

SOME GEOMETRIC ANALYSIS ON GENERIC RICCI SOLITONS

PAOLO MASTROLIA, MARCO RIGOLI, AND MICHELE RIMOLDI

ABSTRACT. We study the geometry of complete generic Ricci solitons with the aid of some geometric–analytical tools extending techniques of the usual Riemannian setting.

1. INTRODUCTION AND MAIN RESULTS

Let (M, \langle, \rangle) be an m -dimensional, complete, connected Riemannian manifold. A *soliton structure* (M, \langle, \rangle, X) on M is the choice (if any) of a smooth vector field X on M and a real constant λ such that

$$(1) \quad \text{Ric} + \frac{1}{2} \mathcal{L}_X \langle, \rangle = \lambda \langle, \rangle,$$

where Ric denotes the Ricci tensor of the metric \langle, \rangle on M and $\mathcal{L}_X \langle, \rangle$ is the Lie derivative of this latter in the direction of X . In what follows we shall refer to λ as to the *soliton constant*. The soliton is called *expanding*, *steady* or *shrinking* if, respectively, $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. If X is the gradient of a potential $f \in C^\infty(M)$, (1) takes the form

$$(2) \quad \text{Ric} + \text{Hess}(f) = \lambda \langle, \rangle,$$

and the Ricci soliton is called a *gradient Ricci soliton*. Both equations (1) and (2) can be considered as perturbations of the Einstein equation

$$\text{Ric} = \lambda \langle, \rangle$$

and reduce to this latter in case X or ∇f are Killing vector fields. When $X = 0$ or f is constant we call the underlying Einstein manifold a *trivial* Ricci soliton.

Since the appearance of the seminal works of R. Hamilton, [7], and G. Perelman, [16], the study of gradient Ricci solitons has become the subject of a rapidly increasing investigation mainly directed towards two goals, *classification* and *triviality*; among the enormous literature on the subject we only quote, as a few examples, the papers [4], [17], [18], [19], [20], [23], which, in some sense, are the most related to the present work.

On the other hand relatively little is known about generic Ricci solitons, that is, when X is not necessarily the gradient of a potential f , and the majority of the results is concerned with the compact case. For instance, it is well known that

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generic expanding and steady compact Ricci solitons are trivial (see e.g. [18]). It is also worth pointing out that on a compact manifold shrinking Ricci solitons always support a gradient soliton structure, [15], and that every complete non-compact shrinking Ricci soliton with $|\text{Ric}|$ bounded supports a gradient soliton structure, [13]. Observe that expanding Ricci solitons which do not support a gradient soliton structure were found by J. Lauret, [9] and P. Baird and L. Danielo, [1]. These spaces exhibit left invariant metrics on Sol and Nil manifolds.

A first important difference is that, in the general case, we cannot make use of the weighted manifold structure $(M, \langle \cdot, \cdot \rangle, e^{-f} d\text{vol})$ which naturally arises when dealing with gradient solitons. The same applies for related concepts such as the Bakry–Emery Ricci tensor, whose boundedness from below with a suitable radial function, together with an additional assumption on the potential function f , gives rise to weighted volume estimates (see [25], [28], [20]). These facts restrict the applicability of analytical tools such as the weak maximum principle for the diffusion operator Δ_f , weighted L^p Liouville–type theorems and *a priori* estimates that have been considered in previous investigations, (see [23], [20] for details). Nevertheless, in the general case the soliton structure is encoded in the geometry of an appropriate operator Δ_X that we shall call the “ X –Laplacian” and that is defined on $u \in C^2(M)$ by

$$\Delta_X u = \Delta u - \langle X, \nabla u \rangle.$$

Clearly, if X is the gradient of some function f , Δ_X reduces to the f –Laplacian. In this spirit, rigidity results and scalar curvature estimates for nonnecessarily gradient Ricci solitons have been recently obtained in [12].

In the present work, assuming a suitable growth condition on the vector field X , we improve Theorems 1.4 and 1.7 of [12] on general solitons making them sharp enough to recover the corresponding results in [20] for gradient solitons. Towards this goal we introduce some analytical tools: a function theoretic version of the Omori–Yau maximum principle for Δ_X , a comparison result for the same operator, and an *a priori* estimate similar to that of Theorem 1.31 in [21]. These results should be useful also in other settings (see Proposition 18 below and its consequences).

From now on we fix an origin $o \in M$ and let $r(x) = \text{dist}(x, o)$. We set B_r and ∂B_r to denote, respectively, the geodesic ball of radius r centered at o and its boundary.

We are now ready to state our

Theorem 1. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold of dimension m with scalar curvature $S(x)$ and $(M, \langle \cdot, \cdot \rangle, X)$ a soliton structure on M with soliton constant λ . Assume*

$$(3) \quad |X| \leq \sqrt{G(r)}$$

where G is a smooth function on $[0, +\infty)$ satisfying

$$(4) \quad \begin{array}{ll} (i) \ G(0) > 0 & (ii) \ G'(t) \geq 0 \text{ on } [0, +\infty) \\ (iii) \ G(t)^{-\frac{1}{2}} \notin L^1(+\infty) & (iv) \ \limsup_{t \rightarrow +\infty} \frac{tG(t^{\frac{1}{2}})}{G(t)} < +\infty. \end{array}$$

Let

$$S_* = \inf_M S.$$

- (i) If $\lambda < 0$ then $m\lambda \leq S_* \leq 0$. Furthermore, if $S(x_0) = S_* = m\lambda$ for some $x_0 \in M$, then (M, \langle, \rangle) is Einstein and X is a Killing field; while if $S(x_0) = S_* = 0$ for some $x_0 \in M$, then (M, \langle, \rangle) is Ricci flat and X is a homothetic vector field.
- (ii) If $\lambda = 0$ then $S_* = 0$. Furthermore, if $S(x_0) = S_* = 0$ for some $x_0 \in M$, then (M, \langle, \rangle) is Ricci flat and X is a Killing field.
- (iii) If $\lambda > 0$ then $0 \leq S_* \leq m\lambda$. Furthermore, if $S(x_0) = S_* = 0$ for some $x_0 \in M$, then (M, \langle, \rangle) is Ricci flat and X is a homothetic vector field, while if $S(x_0) = S_* = m\lambda$ for some $x_0 \in M$, then (M, \langle, \rangle) is compact, Einstein and X is a Killing field.

