. Such a term is zero by Assumption (1) above.
- If T is not trivalent then it involves a product with at least one term of the form:

$$m^M_{(n_1'|1|n_2')}(\iota\otimes\ldots\otimes\iota) \text{ (resp., } m_n^A(\iota\otimes\ldots\otimes\iota))$$

  for $n_1' + n_2' + 1 > 2$ (resp., $n > 2$), which is zero since $\mathbf{M}$ is a dg module (resp., since $\mathbf{A}$ is an algebra).

Therefore, $m_{(n_1|1|n_2)} = 0$ for all $n_1 + n_2 + 1 > 2$, and $\mathbf{M}$ is formal.

To see that p_M is a strict quasi-isomorphism, we again appeal to the Transfer Theorem, [8, Thm. 2.1], which tells us that $(p_M)_{(n_1|1|n_2)}$ is defined recursively as:

$$\begin{aligned}
(p_M)_{(n_1|1|n_2)} \quad := \quad & \sum_{\substack{t+u_1=n_1 \\ q+u_2=n_2}} p_{(t|1|q)}\left(1^{\otimes t}\otimes m_{(u_1|1|u_2)}\otimes 1^{\otimes q}\right)h^{[n_1|1|n_2]} \\
+ \quad & \sum_{\substack{t_1+2+t_2=n_1+1+n_2 \\ n_1\geq t_1+2}} p_{(n_1-1|1|n_2)}\left(1^{\otimes t_1}\otimes m_2\otimes 1^{\otimes t_2}\right)h^{[n_1|1|n_2]} \\
+ \quad & \sum_{\substack{t_1+2+t_2=n_1+1+n_2 \\ n_2\geq t_2+2}} p_{(n_1|1|n_2-1)}\left(1^{\otimes t_1}\otimes m_2\otimes 1^{\otimes t_2}\right)h^{[n_1|1|n_2]},
\end{aligned}$$

where

$$h^{[n_1|1|n_2]} : \mathbf{A}^{\otimes n_1} \otimes \mathbf{M} \otimes \mathbf{A}^{\otimes n_2} \to \mathbf{A}^{\otimes n_1} \otimes \mathbf{M} \otimes \mathbf{A}^{\otimes n_2}$$

can also be defined recursively as:

$$h^{[n_1|1|n_2]} := \sum_{i+1+j=n_1+1+n_2} 1^{\otimes i} \otimes h \otimes (\iota p)^{\otimes j}$$

Noting that $h^{[n_1|1|n_2]} = 1^{\otimes n_1} \otimes h \otimes 1^{\otimes n_2}$ since we are using the identity A_∞ isomorphisms $\mathbf{A} \leftrightarrow H_*(\mathbf{A})$, we see, using Assumptions (1) and (2), that both

$$\begin{aligned}(p_M)_{(1|1|0)} &:= p_{(0|1|0)} \circ m_{(1|1|0)} \circ (1 \otimes h) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(p_M)_{(0|1|1)} &:= p_{(0|1|0)} \circ m_{(0|1|1)} \circ (h \otimes 1) \\ &= 0.\end{aligned}$$

Combined with the fact that $m_n^A = 0$ for all $n \neq 2$ and $m^M_{(n_1|1|n_2)} = 0$ for all $n_1+1+n_2 > 2$, $(p_M)_{(n_1|1|n_2)}$ is then identically 0 by induction for all $(n_1 + 1 + n_2) \geq 2$, as desired. □

3. Khovanov-Seidel Hom algebras and bimodules

In this section, we construct dg bimodules following Khovanov-Seidel in [21]. We begin by describing the topological data needed for the construction of both the Khovanov-Seidel bimodules and their bordered Floer analogues (described in Section 4).

3.1. **Topological data: Bases of curves.**

Let D_m denote the unit disk in the complex plane, equipped with a set,

$$\Delta := \left\{ -1 + \frac{2(j+1)}{m+2} \in D_m \subset \mathbb{C} \,\middle|\, j = 0, \ldots, m \right\},$$

of $m+1$ points equally distributed along the intersection of the real axis with D_m. Label by $\mathbf{j}$ the point at position $-1 + \frac{2(j+1)}{m+2}$.

By a *curve in* D_m we shall always mean the image of a smooth imbedding $\gamma : [0,1] \to D_m$ which is transverse to ∂D_m and satisfies $\gamma^{-1}(\partial D_m \cup \Delta) = \{0,1\}$.

Definition 3.1. A *∂–admissible curve in* D_m is a curve in D_m for which $\gamma(0) = -1$ and $\gamma(1) \in \Delta$.

A ∂–admissible curve is a particular type of admissible curve in the sense of [21, Sec. 3b]. Two ∂–admissible curves c_1 and c_2 are said to be isotopic if there is a homotopy between c_1 and c_2 through ∂–admissible curves.

Notation 3.2. Associated to any curve, $c \subset D_m$, is a canonical section of the interior of c to the real projectivization of the tangent bundle of $D_m \setminus \Delta$. By choosing a lift of this section to a particular $\mathbb{Z}^2$ cover as described in [21, Sec. 3d], one assigns a bigrading to c. We shall denote by $\widetilde{c}$ the data of a curve $c \subset D_m$ equipped with such a choice of bigrading.

Definition 3.3. [21, Sec. 3a] Two curves $c_0, c_1 \subset D_m$ are said to have *minimal geometric intersection* if they satisfy the following conditions:

- c_0 and c_1 intersect transversely,
- $c_0 \cap c_1 \cap \partial D_m = \emptyset$, and

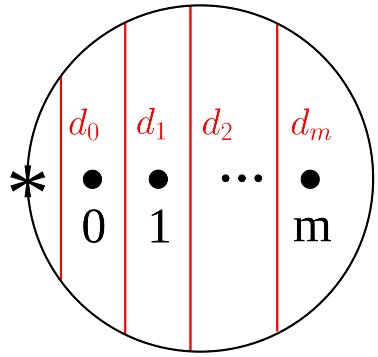


FIGURE 2. The curves d_j, for $j = 0, \ldots, m$, are the intersections of the lines $Re(z) = \left(-1 - \frac{1}{m+2}\right) + \frac{2(j+1)}{m+2}$ with the unit disk in $\mathbb{C}$. By convention, the distinguished point, labeled by a $*$, at $-1 \in \partial D_m$, is the left endpoint for all ∂–admissible curves in D_m.

- If $z_- \neq z_+$ are two points in $c_0 \cap c_1$ not both in Δ, $\alpha_0 \subset c_0$ and $\alpha_1 \subset c_1$ are two arcs with endpoints z_-, z_+ such that $\alpha_0 \cap \alpha_1 = \{z_-, z_+\}$, and K is the connected component of $D_m - (c_0 \cup c_1)$ bounded by $\alpha_0 \cup \alpha_1$, then if K is topologically an open disk, it must contain at least one point of Δ. Informally, we say there are no "trivial bigons" among the connected components of $D_m - (c_0 \cup c_1)$.

Definition 3.4. [21, Sec. 3e] Let $d_0, \ldots, d_m \subset D_m$ be the curves pictured in Figure 2. A ∂–admissible curve in D_m is said to be in *normal form* if it has minimal geometric intersection with d_j for each $j = 0, \ldots, m$.

Definition 3.5. A *basis of ∂–admissible curves in D_m* is a set, $\mathcal{B} = \{c_0, \ldots, c_m\}$, of ∂–admissible curves satisfying the conditions:

- If $\gamma_j : [0,1] \to D_m$ is the imbedding whose image is c_j, then $\gamma(1) = \mathbf{j} \in \Delta$ (the right endpoint of c_j is $\mathbf{j}$), and
- $c_i \cap c_j = \{-1\}$ if $i \neq j$ (distinct curves c_i and c_j intersect only at their left endpoints).

If we, furthermore, specify a lift of each curve, $c_j \in \mathcal{B}$, to a bigraded curve, $\widetilde{c}_j$, we say that we have a *basis, $\widetilde{\mathcal{B}} = \{\widetilde{c}_0, \ldots, \widetilde{c}_m\}$, of ∂–admissible bigraded curves in D_m.*

Unless otherwise specified, from this point forward whenever we write that $\widetilde{\mathcal{B}}$ is a *basis*, we shall always mean that $\widetilde{\mathcal{B}}$ is a *basis of ∂–admissible bigraded curves in normal form in* D_m. Two bases $\mathcal{B} = \{\widetilde{c}_0, \ldots \widetilde{c}_m\}$ and $\mathcal{B}' = \{\widetilde{c}'_0, \ldots, \widetilde{c}'_m\}$ are said to be *equivalent* if there exists an isotopy $\widetilde{c}_i \to \widetilde{c}'_i$ for each $i = 0, \ldots, m$ through ∂–admissible bigraded curves in normal form.

As in [21], we let $\mathcal{G} = \mathrm{Diff}(D_m, \partial D_m; \Delta)$ denote the group of diffeomorphisms f of D_m satisfying $f|_{\partial D_m} = \mathrm{Id}$ and $f(\Delta) = \Delta$ and note that there is a canonical identification of $\pi_0(\mathcal{G})$ with B_{m+1}, the Artin braid group on $m+1$ generators. Under this correspondence, (isotopy classes of) ∂–admissible curves are sent to (isotopy classes of) ∂–admissible curves. Moreover, an (equivalence class of) basis $\widetilde{\mathcal{B}}$ is sent to an (equivalence class of) basis $\sigma(\widetilde{\mathcal{B}})$, after suitably reordering the curves in $\sigma(\widetilde{\mathcal{B}})$.

3.2. **The ring A_m and a braid group action on $D^b(A_m)$.** In [21], Khovanov-Seidel associate to a braid, $\sigma \in B_{m+1}$, a bimodule over a quiver algebra, A_m (defined below). In

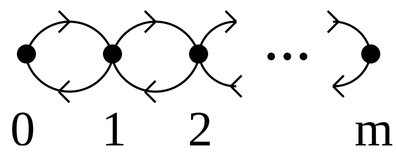


FIGURE 3.

this subsection, we explain how their construction yields a family of algebras and bimodules, one for each choice of basis. Our end goal is the construction of a particular algebra, B^{Kh}, and a bimodule, $\mathcal{M}_\sigma^{Kh}$ over B^{Kh}, from the data of a particular such basis, $\widetilde{\mathcal{Q}}$.

We begin by reviewing the original construction of Khovanov-Seidel in [21]. Let Γ_m be the oriented graph (quiver) whose vertices are labeled $\mathbf{0}, \ldots, \mathbf{m}$ and whose edges are shown in Figure 3. Recall that, given any oriented graph Γ, one defines its path ring as the vector space over $\mathbb{F}$ freely generated by the set of all finite-length paths in Γ, where multiplication is given by concatenation, and the product of two non-composable paths is set to 0. The ring A_m is then defined as a quotient of the path ring of Γ_m by the collection of relations

$$(i-1|i|i+1) = (i+1|i|i-1) = 0, \quad (i|i+1|i) = (i|i-1|i), \quad (0|1|0) = 0$$

for each $0 \leq i \leq m$. In the above, following [21], we have labeled each path in Γ_m by the complete ordered tuple of vertices it traverses. So, for instance, $(i-1|i|i+1)$ denotes the path that starts at vertex $i-1$, moves right to i, then right again to $i+1$. The path ring of Γ_m is further endowed with a grading by setting $\deg(i) = \deg(i|i+1) = 0$ and $\deg(i|i-1) = 1$ for all i. This grading descends to the quotient, A_m, since the relations defining A_m are homogeneous with respect to the grading.[2]

Note that the collection $\{(i)|i \in 0, \ldots, m\}$ of constant paths are mutually orthogonal idempotents, and $\sum_{i=0}^m (i)$ is the identity in A_m. There are corresponding decompositions of A_m as a direct sum of projective left modules $A_m = \bigoplus_{i=0}^m A_m(i)$ (resp., projective right-modules $A_m = \bigoplus_{i=0}^m (i)A_m)$. As in [21], we denote $A_m(i)$ (resp., $(i)A_m$) by P_i (resp., ${}_iP$). Note that P_i (resp., ${}_iP$) is the set of all paths ending at i (resp., beginning at i).

To streamline notation, we henceforth assume that we have fixed $m \geq 0 \in \mathbb{Z}$, and let A denote the algebra A_m.

Khovanov-Seidel go on to associate to each braid $\sigma \in B_{m+1}$ an element of $D^b(A)$, the bounded derived category of A–bimodules, by associating to each elementary Artin braid generator $\sigma_i^{\pm 1}$ (pictured in Figure 4) a dg bimodule $\mathcal{M}_{\sigma_i^\pm}$ and to each braid, $\sigma := \sigma_{i_1}{}^{\pm} \cdots \sigma_{i_k}{}^{\pm}$, decomposed as a product of elementary braid words, the dg bimodule

$$\mathcal{M}_\sigma = \mathcal{M}_{\sigma_{i_1}{}^\pm} \otimes_A \ldots \otimes_A \mathcal{M}_{\sigma_{i_k}^\pm}.$$

They then verify that any two decompositions of σ as a product of elementary Artin braid generators give rise to quasi-isomorphic complexes, and hence $\mathcal{M}_\sigma$ gives rise to a well-defined element in $D^b(A)$.

3.3. **The dg algebra B and the algebra B^{Kh}.**

Now, suppose we are given the data of a ∂–admissible bigraded curve in normal form. Khovanov-Seidel show, in [21, Sec. 4], how to use this data to construct a bounded complex of bigraded projective left modules over the

[2] This *internal grading* corresponds to the second of the two gradings discussed in Notation 3.2. Note that this grading is *not* the grading by path length which appears in [10, 39] and corresponds to the j (quantum) grading of [19]. See Remark 3.21.

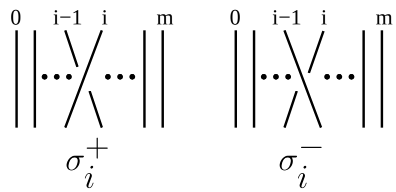


FIGURE 4. The elementary Artin generators, $\sigma_i^{\pm}$

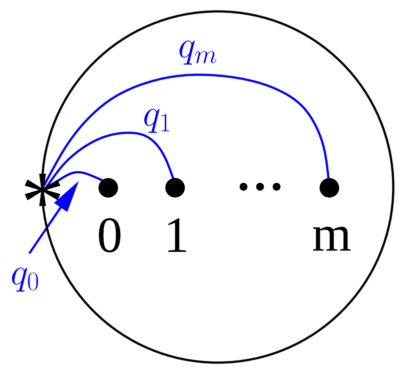


FIGURE 5. The basis $\mathcal{Q} = \{q_0, \ldots, q_m\}$

algebra A. Furthermore, a basis, $\widetilde{\mathcal{B}}$, of such curves yields a dga via Yoneda imbedding (cf. [18, Sec. 2.6]). Recall:

Definition 3.6. Let $(\mathcal{C}_1, \partial_1), (\mathcal{C}_2, \partial_2)$ be two bounded dg left modules over an algebra $\mathbf{A}$. Then the *Hom complex of the pair* $(\mathcal{C}_1, \mathcal{C}_2)$, denoted $\mathrm{Hom}_{\mathbf{A}}(\mathcal{C}_1, \mathcal{C}_2)$, is the bounded complex whose generators are left module morphisms, $F : \mathcal{C}_1 \to \mathcal{C}_2$, and whose differential, D, is given by

$$D(F) := \partial_2 F + F \partial_1.$$

Construction 3.7. *Let* $\widetilde{\mathcal{B}} = \{\widetilde{c}_0, \ldots, \widetilde{c}_m\}$ *be a basis, and let* $L(\widetilde{c}_j)$ *be the bounded complex of projective* A*–modules associated to* $\widetilde{c}_j$*, for each* $j = 0, \ldots, m$*. Then the direct sum,*

$$\bigoplus_{i,j=0}^{m} \mathrm{Hom}_A(L(\widetilde{c}_i), L(\widetilde{c}_j)),$$

is a dga, with multiplication given by composition of A*–bimodule morphisms. We will refer to* $\bigoplus_{i,j=0}^{m} \mathrm{Hom}_A(L(\widetilde{c}_i), L(\widetilde{c}_j))$ *as the* Hom algebra associated to $\widetilde{\mathcal{B}}$*.*

We focus in the present paper on the Hom algebra associated to the basis $\widetilde{\mathcal{Q}} = \{\widetilde{q}_0, \ldots, \widetilde{q}_m\}$ given by (a particular lift of) the collection of curves pictured in Figure 5.[3]

Applying the construction of [21, Sec. 4a], we associate to $\widetilde{q}_j$ the dg bimodule:

$$Q_j := \; 0 \longrightarrow P_0 \xrightarrow{\cdot(0|1)} P_1 \xrightarrow{\cdot(1|2)} \cdots \xrightarrow{\cdot(j-1|j)} P_j \longrightarrow 0 \;,$$

[3] We expect that results similar to those described in Theorems 5.1 and 6.1 hold for other choices of basis, but we do not address that here.

where the differential map " $\cdot(i-1|i)$" denotes "right multiplication by the element $(i-1|i)$." By fixing a lift of the tangent vector to the curve q_0 at a point near $\mathbf{0} \in \Delta$ and declaring this lift to correspond to bigrading $(0,0)$, we obtain a "canonical" bigrading on Q_j satisfying the property that the bigrading of the idempotent $(i) \in P_i$ is $(i,0)$.

Notation 3.8. We shall denote by B the Hom algebra associated to $\widetilde{\mathcal{Q}}$:

$$\bigoplus_{i,j=0}^{m} \mathrm{Hom}_A\left(Q_i, Q_j\right)$$

and by B^{Kh} its homology, $H_*(B)$, considered as an A_∞ algebra via the construction in Proposition 2.5.

We will eventually be interested in $D_\infty(B^{Kh})$–in particular, a braid group action on this category–so we now devote some time to describing the structure of B and B^{Kh}.

Notation 3.9. Let $R_{\mathcal{I}}$ be a bounded complex of elementary projective left A–modules (e.g., one obtained from an admissible curve in normal form in D_m as explained in [21, Sec. 4]):

$$R_{\mathcal{I}} = 0 \to P_{i_0}\{s_0\} \to \ldots \to P_{i_N}\{s_N\} \to 0.$$

Suppose further that $P_{i_0}\{s_0\}$ is in (co)homological grading 0. Then we will use the notation ${}_{\mathcal{I}}R$ to denote the following bounded complex of elementary projective *right* A–modules:

$${}_{\mathcal{I}}R := 0 \leftarrow {}_{i_0}P\{-s_0\}[0] \leftarrow \ldots \leftarrow {}_{i_N}P\{-s_N\}[-N] \leftarrow 0,$$

where, if a map $P_{i_j} \to P_{i_{j+1}}$ in $R_{\mathcal{I}}$ is given by right multiplication by a path $\gamma \in A$, then the corresponding map ${}_{i_j}P \leftarrow {}_{i_{j+1}}P$ in ${}_{\mathcal{I}}R$ is given by *left* multiplication by γ.

Lemma 3.10. *Let $R_{\mathcal{I}}, S_{\mathcal{J}}$ be bounded complexes of elementary projective left A–modules as above. Then* $\mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}}) \cong {}_{\mathcal{I}}R \otimes_A S_{\mathcal{J}}$.

Proof. Each element $\phi \in \mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}})$ can be decomposed as a sum of left A–module maps $\phi_{k,\ell} : P_{i_k}\{s_k\} \to P_{j_\ell}\{s_\ell\}$, each of which is uniquely determined by the image, $\phi_{k,\ell}(i_k)$, of the idempotent, (i_k). We therefore obtain an isomorphism

$$\mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}}) \to {}_{\mathcal{I}}R \otimes_A S_{\mathcal{J}}$$

of $\mathbb{F}$–vector spaces identifying ϕ with the element, $\sum_{k,\ell}((i_k) \otimes \phi_{k,\ell}(i_k))$.