Remark 2. Defining

$$G(t) = At^2 \prod_{j=1}^N (\log^{(j)}(t))^2,$$

for some constant $A > 0$ and $t \gg 1$, where $\log^{(j)}$ stands for the j -th iterated logarithm, and completing appropriately the definition on $[0, +\infty)$, we obtain a family of functions satisfying (4). In particular this is true for

$$G(t) = 1 + t^2 \text{ on } [0, +\infty).$$

Note that by Z.-H. Zhang, [29], in case the soliton is a gradient soliton, that is, $X = \nabla f$ for some potential $f \in C^\infty(M)$, then

$$(5) \quad |\nabla f| \leq c(1 + r(x)),$$

for some constant $c > 0$. Hence for a gradient soliton the upper bound (3) is automatically satisfied. In this way we recover the scalar curvature estimates of Theorem 3 in [23] and Theorem 0.4 in [20].

Remark 3. In general for complete Ricci solitons which are not necessarily gradient Ricci solitons, we have no control on the growth rate of the norm of the soliton field X . Consider now the vector field $Y \in \mathfrak{X}(M \times J_1)$, given by $Y(x, t) = \frac{X(x)}{1-2\lambda t} + \frac{\partial}{\partial t}$, where $J_1 \subseteq \mathbb{R}$ is defined by

$$J_1 = \begin{cases} (\frac{1}{2\lambda}, +\infty) & \text{if } \lambda < 0 \\ \mathbb{R} & \text{if } \lambda = 0 \\ (-\infty, \frac{1}{2\lambda}) & \text{if } \lambda > 0. \end{cases}$$

Our claim is now that the requirement

$$(6) \quad |X| \leq c(1 + r(x)),$$

for some constant $c > 0$, permits to conclude, as in [29], that for every $t_0 < \frac{1}{2\lambda}$, $t_1 > \frac{1}{2\lambda}$ fixed, and for every $t \in \mathbb{R}$ there exist diffeomorphisms $\psi_t : M \times J_2 \rightarrow M \times J_2$ such that $\psi_0 = id_{M \times J_2}$ and $\frac{d}{dt} \psi_t = Y \circ \psi_t$ on $M \times J_2$, where

$$J_2 = \begin{cases} (t_1, +\infty) & \text{if } \lambda < 0 \\ \mathbb{R} & \text{if } \lambda = 0 \\ (-\infty, t_0) & \text{if } \lambda > 0. \end{cases}$$

Toward this aim we have to show that for any fixed $(y, \bar{t}) \in M \times J_2$ the maximal interval (containing 0) $J((y, \bar{t})) = (a((y, \bar{t})), b((y, \bar{t})))$ where the integral curve of Y emanating from (y, \bar{t}) is defined, coincides with \mathbb{R} . In other words, $a((y, \bar{t})) = -\infty$ and $b((y, \bar{t})) = +\infty$. Let us suppose by contradiction that $b((y, \bar{t})) < +\infty$, the other case being similar. By a well-known “escape” lemma (see e.g. Lemma 12.11 in [10]) we then know that the integral curve $\Phi_{(y, \bar{t})} : J((y, \bar{t})) \rightarrow M \times \mathbb{R}$ is a divergent curve. Now, let $\varepsilon = \inf \left\{ s \in J((y, \bar{t})) : \Phi_{(y, \bar{t})}(s) \in ({}^M B_1(y))^c \times J_2 \right\}$ and for every $t < b((y, \bar{t}))$ consider the restriction

$$\gamma = \Phi_{(y, \bar{t})} \Big|_{[\varepsilon, t]} : [\varepsilon, t] \rightarrow M \times \mathbb{R}.$$

Then

$$l(\gamma) = \int_{\varepsilon}^t |\dot{\gamma}(s)| ds.$$

By (6) we have that outside ${}^M B_1(y)$

$$|X(x)| \leq 2cr(x).$$

Let $\tilde{r}((x, t)) = d_{M \times \mathbb{R}}((x, t), (y, \bar{t}))$. Hence we have that in $({}^M B_1(y))^c \times J_2$

$$|Y((x, t))| \leq B\tilde{r}((x, t)),$$

for some constant $B > 0$. Thus

$$\begin{aligned} \tilde{r}(\gamma(t)) &\leq d_{M \times \mathbb{R}}(\gamma(t), (y, \bar{t})) \\ &\leq \tilde{r}(\gamma(\varepsilon)) + d_{M \times \mathbb{R}}(\gamma(\varepsilon), \gamma(t)) \\ &\leq \tilde{r}(\gamma(\varepsilon)) + l(\gamma) \\ &= \tilde{r}(\gamma(\varepsilon)) + \int_{\varepsilon}^t |\dot{\gamma}(s)| ds \\ &= \tilde{r}(\gamma(\varepsilon)) + \int_{\varepsilon}^t |Y(\gamma(s))| ds \\ &\leq \tilde{r}(\gamma(\varepsilon)) + B \int_{\varepsilon}^t \tilde{r}(\gamma(s)) ds. \end{aligned}$$

Writing this in terms of $I(t) = \int_{\varepsilon}^t \tilde{r}(\gamma(s)) ds$ and integrating the resulting differential inequality one obtains

$$\int_{\varepsilon}^t \tilde{r}(\gamma(s)) ds \leq B' e^{Bt},$$

for some constant $B' > 0$. Recalling that γ has to be divergent we thus get that $b((y, \bar{t})) = +\infty$, getting the desired contradiction.

By the definition of Y we have that for every $(x, t) \in M \times J_2$ the diffeomorphisms ψ_t , $t \in J_2$, can be written in the form

$$\psi_t(x, t) = (\varphi_t(x), t),$$

for some diffeomorphisms $\varphi_t : M \rightarrow M$. Moreover we have that $\varphi_0 = id_M$ and $\frac{d}{dt} \varphi_t(x) = \frac{1}{1-2\lambda t} X(x)$. Let now $\langle \cdot, \cdot \rangle(t)$ be defined by

$$\langle \cdot, \cdot \rangle(t) = (1 - 2\lambda t) \varphi_t^* \langle \cdot, \cdot \rangle,$$

We then have that

$$\begin{aligned}
 \frac{d}{dt} \langle , \rangle (t) &= \frac{d}{dt} ((1 - 2\lambda t)\varphi_t^* \langle , \rangle) \\
 &= -2\lambda\varphi_t^* \langle , \rangle + (1 - 2\lambda t)\varphi_t^* (\mathcal{L}_{\frac{X}{1-2\lambda t}} \langle , \rangle) \\
 &= -2\varphi_t^* (\lambda \langle , \rangle - \frac{(1 - 2\lambda t)}{2} \mathcal{L}_{\frac{X}{1-2\lambda t}} \langle , \rangle) \\
 &= -2\varphi_t^* (\lambda \langle , \rangle - \frac{1}{2} \mathcal{L}_X \langle , \rangle) \\
 &= -2\varphi_t^* (\text{Ric}(\langle , \rangle)) = -2(\text{Ric}(\langle , \rangle (t))).
 \end{aligned}$$

Thus for every $t_0 < \frac{1}{2\lambda}$ and $t_1 > \frac{1}{2\lambda}$ we can define a self-similar solution of the Ricci flow $(M, \langle , \rangle (t))$ defined respectively on $(-\infty, t_0)$ if $\lambda > 0$, on \mathbb{R} if $\lambda = 0$ and on $(t_1, +\infty)$ if $\lambda < 0$. In particular, a complete Ricci soliton $(M, \langle , \rangle, X)$ for which (6) holds always corresponds to the “self-similar” solution of the Ricci flow it generates.