To see that the Hom complex differential $D(\phi) := \phi d_{\mathcal{I}} + d_{\mathcal{J}}\phi$ on the left matches the tensor product differential on the right, we simply note that if $\phi = \sum_{k,\ell} \phi_{k,\ell} \in \mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}})$, then for each pair, (k,ℓ), $\phi_{k,\ell} d_{\mathcal{I}}$ is obtained by pre- (i.e., left-) (resp., $d_{\mathcal{J}}\phi$ is obtained by post- (i.e., right-)) multiplying $\phi_{k,\ell}$ by a path γ_k (resp., γ_ℓ). This is precisely the induced differential on the tensor product complex ${}_{\mathcal{I}}R \otimes_A S_{\mathcal{J}}$.

□

Lemma 3.11. *Let $R_{\mathcal{I}}, S_{\mathcal{J}}$ be two bigraded bounded complexes of projective modules obtained from admissible bigraded curves in normal form as explained in* [21, Sec. 4]*. Then the differential on* $\mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}})$ *has degree* $(1,0)$.

Proof. By definition, the differential on each of $R_{\mathcal{I}}$, $S_{\mathcal{J}}$ has degree $(1,0)$, implying that the differential on ${}_{\mathcal{I}}R$ and, hence, the differential on

$$\mathrm{Hom}_A(R_{\mathcal{I}}, S_{\mathcal{J}}) = {}_{\mathcal{I}}R \otimes_A S_{\mathcal{J}},$$

has degree $(1,0)$ as well. □

The following lemma was also obtained independently by Klamt and Stroppel. Compare [22, Thms. 5.7, 7.3] and [23, Thm. 5.7].

Lemma 3.12. *The dg algebra* $B := \bigoplus_{i,j=0}^{m} \mathrm{Hom}_A(Q_i, Q_j)$ *is formal. Furthermore, the algebra*

$$B^{Kh} := H_*(B)$$

has the following explicit description:

$$B^{Kh} := \bigoplus_{i,j=0}^{m} {}_iB_j^{Kh}, \text{ with}$$

$${}_iB_j^{Kh} := \begin{cases} 0 & \text{if } i < j, \\ Span_{\mathbb{F}}\langle {}_i\mathbb{1}_j \rangle & \text{if } i = j, \text{ and} \\ Span_{\mathbb{F}}\langle {}_i\mathbb{1}_j, {}_i\mathbf{x}_j \rangle & \text{if } i > j, \end{cases}$$

where the bigradings on generators are given by:

$$\begin{aligned} gr\,({}_i\mathbb{1}_j) &= (0,0) \quad \text{for all } i \geq j, \\ gr\,({}_i\mathbf{x}_j) &= (-1,1) \quad \text{for all } i > j. \end{aligned}$$

and the multiplication is given by:

$$\begin{aligned} m_2({}_i\mathbb{1}_j \otimes {}_j\mathbb{1}_k) &:= {}_i\mathbb{1}_k \\ m_2({}_i\mathbb{1}_j \otimes {}_j\mathbf{x}_k) &:= {}_i\mathbf{x}_k \\ m_2({}_i\mathbf{x}_j \otimes {}_j\mathbb{1}_k) &:= {}_i\mathbf{x}_k \\ m_2({}_i\mathbf{x}_j \otimes {}_j\mathbf{x}_k) &= 0 \end{aligned}$$

(As usual, $m_2 : {}_iB_j^{Kh} \otimes {}_kB_\ell^{Kh} \to {}_iB_\ell^{Kh}$ *is identically* 0 *when* $j \neq k$*.)*

Proof. We know from [21, Prop. 4.9] that as an $\mathbb{F}$-vector space,

$${}_iB_j^{Kh} \text{ is free of rank } \begin{cases} 0 & \text{when } i < j, \\ 1 & \text{when } i = j, \text{ and} \\ 2 & \text{when } i > j. \end{cases}$$

Furthermore, we claim that the generators of ${}_iB_j^{Kh}$ are represented by the morphisms

$$\begin{cases} (0) + \ldots + (j) & \text{when } i = j, \text{ and} \\ (0) + \ldots + (j) \text{ and } (1|0) + \ldots + (j+1|j) & \text{when } i > j. \end{cases}$$

To see this, let ${}_kP_\ell$ denote the module ${}_kP \otimes_A P_\ell$. Then the complex $\mathrm{Hom}_A(Q_i, Q_j) = {}_iQ \otimes_A Q_j$ is given by:

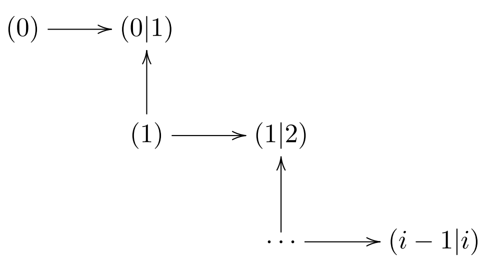
$$\begin{array}{ccccccc}
{}_0P_0 & \longrightarrow & {}_0P_1 & \longrightarrow & \cdots & \longrightarrow & {}_0P_j \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
{}_1P_0 & \longrightarrow & {}_1P_1 & \longrightarrow & \cdots & \longrightarrow & {}_1P_j \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
{}_iP_0 & \longrightarrow & {}_iP_1 & \longrightarrow & \cdots & \longrightarrow & {}_iP_j
\end{array}$$

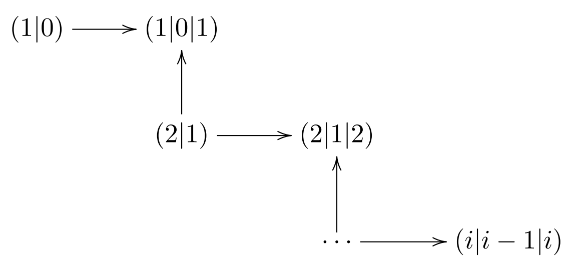
where the horizontal maps ${}_kP_\ell \to {}_kP_{\ell+1}$ are given by right multiplication by $(\ell|\ell{+}1)$ and the vertical maps ${}_kP_\ell \to {}_{k-1}P_\ell$ are given by left multiplication by $(k-1|k)$. Note, furthermore, that, as an $\mathbb{F}$–vector space:

$$ {}_kP_\ell = \begin{cases} \mathrm{Span}_{\mathbb{F}}\langle (k), (k|k-1|k)\rangle & \text{if } k = \ell \neq 0 \\ \mathrm{Span}_{\mathbb{F}}\langle (k)\rangle & \text{if } k = \ell = 0 \\ \mathrm{Span}_{\mathbb{F}}\langle (k|k\pm 1)\rangle & \text{if } \ell = k \pm 1 \\ 0 & \text{otherwise.} \end{cases} $$

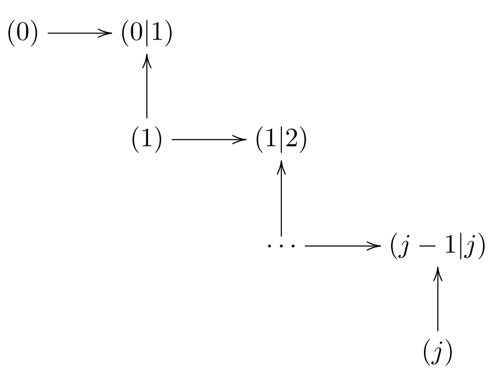
In particular, the chain complex is supported in the three diagonals of the form ${}_kP_k$, ${}_kP_{k-1}$, and ${}_kP_{k+1}$.

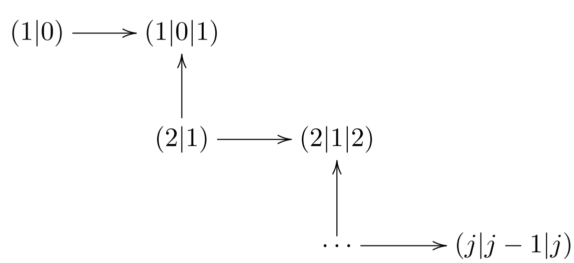
By direct calculation one sees that when $i < j$ the chain complex splits as the direct sum of the two acyclic subcomplexes:





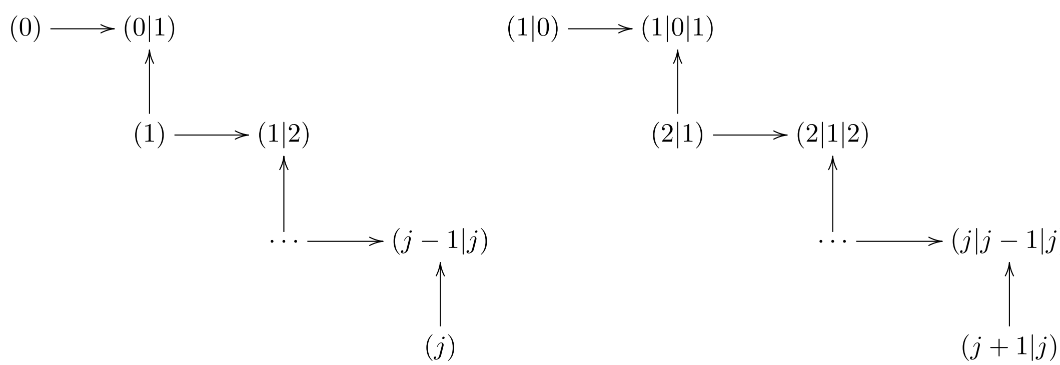
When $i = j$, the chain complex splits in a similar fashion, but the first of the two complexes has homology generated by $(0) + \ldots + (j)$ and the second is acyclic:





When $i > j$, the chain complex again splits, but now both subcomplexes have non-trivial homology, the first generated by $(0) + \ldots + (j)$, and the second generated by $(1|0) + \ldots +$

$(j+1|j)$:



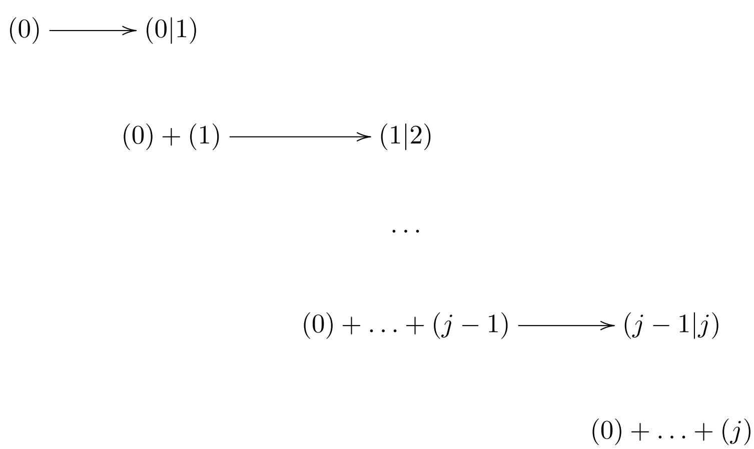
In all three cases, we denote the first subcomplex $\mathcal{C}_{\mathbb{1}}$ and the second subcomplex $\mathcal{C}_{\mathbf{x}}$.

We now note that, as described in Proposition 2.5, $B^{Kh} := H_*(B)$ inherits an A_∞ structure from B. Accordingly, we view B as an A_∞–algebra with differential m_1^B, multiplication m_2^B, and $m_n^B := 0$ for all $n > 2$ and use Proposition 2.5 to give an explicit description of the A_∞ structure on $B^{Kh} := H_*(B)$ from the data of $\mathbb{F}$–linear maps $p : B \to H_*(B)$, $\iota : H_*(B) \to B$, and $h : B \to B$ satisfying

$$p\iota = \mathrm{Id}, \qquad \iota p = \mathrm{Id} + m_1^B h + h m_1^B, \qquad h^2 = 0.$$

We will define $p : {}_iB_j \to {}_iB_j^{Kh}$, $\iota : {}_iB_j^{Kh} \to {}_iB_j$, $h : {}_iB_j \to {}_iB_j$ explicitly in the case $i > j$, leaving the completely analogous cases $i \leq j$ to the reader.

Begin by performing a change of basis on the two subcomplexes comprising ${}_iB_j$, obtaining for the first subcomplex:



and the second:

$$(1|0) \longrightarrow (1|0|1)$$

$$(1|0) + (2|1) \longrightarrow (2|1|2)$$

$$\dots$$

$$(1|0) + \ldots + (j|j-1) \longrightarrow (j|j-1|j)$$

$$(1|0) + \ldots + (j+1|j)$$

Now, define the projection map p on basis elements $\phi \in {}_iB_j$ above, extending $\mathbb{F}$–linearly from the assignment:

$$p(\phi) := \begin{cases} {}_i\mathbb{1}_j & \text{if } \phi = (0) + \ldots + (j), \\ {}_i\mathbf{x}_j & \text{if } \phi = (1|0) + \ldots + (j+1|j), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The homotopy map h is the $\mathbb{F}$–linear extension of:

$$h(\phi) := \begin{cases} \left(m_1^B\right)^{-1}(\phi) & \text{if } \phi \in \mathrm{Im}(m_1^B) \\ 0 & \text{otherwise,} \end{cases}$$

where in the above, $\left(m_1^B\right)^{-1}(\phi)$ is defined to be the (unique) basis element ϕ' satisfying $\partial(\phi') = \phi$.

The inclusion map ι is the $\mathbb{F}$–linear extension of:

- $\iota({}_i\mathbb{1}_j) := (0) + \ldots + (j)$,
- $\iota({}_i\mathbf{x}_j) := (1|0) + \ldots + (j+1|j)$.

One easily checks that p and ι are chain maps and that p, i, h satisfy:

$$p\iota = \mathrm{Id}, \qquad \iota p = \mathrm{Id} + \partial h + h\partial, \qquad ph = h\iota = h^2 = 0.$$

Furthermore, $h\iota = 0$ and $m_2^B(\iota \otimes \iota)\left(B^{Kh}\right)^{\otimes 2} \subseteq \iota\left(B^{Kh}\right)$. An application of Lemma 2.24 then implies that B is formal, as desired.

Verification that the bigradings and multiplication are as stated is a straightforward calculation. □

Remark 3.13. The algebra B^{Kh} is isomorphic to the algebra of lower triangular $(m+1) \times (m+1)$ matrices over $\mathbb{F}[\mathbf{x}]/(\mathbf{x}^2)$ with only 0's and 1's on the main diagonal:

$$B^{Kh} \cong \left\{ \left. \begin{pmatrix} d_0 & 0 & \dots & 0 \\ \phi_{1,0} & d_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \phi_{m,0} & \dots & \phi_{m,m-1} & d_m \end{pmatrix} \right| d_i \in \{0,1\} \right\} \subset M_{m+1}(\mathbb{F}[\mathbf{x}]/(\mathbf{x}^2))$$

We define an algebra isomorphism by sending the generator ${}_i\mathbb{1}_j \in {}_iB_j^{Kh}$ (resp., ${}_i\mathbf{x}_j \in B_j^{Kh}$) to the $(m+1) \times (m+1)$ matrix whose only nonzero matrix entry is a 1 (resp., an $\mathbf{x}$), located in row number i and column number j (where we assume that rows and columns are numbered from 0 to m).

We close our discussion of B^{Kh} with a technical lemma that will prove useful in our construction of the braid group action on $D_\infty\left(B^{Kh}\right)$ (in particular, in the proof of Proposition 3.18).

Lemma 3.14. *Let $\iota : B^{Kh} \to B$, $p : B \to B^{Kh}$, and $h : B \to B$ be the $\mathbb{F}$–linear transformations defined in the proof of Lemma 3.12. The A_∞ morphism of B^{Kh}–modules, $\iota_B : B^{Kh} \to B$, given by*

$$\left(\iota_B\right)_{(n_1|1|n_2)} := \begin{cases} \iota & \text{if } n_1 = n_2 = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

is a quasi-isomorphism. Furthermore, there exists an A_∞ quasi-isomorphism of B^{Kh}–modules, $p_B : B \to B^{Kh}$, whose first few terms are given by:

$$\left(p_B\right)_{(n_1|1|n_2)} := \begin{cases} p & \text{if } n_1 = n_2 = 0, \\ 0 & \text{if } n_1 = 1 \text{ and } n_2 = 0, \text{ and} \end{cases}$$

$\left(p_B\right)_{(0|1|1)} : B \otimes B^{Kh} \to B^{Kh}$ is the bilinear map satisfying

$$\left(p_B\right)_{(0|1|1)}(a \otimes b) :=$$

- ${}_i\mathbb{1}_k$ *if $a = (\ell|\ell+1) \in {}_iB_j$ with $i < j$, $k \leq \ell \leq i$, and $b = {}_j\mathbb{1}_k \in {}_jB^{Kh}_k$ with $j > k$, $i \geq k$,*
- ${}_i\mathbf{x}_k$ *if $a = (\ell|\ell+1) \in {}_iB_j$ with $i < j$, $k+1 \leq \ell \leq i$, and $b = {}_j\mathbf{x}_k \in {}_jB^{Kh}_k$ with $j > k$, $i > k$, and*
- ${}_i\mathbf{x}_k$ *if $a = (\ell|\ell-1|\ell) \in {}_iB_j$ with $i \leq j$, $k+1 \leq \ell \leq i$, and $b = {}_j\mathbb{1}_k \in {}_jB^{Kh}_k$ with $j > k$, $i > k$.*
- 0 *for all other basis elements $a \in B$, $b \in B^{Kh}$ in the proof of Lemma 3.12.*

Proof. Let $m_{(n_1|1|n_2)}$ denote the structure maps for B and $m^{Kh}_{(n_1|1|n_2)}$ denote those (induced by Proposition 2.5) for B^{Kh}, both considered as B^{Kh}–bimodules.

Recall that the "Transfer Theorem" [8, Thm. 2.1] tells us how to extend ι, p to A_∞ quasi-isomorphisms. Explicitly, one defines

$$\left(\iota_B\right)_{(0|1|0)} := \iota, \qquad \left(p_B\right)_{(0|1|0)} := p$$

and constructs higher terms of ι_B, p_B satisfying the A_∞ relations for morphisms. Since ι, p induce isomorphisms on homology, ι_B and p_B will then yield A_∞ quasi-isomorphisms $B \leftrightarrow B^{Kh}$.

We begin by calculating the higher terms of ι_B. But here our work is already done, since ι, p, and h satisfy the assumptions of Lemma 2.24 (see the proof of Lemma 3.12), hence $\left(\iota_B\right)_{(n_1|1|n_2)} = 0$ for all $(n_1 + 1 + n_2) > 1$, as desired.

We now move to the calculation of the higher terms of p_B.

Computation of $\left(p_B\right)_{(1|1|0)}$:

Here we note that $ph = 0$, and $\mathrm{Im}(h)$ and $\mathrm{Im}(m_1^B)$ are both *left* B^{Kh} submodules, so an application of Lemma 2.25, implies that $p : B \to B^{Kh}$ is a *left* module map (and, hence, we can extend p to a left A_∞ morphism with no higher left A_∞ terms). In particular, $\left(p_B\right)_{(1|1|0)} := 0$, as desired.

Computation of $(p_B)_{(0|1|1)}$*:*

Unfortunately, Im(h) and Im(m_1^B) are not *right* B^{Kh} submodules, so we will have to work harder here. The Transfer Theorem ([8, Thm. 2.1]), combined with remarks in the proof of Lemma 2.25, tells us that

$$(p_B)_{(0|1|1)} := p \circ m_{(0|1|1)} \circ (h \otimes 1).$$

We now claim that if ${}_ia_j \in {}_iB_j$ and ${}_jb_k \in {}_jB_k^{Kh}$, then $(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) = 0$ unless the triple i, j, k satisfies the property that $i \leq j$, $j > k$, and $i \geq k$. We can see this by a case-by-case analysis (see the table below, which describes $(p_B)_{(0|1|1)}$ in the various cases). For example, if $j < k$ (first column of table) then ${}_jb_k = 0$, and if $i < k$ (first entry in second column), then $p_{(0|1|0)} := 0$. In both cases, we then have $(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) = 0$. On the other hand, when $i > j \geq k$ or $i = j = k$ (the remaining entries in the table except the top two in the third column), we notice that

$$m_{(0|1|1)}(Im(h) \otimes {}_jb_k) \subseteq Im(h).$$

Since $ph = 0$, we have $(p_B)_{(0|1|1)} = 0$ in these cases as well.