Our next result is

Theorem 4. *Let (M, \langle , \rangle) be a complete Riemannian manifold of dimension $m \geq 3$, scalar curvature $S(x)$ and trace free Ricci tensor T . Suppose that*

$$(7) \quad i) S^* = \sup_M S(x) < +\infty; \quad ii) |W|^* = \sup_M |W| < +\infty,$$

where W is the Weyl tensor of (M, \langle , \rangle) . Let $(M, \langle , \rangle, X)$ be a soliton structure on M with soliton constant λ . Assume

$$(8) \quad |X| \leq \sqrt{G(r)}$$

where G satisfies (4). Then, either (M, \langle , \rangle) is Einstein or $|T|^* = \sup_M |T|$ satisfies

$$(9) \quad |T|^* \geq \frac{1}{2} \left(\sqrt{m(m-1)}\lambda - S^* \frac{m-2}{\sqrt{m(m-1)}} - \sqrt{\frac{m-2}{2(m-1)}} |W|^* \right).$$

In case (M, \langle , \rangle) is conformally flat this result extends Theorem 1.7 of [12] and, since Remark 2 above applies again, it recovers Theorem 0.8 of [20] for gradient Ricci solitons.

As an immediate consequence of Theorem 1 we have

Corollary 5. *Let (M, \langle , \rangle) be a complete Riemannian manifold admitting a shrinking or steady soliton structure $(M, \langle , \rangle, X)$ satisfying (8) and (4). Then any minimal immersion of (M, \langle , \rangle) into \mathbb{R}^n , $n > m = \dim M$ is totally geodesic.*

Proof. Indicating by II the second fundamental tensor of the immersion by Gauss equations we have

$$S(x) = -|II|^2(x).$$

Thus if the immersion is not totally geodesic $S_* < 0$, contradicting (iii) or (ii) of Theorem 1. \square

Because of (5) the above Corollary specializes to

Corollary 6. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold admitting a shrinking or steady gradient soliton structure $(M, \langle \cdot, \cdot \rangle, \nabla f)$. Then any minimal immersion of $(M, \langle \cdot, \cdot \rangle)$ into \mathbb{R}^n , $n > m = \dim M$ is totally geodesic.*

2. SOME BASIC FORMULAS

The aim of this section is to collect and prove some basic formulas for generic Ricci solitons. Some of them have also been obtained in [12] but here we adopt a more direct approach which seems to be clearer and more elegant. In what follows, to perform computations, we shall use the method of the moving frame referring to a local orthonormal coframe. Here we fix the index range $1 \leq i, j, \dots \leq m = \dim M$ and we use the Einstein summation convention throughout.

First of all we recall some commutation formulas.

Lemma 7. *Let X be a vector field on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Then the following commutation formulas hold:*

$$\begin{aligned} (10) \quad & X_{ijk} - X_{ikj} = X_t R_{tijk}; \\ (11) \quad & X_{ijkl} - X_{ikjl} = R_{tijk} X_{tl} + R_{tijk,l} X_t; \\ (12) \quad & X_{ijkl} - X_{ijlk} = R_{tikl} X_{tj} + R_{tjkl} X_{it}. \end{aligned}$$

Proof. Equation (11) follows by taking the covariant derivative of (10).

Equations (10) and (12) are the standard commutation formulas of covariant derivatives acting on tensors. \square

Moreover we have the validity of the following general Ricci identities:

$$\begin{aligned} (13) \quad & R_{ij,k} = R_{ji,k}; \\ (14) \quad & R_{ij,k} - R_{ik,j} = -R_{tijk,t}; \\ (15) \quad & R_{ij,kl} - R_{ij,lk} = R_{tikl} R_{tj} + R_{tjkl} R_{it}; \\ (16) \quad & \frac{1}{2} S_k = R_{ki,i} = R_{ik,i}. \end{aligned}$$

Equation (13) is obvious from the symmetry of the Ricci tensor i.e. $R_{ij} = R_{ji}$.

Equation (14) follows from the second Bianchi's identities, indeed

$$\begin{aligned} R_{ij,k} - R_{ik,j} &= R_{tijk,k} - R_{titk,j} \\ &= R_{tijk,k} + R_{tikl,j} \\ &= -R_{tijk,t}. \end{aligned}$$

Equation (15) can be obtained using the definition of covariant derivative and the structure equations.

Equation (16) follows tracing (14).

We are now ready to prove

Lemma 8. *Let $(M, \langle \cdot, \cdot \rangle, X)$ be a soliton structure on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Then the following identities hold:*

$$(17) \quad \mathbf{R}_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda \delta_{ij},$$

$$(18) \quad \mathbf{S} + X_{ii} = m\lambda,$$

$$(19) \quad \mathbf{S}_j = -X_{ij},$$

$$(20) \quad \mathbf{R}_{lj}X_l = -X_{jii},$$

$$(21) \quad \mathbf{R}_{ij,k} - \mathbf{R}_{ik,j} = -\frac{1}{2}\mathbf{R}_{lijk}X_l + \frac{1}{2}(X_{kij} - X_{jik}),$$

$$(22) \quad \mathbf{R}_{ij,k} - \mathbf{R}_{kj,i} = \frac{1}{2}\mathbf{R}_{ljki}X_l + \frac{1}{2}(X_{kji} - X_{ijk}).$$

Proof. Equation (17) is a rewriting of (1). Equation (18) follows simply by tracing (17).

To obtain (19) take the covariant derivative of (18).

Next, taking the trace of the commutation formula (10) with respect to i and k we get

$$X_{iji} - X_{iij} = \mathbf{R}_{liji}X_l = \mathbf{R}_{lj}X_l.$$

Using this latter and equation (17) we compute

$$\begin{aligned} \mathbf{R}_{ij,i} &= -\frac{1}{2}(X_{iji} + X_{jii}) \\ &= -\frac{1}{2}(X_{iji} - X_{iij} + X_{iij} + X_{jii}) \\ &= -\frac{1}{2}\mathbf{R}_{lj}X_l - \frac{1}{2}(X_{iij} + X_{jii}). \end{aligned}$$

It follows by (16) that

$$\mathbf{S}_j = -\mathbf{R}_{lj}X_l - (X_{iij} + X_{jii})$$

and using (19) we deduce (20).