We are therefore left to compute $(p_B)_{(0|1|1)}$ when $i \leq j, j > k$, and $i \geq k$ (the starred entries of the table). There are three subcases.

$(p_B)_{(0\|1\|1)}\left({}_ia_j \otimes {}_jb_k\right)$	$j < k$	$j = k$	$j > k$
$i < j$	0	0	*
$i = j$	0	0	*
$i > j$	0	0	0

Case 1: $i < j$**,** $j > k$**, and** $i = k$

Here, we notice that for basis elements ${}_ia_j, {}_jb_k$, we have $(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) \neq 0$ iff ${}_ia_j = (i|i+1)$ and ${}_jb_k = {}_j\mathbb{1}_k$.

In this case,

$$\begin{aligned} (p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) &:= p\left[(0) + \ldots + (i)\right] \\ &= {}_i\mathbb{1}_k. \end{aligned}$$

Case 2: $i < j$**,** $j > k$**, and** $i > k$

Again, we notice that for basis elements ${}_ia_j, {}_jb_k$, we have $(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) \neq 0$ iff either

- ${}_ia_j = (\ell|\ell+1)$ for $k \leq \ell \leq i$ and ${}_jb_k = {}_j\mathbb{1}_k$, in which case
$$(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) := p\left[(0) + \ldots + (k)\right] = {}_i\mathbb{1}_k.$$
- ${}_ia_j = (\ell|\ell+1)$ for $k+1 \leq \ell \leq i$ and ${}_jb_k = {}_j\mathbf{x}_k$, in which case
$$(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) := p\left[(1|0) + \ldots + (k+1|k)\right] = {}_i\mathbf{x}_k.$$
- ${}_ia_j = (\ell+1|\ell|\ell+1)$ for $k \leq \ell \leq i$ and ${}_jb_k = {}_j\mathbb{1}_k$, in which case
$$(p_B)_{(0|1|1)}\left({}_ia_j \otimes {}_jb_k\right) := p\left[(1|0) + \ldots + (k+1|k)\right] = {}_i\mathbf{x}_k.$$

Case 3: $i = j > k$

An analysis similar to the previous cases allows us to conclude that $p_{(0|1|1)}({}_ia_j \otimes {}_jb_k) = 0$ on basis elements ${}_ia_j, {}_jb_k$ except when ${}_ia_j = (\ell|\ell-1|\ell)$ for $k+1 \leq \ell \leq i$ and ${}_jb_k = {}_j\mathbb{1}_k$. In these cases, we have:

$$p_{(0|1|1)}\left[{}_ia_j \otimes {}_jb_k\right] = {}_i\mathbf{x}_k.$$

Armed with the above calculations, we define $p_{(0|1|1)} : {}_iB_j \otimes {}_jB^{Kh}_k \to {}_iB_k$ in the case $i \leq j, j > k, i \geq k$ to be the unique bilinear map assigning the values above to the basis elements described and 0 to all other basis elements. The desired conclusion follows.

□

3.4. **A braid group action on $D_\infty(B^{Kh})$.** Khovanov-Seidel's braid group action on $D^b(A)$ induces a braid group action on $D_\infty(B^{Kh})$, via the following:

Proposition 3.15. *There is an equivalence of triangulated categories*

$$D(A) \leftrightarrow D(B) \leftrightarrow D_\infty(B) \leftrightarrow D_\infty(B^{Kh}).$$

Proof. Lemmas 2.22 and 3.12 together imply the equivalence $D_\infty(B) \leftrightarrow D_\infty(B^{Kh})$ and [28, Proposition 2.4.1] implies the equivalence of $D(B)$ with $D_\infty(B)$.

To see that $D(A) \leftrightarrow D(B)$, we will show that the functors $\mathcal{F} : D(A) \to D(B)$ and $\mathcal{G} : D(B) \to D(A)$ given by

$$\begin{aligned}\mathcal{F}(M) &:= Q^* \otimes_A M = \mathrm{Hom}_A(Q, M)\\ \mathcal{G}(N) &:= Q \otimes_B N\end{aligned}$$

where $Q := \bigoplus_{i=0}^m Q_i$ and $Q^* := \mathrm{Hom}_A(Q, A) = \bigoplus_{i=0}^m {}_iQ$ are well-defined mutually inverse equivalences of triangulated categories.

Since each ${}_iQ \subset Q^*$ is a complex of projective right modules over A, the functor $Q^* \otimes_A -$ is exact, so $\mathcal{F}$ is clearly well-defined. To prove that $\mathcal{G}$ is also well-defined, we will show that the right dg B-module $\mathrm{Hom}_A(P_i, Q) \subset Q = \mathrm{Hom}_A(A, Q) = \bigoplus_{i=0}^m \mathrm{Hom}_A(P_i, Q)$ is homotopy equivalent to a semi-free dg B-module, and so tensoring with this dg B-module is exact.

Let $MC({}_i\mathbb{1}_{i-1})$ denote the mapping cone of the chain map ${}_i\mathbb{1}_{i-1} \colon Q_i \to Q_{i-1}$ defined by ${}_i\mathbb{1}_{i-1} := (0) + \ldots + (i-1) \in \mathrm{Hom}_A(Q_i, Q_{i-1})$. There is an A–linear chain map $\iota\colon P_i \to MC({}_i\mathbb{1}_{i-1})$ given by the inclusion of P_i into Q_i, and an A–linear chain map $p\colon MC({}_i\mathbb{1}_{i-1}) \to P_i$ given by

$$p(\phi) := \begin{cases} \phi & \text{if } \phi \in P_i \subset Q_i, \text{ and}\\ -\phi(i-1|i) & \text{if } \phi \in P_{i-1} \subset Q_{i-1}, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

We leave it to the reader to verify that

$$p\iota = \mathrm{Id} \qquad \text{and} \qquad \iota p = \mathrm{Id} + \partial h + h\partial,$$

where ∂ is the differential in $MC({}_i\mathbb{1}_{i-1})$ and $h\colon MC({}_i\mathbb{1}_{i-1}) \to MC({}_i\mathbb{1}_{i-1})$ is the A–linear map $h := {}_{i-1}1_i \colon Q_{i-1} \to Q_i$ defined by ${}_{i-1}\mathbb{1}_i := (0) + \ldots + (i-1) \in \mathrm{Hom}_A(Q_{i-1}, Q_i)$.

Thus P_i is homotopy equivalent to the mapping cone of the chain map ${}_i\mathbb{1}_{i-1} \colon Q_i \to Q_{i-1}$, and consequently, $\mathrm{Hom}_A(P_i, Q)$ is homotopy equivalent to the mapping cone of the induced chain map ${}_{i-1}f_i \colon {}_{i-1}B \to {}_iB$, where ${}_iB := \mathrm{Hom}_A(Q_i, Q) = ({}_i\mathbb{1}_i)B$. Since $MC({}_{i-1}f_i)$ is semi-free (because ${}_{i-1}B$ and ${}_iB$ are semi-free), the functor $(\mathrm{Hom}_A(P_i, Q) \otimes_B -) \cong (MC({}_{i-1}f_i) \otimes_B -)$ is exact, as desired.

It remains to show that the functors $\mathcal{F}$ and $\mathcal{G}$ are inverses of each other. Clearly, the composition $\mathcal{F} \circ \mathcal{G}$ is isomorphic to the identity functor of $D(B)$ because $Q^* \otimes_A Q \cong \mathrm{Hom}_A(Q, Q) = B$ by Lemma 3.10. To show that the composition $\mathcal{G} \circ \mathcal{F}$ is isomorphic to the identity functor of $D(A)$, we will show that the map

$$\psi\colon Q \otimes_B Q^* \longrightarrow A$$

defined by $\psi(q \otimes f) := f(q) \in A$ for $f \in Q^*$ and $q \in Q$ is an isomorphism of dg bimodules. We first note that the differential in $Q \otimes_B Q^*$ is trivial because the differential in Q (resp.,

Q^*) is given by right (resp., left) multiplication with the element

$$b := \sum_{i=1}^{m} \big((0|1) + \ldots + (i-1|i)\big) \in \bigoplus_{i=1}^{m} \mathrm{Hom}_A(Q_i, Q_i) \subset B;$$

and so the differential in $Q \otimes_B Q^*$ is equal to $b \otimes \mathrm{Id} + \mathrm{Id} \otimes b = 2(b \otimes \mathrm{Id}) = 0$. Since the differential in A is trivial as well, it thus suffices to show that ψ is a homotopy equivalence.

However, we have already seen that Q is homotopy equivalent to a sum of complexes of the form ${}_iB \to {}_{i-1}B$ where ${}_iB = \mathrm{Hom}_A(Q_i, Q)$, and an analogous argument shows that Q^* is homotopy equivalent to a sum of complexes of the form $B_{i-1} \to B_i$ where $B_i := \mathrm{Hom}_A(Q, Q_i)$, and B is homotopy equivalent to a sum of complexes of the form ${}_iB_{j-1} \to ({}_{i-1}B_{j-1} \oplus {}_iB_j) \to {}_{i-1}B_j$ where ${}_iB_j := \mathrm{Hom}_A(Q_i, Q_j)$. Moreover, one can check that under these various homotopy equivalences, the map ψ corresponds to the canonical map from $({}_iB \to {}_{i-1}B) \otimes_B (B_{j-1} \to B_j)$ to ${}_iB_{j-1} \to ({}_{i-1}B_{j-1} \oplus {}_iB_j) \to {}_{i-1}B_j$, and now the fact that ψ is a homotopy equivalence follows from the identities

$${}_iB \otimes_B B_j = ({}_i\mathbb{1}_i)B \otimes_B B({}_j\mathbb{1}_j) = ({}_i\mathbb{1}_i)B({}_j\mathbb{1}_j) = {}_iB_j.$$

□

To understand the braid group action on $D_\infty(B^{Kh})$, recall (see [21, Sec. 2d]) that Khovanov-Seidel associate

- to the elementary Artin generator σ_k^+ the dg A–bimodule

$$\mathcal{M}_{\sigma_k^+} := \; 0 \longrightarrow P_k \otimes {}_kP \xrightarrow{\beta_k} A \longrightarrow 0 \;,$$

  where β_k is the A–bimodule map specified by $\beta_k((k) \otimes (k)) = (k)$, and
- to the elementary Artin generator σ_k^- the dg A–bimodule

$$\mathcal{M}_{\sigma_k^-} := \; 0 \longrightarrow A \xrightarrow{\gamma_k} P_k \otimes {}_kP\{-1\} \longrightarrow 0 \;,$$

  where

$$\gamma_k(1) = (k-1|k) \otimes (k|k-1) + (k+1|k) \otimes (k|k+1) + (k) \otimes (k|k-1|k) + (k|k-1|k) \otimes (k).$$

  Here, "1" denotes the identity element $1 = \sum_{i=0}^m (i)$.

We can therefore understand the induced braid group action on $D_\infty(B^{Kh})$ by understanding the images of $\mathcal{M}_{\sigma_k^\pm}$ under the derived equivalence $D_\infty(A) \to D_\infty(B) \to D_\infty(B^{Kh})$.

Accordingly, we denote by $\widetilde{\mathcal{M}}_{\sigma_k^+}$ (resp., $\widetilde{\mathcal{M}}_{\sigma_k^-}$) the mapping cone

$$0 \longrightarrow \mathrm{Hom}_A\left(\textstyle\bigoplus_{i=0}^m Q_i, P_k\right) \otimes \mathrm{Hom}_A\left(P_k, \textstyle\bigoplus_{j=0}^m Q_j\right) \xrightarrow{\widetilde{\beta}_k} B \longrightarrow 0$$

(resp.,

$$0 \longrightarrow B \xrightarrow{\widetilde{\gamma}_k} \mathrm{Hom}_A\left(\textstyle\bigoplus_{i=0}^m Q_i, P_k\right) \otimes \mathrm{Hom}_A\left(P_k, \textstyle\bigoplus_{j=0}^m Q_j\right)\{-1\} \longrightarrow 0 \;),$$

considered as a B^{Kh}-B^{Kh} dg bimodule.

After an application of Lemma 3.10:

$$\mathrm{Hom}_A\left(\bigoplus_{i=0}^m Q_i, P_k\right) \otimes \mathrm{Hom}_A\left(P_k, \bigoplus_{j=0}^m Q_j\right) = \left(\bigoplus_{i=0}^m {}_iQ\right) \otimes_A P_k \otimes {}_kP \otimes_A \left(\bigoplus_{i=0}^m Q_j\right),$$

the induced maps $\widetilde{\beta}_k$, $\widetilde{\gamma}_k$ can be described as $\widetilde{\beta}_k = \mathrm{Id} \otimes \beta_k \otimes \mathrm{Id}$ and $\widetilde{\gamma}_k = \mathrm{Id} \otimes \gamma_k \otimes \mathrm{Id}$.

To further streamline notation, we set

$$\widetilde{P}_k := \mathrm{Hom}_A\left(\bigoplus_{i=0}^{m} Q_i, P_k\right)$$

and

$${}_k\widetilde{P} := \mathrm{Hom}_A\left(P_k, \bigoplus_{j=0}^{m} Q_j\right).$$

We will also find it convenient to replace the mapping cones $\widetilde{\mathcal{M}}_{\sigma_k^\pm}$ with simpler, quasi-isomorphic, mapping cones. We do this by replacing each bimodule B and $\widetilde{P}_k \otimes {}_k\widetilde{P}$ by its homology and the maps $\widetilde{\beta}_k, \widetilde{\gamma}_k$ by the induced maps on homology.

We already understand the structure of $B^{Kh} = H_*(B)$ (Lemma 3.12). The homology of $\widetilde{P}_k$ (resp., ${}_k\widetilde{P}$) is described by:

Lemma 3.16. *$\widetilde{P}_k$ (resp., ${}_k\widetilde{P}$) is formal as a left (resp., right) B^{Kh} module.*

Furthermore, $P_k^{Kh} := H_*\left(\widetilde{P}_k\right)$ and ${}_kP^{Kh} := H_*\left({}_k\widetilde{P}\right)$ have the following explicit descriptions.

$$P_k^{Kh} = Span_{\mathbb{F}}\langle \mathbf{u}^*, \mathbf{v}^*\rangle, \qquad {}_kP^{Kh} = Span_{\mathbb{F}}\langle \mathbf{u}, \mathbf{v}\rangle$$

where the bigradings of $\mathbf{u}^*, \mathbf{v}^*, \mathbf{u}, \mathbf{v}$ are given by:

$$gr(\mathbf{u}^*) = (0,1), \qquad gr(\mathbf{v}^*) = (1,0), \qquad gr(\mathbf{u}) = (0,0), \qquad gr(\mathbf{v}) = (-1,1),$$

and left multiplication by a generator $\theta \in B^{Kh}$ on P_k^{Kh} is given by:

$$\theta\cdot\mathbf{u}^* = \begin{cases} \mathbf{u}^* & \text{if } \theta = {}_k\mathbb{1}_k, \\ 0 & \text{otherwise.}\end{cases} \qquad \theta\cdot\mathbf{v}^* = \begin{cases} \mathbf{v}^* & \text{if } \theta = {}_{k-1}\mathbb{1}_{k-1} \\ \mathbf{u}^* & \text{if } \theta = {}_k\mathbf{x}_{k-1}, \\ 0 & \text{otherwise.}\end{cases}$$

and right multiplication by a generator $\theta \in B^{Kh}$ on ${}_kP^{Kh}$ is given by:

$$\mathbf{u}\cdot\theta = \begin{cases} \mathbf{u} & \text{if } \theta = {}_k\mathbb{1}_k \\ \mathbf{v} & \text{if } \theta = {}_k\mathbf{x}_{k-1} \\ 0 & \text{otherwise.}\end{cases} \qquad \mathbf{v}\cdot\theta = \begin{cases} \mathbf{v} & \text{if } \theta = {}_{k-1}\mathbb{1}_{k-1} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Lemma 3.10, $\mathrm{Hom}_A(Q_i, P_k)$ is given by the complex ${}_iQ \otimes_A P_k$ and $\mathrm{Hom}_A(P_k, Q_i)$ by ${}_kP \otimes_A Q_i$, where

$${}_iQ := {}_0P \xleftarrow{(0|1)\cdot} {}_1P \xleftarrow{(1|2)\cdot} \cdots \xleftarrow{(i-1|i)\cdot} {}_iP$$

This implies that $\widetilde{P}_k, {}_k\widetilde{P}$ are given by:

$$\widetilde{P}_k := \bigoplus_{i=0}^{m} {}_0P_k \xleftarrow{(0|1)\cdot} {}_1P_k \xleftarrow{(1|2)\cdot} \cdots \xleftarrow{(i-1|i)\cdot} {}_iP_k$$

$${}_k\widetilde{P} := \bigoplus_{i=0}^{m} {}_kP_0 \xrightarrow{\cdot(0|1)} {}_kP_1 \xrightarrow{\cdot(1|2)} \cdots \xrightarrow{\cdot(i-1|i)} {}_kP_i$$

We see from above that ${}_iQ \otimes_A P_k$ is:

- 0 when $i < k-1$,
- rank one, generated by $(k-1|k) \in {}_{k-1}P_k$, with 0 differential, when $i = k-1$,
- a direct sum of $\mathrm{Span}\langle (k|k-1|k)\rangle \subset {}_kP_k$ and the acyclic subcomplex

$$(k-1|k) \leftarrow (k) \subset \{{}_{k-1}P_k \leftarrow {}_kP_k\}$$

  when $i = k$, and

- a direct sum of the two acyclic subcomplexes

$$(k-1|k) \leftarrow (k) \subset \{{}_{k-1}P_k \leftarrow {}_kP_k\} \text{ and } (k|k-1|k) \leftarrow (k+1|k) \subset \{{}_kP_k \leftarrow {}_{k+1}P_k\}$$

  when $i > k$.

To show formality of $\widetilde{P}_k$, we use Lemma 2.25 to show that all induced multiplications

$$m_{(n-1|1|0)} : \left(B^{Kh}\right)^{\otimes n-1} \otimes H_*(\mathrm{Hom}_A(Q_i, P_k)) \to H_*(\mathrm{Hom}_A(Q_j, P_k))$$

vanish for $n > 2$.

When $i \leq k-1$, $\mathrm{Hom}_A(Q_i, P_k)$ has trivial differential, so the maps ι_i, p_i, h_i are clear. In the case $i \geq k$, we define:

$$\begin{aligned} \iota_i &: H_*\left(\mathrm{Hom}_A(Q_i, P_k)\right) \to \mathrm{Hom}_A(Q_i, P_k), \\ p_i &: \mathrm{Hom}_A(Q_i, P_k) \to H_*\left(\mathrm{Hom}_A(Q_i, P_k)\right), \text{and} \\ h_i &: \mathrm{Hom}_A(Q_i, P_k) \to \mathrm{Hom}_A(Q_i, P_k)[-1] \end{aligned}$$

as follows.

Let θ denote any generator of $\mathrm{Hom}_A(Q_i, P_k)$, let $\mathbf{u}^*$ denote the lone generator of $H_*(\mathrm{Hom}_A(Q_k, P_k))$, and let ∂ denote the differential on the complex $\mathrm{Hom}_A(Q_i, P_k)$. Note that $H_*(\mathrm{Hom}_A(Q_i, P_k)) = 0$ for $i > k$. Then we define ι_i, p_i, h_i to be the $\mathbb{F}$–linear extensions of:

$$\begin{aligned} \iota_k(\mathbf{u}^*) &:= (k|k-1|k) \\ \iota_{i>k} &:= 0 \end{aligned}$$

$$p_i(\theta) := \begin{cases} \mathbf{u}^* & \text{if } i = k \text{ and } \theta = (k|k-1|k) \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_i(\theta) := \begin{cases} \partial^{-1}(\theta) & \text{if } \theta \in \mathrm{Im}(\partial), \\ 0 & \text{otherwise.} \end{cases}$$

In the above, $\partial^{-1}(\theta)$ is defined to be the (unique) basis element θ' satisfying $\partial(\theta') = \theta$.