To prove (21) observe that taking the covariant derivative of (17) yields

$$\mathbf{R}_{ij,k} + \frac{1}{2}(X_{ijk} + X_{jik}) = 0$$

and

$$\mathbf{R}_{ik,j} + \frac{1}{2}(X_{ikj} + X_{kij}) = 0.$$

Taking the difference and using (17) we have

$$\begin{aligned} \mathbf{R}_{ij,k} - \mathbf{R}_{ik,j} &= -\frac{1}{2}(X_{ijk} + X_{jik} - X_{ikj} - X_{kij}) \\ &= -\frac{1}{2}(X_{ijk} - X_{ikj}) + \frac{1}{2}(X_{kij} - X_{jik}) \\ &= -\frac{1}{2}\mathbf{R}_{lijk}X_l + \frac{1}{2}(X_{kij} - X_{jik}), \end{aligned}$$

that is, (21).

Similarly, using the commutation formula (10) we obtain

$$\begin{aligned} 2(\mathbf{R}_{ij,k} - \mathbf{R}_{kj,i}) &= -(X_{ijk} + X_{jik} - X_{kji} - X_{jki}) \\ &= X_{kji} - X_{ijk} + X_l \mathbf{R}_{ljki}, \end{aligned}$$

that is, (22). □

Lemma 9. *Let $(M, \langle \cdot, \cdot \rangle, X)$ be a soliton structure on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, then*

$$(23) \quad \Delta_X R_{ik} = 2\lambda R_{ik} - 2R_{ijk}s R_{js} + \frac{1}{2}R_{is}(X_{sk} - X_{ks}) + \frac{1}{2}R_{sk}(X_{si} - X_{is}),$$

$$(24) \quad \Delta_X S = 2\lambda S - 2|Ric|^2.$$

Proof. To prove Equation (23) first observe that by (21)

$$R_{ki,j} - R_{kj,i} = \frac{1}{2}R_{lkji}X_l + \frac{1}{2}(X_{jki} - X_{ikj}),$$

and thus taking the covariant derivative we obtain the commutation relations

$$(25) \quad R_{ik,jt} - R_{jk,it} = \frac{1}{2}(R_{ijkl,t}X_l + R_{ijkl}X_{lt}) + \frac{1}{2}(X_{jkit} - X_{ikjt}).$$

Moreover, contracting the commutation relations (15) for the second covariant derivative of R_{jk}

$$(26) \quad \begin{aligned} R_{jk,ij} &= R_{jk,ji} + R_{sjij}R_{sk} + R_{skij}R_{js} \\ &= R_{jk,ji} + R_{si}R_{sk} + R_{skij}R_{js}. \end{aligned}$$

We now use (25) to obtain

$$\Delta R_{ik} = R_{ik,jj} = R_{jk,ij} + \frac{1}{2}(R_{ijkl,j}X_l + R_{ijkl}X_{lj}) + \frac{1}{2}(X_{jkij} - X_{ikjj}).$$

On the other hand, from the second Bianchi identities

$$\begin{aligned} R_{ijkl,j}X_l &= -R_{ijlj,k}X_l - R_{ijjk,l}X_l \\ &= -R_{il,k}X_l + R_{ik,l}X_l, \end{aligned}$$

and inserting into the above, with the aid of (26), yields

$$\begin{aligned} \Delta R_{ik} &= R_{jk,ij} + \frac{1}{2}(R_{ik,l} - R_{il,k})X_l + \frac{1}{2}R_{ijkl}X_{lj} + \frac{1}{2}(X_{jkij} - X_{ikjj}) \\ &= R_{jk,ji} + R_{si}R_{sk} + R_{skij}R_{js} + \frac{1}{2}(R_{ik,l} - R_{il,k})X_l \\ &\quad + \frac{1}{2}R_{ijkl}X_{lj} + \frac{1}{2}(X_{jkij} - X_{ikjj}). \end{aligned}$$

Hence, from (16) and (17)

$$\begin{aligned}
 \Delta R_{ik} &= \frac{1}{2}S_{ki} + R_{si}R_{sk} + R_{skij}R_{sj} - \frac{1}{2}R_{skij}X_{sj} \\
 &\quad + \frac{1}{2}(R_{ik,l} - R_{il,k})X_l + \frac{1}{2}(X_{jki} - X_{ikj}) \\
 &= \frac{1}{2}S_{ki} + R_{si}R_{sk} + 2R_{skij}R_{sj} - R_{skij}(-R_{sj} + \lambda\delta_{sj} - \frac{1}{2}X_{js}) \\
 &\quad + \frac{1}{2}(R_{ik,l} - R_{il,k})X_l + \frac{1}{2}(X_{jki} - X_{ikj}) \\
 &= \frac{1}{2}S_{ki} + R_{si}R_{sk} + 2R_{skij}R_{sj} + \lambda R_{ik} + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}(R_{ik,s} - R_{is,k})X_s \\
 &\quad + \frac{1}{2}(X_{jki} - X_{ikj}).
 \end{aligned}$$

We shall now deal with the sum

$$(27) \quad Z = \frac{1}{2}S_{ik} + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{1}{2}X_{skis}.$$

Towards this aim we first observe that taking the covariant derivative of (19) gives

$$\frac{1}{2}S_{ik} = -\frac{1}{2}X_{ssik}.$$

Using this information in (27) we obtain

$$\begin{aligned}
 Z &= -\frac{1}{2}X_{ssik} + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{1}{2}X_{skis} \\
 &= \frac{1}{2}(X_{skis} - X_{ssik}) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s \\
 &= \frac{1}{2}(X_{skis} - X_{sksi} + X_{sksi} - X_{sski}) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s \\
 &= \frac{1}{2}(R_{tsis}X_{tk} + R_{tkis}X_{st} + R_{tsks}X_{ti} + R_{tsks,i}X_t) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s \\
 &= \frac{1}{2}(R_{ti}X_{tk} + R_{tkis}X_{st} + R_{tk}X_{ti} + R_{tk,i}X_t) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s \\
 &= \frac{1}{2}(R_{ti}X_{tk} + R_{tkis}X_{st} + R_{tk}X_{ti}) + \frac{1}{2}(R_{sk,i} - R_{si,k})X_s + R_{sk}R_{si} \\
 &= \frac{1}{2}(R_{si}X_{sk} + R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s + R_{sk}R_{si} \\
 &= R_{si}(R_{sk} + \frac{1}{2}X_{sk}) + \frac{1}{2}(R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s \\
 &= R_{si}(-\frac{1}{2}X_{sk} + \lambda\delta_{sk}) + \frac{1}{2}(R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s \\
 &= -\frac{1}{2}R_{si}X_{ks} + \lambda R_{ik} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s.
 \end{aligned}$$