It is now straightforward to verify that

1. $p_i h_i = 0$ for all i, and
2. $\mathrm{Im}(h_i)$ and $\mathrm{Im}(\partial)$ are left B^{Kh}-submodules.

Therefore $\widetilde{P}_k$ is formal by Lemma 2.25.

To see that ${}_k\widetilde{P}$ is also formal, we perform a very similar computation, observing that ${}_k\widetilde{P}$ satisfies the assumptions of Lemma 2.24 as a *right* B^{Kh}–module, hence is formal.

Now, we simply note that $H_*(\widetilde{P}_k)$ is rank 2, generated by

- $\mathbf{u}^* := p_k(k|k-1|k) \in {}_kP_k \subset \mathrm{Hom}_A\left(Q_k, P_k\right)$ and
- $\mathbf{v}^* := p_{k-1}(k-1|k) \in {}_{k-1}P_k \subset \mathrm{Hom}_A\left(Q_{k-1}, P_k\right)$,

as is $H_*({}_k\widetilde{P})$, generated by

- $\mathbf{u} := p_k(k) \in {}_kP_k \subset \mathrm{Hom}_A\left(P_k, Q_k\right)$ and
- $\mathbf{v} := p_{k-1}(k|k-1) \in {}_kP_{k-1} \subset \mathrm{Hom}_A\left(P_k, Q_{k-1}\right)$.

Recalling (see the proof of Lemma 3.12) that the generators ${}_i\mathbb{1}_j$ (for $i \geq j$) and ${}_i\mathbf{x}_j$ (for $i > j$) of B^{Kh} are represented by $(0) + \ldots + (j)$ and $(1|0) + \ldots + (j+1|j)$, we see that the multiplication is also as claimed. □

We now have the proposed model

$$MC\left(\ P_k^{Kh} \otimes {}_kP^{Kh} \xrightarrow{\beta_k^{Kh}} B^{Kh}\ \right)$$

for $\mathcal{M}^{Kh}_{\sigma_k^+}$ and the model

$$MC\left(B^{Kh} \xrightarrow{\gamma_k^{Kh}} P_k^{Kh} \otimes_k P^{Kh}\{-1\} \right)$$

for $\mathcal{M}^{Kh}_{\sigma_k^-}$, where β_k^{Kh} and γ_k^{Kh} are the A_∞ morphisms on homology induced by $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$.

To understand the induced maps on homology, we must explicitly understand the quasi-isomorphisms $B \leftrightarrow B^{Kh}$ and $\widetilde{P}_k \otimes_k \widetilde{P} \leftrightarrow P_k^{Kh} \otimes_k P^{Kh}$.

Explicitly, if $\iota_P \otimes \iota'_P : P_k^{Kh} \otimes_k P^{Kh} \to \widetilde{P}_k \otimes_k \widetilde{P}$ and $p_B : B \to B^{Kh}$ are A_∞ quasi-isomorphisms, then the induced A_∞ morphism on homology is given by:

$$\beta_k^{Kh} = p_B \circ \widetilde{\beta}_k \circ (\iota_P \otimes \iota'_P) : P_k^{Kh} \otimes_k P^{Kh} \to B^{Kh}.$$

Furthermore, (cf. [37, Cor. 3.16]), the mapping cones satisfy:

$$\left(0 \longrightarrow P_k^{Kh} \otimes_k P^{Kh} \xrightarrow{\beta_k^{Kh} = p_B \circ \widetilde{\beta}_k \circ (\iota_P \otimes \iota'_P)} B^{Kh} \longrightarrow 0 \right) =$$

$$\left(0 \longrightarrow \widetilde{P}_k \otimes_k \widetilde{P} \xrightarrow{\widetilde{\beta}_k} B \longrightarrow 0 \right)$$

as elements of $D_\infty(B^{Kh})$.

Similarly, if $\iota_B : B^{Kh} \to B$ and $p_P : \widetilde{P}_k \otimes_k \widetilde{P} \to P_k^{Kh} \otimes_k P^{Kh}$ are A_∞ quasi-isomorphisms, then:

$$\left(0 \longrightarrow B^{Kh} \xrightarrow{\gamma_k^{Kh} = (p_P \otimes p'_P) \circ \widetilde{\beta}_k \circ \iota_B} P_k^{Kh} \otimes_k P^{Kh}\{-1\} \longrightarrow 0 \right) =$$

$$\left(0 \longrightarrow B \xrightarrow{\widetilde{\gamma}_k} \widetilde{P}_k \otimes_k \widetilde{P}\{-1\} \longrightarrow 0 \right)$$

as elements of $D_\infty(B^{Kh})$.

Proposition 3.17. *The image of $\mathcal{M}_{\sigma_k^-} \in D_\infty(A)$ under the derived equivalence $D_\infty(A) \to D_\infty\left(B^{Kh}\right)$ is $MC\left(\gamma_k^{Kh}\right)$, where*

$$\gamma_k^{Kh} : B^{Kh} \to P_k^{Kh} \otimes_k P^{Kh}\{-1\}$$

is the $\mathbb{F}$–linear B^{Kh}–bimodule map (i.e., strict A_∞ morphism) determined by

$$_i\mathbb{1}_i \mapsto \begin{cases} \mathbf{v}^* \otimes \mathbf{v} & \text{when } i = k-1, \\ \mathbf{u}^* \otimes \mathbf{u} & \text{when } i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, we define $\mathcal{M}^{Kh}_{\sigma_k^-} := MC\left(\gamma_k^{Kh}\right)$

Proof. We must compute the terms of the induced A_∞ morphism $\gamma_k^{Kh} := (p_P \otimes p'_P) \circ \widetilde{\beta}_k \circ \iota_B$, as described above.

We begin by noting that the $(n_1|1|n_2)$ map of the A_∞ morphism γ_k^{Kh}, i.e., the map

$$\left(\gamma_k^{Kh}\right)_{(n_1|1|n_2)} : \left(B^{Kh}\right)^{\otimes n_1} \otimes B^{Kh} \otimes \left(B^{Kh}\right)^{\otimes n_2} \to \left(P_k^{Kh} \otimes_k P^{Kh}\right)\{-1\}$$

is degree $(-(n_1+n_2), 0)$ with respect to the bigrading. This follows from the A_∞ relations for morphisms, combined with Lemma 3.11.

An examination of the bigradings of elements of B^{Kh} and $P_k^{Kh} \otimes_k P^{Kh}$ then immediately implies that $\left(\gamma_k^{Kh}\right)_{(n_1|1|n_2)} = 0$ unless $n_1 = n_2 = 0$, so γ_k^{Kh} is a strict A_∞ isomorphism, as desired. A quick way to see this is to notice that the sum of the two gradings associated to each element in B^{Kh} and $\left(P_k^{Kh} \otimes_k P^{Kh}\right)\{-1\}$ is 0, and $(\gamma_k)_{(n_1|1|n_2)}$ is degree $-(n_1+n_2)$ on this sum.

It is now easy to verify that

$$\left(\gamma_k^{Kh}\right) := \left(\gamma_k^{Kh}\right)_{(0|1|0)} = (p_P \otimes p'_P)_{(0|1|0)} \circ \widetilde{\gamma}_k \circ (\iota_B)_{(0|1|0)}$$

is as described. In particular, γ_k^{Kh} is determined by its behavior on the $(m+1)$ idempotents ${}_i\mathbb{1}_i \in B^{Kh}$, since it is a B^{Kh}–bimodule map.

For example:

$$\begin{aligned} \gamma_k^{Kh}\left({}_{k-1}\mathbb{1}_{k-1}\right) &:= (p_P)_{(0|1|0)} \circ \widetilde{\gamma}_k \circ (\iota_B)_{(0|1|0)}\left({}_{k-1}\mathbb{1}_{k-1}\right) \\ &= (p_P)_{(0|1|0)} \circ \widetilde{\gamma}_k\left[(0) + \ldots + (k-1)\right] \\ &= (p_P)_{(0|1|0)}\left[(k-1|k) \otimes (k|k-1)\right] \\ &= \mathbf{v}^* \otimes \mathbf{v} \end{aligned}$$

We leave the remaining similarly straightforward computations to the reader. □

Proposition 3.18. *The image of $\mathcal{M}_{\sigma_k^+} \in D_\infty(A)$ under the derived equivalence $D_\infty(A) \to D_\infty\left(B^{Kh}\right)$ is $MC\left(\beta_k^{Kh}\right)$, where the terms of the A_∞ morphism β_k^{Kh} are given as follows.*

$$\left(\beta_k^{Kh}\right)_{(n_1|1|n_2)} : \left(B^{Kh}\right)^{\otimes n_1} \otimes \left(P_k^{Kh} \otimes_k P^{Kh}\right) \otimes \left(B^{Kh}\right)^{\otimes n_2} \to B^{Kh}$$

is identically zero unless $n_1 + n_2 = 1$.

When $n_1 = 1, n_2 = 0$:

$$\left(\beta_k^{Kh}\right)_{(1|1|0)} : B^{Kh} \otimes \left(P_k^{Kh} \otimes_k P^{Kh}\right) \to B^{Kh}$$

is the trilinear map satisfying:

$$\left(\beta_k^{Kh}\right)_{(1|1|0)} : \begin{cases} \left[{}_i\mathbb{1}_k \otimes (\mathbf{u}^* \otimes \mathbf{u})\right] \mapsto {}_i\mathbf{x}_k & (i \geq k+1) \\ \left[{}_i\mathbb{1}_{k-1} \otimes (\mathbf{v}^* \otimes \mathbf{u})\right] \mapsto {}_i\mathbb{1}_k & (i \geq k) \\ \left[{}_i\mathbf{x}_{k-1} \otimes (\mathbf{v}^* \otimes \mathbf{u})\right] \mapsto {}_i\mathbf{x}_k & (i \geq k+1) \\ \left[{}_i\mathbb{1}_{k-1} \otimes (\mathbf{v}^* \otimes \mathbf{v})\right] \mapsto {}_i\mathbf{x}_{k-1} & (i \geq k) \end{cases}$$

and $\left(\beta_k^{Kh}\right)_{(1|1|0)}(b \otimes \theta) = 0$ for all other basis elements $b \in B^{Kh}, \theta \in \left(P_k^{Kh} \otimes_k P^{Kh}\right)$.

When $n_1 = 0, n_2 = 1$:

$$\left(\beta_k^{Kh}\right)_{(0|1|1)} : \left(P_k^{Kh} \otimes_k P^{Kh}\right) \otimes B^{Kh} \to B^{Kh}$$

is the trilinear map satisfying:

$$\left(\beta_k^{Kh}\right)_{(0|1|1)} : \begin{cases} \left[(\mathbf{u}^* \otimes \mathbf{u}) \otimes {}_k\mathbb{1}_j\right] \mapsto {}_k\mathbf{x}_j & (j \leq k-1) \\ \left[(\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k\mathbb{1}_j\right] \mapsto {}_{k-1}\mathbb{1}_j & (j \leq k-1) \\ \left[(\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k\mathbf{x}_j\right] \mapsto {}_{k-1}\mathbf{x}_j & (j \leq k-2) \\ \left[(\mathbf{v}^* \otimes \mathbf{v}) \otimes {}_{k-1}\mathbb{1}_j\right] \mapsto {}_{k-1}\mathbf{x}_j & (j \leq k-2) \end{cases}$$

and $\left(\beta_k^{Kh}\right)_{(0|1|1)}(\theta \otimes b) = 0$ for all other basis elements $b \in B^{Kh}, \theta \in \left(P_k^{Kh} \otimes_k P^{Kh}\right)$.

Accordingly, we define $\mathcal{M}_{\sigma_k^+}^{Kh} := MC\left(\beta_k^{Kh}\right)$

Proof. As in the proof of Proposition 3.17, the $(n_1|1|n_2)$ map of the A_∞ morphism β_k^{Kh} is degree $(-(n_1+n_2),0)$ with respect to the bigrading.

In this case, however, we see that the sum of the two gradings for each element in $P_k^{Kh} \otimes_k P^{Kh}$ is 1, while the sum of the two gradings associated to each element in B^{Kh} is 0. Since $\left(\beta_k^{Kh}\right)_{(n_1|1|n_2)}$ is degree $-(n_1+n_2)$ on this sum, we conclude that $\left(\beta_k^{Kh}\right)_{(n_1|1|n_2)} = 0$ unless $-(n_1+n_2) = -1$, as claimed.

To calculate $\left(\beta_k^{Kh}\right)_{(n_1|1|n_2)}$ in the relevant cases $(n_1 = 1, n_2 = 0)$ and $(n_1 = 0, n_2 = 1)$, we recall that $\beta_k^{Kh} : P_k^{Kh} \otimes_k P^{Kh} \to B^{Kh}$ is given by the composition

$$P_k^{Kh} \otimes_k P^{Kh} \xrightarrow{\iota \otimes \iota'} \widetilde{P}_k \otimes_k \widetilde{P} \xrightarrow{\widetilde{\beta}_k} B \xrightarrow{p} B^{Kh} .$$

Calculation of $\left(\beta_k^{Kh}\right)_{(1|1|0)}$:

Since $\widetilde{\beta}_k$ is, by definition, a strict A_∞ morphism, we see that

$$\left(\beta_k^{Kh}\right)_{(1|1|0)} := p_{(1|1|0)} \circ \widetilde{\beta}_k \circ \left(\iota_{(0|1|0)} \otimes \iota'_{(0|1|0)}\right) + p_{(0|1|0)} \circ \widetilde{\beta}_k \circ \left(\iota_{(1|1|0)} \otimes \iota'_{(0|1|0)}\right).$$

Furthermore, we showed during the proof of Lemma 3.14 that $p_{(1|1|0)} := 0$, so the first term above also vanishes, leaving:

$$\left(\beta_k^{Kh}\right)_{(1|1|0)} := p_{(0|1|0)} \circ \widetilde{\beta}_k \circ \left(\iota_{(1|1|0)} \otimes \iota'_{(0|1|0)}\right).$$

Another application of the Transfer Theorem [8, Thm. 2.1] tells us that on basis elements $b \in B^{Kh}$ and $\theta \in P_k^{Kh}$, we have

$$\iota_{(1|1|0)}\left[b \otimes \theta\right] = \begin{cases} (k+1|k) \in \mathrm{Hom}_A(Q_i, P_k) & \text{when } b = {}_i\mathbb{1}_k,\ \theta = \mathbf{u}^*, \text{ and } i \geq k+1, \\ (k) \in \mathrm{Hom}_A(Q_i, P_k) & \text{when } b = {}_i\mathbb{1}_k,\ \theta = \mathbf{v}^*, \text{ and } i \geq k, \\ (k+1|k) \in \mathrm{Hom}_A(Q_i, P_k) & \text{when } b = {}_i\mathbf{x}_k,\ \theta = \mathbf{v}^*, \text{ and } i \geq k+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Composing the above with $p_{(0|1|0)} \circ \widetilde{\beta}_k$ yields the desired result. We perform this computation in one case, leaving the small number of remaining (similarly straightforward) computations to the reader. Assume $i \geq k+1$. Then:

$$\begin{aligned} \left(\beta_k^{Kh}\right)_{(1|1|0)}\left({}_i\mathbb{1}_k \otimes (\mathbf{u}^* \otimes \mathbf{u})\right) &:= p_{(0|1|0)} \circ \widetilde{\beta}_k \circ \left[\iota_{(1|1|0)}({}_i\mathbb{1}_k \otimes \mathbf{u}^*) \otimes \iota'_{(0|1|0)}(\mathbf{u})\right] \\ &= p_{(0|1|0)} \circ \widetilde{\beta}_k \left[(k+1|k) \otimes (k)\right] \\ &= p_{(0|1|0)}[(k+1|k)] \\ &= {}_i\mathbf{x}_k \end{aligned}$$

Calculation of $\left(\beta_k^{Kh}\right)_{(0|1|1)}$: Similarly, we have:

$$\left(\beta_k^{Kh}\right)_{(0|1|1)} := p_{(0|1|1)} \circ \widetilde{\beta}_k \circ \left(\iota_{(0|1|0)} \otimes \iota'_{(0|1|0)}\right) + p_{(0|1|0)} \circ \widetilde{\beta}_k \circ \left(\iota_{(0|1|0)} \otimes \iota'_{(0|1|1)}\right),$$

and an application of Lemma 2.24 (see the proof of Lemma 3.16) implies that $\iota'_{(0|1|1)} := 0$, leaving:

$$\left(\beta_k^{Kh}\right)_{(0|1|1)} := p_{(0|1|1)} \circ \widetilde{\beta}_k \circ \left(\iota_{(0|1|0)} \otimes \iota'_{(0|1|0)}\right).$$

Referring to Lemma 3.14, we again perform a sample computation, leaving the remaining computations to the reader. Assume $j \leq k-1$. Then:

$$\begin{aligned} \left(\beta_k^{Kh}\right)_{(0|1|1)}((\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k\mathbb{1}_j) &:= p_{(0|1|1)} \circ \left(\widetilde{\beta}_k[(k-1|k) \otimes (k)] \otimes {}_k\mathbb{1}_j\right) \\ &= p_{(0|1|1)}\left[(k-1|k) \otimes {}_k\mathbb{1}_j\right] \\ &= {}_{k-1}\mathbb{1}_j \end{aligned}$$

□

Now, if we have a general braid group element $\sigma \in B_{m+1}$ that decomposes as $\sigma = \sigma_{k_1}^{\pm} \cdots \sigma_{k_n}^{\pm}$, [21] associates to $\sigma \in B_{m+1}$ the dg bimodule:

$$\mathcal{M}_\sigma := \mathcal{M}_{\sigma_{k_1}^{\pm}} \otimes_A \ldots \otimes_A \mathcal{M}_{\sigma_{k_n}^{\pm}}$$

over the algebra A (or, rather, its equivalence class in $D^b(A)$).

Considered as an element of $D_\infty(A)$, we can alternatively describe $\mathcal{M}_\sigma$ in terms of an A_∞ tensor product, by the following.

Definition 3.19. [20, Defn. 1] Given rings $\mathbf{A}, \mathbf{B}$, an $\mathbf{A}$-$\mathbf{B}$ bimodule $\mathbf{M}$ is called *sweet* if it is finitely-generated and projective as a left $\mathbf{A}$ module and as a right $\mathbf{B}$ module.

Remark 3.20. The tensor product $\mathbf{N} \otimes_{\mathbf{A}} \mathbf{M}$ of an $\mathbf{A}'$-$\mathbf{A}$ bimodule $\mathbf{N}$ with an $\mathbf{A}$-$\mathbf{B}$ bimodule is a sweet $\mathbf{A}'$-$\mathbf{B}$ bimodule.

Since each $\mathcal{M}_{\sigma_k^{\pm}}$ is a bounded complex of *sweet* bimodules over A whose higher multiplications are all trivial, the ordinary tensor product above agrees with the A_∞ tensor product in $D_\infty(A)$. In other words,

$$\mathcal{M}_\sigma := \mathcal{M}_{\sigma_{k_1}^{\pm}} \widetilde{\otimes}_A \ldots \widetilde{\otimes}_A \mathcal{M}_{\sigma_{k_n}^{\pm}}.$$

Since A_∞ tensor products are sent to A_∞ tensor products under the derived equivalence $D_\infty(A) \leftrightarrow D_\infty(B) \leftrightarrow D_\infty(B^{Kh})$, we see that the element of $D_\infty(B^{Kh})$ associated to a general braid $\sigma = \sigma_{k_1}^{\pm} \cdots \sigma_{k_n}^{\pm} \in B_{m+1}$ is given by:

$$\mathcal{M}_\sigma^{Kh} := \mathcal{M}_{\sigma_{k_1}^{\pm}}^{Kh} \widetilde{\otimes}_{B^{Kh}} \ldots \widetilde{\otimes}_{B^{Kh}} \mathcal{M}_{\sigma_{k_n}^{\pm}}^{Kh}.$$

Remark 3.21. The B^{Kh} modules described here (and, more generally, any A_∞ module over the Hom algebra of a basis of curves) are equipped with three gradings:

(1) a (co)homological grading,
(2) an internal grading counting steps to the left in the path algebra, A_m, which corresponds to the power of t under the identification of the Khovanov-Seidel construction with a categorification of the Burau representation (see [21, Sec. 2e]),
(3) a grading by path length in the path algebra, A_m, which corresponds to Khovanov's j (quantum) grading if one identifies the Khovanov-Seidel quiver algebra A_m with the algebra $A^{1,m}$ appearing in [10, 39].