Substituting the above into the expression for ΔR_{ik} we deduce

$$\begin{aligned}\Delta R_{ik} &= -\frac{1}{2}R_{si}X_{ks} + 2\lambda R_{ik} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s \\ &\quad + 2R_{skij}R_{sj} + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s - \frac{1}{2}X_{ikss}.\end{aligned}$$

Now we note that by (11) and (12),

$$\begin{aligned}X_{ikss} - X_{issk} &= X_{ikss} - X_{isks} + X_{isks} - X_{issk} \\ &= R_{tiks}X_{ts} + R_{tik,s}X_t + R_{tik}X_{ts} + R_{tsk}X_{it},\end{aligned}$$

that is,

$$X_{ikss} = X_{issk} + R_{tik}X_{ts} + R_{tik,s}X_t + R_{tik}X_{ts} + R_{tk}X_{it}.$$

Moreover we know, taking the covariant derivative of (20), that

$$X_{issk} = -R_{ti,k}X_t - R_{ti}X_{tk}.$$

Thus,

$$X_{ikss} = -R_{ti,k}X_t - R_{ti}X_{tk} + R_{tik}X_{ts} + R_{tik,s}X_t + R_{tik}X_{ts} + R_{tk}X_{it},$$

and substituting above, using the first Bianchi identities and (14),

$$\begin{aligned}\Delta R_{ik} &= 2\lambda R_{ik} - 2R_{ijks}R_{js} - \frac{1}{2}R_{si}X_{ks} + \frac{1}{2}R_{tkis}X_{st} \\ &\quad + \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s \\ &\quad - \frac{1}{2}(-R_{it,k}X_t - R_{it}X_{tk} + R_{tik}X_{ts} + R_{tik,s}X_t + R_{tik}X_{ts} + R_{tk}X_{it}) \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} - \frac{1}{2}R_{is}X_{ks} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s \\ &\quad + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s + \frac{1}{2}R_{is,k}X_s + \frac{1}{2}R_{is}X_{sk} \\ &\quad - \frac{1}{2}R_{tik}X_{ts} - \frac{1}{2}R_{tik,s}X_t - \frac{1}{2}R_{tik}X_{ts} - \frac{1}{2}R_{sk}X_{is} \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} + R_{ik,s}X_s + \frac{1}{2}R_{is}(X_{sk} - X_{ks}) + \frac{1}{2}R_{sk}(X_{si} - X_{is}) \\ &\quad + R_{tkis}X_{st} - R_{tik}X_{ts} + \frac{1}{2}R_{tik,t} - \frac{1}{2}R_{tik,t}X_s,\end{aligned}$$

implying (23). We conclude observing that (24) is easily obtained by tracing (23). \square

Our next step is to compute the X -Laplacian of the square norm of the traceless Ricci tensor $T = \text{Ric} - \frac{S}{m}$ using Lemma 9. The following result recovers Proposition 2.6 in [12] but the proof is more straightforward. Before stating the lemma we recall the splitting of the Riemann curvature tensor:

$$(28) \quad \begin{aligned} \mathbf{R}_{ijks} &= \mathbf{W}_{ijks} + \frac{1}{m-2}(\mathbf{R}_{ik}\delta_{js} - \mathbf{R}_{is}\delta_{jk} + \mathbf{R}_{js}\delta_{ik} - \mathbf{R}_{jk}\delta_{is}) \\ &\quad - \frac{S}{(m-1)(m-2)}(\delta_{ik}\delta_{js} - \delta_{is}\delta_{jk}), \end{aligned}$$

where \mathbf{W}_{ijks} are the components of the Weyl curvature tensor.

Lemma 10. *Let $(M, \langle \cdot, \cdot \rangle, X)$ be a Ricci soliton structure on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Then*

$$(29) \quad \frac{1}{2}\Delta_X |\mathbf{T}|^2 = |\nabla \mathbf{T}|^2 + 2\left(\lambda - \frac{m-2}{m(m-1)}S\right)|\mathbf{T}|^2 + \frac{4}{m-2}\text{Tr}(\mathbf{T}^3) - 2\mathbf{T}_{ik}\mathbf{T}_{sj}\mathbf{W}_{ksij}.$$

Proof. From the definition of \mathbf{T} we have

$$(30) \quad |\mathbf{T}|^2 = |\text{Ric}|^2 - \frac{S^2}{m},$$

$$(31) \quad |\nabla \mathbf{T}|^2 = |\nabla \text{Ric}|^2 - \frac{|\nabla S|^2}{m}.$$

Using (23) we deduce

$$(32) \quad \begin{aligned} \Delta_X |\text{Ric}|^2 &= 2(\Delta \mathbf{R}_{ik})\mathbf{R}_{ik} + 2|\nabla \text{Ric}|^2 - 2\mathbf{R}_{ik}\mathbf{R}_{ik,s}X_s \\ &= 2\mathbf{R}_{ik}\Delta_X \mathbf{R}_{ik} + 2|\nabla \text{Ric}|^2 \\ &= 4\lambda |\text{Ric}|^2 - 4\mathbf{R}_{ijks}\mathbf{R}_{ik}\mathbf{R}_{js} + \mathbf{R}_{is}\mathbf{R}_{ik}(X_{sk} - X_{ks}) \\ &\quad + \mathbf{R}_{sk}\mathbf{R}_{ik}(X_{si} - X_{is}) + 2|\nabla \text{Ric}|^2 \\ &= 4\lambda |\text{Ric}|^2 - 4\mathbf{R}_{ijks}\mathbf{R}_{ik}\mathbf{R}_{js} + 2|\nabla \text{Ric}|^2, \end{aligned}$$

where in the last inequality we have used the fact that

$$\mathbf{R}_{is}\mathbf{R}_{ik}(X_{sk} - X_{ks}) = \mathbf{R}_{sk}\mathbf{R}_{ik}(X_{si} - X_{is}) = 0.$$

Indeed, both quantities are products of a term symmetric in s, k with another antisymmetric in s, k . Moreover, using (24),

$$(33) \quad \begin{aligned} \Delta_X S^2 &= 2|\nabla S|^2 + 2S\Delta_X S \\ &= 2|\nabla S|^2 + 4\lambda S^2 - 4S|\text{Ric}|^2. \end{aligned}$$