The first two of these gradings constitute the bigrading described in [21, Sec. 3d] and discussed throughout this section.

For the benefit of those readers interested in the *trigradings* of generators of B^{Kh}, P_k^{Kh}, and ${}_kP^{Kh}$, we record them here:

- $gr({}_i\mathbb{1}_j) = (0,0,0)$ for ${}_i\mathbb{1}_j \in {}_iB_j$ for all $i, j \in \{0,\ldots,m\}$,
- $gr({}_i\mathbf{x}_j) = (-1,1,1)$ for ${}_i\mathbf{x}_j \in {}_iB_j$ for all $i > j \in \{0,\ldots,m\}$,
- $gr(\mathbf{v}^*) = (1,0,1)$ and $gr(\mathbf{u}^*) = (0,1,2)$ for $\mathbf{v}^*, \mathbf{u}^* \in P_k^{Kh}$ for all $k \in \{1,\ldots,m\}$, and
- $gr(\mathbf{v}) = (-1,1,1)$ and $gr(\mathbf{u}) = (0,0,0)$ for $\mathbf{v}, \mathbf{u} \in {}_kP^{Kh}$ for all $k \in \{1,\ldots,m\}$.

3.5. B^{Kh} **and Fukaya categories.** For completeness, and to motivate the constructions in the next section, we briefly outline (without proofs) a geometric interpretation of the algebra B^{Kh} and the bimodules $\mathcal{M}^{Kh}_{\sigma_i^{\pm}}$, in terms of the Fukaya category of a suitable Lefschetz fibration [35, 37].

Namely, denote by p a polynomial of degree $m+1$ whose roots are exactly the points of Δ, and consider the complex surface $S = \{(x, y, z) \in \mathbb{C}^3 \,|\, x^2 + y^2 = p(z)\}$. The projection to the z coordinate defines a Lefschetz fibration $\pi_S : S \to \mathbb{C}$, whose generic fiber is an affine conic, and whose $m+1$ vanishing cycles are all isotopic to each other. The basis of arcs $\mathcal{Q} = \{q_0, \dots, q_m\}$ of Figure 5 then determines a collection of Lefschetz thimbles $Q_0^S, \dots, Q_m^S$ (i.e., Lagrangian disks in S whose boundaries are the vanishing cycles in the fiber $\pi_S^{-1}(-1)$). These form an exceptional collection which generates the Fukaya category $\mathcal{F}(\pi_S)$ of the Lefschetz fibration π_S [37].

Perturbing the symplectic structure slightly, we can ensure that the vanishing cycles (which are Hamiltonian isotopic loops in $\pi_S^{-1}(-1) \simeq \mathbb{C}^*$) are mutually transverse and intersect in a suitable manner (i.e., they pairwise intersect in exactly two points, and the intersection points are arranged in a configuration which forces the vanishing of higher products on Floer complexes within the ordered collection).

The Floer complexes which determine morphisms from Q_i^S to Q_j^S whenever $i > j$ then have rank 2, while by definition these morphism spaces have rank 1 for $i = j$ and 0 for $i < j$ [35]. (Note: our ordering convention for bases of arcs is the opposite of Seidel's.) Moreover, an easy calculation in Floer homology then shows that

$$B^S := \bigoplus_{i,j=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_S)}(Q_i^S, Q_j^S)$$

is isomorphic to B^{Kh} (viewing both as A_∞-algebras, in which m_n happens to vanish for $n \neq 2$). The categories of modules over $\mathcal{F}(\pi_S)$ and B^{Kh} are therefore equivalent.

In fact, the B^{Kh}-module P_k^{Kh} has a geometric counterpart via this equivalence, namely a Lagrangian sphere P_k^S in S which projects under π_S to a line segment connecting two consecutive points of Δ. Indeed, P_k^S intersects Q_{k-1}^S and Q_k^S in one point each, and is disjoint from the other Q_i^S; it is then not hard to check that $\bigoplus_i \mathrm{Hom}_{\mathcal{F}(\pi_S)}(Q_i^S, P_k^S) \simeq P_k^{Kh}$ as an A_∞-module over $B^S \simeq B^{Kh}$). See Chapter 20 of [37] for more about the symplectic geometry of S.

Elements of the braid group B_{m+1} acting on (D_m, Δ) lift to symplectic automorphisms of S preserving the fiber $\pi_S^{-1}(-1)$; specifically, the Artin generator σ_k lifts to the Dehn twist about the Lagrangian sphere P_k^S. Denoting again by σ the symplectic automorphism of S which corresponds to a braid $\sigma \in B_{m+1}$, we associate to it the A_∞-bimodule

$$\mathcal{M}_\sigma^S = \bigoplus_{i,j=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_S)}(Q_i^S, \sigma(Q_j^S))$$

over $B^S \simeq B^{Kh}$. It then follows from Seidel's long exact sequence for Dehn twists [36] that the bimodules $\mathcal{M}^S_{\sigma_k^{\pm}}$ and $\mathcal{M}^{Kh}_{\sigma_k^{\pm}}$ associated to Artin generators (or their inverses) are quasi-isomorphic.

4. Bordered Floer algebras and bimodules

We now consider the analogues in bordered Floer homology of the Khovanov-Seidel bimodules described in Section 3. We follow Lipshitz-Ozsváth-Thurston in [27, 28, 29] and Zarev in [42], using a symplectic reinterpretation of their work due to the first author [3].

4.1. **The bordered Floer algebra.** Denote by Σ the double cover of D_m branched at the $m+1$ points of Δ (with covering map $\pi_\Sigma : \Sigma \to D_m$). We make Σ a *parametrized surface* by equipping it with two marked points $z_\pm$ on its boundary (the two preimages by π_Σ of a point in ∂D_m) and the collection of arcs $\mathcal{Q}_\Sigma = \{Q_0^\Sigma, \dots, Q_m^\Sigma\}$, where $Q_k^\Sigma = \pi_\Sigma^{-1}(q_k)$.

In the language introduced by Lipshitz, Ozsváth and Thurston [27], the parametrized surface $(\Sigma, z_\pm, \mathcal{Q}_\Sigma)$ is described combinatorially by a *(twice) pointed matched circle* (or pair of circles when m is odd), $\mathcal{Z}_\mathcal{Q}$. This consists of a pair of oriented intervals (the two components of $\partial\Sigma \setminus \{z_\pm\}$), each carrying $m+1$ distinguished points (the end points of disjoint pushoffs of the Q_k^Σ), labeled successively in decreasing order $m, \dots, 1, 0$ along each interval (according to the manner in which the end points of the 1-handles Q_k^Σ match up).

Recall that the 1–moving strands algebra $\mathcal{A}(\mathcal{Z}_\mathcal{Q}, 1)$,[4] which we also denote by B^{HF} for consistency with the preceding sections, can be described as:

$$\mathcal{A}(\mathcal{Z}_\mathcal{Q}, 1) = \bigoplus_{i,j=0}^{m} {}_iB_j^{HF},$$

where

$${}_iB_j^{HF} = \mathrm{Span}_\mathbb{F} \left\{ \begin{array}{ll} 0 & \text{if } i < j, \\ {}_i\mathbb{1}_i & \text{if } i = j, \\ {}_i\rho_j, {}_i\sigma_j & \text{if } i > j \end{array} \right\},$$

and the multiplication $m_2^{HF} : {}_iB_j^{HF} \otimes {}_jB_k^{HF} \to {}_iB_k^{HF}$ is defined by

- $m_2^{HF}({}_i\mathbb{1}_i \otimes a) = m_2^{HF}(a \otimes {}_j\mathbb{1}_j) = a$ for all $a \in {}_iB_j^{HF}$, and
- $m_2^{HF}({}_i\rho_j \otimes {}_j\rho_k) = {}_i\rho_k$ and $m_2^{HF}({}_i\sigma_j \otimes {}_j\sigma_k) = {}_i\sigma_k$, but $m_2^{HF}({}_i\rho_j \otimes {}_j\sigma_k) = m_2^{HF}({}_i\sigma_j \otimes {}_j\rho_k) = 0$.

As usual, the multiplication map $m_2^{HF} : {}_iB_j^{HF} \otimes {}_kB_\ell^{HF} \to {}_iB_\ell^{HF}$ is zero unless $j = k$. We also set $m_n^{HF} = 0$ for $n \neq 2$.

Remark 4.1. Let $\mathbb{F}\rho \oplus \mathbb{F}\sigma$ denote the $\mathbb{F}$-algebra generated by two orthogonal idempotents ρ and σ, and let $1 := \rho + \sigma$ be its identity element. As we did in the previous section for B^{Kh} (Remark 3.13), we can interpret B^{HF} as the algebra of all lower triangular $(m+1) \times (m+1)$ matrices over $\mathbb{F}\rho \oplus \mathbb{F}\sigma$ which have only 0's and 1's on the main diagonal:

$$B^{HF} \cong \left\{ \left. \begin{pmatrix} d_0 & 0 & \dots & 0 \\ \phi_{1,0} & d_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \phi_{m,0} & \dots & \phi_{m,m-1} & d_m \end{pmatrix} \right| d_i \in \{0, 1\} \right\} \subset M_{m+1}(\mathbb{F}\rho \oplus \mathbb{F}\sigma)$$

We identify the generator ${}_i\rho_j \in {}_iB_j^{HF}$ (resp., ${}_i\sigma_j \in {}_iB_j^{HF}$) with the $(m+1) \times (m+1)$ matrix whose only nonzero matrix entry is a ρ (resp., a σ), located in row number i and column number j; and we identify the generator ${}_i\mathbb{1}_i \in {}_iB_i^{HF}$ with the $(m+1) \times (m+1)$ matrix whose only nonzero entry is a 1, located on the diagonal in row number i. (Here we assume that rows and columns are numbered from 0 to m).

The 1-moving strands algebra has a more geometric interpretation in terms of the arcs $Q_0^\Sigma, \dots, Q_m^\Sigma$ on the surface Σ. Namely, these arcs (or small isotopic deformations of them) are objects of (and in fact generate) the "partially wrapped" Fukaya category of Σ relatively to the two marked points $z_\pm$ (see [2, 3]). In this category, the morphism spaces $\hom(Q_i^\Sigma, Q_j^\Sigma)$

[4] Here we use the notation convention from [42], which differs by a shift from the one in [27]. See the note in [42, Sec. 2.2].

are the Floer complexes generated by intersections between suitably perturbed copies of the arcs (pushing the end points by Hamiltonian isotopies so that they lie in a specific position along the components of $\partial\Sigma \setminus \{z_\pm\}$). In our case, $\{z_\pm\}$ is a fiber of the covering map π_Σ, which is in fact a Lefschetz fibration. The partially wrapped Fukaya category is then equivalent to $\mathcal{F}(\pi_\Sigma)$, Seidel's Fukaya category of the Lefschetz fibration π_Σ (see the Remark in section 4 of [2]), and the Q_i^Σ are nothing but the Lefschetz thimbles associated to the basis of arcs $\mathcal{Q}$ of Figure 5.

The Floer complexes which determine morphisms from Q_i^Σ to Q_j^Σ whenever $i > j$ then have rank 2 (the vanishing cycles consist of the same two points), while by definition these morphism spaces have rank 1 for $i = j$ and 0 for $i < j$ [35]. (As before, our ordering convention for bases of arcs is the opposite of Seidel's.) An easy calculation in Floer homology shows that

$$B^\Sigma := \bigoplus_{i,j=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_\Sigma)}(Q_i^\Sigma, Q_j^\Sigma)$$

is isomorphic to B^{HF}, viewing both as A_∞-algebras in which m_n happens to vanish for $n \neq 2$ (cf. [2, 3]). The categories of modules over $\mathcal{F}(\pi_\Sigma)$ and B^{HF} are therefore equivalent.

4.2. **Bordered Floer bimodules.** Elements of the braid group B_{m+1} acting on (D_m, Δ) lift to elements of the mapping class group of the double cover Σ; specifically, the Artin generator σ_k lifts to the Dehn twist about the simple closed curve $P_k^\Sigma = \pi_\Sigma^{-1}(p_k)$, where p_k is the line segment in D_m joining the two points labeled $k-1$ and k (see Figure 7). We denote by $\hat{\sigma}$ the mapping class group element which lifts a braid $\sigma \in B_{m+1}$. With this understood, there are two natural ways of associating an A_∞-bimodule over B^{HF} to a braid σ.

On one hand, Lipshitz, Ozsváth and Thurston [28] associate to the element $\hat{\sigma}$ of the mapping class group a bimodule $\widehat{CFDA}(\hat{\sigma})$ over the strands algebra, defined in terms of a suitable Heegaard diagram for the "mapping cylinder" of $\hat{\sigma}$, i.e. the 3-manifold $\Sigma \times [0,1]$ equipped with parametrizations of the two boundary components which differ by the action of $\hat{\sigma}$ (see [27, 28] for details). We denote by $\mathcal{M}_\sigma^{HF}$ the 1-moving strand part of $\widehat{CFDA}(\hat{\sigma})$; this is an A_∞-bimodule over B^{HF} (in fact a "type DA" bimodule, which has nicer algebraic properties).

On the other hand, $\hat{\sigma}$ acts on the Fukaya category of π_Σ, and the A_∞-functor induced by $\hat{\sigma}$ naturally yields a bimodule over $\mathcal{F}(\pi_\Sigma)$, hence over B^Σ. More concretely, following [2] (see also [29]) we set

$$\mathcal{M}_\sigma^\Sigma := \bigoplus_{i,j=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_\Sigma)}\left(Q_i^\Sigma, \hat{\sigma}(Q_j^\Sigma)\right),$$

which is naturally an A_∞-bimodule over $B^\Sigma \simeq B^{HF}$.

Lemma 4.2. *The A_∞-bimodules $\mathcal{M}_\sigma^\Sigma$ and $\mathcal{M}_\sigma^{HF}$ are quasi-isomorphic.*

Proof. It is known [28] that the bordered bimodule $\widehat{CFDA}(\mathrm{id})$ is quasi-isomorphic to the strands algebra viewed as a bimodule over itself; therefore $\mathcal{M}_{\mathrm{id}}^{HF} \simeq B^{HF} \simeq B^\Sigma \simeq \mathcal{M}_{\mathrm{id}}^\Sigma$ (as bimodules). We now give a more geometric interpretation, still in the case $\sigma = \mathrm{id}$.

Following the terminology in [29], denote by AZ the bordered Heegaard diagram depicted in Figure 6, in which the α-arcs and the β-arcs are obtained from Q_k^Σ by pushing the end points along the boundary of Σ, in such a manner that the end points of the α-arcs all lie *before* those of the β-arcs along the oriented intervals $\partial\Sigma \setminus \{z_\pm\}$. Then the 1-moving strand part of the A_∞-bimodule $\widehat{CFAA}(AZ)$ is quasi-isomorphic to $\mathcal{M}_{\mathrm{id}}^{HF} \simeq B^{HF}$; in fact,

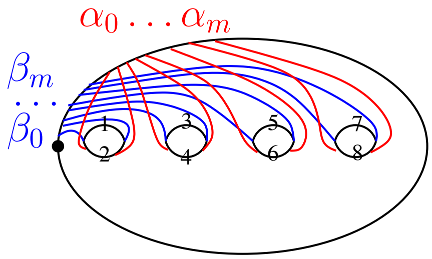


FIGURE 6. A Heegaard diagram for the identity mapping class on Σ (the left and right hand side pictures are glued according to the numbers). Note that the α and β arcs are perturbed copies of the arcs Q_k^Σ.

$\widehat{CFAA}(AZ) \simeq \widehat{CFDA}(\mathrm{id}) \simeq \mathcal{A}(\mathcal{Z}_\mathcal{Q})$ [3, 42, 29]. Thus it is enough to show that the 1-moving strand part of $\widehat{CFAA}(AZ)$ is quasi-isomorphic to $\mathcal{M}_{\mathrm{id}}^\Sigma = B^\Sigma$.

To understand this, recall that morphisms in $\mathcal{F}(\pi_\Sigma)$ are computed by perturbing the arcs to the same positions used in the Heegaard diagram AZ. Hence, the generators of $\mathrm{Hom}(Q_i^\Sigma, Q_j^\Sigma)$ are precisely the intersection points between β_i and α_j, i.e. the generators of the 1-moving strand type AA bimodule. Moreover, the structure maps $m_{(k|1|\ell)}$ count:

- in the case of the type AA bordered Floer bimodule $\widehat{CFAA}(AZ)$, holomorphic strips in Σ connecting two generators of the Heegaard-Floer complex, and with k (resp. ℓ) additional strip-like ends corresponding to chords between β (resp. α) arcs;
- in the case of $\mathcal{M}_{\mathrm{id}}^\Sigma$ (bimodule over the Fukaya category), rigid holomorphic polygons bounded by $k+1$ successively perturbed copies of the β-arcs and $\ell+1$ successively perturbed copies of the α-arcs.

However, there is a natural correspondence between these two types of objects; see Proposition 6.5 of [3] and its proof for details.

In the case of an arbitrary braid σ, denote by $\hat\sigma(AZ)$ the bordered Heegaard diagram obtained from AZ by having $\hat\sigma$ act on the α-arcs (leaving the β-arcs unchanged). From the perspective of Heegaard-Floer theory, the bordered 3-manifold represented by $\hat\sigma(AZ)$ differs from that corresponding to AZ by a reparametrization of its α-boundary via the action of $\hat\sigma$, or equivalently, by attaching the mapping cylinder of $\hat\sigma$. Thus

$$\widehat{CFAA}(\hat\sigma(AZ)) \simeq \widehat{CFAA}(AZ) \widetilde{\otimes} \widehat{CFDA}(\hat\sigma) \simeq \widehat{CFDA}(\hat\sigma).$$

Hence $\mathcal{M}_\sigma^{HF}$ is quasi-isomorphic to the 1-moving strands part of $\widehat{CFAA}(\hat\sigma(AZ))$. On the other hand, by the same argument as above this latter bimodule is quasi-isomorphic to $\mathcal{M}_\sigma^\Sigma = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{F}(\pi_\Sigma)}(Q_i^\Sigma, \hat\sigma(Q_j^\Sigma))$. □

If a braid σ can be expressed in terms of the Artin generators as $\sigma = \sigma_{k_1}^\pm \dots \sigma_{k_n}^\pm$, then its lift can be written as $\hat\sigma = \hat\sigma_{k_1}^\pm \dots \hat\sigma_{k_n}^\pm$, and the pairing theorem for CFDA bimodules [27, 28] implies that

$$\mathcal{M}_\sigma^{HF} \simeq \mathcal{M}_{\sigma_{k_1}^\pm}^{HF} \widetilde{\otimes}_{B^{HF}} \dots \widetilde{\otimes}_{B^{HF}} \mathcal{M}_{\sigma_{k_n}^\pm}^{HF}.$$

Thus it is enough to understand the bimodules $\mathcal{M}_{\sigma_k^\pm}^{HF} \simeq \mathcal{M}_{\sigma_k^\pm}^\Sigma$ associated to the Artin generators and their inverses. We do this working in the category $\mathcal{F}(\pi_\Sigma)$. Recall that morphism spaces in that category are defined by Lagrangian Floer theory after a suitable perturbation (so the end points of arcs lie in the correct order along the boundary of Σ); in particular they are generated by intersection points.