Hence, by (30), (31) (32) and (33) we compute

$$\begin{aligned} \frac{1}{2}\Delta_X |\mathbf{T}|^2 &= \frac{1}{2}\Delta_X |\text{Ric}|^2 - \frac{1}{2m}\Delta_X S^2 \\ &= 2\lambda |\text{Ric}|^2 - 2\mathbf{R}_{ijks}\mathbf{R}_{ik}\mathbf{R}_{js} + |\nabla \text{Ric}|^2 - \frac{1}{m}|\nabla S|^2 - \frac{2}{m}\lambda S^2 + \frac{2}{m}S|\text{Ric}|^2 \\ &= 2\lambda |\mathbf{T}|^2 - 2\mathbf{R}_{ijks}\mathbf{R}_{ik}\mathbf{R}_{js} + \frac{2}{m}S|\text{Ric}|^2 + |\nabla \mathbf{T}|^2. \end{aligned}$$

Observe now that, using the splitting of the curvature tensor (28),

$$(34) \quad \begin{aligned} R_{ijks}R_{ik}R_{js} &= W_{ijks}R_{ik}R_{js} + \frac{(2m-1)}{(m-1)(m-2)}S|\text{Ric}|^2 \\ &\quad - \frac{2}{m-2}\text{Tr}(\text{Ric}^3) - \frac{S^3}{(m-1)(m-2)}. \end{aligned}$$

Moreover, since all the traces of the Weyl tensor vanish we have that

$$(35) \quad W_{ijks}R_{ik}R_{js} = W_{ijks}T_{ik}T_{js}.$$

Using (34) and (35) in the above computation we obtain (29). \square

3. PROOF OF THE RESULTS

To prove our main results we first need to introduce some auxiliary analytical lemmas. We adapt the proof of Theorem 1.9 of [21] to show the validity of the next

Lemma 11. *Assume on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ the existence of a non-negative C^2 function γ satisfying the following conditions*

$$(36) \quad \gamma(x) \rightarrow +\infty \text{ as } r(x) \rightarrow \infty,$$

$$(37) \quad \exists A > 0 \text{ such that } |\nabla\gamma| \leq A\gamma^{\frac{1}{2}} \text{ off a compact set,}$$

$$(38) \quad \exists B > 0 \text{ such that } \Delta_X\gamma \leq B\gamma^{\frac{1}{2}}G\left(\gamma^{\frac{1}{2}}\right)^{\frac{1}{2}} \text{ off a compact set,}$$

with $X \in \mathfrak{X}(M)$, G as in (4), A, B positive constants. Then, given any function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence $\{z_k\}_k \subset M$ such that

$$(39) \quad (i) u(z_k) > u^* - \frac{1}{k}, \quad (ii) |\nabla u(z_k)| < \frac{1}{k}, \quad (iii) \Delta_X u(z_k) < \frac{1}{k},$$

for each $k \in \mathbb{N}$, i.e. the full Omori-Yau maximum principle for Δ_X holds on $(M, \langle \cdot, \cdot \rangle)$.

Remark 12. For $X \equiv 0$ we recover Theorem 1.9 of [21].

Proof. We define the function

$$\psi(t) = \exp \int_0^t \frac{ds}{\sqrt{G(s)}}$$

and we note that, by the properties of G , ψ is well defined, smooth, positive and it satisfies

$$(40) \quad \psi(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty,$$

$$(41) \quad 0 < \frac{\psi'(t)}{\psi(t)} \leq \frac{C}{t^{\frac{1}{2}}\sqrt{G(t^{\frac{1}{2}})}},$$

for some constant $C > 0$. Next, we fix a point $p \in M$ and, for each $k \in \mathbb{N}$ and u as in the statement of the lemma, we define

$$(42) \quad f_k(x) = \frac{u(x) - u(p) + 1}{\psi(\gamma(x))^{\frac{1}{k}}}.$$

Then, $f_k(p) > 0$. From $u^* < +\infty$ and $\psi(\gamma(x)) \rightarrow +\infty$ as $r(x) \rightarrow +\infty$ we deduce that $\limsup_{r(x) \rightarrow +\infty} f_k(x) \leq 0$. Thus f_k attains a positive absolute maximum at some $x_k \in M$. In this way we produce a sequence $\{x_k\}$. Proceeding as in the proof of Theorem 1.9 of [21], (up to passing to a subsequence if needed)

$$(43) \quad u(x_k) \rightarrow u^* \quad \text{as } k \rightarrow +\infty.$$

If $\{x_k\}$ remains in a compact set, then $x_k \rightarrow \bar{x} \in M$ as $k \rightarrow +\infty$ (up to passing to a subsequence) and u attains its absolute maximum.

At \bar{x} we have

$$(44) \quad u(\bar{x}) = u^*; \quad |\nabla u(\bar{x})| = 0; \quad \Delta_X u(\bar{x}) = \Delta u(\bar{x}) - \langle X, \nabla u \rangle(\bar{x}) \leq 0.$$

In this case we let $z_k = \bar{x}$ for any $k \in \mathbb{N}$. We now consider the case $x_k \rightarrow \infty$ so that $\gamma(x_k) \rightarrow +\infty$. Since f_k attains a positive maximum at x_k we have

$$(45) \quad (i) \nabla \log f_k(x_k) = 0; \quad (ii) \Delta_X \log f_k(x_k) \leq 0.$$

From (45) (i), since

$$\nabla(\log f_k) = \nabla \left(\log \left(\frac{u(x) - u(p) + 1}{\psi(\gamma(x))^{\frac{1}{k}}} \right) \right) = \frac{\nabla u(x)}{u(x) - u(p) + 1} - \frac{1}{k} \frac{\psi'(\gamma(x))}{\psi(\gamma(x))} \nabla \gamma(x),$$

we have

$$0 = \nabla(\log f_k)(x_k) = \frac{\nabla u(x_k)}{u(x_k) - u(p) + 1} - \frac{1}{k} \frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \nabla \gamma(x_k),$$

that is,

$$(46) \quad \nabla u(x_k) = \frac{1}{k} \frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \nabla \gamma(x_k) (u(x_k) - u(p) + 1).$$

Reasoning as in [21], page 9, we deduce

$$(47) \quad \Delta u(x_k) \leq \frac{u(x_k) - u(p) + 1}{k} \left\{ \frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \Delta \gamma(x_k) + \frac{1}{k} \left[\frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \right]^2 |\nabla \gamma(x_k)|^2 \right\}$$

A computation using (45) (i), (37) and (41) shows that

$$(48) \quad |\nabla u|(x_k) \leq \frac{a}{k} \frac{u(x_k) - u(p) + 1}{\sqrt{G(\gamma^{\frac{1}{2}})}}$$

Therefore, using (46), (47), (41) and the assumptions (37) and (38) we obtain (49)

$$\begin{aligned}
\Delta_X u(x_k) &= \Delta u(x_k) - \langle X, \nabla u \rangle(x_k) \\
&\leq \frac{u(x_k) - u(p) + 1}{k} \left\{ \frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \Delta \gamma(x_k) + \frac{1}{k} \left[\frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \right]^2 |\nabla \gamma(x_k)|^2 \right\} \\
&\quad - \frac{1}{k} (u(x_k) - u(p) + 1) \frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \langle \nabla \gamma(x_k), X(x_k) \rangle \\
&= \frac{u(x_k) - u(p) + 1}{k} \left[\frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \Delta_X \gamma(x_k) + \frac{1}{k} \left[\frac{\psi'(\gamma(x_k))}{\psi(\gamma(x_k))} \right]^2 |\nabla \gamma|^2(x_k) \right] \\
&\leq \frac{u(x_k) - u(p) + 1}{k} \left[C(\gamma(x_k) G(\gamma(x_k)^{\frac{1}{2}}))^{-\frac{1}{2}} B \gamma(x_k)^{\frac{1}{2}} G(\gamma(x_k)^{\frac{1}{2}})^{\frac{1}{2}} \right. \\
&\quad \left. + \frac{1}{k} C^2(\gamma(x_k) G(\gamma(x_k)^{\frac{1}{2}}))^{-1} A^2 \gamma(x_k) \right] \\
&= \frac{u(x_k) - u(p) + 1}{k} \left[CB + \frac{1}{k} C^2 A^2 G(\gamma(x_k)^{\frac{1}{2}})^{-1} \right].
\end{aligned}$$

and the RHS tends to zero as $k \rightarrow +\infty$. In this case we choose $z_k = x_k$ and (43), (48), (49) together with $u^* < +\infty$ show the validity of the lemma. \square

The next result follows from the easily verified formula

$$(50) \quad \Delta_X \varphi(u) = \varphi'(u) \Delta_X u + \varphi''(u) |\nabla u|^2$$

valid for $u \in C^2(M)$, $\varphi \in C^2(\mathbb{R})$ and Lemma 11 with the same reasoning as in the proof of Theorem 1.31 in [21]; see also Theorem 2.6 in [26]. Precisely we have

Lemma 13. *In the assumptions of Lemma 11 let $u \in C^2(M)$ be a solution of the differential inequality*

$$(51) \quad \Delta_X u \geq \rho(u, |\nabla u|)$$

with $\rho(t, y)$ continuous in t , C^2 in y and such that

$$\frac{\partial^2 \rho}{\partial y^2}(t, y) \geq 0 \quad \forall y \geq 0.$$

Let $f(t) = \rho(t, 0)$. Then a sufficient condition to guarantee that $u^* = \sup_M u < +\infty$, is the existence of a continuous function F , positive on $[a, +\infty)$ for some $a \in \mathbb{R}$ and satisfying

$$\begin{aligned}
&\left\{ \int_a^t F(s) ds \right\}^{-\frac{1}{2}} \in L^1(+\infty), \\
&\limsup_{t \rightarrow +\infty} \frac{\int_a^t F(s) ds}{t F(t)} < +\infty, \\
&\liminf_{t \rightarrow +\infty} \frac{f(t)}{F(t)} > 0, \\
&\liminf_{t \rightarrow +\infty} \frac{\left\{ \int_a^t F(s) ds \right\}^{\frac{1}{2}}}{F(t)} \frac{\partial \rho}{\partial y} \Big|_{(t, 0)} > -\infty.
\end{aligned}$$

Furthermore in this case

$$f(u^*) \leq 0.$$

In the next result we shall use the following generalized Bochner formula which can be easily deduced from the usual one, see [18]. For $X \in \mathfrak{X}(M)$ and $u \in C^3(M)$,

$$(52) \quad \frac{1}{2} \Delta_X |\nabla u|^2 = |Hess(u)|^2 + \langle \nabla \Delta_X u, \nabla u \rangle + Ric_X(\nabla u, \nabla u)$$

with

$$(53) \quad Ric_X = Ric + \frac{1}{2} \mathcal{L}_X \langle \cdot, \cdot \rangle.$$

Note that, when $X = \nabla f$, $f \in C^\infty(M)$ then $Ric_{\nabla f} = Ric_f$, the usual Bakry-Emery Ricci tensor

$$Ric_f = Ric + Hess(f).$$

We are now ready to state our next lemma which essentially recovers Theorem 2.1 (ii) in [24] by Z. Qian.

Lemma 14. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold of dimension m and $X \in \mathfrak{X}(M)$ a vector field satisfying the growth condition*

$$(54) \quad |X| \leq \sqrt{G(r)}$$

for some positive nondecreasing function $G \in C^1([0, +\infty))$. Suppose that

$$(55) \quad Ric_X(\nabla r, \nabla r) \geq -(m-1)\overline{G}(r),$$

for some C^1 positive function \overline{G} on $[0, +\infty)$ such that

$$(56) \quad \inf_{[0, +\infty)} \frac{\overline{G}'}{\overline{G}^{\frac{3}{2}}} > -\infty.$$

Then there exists $A = A(m) > 0$ sufficiently large such that

$$(57) \quad \Delta_X r \leq A\sqrt{\overline{G}} + \sqrt{\overline{G}}$$

pointwise in $M \setminus (\{o\} \cup cut(o))$ and weakly on all of M .

Proof. Let h be the solution on \mathbb{R}_0^+ of the Cauchy problem

$$(58) \quad \begin{cases} h'' - \overline{G}h = 0 \\ h(0) = 0, \quad h'(0) = 1. \end{cases}$$

Note that $h > 0$ on \mathbb{R}^+ since $\overline{G} \geq 0$. Fix $x \in M \setminus (cut(o) \cup \{o\})$, with $cut(o)$ the cut locus of o , and let $\gamma : [0, l] \rightarrow M$, $l = length(\gamma)$, be a minimizing geodesic with $\gamma(0) = o$, $\gamma(l) = x$. Note that $\overline{G}(r \circ \gamma)(t) = \overline{G}(t)$. From Bochner formula applied to the distance function r (outside $cut(o) \cup \{o\}$) we have

$$(59) \quad 0 = |Hess(r)|^2 + \langle \nabla r, \nabla \Delta r \rangle + Ric(\nabla r, \nabla r)$$

so that, using the inequality

$$|Hess(r)|^2 \geq \frac{(\Delta r)^2}{m-1},$$

it follows that the function $\varphi(t) = (\Delta r) \circ \gamma(t)$, $t \in (0, l]$, satisfies the Riccati differential inequality

$$(60) \quad \varphi' + \frac{1}{m-1}\varphi^2 \leq -\text{Ric}(\nabla r \circ \gamma, \nabla r \circ \gamma)$$

on $(0, l]$. With h as in (58) and using the definition of Ric_X , (55) and (60) we compute

$$\begin{aligned} (h^2\varphi)' &= 2hh'\varphi + h^2\varphi' \\ &\leq 2hh'\varphi - \frac{h^2\varphi^2}{m-1} + h^2(m-1)\overline{G}(t) + h^2\langle \nabla_{\nabla r} X, \nabla r \rangle \circ \gamma \\ &= -\left(\frac{h\varphi}{\sqrt{m-1}} - \sqrt{m-1}h'\right)^2 + (m-1)(h')^2 + h^2(m-1)\overline{G}(t) \\ &\quad + h^2(\langle X, \dot{\gamma} \rangle \circ \gamma)'. \end{aligned}$$