Focusing first on $\mathcal{M}^{HF}_{\sigma_k^+}$, and recalling that $\hat{\sigma}_k^+$ is the positive Dehn twist about P_k^Σ, Seidel's exact triangle for Lagrangian Floer homology [36] tells us that, for each $i, j \in \{0, \dots, m\}$, $\mathrm{Hom}\left(Q_i^\Sigma, \hat{\sigma}_k^+(Q_j^\Sigma)\right)$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathrm{Hom}\left(Q_i^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_j^\Sigma\right) \xrightarrow{\beta_k^{HF}} \mathrm{Hom}\left(Q_i^\Sigma, Q_j^\Sigma\right) \longrightarrow 0\ ,$$

where β_k^{HF} is the Floer product map (cf. [36]) induced by counting holomorphic triangles in Σ whose sides lie on (suitable perturbations of) $Q_i^\Sigma, P_k^\Sigma, Q_j^\Sigma$, appearing in counterclockwise order around the boundary. Moreover, these quasi-isomorphisms are compatible with Floer products, in the sense that in $D_\infty(B^{HF})$ the bimodule $\mathcal{M}^{HF}_{\sigma_k^+}$ is equivalent to the complex of bimodules obtained by taking the direct sum of the above complexes over all i, j.

In analogy to the previous section, we introduce the A_∞-modules

$$P_k^{HF} := \bigoplus_{i=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_\Sigma)}(Q_i^\Sigma, P_k^\Sigma) \qquad \text{and} \qquad {}_kP^{HF} := \bigoplus_{j=0}^{m} \mathrm{Hom}_{\mathcal{F}(\pi_\Sigma)}(P_k^\Sigma, Q_j),$$

which allows us to write

$$\mathcal{M}^{HF}_{\sigma_k^+} \simeq \left\{ 0 \longrightarrow P_k^{HF} \otimes {}_kP^{HF} \xrightarrow{\beta_k^{HF}} B^{HF} \longrightarrow 0 \right\}$$

Like the linear term described above, the higher terms

$$\begin{aligned}&(\beta_k^{HF})_{(n_1|1|n_2)}:\\ &\bigoplus_{\substack{i_0,\dots,i_{n_1}\\ j_0,\dots,j_{n_2}}} \mathrm{Hom}(Q^\Sigma_{i_{n_1}}, Q^\Sigma_{i_{n_1-1}}) \otimes \cdots \otimes \mathrm{Hom}(Q^\Sigma_{i_1}, Q^\Sigma_{i_0}) \otimes \mathrm{Hom}(Q^\Sigma_{i_0}, P_k^\Sigma) \otimes\\ &\qquad \otimes \mathrm{Hom}(P_k^\Sigma, Q^\Sigma_{j_0}) \otimes \cdots \otimes \mathrm{Hom}(Q^\Sigma_{j_{n_2-1}}, Q^\Sigma_{j_{n_2}}) \longrightarrow \bigoplus_{i_{n_1}, j_{n_2}} \mathrm{Hom}(Q^\Sigma_{i_{n_1}}, Q^\Sigma_{j_{n_2}})\end{aligned}$$

of the A_∞-bimodule homomorphism β_k^{HF} count rigid holomorphic polygons in Σ whose sides lie on (suitable perturbations of) $Q^\Sigma_{i_{n_1}}, \dots, Q^\Sigma_{i_0}, P_k^\Sigma, Q^\Sigma_{j_0}, \dots, Q^\Sigma_{j_{n_2}}$ in that order.

Similarly, $\mathcal{M}^{HF}_{\sigma_k^-}$ is equivalent in $D_\infty\left(B^{HF}\right)$ to the direct sum of the complexes

$$0 \longrightarrow \mathrm{Hom}\left(Q_i^\Sigma, Q_j^\Sigma\right) \xrightarrow{\gamma_k^{HF}} \mathrm{Hom}\left(Q_i^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_j^\Sigma\right) \longrightarrow 0\ ,$$

where γ_k^{HF} is induced by counting holomorphic triangles in Σ whose sides lie on (suitable perturbations of) $P_k^\Sigma, Q_i^\Sigma, Q_j^\Sigma$, appearing in counterclockwise order around the boundary. Thus, in $D_\infty(B^{HF})$ we have

$$\mathcal{M}^{HF}_{\sigma_k^-} \simeq \left\{ 0 \longrightarrow B^{HF} \xrightarrow{\gamma_k^{HF}} P_k^{HF} \otimes {}_kP^{HF} \longrightarrow 0 \right\}$$

where the higher terms of the A_∞-bimodule homomorphism γ_k^{HF} again count rigid holomorphic polygons in Σ.

We remark that, in our very simple setting, these counts are equivalent (by the Riemann mapping theorem) to counts of topological immersed triangles in Σ with the stated boundary conditions, and satisfying a local convexity condition at their corners.

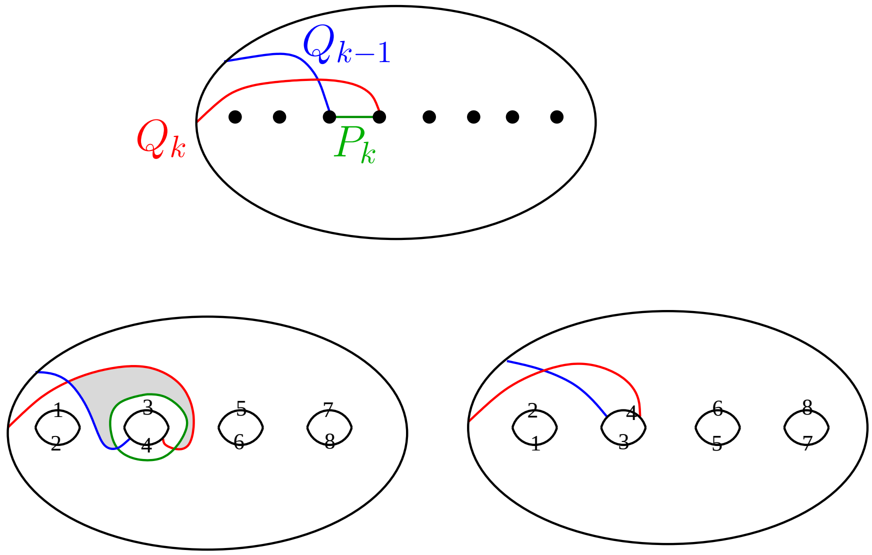


FIGURE 7. The top row above shows curves p_k, q_{k-1}, and q_k in the disk D_m, while the bottom row shows their lifts to Lagrangians in the double branched cover Σ (the figures on the left and right are identified according to the numbers). The shaded triangle gives rise to a non-trivial multiplication map $m_{(1|1|0)} : \mathrm{Hom}(Q_k^\Sigma, Q_{k-1}^\Sigma) \otimes \mathrm{Hom}(Q_{k-1}^\Sigma, P_k^\Sigma) \to \mathrm{Hom}(Q_k^\Sigma, P_k^\Sigma)$.

4.3. **Explicit calculations.** We now make the above story more explicit, by determining the left (resp., right) A_∞-modules P_k^{HF} (resp., ${}_kP^{HF}$) and the maps β_k^{HF} and γ_k^{HF}. Since P_k^Σ intersects Q_{k-1}^Σ and Q_k^Σ transversely once each and is disjoint from all the other Q_j^Σ, the vector spaces underlying these modules have rank 2. The multiplication maps

$$m_{(n|1|0)} : \left(B^{HF}\right)^{\otimes n} \otimes P_k^{HF} \to P_k^{HF} \qquad \text{and} \qquad m_{(0|1|n)} : {}_kP^{HF} \otimes \left(B^{HF}\right)^{\otimes n} \to {}_kP^{HF}$$

are given by counting holomorphic $(n+2)$–gons in Σ as in Figure 7. Again letting the two generators of P_k^{HF} (resp., of ${}_kP^{HF}$) be denoted by $\mathbf{u}^*, \mathbf{v}^*$ (resp., by $\mathbf{u}$, $\mathbf{v}$) and letting θ represent an element of B^{HF}, it is easily verified (see Figure 8) that the $m_{(1|1|0)}$ (resp., $m_{(0|1|1)}$) multiplication is given by:

$$\theta \cdot \mathbf{u}^* = \begin{cases} \mathbf{u}^* & \text{if } \theta = {}_k\mathbb{1}_k, \\ 0 & \text{otherwise.} \end{cases} \qquad \theta \cdot \mathbf{v}^* = \begin{cases} \mathbf{v}^* & \text{if } \theta = {}_{k-1}\mathbb{1}_{k-1}, \\ \mathbf{u}^* & \text{if } \theta = {}_k\rho_{k-1} \text{ or } {}_k\sigma_{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(resp., given by:

$$\mathbf{u} \cdot \theta = \begin{cases} \mathbf{u} & \text{if } \theta = {}_k\mathbb{1}_k, \\ \mathbf{v} & \text{if } \theta = {}_k\rho_{k-1} \text{ or } {}_k\sigma_{k-1}, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathbf{v} \cdot \theta = \begin{cases} \mathbf{v} & \text{if } \theta = {}_{k-1}\mathbb{1}_{k-1}, \\ 0 & \text{otherwise.} \end{cases} \Bigg)$$

The multiplications $m_{(1|1|0)}$ and $m_{(0|1|1)}$ are associative. Moreover, the higher multiplications are all identically zero. One way to see the vanishing of $m_{(n|1|0)}$ is to observe that, for any sequence $i_n \geq \cdots \geq i_1 \geq i_0$ $(n \geq 2)$, and perturbing $Q_{i_0}^\Sigma, \ldots, Q_{i_n}^\Sigma$ so that their end points are in counterclockwise order along the boundary of Σ (but preserving minimal

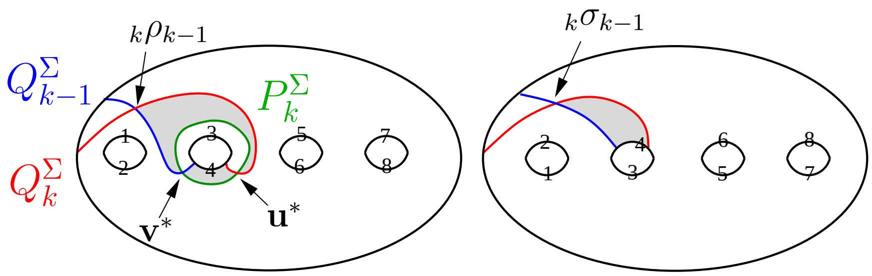


FIGURE 8. The holomorphic triangles giving rise to the nontrivial multiplication maps $m_{(1|1|0)} : \mathrm{Hom}(Q^\Sigma_k, Q^\Sigma_{k-1}) \otimes \mathrm{Hom}(Q^\Sigma_{k-1}, P^\Sigma_k) \to \mathrm{Hom}(Q^\Sigma_k, P^\Sigma_k)$. The other nontrivial multiplication maps can be seen in a similar manner.

intersection otherwise), there are no convex $(n+2)$-gons with edges lying successively on $Q^\Sigma_{i_n}, \dots, Q^\Sigma_{i_0}, P^\Sigma_k$ (and similarly for the vanishing of $m_{(0|1|n)}$).

A more conceptual explanation is that it is possible to find a trivialization of the tangent bundle of Σ and graded lifts [37] of the Lagrangians $P^\Sigma_k, Q^\Sigma_0, \dots, Q^\Sigma_m$, and hence a $\mathbb{Z}$-grading by Maslov index on B^{HF} and the modules P^{HF}_k, ${}_kP^{HF}$, with the following properties:

- all the generators of B^{HF} have degree 0;
- the generators $\mathbf{u}^*$, $\mathbf{v}^*$ of P^{HF}_k have the same degree.
- the generators $\mathbf{u}$, $\mathbf{v}$ of ${}_kP^{HF}$ have the same degree.

Not all degrees can be taken to be zero: in fact $\deg \mathbf{u} + \deg \mathbf{u}^* = \deg \mathbf{v} + \deg \mathbf{v}^* = 1$.

Since the maps $m_{(n|1|0)}$ and $m_{(0|1|n)}$ are compatible with the grading and have degree $1-n$, this forces their vanishing unless $n = 1$.

We now turn to the A_∞ morphisms β^{HF}_k and γ^{HF}_k. The calculations are simplified by constraints arising from the Maslov $\mathbb{Z}$-grading.

First, we observe that β^{HF}_k is a degree-preserving A_∞-homomorphism of bimodules. Namely, since $(\beta^{HF}_k)_{(n_1|1|n_2)}$ corresponds to a Floer product of order (n_1+n_2+2) in $\mathcal{F}(\pi_\Sigma)$, it has degree $-(n_1+n_2)$. However, $P^{HF}_k \otimes {}_kP^{HF}$ is concentrated in degree 1, while all the generators of B^{HF} have degree 0. Therefore, the only non-trivial terms in β^{HF}_k are those of degree -1, namely $(\beta^{HF}_k)_{(1|1|0)}$ and $(\beta^{HF}_k)_{(0|1|1)}$. In particular the linear term $\beta^{HF}_k : \mathrm{Hom}\left(Q^\Sigma_i, P^\Sigma_k\right) \otimes \mathrm{Hom}\left(P^\Sigma_k, Q^\Sigma_j\right) \to \mathrm{Hom}\left(Q^\Sigma_i, Q^\Sigma_j\right)$ vanishes identically.

Similarly, γ^{HF}_k, which is an A_∞-refinement of the pair of pants coproduct in Floer homology, has degree $\dim_{\mathbb{C}}(\Sigma) = 1$ with respect to the Maslov $\mathbb{Z}$-grading. Hence, the map $(\gamma^{HF}_k)_{(n_1|1|n_2)}$ has degree $1-(n_1+n_2)$ and, for degree reasons, it must vanish identically unless $n_1+n_2 = 0$. Thus, the only nontrivial term of γ^{HF}_k is the linear one.

The calculations are further simplified by recalling that

- $\mathrm{Hom}\left(Q^\Sigma_i, P^\Sigma_k\right) = \mathrm{Hom}\left(P^\Sigma_k, Q^\Sigma_i\right) = 0$ whenever $i \neq k, k-1$ and
- $\mathrm{Hom}\left(Q^\Sigma_i, Q^\Sigma_j\right) = 0$ whenever $i < j$.

Lemma 4.3. *$\gamma^{HF}_k : B^{HF} \to P^{HF}_k \otimes {}_kP^{HF}$ is the bimodule map determined by*

$$ {}_i\mathbb{1}_i \mapsto \begin{cases} \mathbf{v}^* \otimes \mathbf{v} & \text{when } i = k-1, \\ \mathbf{u}^* \otimes \mathbf{u} & \text{when } i = k, \text{ and} \\ 0 & \text{otherwise} \end{cases} $$

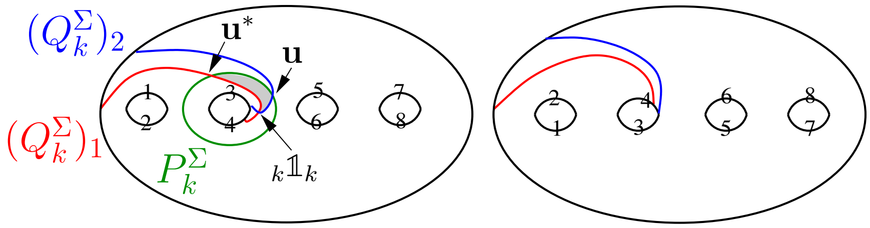


FIGURE 9. The above diagram verifies both that the linear part of

$$\beta_k^{HF} : \mathrm{Hom}\left(Q_k^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_k^\Sigma\right) \to \mathrm{Hom}\left(Q_k^\Sigma, Q_k^\Sigma\right)$$

is zero, and that the map

$$\gamma_k^{HF} : \mathrm{Hom}\left(Q_k^\Sigma, Q_k^\Sigma\right) \to \mathrm{Hom}\left(Q_k^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_k^\Sigma\right)$$

sends ${}_k\mathbb{1}_k \in \mathrm{Hom}\left(Q_k^\Sigma, Q_k^\Sigma\right)$ to $\mathbf{u}^* \otimes \mathbf{u} \in \mathrm{Hom}\left(Q_k^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_k^\Sigma\right)$. By definition, these maps count holomorphic triangles with boundary on P_k^Σ and on two perturbed copies of Q_k^Σ, denoted by $(Q_k^\Sigma)_1$ and $(Q_k^\Sigma)_2$ in the picture; in counterclockwise order, the successive edges must lie on $(Q_k^\Sigma)_1, P_k^\Sigma, (Q_k^\Sigma)_2$ for β_k^{HF}, and on $P_k^\Sigma, (Q_k^\Sigma)_1, (Q_k^\Sigma)_2$ for γ_k^{HF}. Hence, the shaded topological triangle does not contribute to β_k^{HF}, because its boundary has the incorrect orientation, hence it does not admit a holomorphic representative. However, it does contribute to the map γ_k^{HF}. Computations for the pairs $(i,j) = (k,k-1), (k-1,k-1)$ are similarly straightforward.

and by associativity with respect to the multiplication. Moreover, the higher order maps $(\gamma_k^{HF})_{(n_1|1|n_2)}$ *vanish identically for* $(n_1, n_2) \neq (0,0)$.

Proof. The map $\gamma_k^{HF} : \mathrm{Hom}\left(Q_i^\Sigma, Q_j^\Sigma\right) \to \mathrm{Hom}\left(Q_i^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_j^\Sigma\right)$ is 0 unless $(i,j) = (k,k), (k-1,k-1)$, or $(k,k-1)$, since in all other cases either the domain or the target is zero. The nontrivial cases are then determined by counting immersed triangles in Σ; the case $(i,j) = (k,k)$ is shown in Figure 9. By inspection, we see that γ_k^{HF} is given by:

- When $(i,j) = (k,k)$ or $(k-1,k-1)$, γ_k^{HF} sends the unique generator of $\mathrm{Hom}\left(Q_i^\Sigma, Q_j^\Sigma\right)$ to the unique generator of $\mathrm{Hom}\left(Q_i^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_j^\Sigma\right)$, and
- When $(i,j) = (k,k-1)$, γ_k^{HF} sends both ${}_k\rho_{k-1}$ and ${}_k\sigma_{k-1} \in \mathrm{Hom}\left(Q_i^\Sigma, Q_j^\Sigma\right)$ to the unique generator of $\mathrm{Hom}\left(Q_i^\Sigma, P_k^\Sigma\right) \otimes \mathrm{Hom}\left(P_k^\Sigma, Q_j^\Sigma\right)$.

The vanishing of the higher maps follows from the degree argument explained above. □

The story for β_k^{HF} is slightly more complicated, because the maps

$$(\beta_k^{HF})_{(1|1|0)} : \mathrm{Hom}(Q_{i_1}^\Sigma, Q_{i_0}^\Sigma) \otimes \mathrm{Hom}(Q_{i_0}^\Sigma, P_k^\Sigma) \otimes \mathrm{Hom}(P_k^\Sigma, Q_j^\Sigma) \longrightarrow \mathrm{Hom}(Q_{i_1}^\Sigma, Q_j^\Sigma)$$

and

$$(\beta_k^{HF})_{(0|1|1)} : \mathrm{Hom}(Q_i^\Sigma, P_k^\Sigma) \otimes \mathrm{Hom}(P_k^\Sigma, Q_{j_0}^\Sigma) \otimes \mathrm{Hom}(Q_{j_0}^\Sigma, Q_{j_1}^\Sigma) \longrightarrow \mathrm{Hom}(Q_{i_1}^\Sigma, Q_j^\Sigma),$$

which count holomorphic 4-gons in Σ, depend on the choice of Hamiltonian perturbations used to resolve triple intersections at the branch points of π_Σ. (Of course, the behavior of Lagrangian Floer homology under Hamiltonian isotopies guarantees that the maps obtained from different choices are homotopic.) To fix a convention, we perturb P_k^Σ away from the branch points of π_Σ in such a way that its intersections with Q_k^Σ and Q_{k-1}^Σ occur on the sheet of the double cover that contains the generators ${}_i\rho_j$. With this understood, we have:

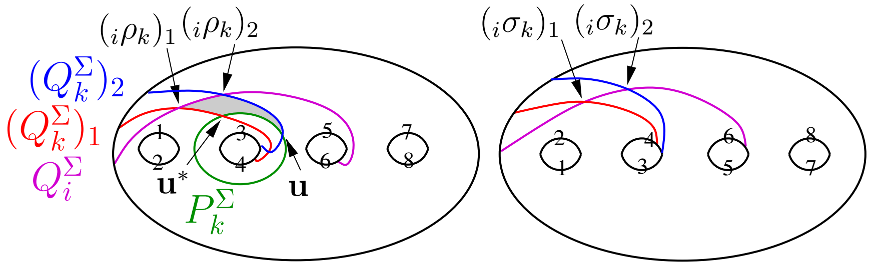


FIGURE 10. The above diagram verifies that

$$(\beta_k^{HF})_{(1|1|0)}({}_i\rho_k, \mathbf{u}^* \otimes \mathbf{u}) = {}_i\rho_k \quad \text{and} \quad (\beta_k^{HF})_{(1|1|0)}({}_i\sigma_k, \mathbf{u}^* \otimes \mathbf{u}) = 0$$

for $i > k$. By definition, $\beta^{HF}_{k\ (1|1|0)} : \mathrm{Hom}(Q_i^\Sigma, Q_k^\Sigma) \otimes \mathrm{Hom}(Q_k^\Sigma, P_k^\Sigma) \otimes \mathrm{Hom}(P_k^\Sigma, Q_k^\Sigma) \to \mathrm{Hom}(Q_i^\Sigma, Q_k^\Sigma)$ counts rigid holomorphic 4-gons with successive edges, in counterclockwise order, on perturbed copies of Q_i^Σ, Q_k^Σ (denoted $(Q_k^\Sigma)_1$), P_k^Σ, and Q_k^Σ again (denoted $(Q_k^\Sigma)_2$). The only contribution comes from the shaded region.

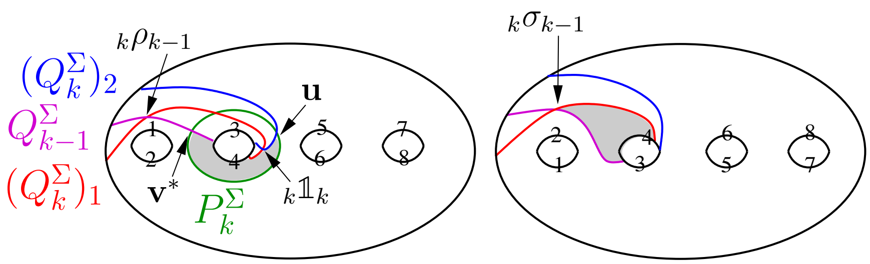


FIGURE 11. The above diagram verifies that

$$(\beta_k^{HF})_{(1|1|0)}({}_k\rho_{k-1}, \mathbf{v}^* \otimes \mathbf{u}) = 0 \quad \text{and} \quad (\beta_k^{HF})_{(1|1|0)}({}_k\sigma_{k-1}, \mathbf{v}^* \otimes \mathbf{u}) = {}_k\mathbb{1}_k.$$

Lemma 4.4. *The only nontrivial terms of β_k^{HF} are:*

$$(\beta_k^{HF})_{(1|1|0)} : \begin{cases} ({}_i\rho_k,\ \mathbf{u}^* \otimes \mathbf{u}) & \mapsto & {}_i\rho_k & (i \geq k+1) \\ ({}_k\sigma_{k-1},\ \mathbf{v}^* \otimes \mathbf{u}) & \mapsto & {}_k\mathbb{1}_k & \\ ({}_i\rho_{k-1},\ \mathbf{v}^* \otimes \mathbf{u}) & \mapsto & {}_i\rho_k & (i \geq k+1) \\ ({}_i\sigma_{k-1},\ \mathbf{v}^* \otimes \mathbf{u}) & \mapsto & {}_i\sigma_k & (i \geq k+1) \\ ({}_i\rho_{k-1},\ \mathbf{v}^* \otimes \mathbf{v}) & \mapsto & {}_i\rho_{k-1} & (i \geq k) \end{cases}$$

$$\textit{and} \quad (\beta_k^{HF})_{(0|1|1)} : \begin{cases} (\mathbf{u}^* \otimes \mathbf{u},\ {}_k\rho_j) & \mapsto & {}_k\rho_j & (j \leq k-1) \\ (\mathbf{v}^* \otimes \mathbf{u},\ {}_k\sigma_{k-1}) & \mapsto & {}_{k-1}\mathbb{1}_{k-1} & \\ (\mathbf{v}^* \otimes \mathbf{u},\ {}_k\rho_j) & \mapsto & {}_{k-1}\rho_j & (j \leq k-2) \\ (\mathbf{v}^* \otimes \mathbf{u},\ {}_k\sigma_j) & \mapsto & {}_{k-1}\sigma_j & (j \leq k-2) \\ (\mathbf{v}^* \otimes \mathbf{v},\ {}_{k-1}\rho_j) & \mapsto & {}_{k-1}\rho_j & (j \leq k-2) \end{cases}$$

Proof. By definition, $(\beta_k^{HF})_{(1|1|0)}$ counts rigid holomorphic 4-gons in Σ whose successive edges, in counterclockwise order, lie on suitably perturbed copies of the following Lagrangians: Q_i^Σ; either Q_k^Σ (for $\mathbf{u}^*$) or Q_{k-1}^Σ (for $\mathbf{v}^*$); P_k^Σ; and either Q_k^Σ (for $\mathbf{u}$) or Q_{k-1}^Σ (for $\mathbf{v}$). The count depends on the perturbations, so we have to be more specific.

Since we are working in the Fukaya category $\mathcal{F}(\pi_\Sigma)$, the various arcs must be perturbed by Hamiltonian isotopies which ensure that their end points are suitably ordered along $\partial\Sigma$; these perturbations are responsible for the intersection points corresponding to the generators ${}_i\rho_k$ and ${}_i\sigma_k$ (resp. ${}_i\rho_{k-1}$, ${}_i\sigma_{k-1}$), which we take to lie close to the boundary of Σ. By contrast, the intersection points corresponding to the generators $\mathbf{u}^*, \mathbf{u}$ and ${}_k\mathbb{1}_k$ normally all lie at the k-th branch point of π_Σ, and perturbations are needed to avoid triple intersections. As mentioned above, we achieve this by choosing a Hamiltonian which pushes P_k^Σ slightly towards the "ρ" side of the surface. Likewise for $\mathbf{v}^*, \mathbf{v}$ and ${}_{k-1}\mathbb{1}_{k-1}$.

With this understood, the calculation simply becomes a matter of drawing the relevant diagrams and looking for immersed four-gons with locally convex corners. The first two cases are shown on Figures 10 and 11; the others are similar. □

As a consistency check, it is not hard to verify that the map β_k^{HF} is indeed an A_∞-homomorphism, namely for all $a_1, a_2 \in B^{HF}$ and $m \in P_k^{HF} \otimes {}_k P^{HF}$ we have the identities

$$\beta^{HF}_{k\ (1|1|0)}(a_1a_2, m) + \beta^{HF}_{k\ (1|1|0)}(a_1, a_2m) + a_1\beta^{HF}_{k\ (1|1|0)}(a_2, m) = 0,$$

$$\beta^{HF}_{k\ (0|1|1)}(m, a_1a_2) + \beta^{HF}_{k\ (0|1|1)}(ma_1, a_2) + \beta^{HF}_{k\ (0|1|1)}(m, a_1)a_2 = 0,$$

$$a_1\beta^{HF}_{k\ (0|1|1)}(m, a_2) + \beta^{HF}_{k\ (0|1|1)}(a_1m, a_2) + \beta^{HF}_{k\ (1|1|0)}(a_1, m)a_2 + \beta^{HF}_{k\ (1|1|0)}(a_1, ma_2) = 0.$$

5. A spectral sequence from the Khovanov-Seidel to the bordered Floer algebra

In Sections 3 and 4 we showed how to use the data of a basis, $\widetilde{\mathcal{Q}}$, to construct

- a graded algebra, B^{Kh}, using a construction of Khovanov-Seidel in [21] and
- a (graded) algebra B^{HF}, using ideas of Lipshitz-Ozsváth-Thurston in [27] as generalized by Zarev in [42] and reinterpreted by the first author in [3].

In this section, we establish the existence of a spectral sequence connecting B^{Kh} and B^{HF}. Explicitly, we prove:

Theorem 5.1. *Let*

$$B^{Kh} := H_*\left(\bigoplus_{i,j=0}^{m} \mathrm{Hom}_A(Q_i, Q_j)\right)$$

be the homology of the Hom algebra associated to the basis $\widetilde{\mathcal{Q}}$ and let $B^{HF} := \mathcal{A}(\mathcal{Z}_\mathcal{Q}, 1)$ be the 1–moving strands algebra associated to the arc diagram, $\mathcal{Z}_\mathcal{Q}$. There exists a filtration on B^{HF} whose associated graded algebra is isomorphic, as an ungraded algebra, to B^{Kh}. Accordingly, one obtains a spectral sequence whose E^1 page is isomorphic to B^{Kh} and whose E^∞ page is isomorphic to B^{HF}.

Remark 5.2. The observant reader will at this point notice that the spectral sequence described in the statement of Theorem 5.1 must be somewhat unusual, since B^{HF} is not a *dg algebra* but an *algebra*; hence, the induced differential on the associated graded page is necessarily trivial and the associated spectral sequence on $\mathbb{F}$–vector spaces collapses immediately. This should perhaps not be surprising, as we have $\dim\left({}_iB_j^{Kh}\right) = \dim\left({}_iB_j^{HF}\right)$ for each $i, j \in \{0, \ldots, m\}$. On the other hand, B^{Kh} and B^{HF} are not isomorphic as algebras.

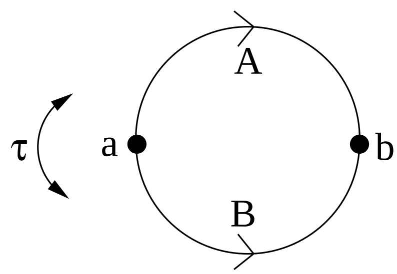


FIGURE 12. A $\mathbb{Z}/2\mathbb{Z}$–equivariant chain complex for S^1.

The filtration serves only to alter the multiplicative structure on the underlying algebra and not to change the dimensions of the underlying $\mathbb{F}$–vector spaces.

We pave the way for a proof of Theorem 5.1 by focusing first on a "toy model" given by the following two lemmas. Though not logically necessary for the proof of Theorem 5.1, we include them in order to motivate the definition of the filtration yielding the spectral sequence from B^{Kh} and B^{HF}.

Lemma 5.3. *There exists a filtered differential algebra, $\mathcal{C}$, whose associated graded homology algebra is isomorphic to $H^*(S^1)$ and whose total homology algebra is isomorphic to $H^*(S^0)$. Furthermore, the associated graded complex and the total complex of $\mathcal{C}$ are formal A_∞ algebras.*

Proof. We construct $\mathcal{C}$ using a $\mathbb{Z}/2\mathbb{Z}$–equivariant cochain complex for $H^*(S^1)$. Specifically, identify S^1 with the unit circle in $\mathbb{C}$ and give it the structure of a simplicial complex by placing two 0–simplices labeled $\mathbf{a}$ and $\mathbf{b}$ at -1 and 1, respectively, and two 1–simplices labeled $\mathbf{A}$ and $\mathbf{B}$ along the arcs $\left\{e^{i\theta}|\theta \in [\pi, 0]\right\}$ and $\left\{e^{i\theta}|\theta \in [-\pi, 0]\right\}$, respectively, as in Figure 12. Let $\mathbf{a}^*$ (resp. $\mathbf{b}^*$, $\mathbf{A}^*$, $\mathbf{B}^*$) represent the $\mathbb{Z}/2\mathbb{Z}$ cochain that assigns 1 to $\mathbf{a}$ (resp., $\mathbf{b}$, $\mathbf{A}$, $\mathbf{B}$) and 0 to all other simplices in the basis.

The filtered differential algebra, $\mathcal{C}$, is generated by $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{A}^*$, and $\mathbf{B}^*$ with multiplication given by the cup product on cochains (cf. [15]):

(2)

$\cup$	$\mathbf{a}^*$	$\mathbf{b}^*$	$\mathbf{A}^*$	$\mathbf{B}^*$
$\mathbf{a}^*$	$\mathbf{a}^*$	0	$\mathbf{A}^*$	$\mathbf{B}^*$
$\mathbf{b}^*$	0	$\mathbf{b}^*$	0	0
$\mathbf{A}^*$	0	$\mathbf{A}^*$	0	0
$\mathbf{B}^*$	0	$\mathbf{B}^*$	0	0

There are two commuting differentials, δ and ∂_τ, on $\mathcal{C}$, giving $\mathcal{C}$ the structure of a differential algebra:

- δ is the standard coboundary map on the simplicial cochain complex (hence satisfies the Leibniz rule with respect to the cup product multiplication), and
- $\partial_\tau = \mathbb{1} + \tau$, where τ is the involution on the cochain complex induced by complex conjugation on $\mathbb{C}$. One easily checks that ∂_τ satisfies the Leibniz rule with respect to the cup product multiplication.

We have the following two-step filtration $\mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1$:

$$0 \subseteq \ker(\partial_\tau) \subseteq \mathcal{C}$$

on $(\mathcal{C}, \delta + \partial_\tau)$. This gives $\mathcal{C}$ the structure of a filtered algebra, since $\mathcal{F}_i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$ for all i, j.[5] Furthermore, the associated graded complex is $(\mathcal{C}, \delta)$, with homology $H^*(S^1)$ and the homology of the total complex $(\mathcal{C}, \delta + \partial_\tau)$ is the cohomology of the fixed point set of τ, i.e., $H^*(S^0)$.

We now use Proposition 2.5 to compute the A_∞ structure on the associated graded complex of $\mathcal{C}$, defining maps $\iota : H^*(S^1) \to (\mathcal{C}, \delta)$, $p : (\mathcal{C}, \delta) \to H^*(S^1)$ and $h : (\mathcal{C}, \delta) \to (\mathcal{C}, \delta)$ satisfying the conditions in Equation 1.

Let $\mathbf{1}$ denote the generator of $H^0(S^1)$ and $\mathbf{x}$ denote the generator of $H^1(S^1)$. Then we define

$$\begin{aligned} \iota(\mathbf{1}) &:= \mathbf{a}^* + \mathbf{b}^* \\ \iota(\mathbf{x}) &:= \mathbf{A}^*, \end{aligned}$$

$$\begin{aligned} p(\mathbf{a}^*) &:= \mathbf{1} \\ p(\mathbf{A}^*) = p(\mathbf{B}^*) &:= \mathbf{x} \\ p(\mathbf{b}^*) &:= 0, \end{aligned}$$

and

$$\begin{aligned} h(\mathbf{B}^*) &:= \mathbf{b}^* \\ h(\mathbf{a}^*) = h(\mathbf{b}^*) = h(\mathbf{A}^*) &:= 0 \end{aligned}$$

An application of Lemma 2.24 then implies that the associated graded algebra is formal.

We proceed similarly for $(\mathcal{C}, \delta + \partial_\tau)$. Let ρ, σ denote the two generators of $H^*(S^0)$ corresponding to the two connected components of S^0. We define:

$$\begin{aligned} \iota(\rho) &:= \mathbf{a}^* + \mathbf{A}^* \\ \iota(\sigma) &:= \mathbf{b}^* + \mathbf{A}^*, \end{aligned}$$

$$\begin{aligned} p(\mathbf{a}^*) &:= \rho \\ p(\mathbf{b}^*) &:= \sigma, \\ p(\mathbf{A}^*) = p(\mathbf{B}^*) &:= 0 \end{aligned}$$

and

$$\begin{aligned} h(\mathbf{B}^*) &:= \mathbf{A}^* \\ h(\mathbf{a}^*) = h(\mathbf{b}^*) = h(\mathbf{A}^*) &:= 0, \end{aligned}$$

Once again, an application of Lemma 2.24 implies that the total algebra of $\mathcal{C}$ is formal. □

As noted in the proof of Lemma 5.3, we have simple descriptions of $H^*(S^1)$ and $H^*(S^0)$ as $\mathbb{F}$–algebras:

$$H^*(S^1) \cong \mathbb{F}[\mathbf{x}]/\mathbf{x}^2$$

and

$$H^*(S^0) := \mathrm{Span}_{\mathbb{F}}\langle \rho, \sigma \rangle,$$

[5]The only non-trivial check that must be performed is that $\mathcal{F}_0 \cdot \mathcal{F}_0 \subseteq \mathcal{F}_0$, but this follows from the fact that ∂_τ satisfies the Leibniz rule.

with multiplication given by

$$\begin{aligned} m_2(\rho\otimes\rho) &= \rho \\ m_2(\sigma\otimes\sigma) &= \sigma \\ m_2(\rho\otimes\sigma)=m_2(\sigma\otimes\rho) &= 0. \end{aligned}$$

Furthermore, the filtration on the filtered differential algebra $\mathcal{C}$ defined in the proof of Lemma 5.3 induces a filtration on $H^*(S^0)$. Accordingly, we have:

Lemma 5.4. *Consider the following filtration, $\mathcal{F}_{-1}\subseteq\mathcal{F}_0\subseteq\mathcal{F}_1$, on $H^*(S^0)$:*

$$0\subseteq Span_{\mathbb{F}}\langle\rho+\sigma\rangle\subseteq H^*(S^0).$$

With respect to this filtration, $H^*(S^0)$ is a well-defined filtered (differential) algebra with associated graded algebra isomorphic to $H^*(S^1)$.

Proof. The claim follows immediately from the observation that the A_∞ quasi-isomorphism $\iota: H^*(S^0)\to\mathcal{C}$ guaranteed by Lemma 2.24 is filtered, hence induces a filtered A_∞ quasi-isomorphism.

However, we find it instructive to give a more direct proof.

First, $H^*(S^0)$ is easily seen to be a well-defined filtered (A_∞) algebra (Definition 2.15) with respect to the above choice of filtration. The only non-trivial check that must be performed is that $m_2((\rho+\sigma)\otimes(\rho+\sigma))\subseteq\mathcal{F}_0$, which follows since $1:=\rho+\sigma$ is the identity element of $H^*(S^0)$. Recalling that the multiplication on the associated graded is given by

$$m_2:\mathcal{F}_r/\mathcal{F}_{r-1}\otimes\mathcal{F}_s/\mathcal{F}_{s-1}\to\mathcal{F}_{r+s}/\mathcal{F}_{r+s-1},$$

we see immediately that 1 is also the multiplicative identity in $\mathrm{gr}(H^*(S^0))$, since it lies in filtration level 0.

The underlying $\mathbb{F}$–vector space of the associated graded algebra $\mathrm{gr}(H^*(S^0))$ can be described by:

$$\mathcal{F}_n/\mathcal{F}_{n-1}:=\begin{cases}\mathrm{Span}_{\mathbb{F}}\langle 1\rangle & \text{if } n=0,\\ \mathrm{Span}_{\mathbb{F}}\langle\rho\rangle & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore,

$$m_2(\rho\otimes\rho)=\rho=0\in\mathcal{F}_2/\mathcal{F}_1.$$

Hence, $\mathrm{gr}(H^*(S^0))$ is isomorphic to $H^*(S^1)$, by identifying $1,\rho\in\mathrm{gr}(H^*(S^0))$ with $1,\mathbf{x}\in H^*(S^1)$. □

We now proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. Recalling (see Remark 4.1) that B^{HF} is isomorphic to the algebra of lower triangular $(m+1)\times(m+1)$ matrices over $H^*(S^0)$ with only 0's and 1's on the diagonal, we define the desired filtration, $\mathcal{F}_{-1}\subseteq\mathcal{F}_0\subseteq\mathcal{F}_1$, on B^{HF} as follows:

$$0\subseteq\left\{M\in B^{HF}\,\middle|\,\phi_{i,j}\in\{0,1\}\ \forall\ i>j\right\}\subseteq B^{HF}$$

We now claim that the associated graded algebra, $\mathrm{gr}\left(B^{HF}\right)$, is isomorphic to B^{Kh}. To see this, note that

$$\mathcal{F}_n/\mathcal{F}_{n-1}:=\begin{cases}\{M\in B^{HF}\,|\,\phi_{i,j}\in\{0,1\}\ \forall\ i>j\} & \text{when } n=0,\\ \{M\in B^{HF}\,|\,\phi_{i,j}\in\{0,\rho\}\ \forall\ i>j, \text{ and } d_k=0\ \forall\ k\} & \text{when } n=1, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $\mathrm{gr}(B^{HF})$ is isomorphic to the algebra of $(m+1)\times(m+1)$ lower triangular matrices over $\mathrm{gr}(H^*(S^0))$ with only 0's and 1's on the diagonal, where the filtration on

$H^*(S^0)$ is the one described in Lemma 5.4. Hence, Lemma 5.4 tells us that $\mathrm{gr}(B^{HF})$ is isomorphic to B^{Kh} as an $\mathbb{F}$–algebra, as desired. □

6. A spectral sequence from the Khovanov-Seidel to the bordered Floer bimodules

In analogy to Theorem 5.1, we prove the following theorem relating the Hom modules described in Section 3 to the bordered Floer modules described in Section 4.

Recall that $\widetilde{\mathcal{Q}}$ is the basis (of ∂–admissible bigraded curves in normal form) pictured in Figure 5.

Theorem 6.1. *Let $\sigma \in B_{m+1}$ be a braid, $\mathcal{M}^{Kh}_\sigma$ the bimodule associated to the pair $(\widetilde{\mathcal{Q}}, \sigma)$ in Section 3, and $\mathcal{M}^{HF}_\sigma$ the bordered Floer bimodule associated to the pair $(\mathcal{Q}, \sigma)$ in Section 4. There exists a filtration on $\mathcal{M}^{HF}_\sigma$ whose associated graded bimodule is isomorphic (as an ungraded A_∞ bimodule over B^{Kh}) to $\mathcal{M}^{Kh}_\sigma$. Accordingly, one obtains a spectral sequence whose E^1 page is isomorphic to $\mathcal{M}^{Kh}_\sigma$ and whose E^∞ page is isomorphic to $\mathcal{M}^{HF}_\sigma$.*

Note that Theorem 5.1 is Theorem 6.1 in the special case $\sigma = \mathrm{Id}$. The proof of Theorem 6.1 proceeds in two steps. We begin by giving an explicit construction of the filtration in the special case where σ is one of the elementary Artin braid generators, $\{\sigma_k^\pm | k = 1, \ldots, m\}$ (Proposition 6.2). Then in the general case, $\sigma = \sigma^\pm_{k_1} \cdots \sigma^\pm_{k_n}$, we explain how to construct a filtration and appropriate spectral sequence on the A_∞ module formed as the A_∞ tensor product

$$\mathcal{M}^{HF}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{HF}} \ldots \widetilde{\otimes}_{B^{HF}} \mathcal{M}^{HF}_{\sigma^\pm_{k_n}}.$$

Proposition 6.2. *Let $\sigma^\pm_k \in B_{m+1}$ be an elementary Artin braid generator, $\mathcal{M}^{Kh}_{\sigma^\pm_k}$ the bimodule associated to the pair $(\widetilde{\mathcal{Q}}, \sigma^\pm_k)$ in Section 3, and $\mathcal{M}^{HF}_{\sigma^\pm_k}$ the bordered-Floer bimodule associated to the pair $(\mathcal{Q}, \sigma^\pm_k)$ in Section 4. There exists a filtration on $\mathcal{M}^{HF}_{\sigma^\pm_k}$ whose associated graded bimodule is isomorphic (as an ungraded A_∞ bimodule over B^{Kh}) to $\mathcal{M}^{Kh}_{\sigma^\pm_k}$. Accordingly, one obtains a spectral sequence whose E^1 page is isomorphic to $\mathcal{M}^{Kh}_{\sigma^\pm_k}$ and whose E^∞ page is isomorphic to $\mathcal{M}^{HF}_{\sigma^\pm_k}$.*

Proof of Proposition 6.2. Guided by the models $\mathcal{M}^{Kh}_{\sigma^\pm_k}$ and $\mathcal{M}^{HF}_{\sigma^\pm_k}$ constructed in Sections 3 and 4, we turn now to constructing filtrations on the filtered bimodules $\mathcal{M}^{HF}_{\sigma^\pm_k}$ (over the filtered algebra B^{HF}) with the desired properties.

We begin by defining, for each $k \in \{0, \ldots, m\}$, filtrations on P^{HF}_k and ${}_kP^{HF}$. Since:

1. we have already defined (Theorem 5.1) a filtration on B^{HF},
2. the tensor product of two filtered A_∞ modules inherits the structure of a filtered A_∞ module,
3. the mapping cone of two filtered A_∞ modules inherits the structure of a filtered A_∞ module, and
4. we have

$$\mathcal{M}^{HF}_{\sigma^+_k} := MC\left(\beta^{HF}_k : (P^{HF}_k \otimes {}_kP^{HF}) \to B^{HF}\right)$$

and

$$\mathcal{M}^{HF}_{\sigma^-_k} := MC\left(\gamma^{HF}_k : B^{HF} \to (P^{HF}_k \otimes {}_kP^{HF})\{-1\}\right),$$

this will induce a filtration on each $\mathcal{M}^{HF}_{\sigma_k^\pm}$, as desired.

Recalling that $P_k^{HF} := \mathrm{Span}_{\mathbb{F}}\langle \mathbf{u}^*, \mathbf{v}^*\rangle$ (resp., ${}_kP^{HF} := \mathrm{Span}_{\mathbb{F}}\langle \mathbf{u}, \mathbf{v}\rangle$), we define the filtration, $\mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1$, on P_k^{HF} to be $0 \subseteq \mathrm{Span}\langle \mathbf{v}^*\rangle \subseteq P_k^{HF}$ (resp., on ${}_kP^{HF}$ to be $0 \subseteq \mathrm{Span}\langle \mathbf{u}\rangle \subseteq {}_kP^{HF}$).

Verification that β_k^{HF} and γ_k^{HF} are filtered A_∞ morphisms with respect this choice of filtration is a straightforward check of a small number of cases, and is left to the reader.

We now must show that the associated graded (homology) of $\mathcal{M}^{HF}_{\sigma_k^\pm}$ is isomorphic to $\mathcal{M}^{Kh}_{\sigma_k^\pm}$ as a $\left(\mathrm{gr}(B^{HF}) = B^{Kh}\right)$–bimodule.

Since we have already shown (in the proof of Theorem 5.1) that the multiplication on $\mathrm{gr}(B^{HF})$ matches the multiplication on B^{Kh}, all that remains to show is

(1) that the multiplication of $\mathrm{gr}(B^{HF})$ on $\mathrm{gr}\left(P_k^{HF} \otimes {}_kP^{HF}\right)$ matches the multiplication of B^{Kh} on $P_k^{Kh} \otimes {}_kP^{Kh}$ and
(2) that the maps induced by γ_k^{HF} and β_k^{HF} on $\mathrm{gr}(B^{HF})$ and $\mathrm{gr}\left(P_k^{HF} \otimes {}_kP^{HF}\right)$ match the maps γ_k^{Kh} and β_k^{Kh}.

Seeing that the multiplication of $\mathrm{gr}(B^{HF})$ on $\mathrm{gr}\left(P_k^{HF} \otimes {}_kP^{HF}\right)$ matches the multiplication of B^{Kh} on $P_k^{Kh} \otimes {}_kP^{Kh}$ is a simple check of a small number of cases, bearing in mind that under the isomorphism $\mathrm{gr}(B^{HF}) \leftrightarrow B^{Kh}$, we have the identification ${}_i\rho_j \leftrightarrow {}_i\mathbf{x}_j$.

The map induced by γ_k^{HF} on $\mathrm{gr}(B^{HF})$ is quickly seen to match the map γ_k^{Kh}, since γ_k^{HF} is a filtered morphism with no higher terms, and the descriptions of γ_k^{Kh} (Proposition 3.17) and γ_k^{HF} (Lemma 4.3) are identical.

Verifying that the map induced by β_k^{HF} on $\mathrm{gr}\left(P_k^{HF} \otimes {}_kP^{HF}\right)$ matches the map β_k^{Kh} is a bit more involved but, again, requires only a handful of checks. We perform a couple here, leaving the rest to the reader.

Lemma 4.4 tells us that when $i \geq k+1$:

$$\left(\beta_k^{HF}\right)_{(1|1|0)}[{}_i\rho_k \otimes (\mathbf{u}^* \otimes \mathbf{u})] := {}_i\rho_k.$$

But viewed as elements of the associated graded, we have ${}_i\rho_k \in \mathcal{F}_1/\mathcal{F}_0(B^{HF})$ and $\mathbf{u}^* \otimes \mathbf{u} \in \mathcal{F}_1/\mathcal{F}_0\left(P_k^{HF} \otimes {}_kP^{HF}\right)$, and thus the induced associated graded map is:

$$\left(\beta_k^{HF}\right)_{(1|1|0)}[{}_i\rho_k \otimes (\mathbf{u}^* \otimes \mathbf{u})] := {}_i\rho_k = 0 \in \mathcal{F}_2/\mathcal{F}_1(B^{HF}).$$

Under the identification $\left({}_i\rho_k \in \mathrm{gr}\left(B^{HF}\right)\right) \leftrightarrow \left({}_i\mathbf{x}_k \in B^{Kh}\right)$, this agrees with Proposition 3.18, which says:

$$\left(\beta_k^{Kh}\right)_{(1|1|0)}[{}_i\mathbf{x}_k \otimes (\mathbf{u}^* \otimes \mathbf{u})] := 0.$$

Lemma 4.4 also tells us that when $j \leq k-1$:

$$\left(\beta_k^{HF}\right)_{(0|1|1)}[(\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k(\rho+\sigma)_j] := {}_{k-1}(\rho+\sigma)_j.$$

Since $(\mathbf{v}^* \otimes \mathbf{u}), {}_k(\rho+\sigma)_j$, and ${}_{k-1}(\rho+\sigma)_j$ are all in $\mathcal{F}_0/\mathcal{F}_{-1}$, the induced map on the associated graded is still:

$$\left(\beta_k^{HF}\right)_{(0|1|1)}[(\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k(\rho+\sigma)_j] := {}_{k-1}(\rho+\sigma)_j.$$

Under the identification $\left({}_i\mathbb{1}_j := {}_i(\rho+\sigma)_j \in \mathrm{gr}(B^{HF})\right) \leftrightarrow {}_i\mathbb{1}_j \in B^{Kh}$, this agrees with Proposition 3.18 which says:

$$\left(\beta_k^{Kh}\right)_{(0|1|1)}[(\mathbf{v}^* \otimes \mathbf{u}) \otimes {}_k\mathbb{1}_j] := {}_{k-1}\mathbb{1}_j.$$

□

Proof of Theorem 6.1. Now that we have a filtration on the A_∞ bimodule $\mathcal{M}^{HF}_{\sigma_k^\pm}$ yielding a spectral sequence from $\mathcal{M}^{Kh}_{\sigma_k^\pm}$ to $\mathcal{M}^{HF}_{\sigma_k^\pm}$ for each elementary Artin generator, $\sigma_k^\pm$, we would like to construct a filtered A_∞ bimodule $\mathcal{M}^{HF}_\sigma$ and corresponding spectral sequence $\mathcal{M}^{Kh}_\sigma \to \mathcal{M}^{HF}_\sigma$ for every $\sigma \in B_{m+1}$.

We begin with a decomposition $\sigma = \sigma^\pm_{k_1} \cdots \sigma^\pm_{k_n}$ and define

$$\mathcal{M}^{HF}_\sigma := \mathcal{M}^{HF}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{HF}} \ldots \widetilde{\otimes}_{B^{HF}} \mathcal{M}^{HF}_{\sigma^\pm_{k_n}},$$

which has the structure of a filtered A_∞ bimodule, by Lemma 2.18.

We then check that the associated graded complex of $\mathcal{M}^{HF}_\sigma$ is equivalent to $\mathcal{M}^{Kh}_\sigma$ in $D_\infty\left(B^{Kh}\right)$, i.e.:

$$\begin{aligned} \mathrm{gr}\left(\mathcal{M}^{HF}_\sigma\right) &\sim \mathcal{M}^{Kh}_\sigma \\ \mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{HF}} \ldots \widetilde{\otimes}_{B^{HF}} \mathcal{M}^{HF}_{\sigma^\pm_{k_n}}\right) &\sim \mathcal{M}^{Kh}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{Kh}} \ldots \widetilde{\otimes}_{B^{Kh}} \mathcal{M}^{Kh}_{\sigma^\pm_{k_n}} \end{aligned}$$

in $D_\infty(B^{Kh})$.

Lemma 2.21 tells us that

$$\mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{HF}} \ldots \widetilde{\otimes}_{B^{HF}} \mathcal{M}^{HF}_{\sigma^\pm_{k_n}}\right) \sim \mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_1}}\right) \widetilde{\otimes}_{\mathrm{gr}(B^{HF})} \ldots \widetilde{\otimes}_{\mathrm{gr}(B^{HF})} \mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_n}}\right)$$

as bimodules over $\mathrm{gr}(B^{HF})$. Therefore, they are equivalent in $D_\infty\left(B^{Kh}\right)$, since $\mathrm{gr}(B^{HF})$ is isomorphic to B^{Kh} (Theorem 5.1). Furthermore, we also know (Proposition 6.2) that $\mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_i}}\right) \sim \mathcal{M}^{Kh}_{\sigma^\pm_{k_i}}$ in $D_\infty(B^{Kh})$, so we have

$$\mathrm{gr}\left(\mathcal{M}^{HF}_\sigma\right) = \mathrm{gr}\left(\mathcal{M}^{HF}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{HF}} \ldots \widetilde{\otimes}_{B^{HF}} \mathcal{M}^{HF}_{\sigma^\pm_{k_n}}\right) \sim \mathcal{M}^{Kh}_{\sigma^\pm_{k_1}} \widetilde{\otimes}_{B^{Kh}} \ldots \widetilde{\otimes}_{B^{Kh}} \mathcal{M}^{Kh}_{\sigma^\pm_{k_n}} = \mathcal{M}^{Kh}_\sigma,$$

as desired.

□

References

[1] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. Categorification of the Kauffman bracket skein module of I-bundles over surfaces. *Algebr. Geom. Topol.*, 4:1177–1210 (electronic), 2004.

[2] Denis Auroux. Fukaya categories and bordered Heegaard-Floer homology. In *Proc. International Congress of Mathematicians, Vol. II (Hyderabad, 2010)*, pages 917–941. Hindustan Book Agency, 2010. math.GT/1003.2962.

[3] Denis Auroux. Fukaya categories of symmetric products and bordered Heegaard-Floer homology. *J. Gökova Geom. Topol. GGT*, 4:1–54, 2010. math.GT/1001.4323.

[4] Denis Auroux, J. Elisenda Grigsby, and Stephan M. Wehrli. Sutured Khovanov homology, Hochschild homology, and the Ozsváth-Szabó spectral sequence. in preparation.

[5] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, Srikanth B. Iyengar, and Claudia Miller. Homology of perfect complexes. *Adv. Math.*, 223(2010):1731–1781, 2010.

[6] John Baldwin. On the spectral sequence from Khovanov homology to Heegaard Floer homology. math.GT/0809.3293, 2008.

[7] John A. Baldwin and Olga Plamenevskaya. Khovanov homology, open books, and tight contact structures. *Adv. Math.*, 224(6):2544–2582, 2010.

[8] Alexander Berglund. A-infinity algebras and homological perturbation theory. 2010.

[9] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[10] Yanfeng Chen and Mikhail Khovanov. An invariant of tangle cobordisms via subquotients of arc rings. math.QA/0610054, 2006.

[11] J. Elisenda Grigsby and Stephan M. Wehrli. Khovanov homology, sutured Floer homology and annular links. *Algebr. Geom. Topol.*, 10(4):2009–2039, 2010.
[12] J. Elisenda Grigsby and Stephan M. Wehrli. On gradings in Khovanov homology and sutured Floer homology. math.GT/1010.3727, 2010.
[13] J. Elisenda Grigsby and Stephan M. Wehrli. On the colored Jones polynomial, sutured Floer homology, and knot Floer homology. *Adv. Math.*, 223(6):2114–2165, 2010.
[14] J. Elisenda Grigsby and Stephan M. Wehrli. On the naturality of the spectral sequence from Khovanov homology to Heegaard Floer homology. *Int. Math. Res. Not. IMRN*, (21):4159–4210, 2010.
[15] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
[16] T. V. Kadeishvili. The algebraic structure in the homology of an $A(\infty)$-algebra. *Soobshch. Akad. Nauk Gruzin. SSR*, 108(2):249–252 (1983), 1982.
[17] Bernhard Keller. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35, 2001.
[18] Bernhard Keller. A-infinity algebras, modules and functor categories. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 67–93. Amer. Math. Soc., Providence, RI, 2006.
[19] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.
[20] Mikhail Khovanov. A functor-valued invariant of tangles. *Alg. Geom. Topol.*, 2:665–741, 2002.
[21] Mikhail Khovanov and Paul Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271 (electronic), 2002.
[22] Angela Klamt. A_∞ structures on the algebra of extensions of Verma modules in the parabolic category O. Diploma thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2010. math.RT/1104.0102.
[23] Angela Klamt and Catharina Stroppel. On the Ext algebras of parabolic Verma modules and A_∞-structures. math.RT/1106.5406, 2011.
[24] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.
[25] P.B. Kronheimer and T.S. Mrowka. Khovanov homology is an unknot-detector. math.GT/1005.4346, 2010.
[26] Kenji Lefèvre-Hasegawa. *Sur les A_∞-catégories*. PhD thesis, Université Paris 7, 2003. math.CT/0310337.
[27] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Bordered Heegaard Floer homology: Invariance and pairing. math.GT/0810.0687, 2008.
[28] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Bimodules in bordered Heegaard Floer homology. math.GT/1003.0598, 2010.
[29] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Heegaard Floer homology as morphism spaces. math.GT/1005.1248, 2010.
[30] Robert Lipshitz, Peter S. Ozsváth , and Dylan P. Thurston. Bordered Floer homology and the spectral sequence of a branched double cover I. math.GT/1011.0499, 2010.
[31] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.
[32] Olga Plamenevskaya. Transverse knots and Khovanov homology. *Math. Res. Lett.*, 13(4):571–586, 2006.
[33] Lawrence P. Roberts. On knot Floer homology in double branched covers. math.GT/0706.0741, 2007.
[34] Lawrence P. Roberts. Notes on the Heegaard-Floer link surgery spectral sequence. math.GT/0808.2817, 2008.
[35] Paul Seidel. Vanishing cycles and mutation. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 65–85. Birkhäuser, Basel, 2001.
[36] Paul Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42(5):1003–1063, 2003.
[37] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
[38] Paul Seidel and Ivan Smith. Localization for involutions in Floer cohomology. *Geom. Funct. Anal.*, 20(6):1464–1501, 2010.
[39] Catharina Stroppel. Parabolic category $\mathcal{O}$, perverse sheaves on Grassmanians, Springer fibers and Khovanov homology. *Compositio Math.*, pages 954–992, 2009.
[40] Zoltán Szabó. A geometric spectral sequence in Khovanov homology. math.GT/1010.4252, 2010.
[41] Liam Watson. Surgery obstructions from Khovanov homology. math.GT/0807.1341, 2008.
[42] Rumen Zarev. Bordered Floer homology for sutured manifolds. math.GT/0908.1106, 2009.

UC Berkeley, Department of Mathematics, 970 Evans Hall # 3840, Berkeley CA 94720, USA
E-mail address: auroux@math.berkeley.edu

Boston College; Mathematics Department; 301 Carney Hall; Chestnut Hill, MA 02467, USA
E-mail address: grigsbyj@bc.edu

Syracuse University; Mathematics Department; 215 Carnegie; Syracuse, NY 13244, USA
E-mail address: smwehrli@syr.edu