We let

$$\varphi_{\overline{G}}(t) = (m-1)\frac{h'}{h}(t)$$

so that, using (58)

$$(h^2\varphi_{\overline{G}})' = (m-1)(h')^2 + (m-1)\overline{G}(t)h^2.$$

Inserting this latter into the above inequality we obtain

$$(h^2\varphi)' \leq (h^2\varphi_{\overline{G}})' + h^2(\langle X, \dot{\gamma} \rangle \circ \gamma)'$$

Integrating on $[0, r]$ and using (58) yields

$$(61) \quad h^2(r)\varphi(r) \leq h^2(r)\varphi_{\overline{G}}(r) + \int_0^r h^2(\langle X, \dot{\gamma} \rangle \circ \gamma)'.$$

Next, we define that

$$\varphi_X = (\Delta_X r) \circ \gamma = (\Delta r) \circ \gamma - \langle X, \nabla r \rangle \circ \gamma = \varphi - \langle X, \nabla r \rangle \circ \gamma.$$

Thus, using (61), (58) and integrating by parts we compute

$$\begin{aligned} h^2\varphi_X(r) &\leq h^2\varphi_{\overline{G}}(r) - h^2\langle X, \nabla r \rangle \circ \gamma(r) + \int_0^r h^2(\langle X, \dot{\gamma} \rangle \circ \gamma)' dt \\ &= h^2\varphi_{\overline{G}} - h^2\langle X, \nabla r \rangle \circ \gamma(r) + h^2(\langle X, \dot{\gamma} \rangle \circ \gamma) \Big|_0^r - \int_0^r (h^2)'(\langle X, \dot{\gamma} \rangle \circ \gamma) dt \\ &= h^2\varphi_{\overline{G}}(r) - \int_0^r (h^2)'(\langle X, \dot{\gamma} \rangle \circ \gamma) dt, \end{aligned}$$

that is,

$$(62) \quad h^2\varphi_X(r) \leq h^2\varphi_{\overline{G}}(r) - \int_0^r (h^2)'(\langle X, \dot{\gamma} \rangle \circ \gamma) dt$$

on $(0, l]$. Observe now that, by the Cauchy–Schwarz inequality and (54),

$$-\langle X, \dot{\gamma} \rangle \leq |X| \leq \sqrt{G(r)}.$$

Inserting this into (62), using the fact that $(h^2)' \geq 0$, integrating by parts and recalling that $G' \geq 0$, we obtain

$$h^2 \varphi_X(r) \leq h^2 \varphi_{\bar{G}}(r) + h^2 \sqrt{G(r)} - \int_0^r \frac{h^2 G'(t)}{2\sqrt{G(t)}} dt \leq h^2 \varphi_{\bar{G}}(r) + h^2 \sqrt{G(r)}$$

on $(0, l]$. It follows that

$$\varphi_X(r) \leq \varphi_{\bar{G}}(r) + \sqrt{G(r)}$$

on $(0, l]$. In particular

$$(63) \quad \Delta_X r(x) \leq (m-1) \frac{h'(r(x))}{h(r(x))} + \sqrt{G(r(x))}$$

pointwise on $M \setminus (\{0\} \cup \text{cut}(o))$. Proceeding as in Theorem 2.4 of [22] one shows that (63) holds weakly on all of M . Moreover, fixing $\bar{A} \in \mathbb{R}_+$ and defining

$$\bar{h}(t) = \frac{\bar{A}^{-1}}{\sqrt{G(0)}} \left\{ e^{\bar{A} \int_0^t \sqrt{G(s)} ds} - 1 \right\},$$

we have $\bar{h}(0) = 0$, $\bar{h}'(0) = 1$ and

$$\bar{h}'' - \bar{G} \bar{h} \geq \frac{\bar{G}}{\sqrt{G(0)}} \left[e^{\bar{A} \int_0^t \sqrt{G(s)} ds} \left(\inf_{[0, +\infty)} \frac{\bar{G}'^{\frac{3}{2}}}{2\bar{G}}(t) + \bar{A} - \frac{1}{\bar{A}} \right) \right] \geq 0$$

by (56), for \bar{A} sufficiently large. So by Sturm comparison (see e.g. Lemma 2.1 in [22]),

$$\frac{h'}{h} \leq \frac{\bar{h}'}{\bar{h}} = D\sqrt{\bar{G}} \frac{\exp \left[D \int_0^t \sqrt{\bar{G}} ds \right]}{\exp \left[D \int_0^t \sqrt{\bar{G}} ds \right] - 1} \leq D\sqrt{\bar{G}}$$

for some constant $D > 0$ sufficiently large. Thus from (63) we finally obtain

$$\begin{aligned} \Delta_X r &\leq (m-1)D\sqrt{\bar{G}} + \sqrt{G} \\ &\leq A\sqrt{\bar{G}} + \sqrt{G} \end{aligned}$$

for some constant $A = A(m) > 0$, pointwise in $M \setminus (\{o\} \cup \text{cut}(o))$ and weakly on all of M . \square

Remark 15. In case G is non-increasing, inspection of the above proof yields the estimate

$$\Delta_X r \leq A\sqrt{\bar{G}} + \sqrt{G(0)}$$

instead of (57).

Remark 16. Since with the above notation $\varphi = \varphi_X + \langle X, \nabla r \rangle \circ \gamma$, from (57) and (54) we obtain $\Delta r \leq B(\sqrt{\bar{G}} + \sqrt{G})$ for some $B \geq \max\{A, 2\}$. This observation shows that if (M, \langle, \rangle, X) is a soliton structure on the complete manifold (M, \langle, \rangle) satisfying (3) with G as in (4), applying Lemma 11 with $\gamma(r) = r(x)^2$, we have the validity of the full Omori-Yau maximum principle (and in particular, (M, \langle, \rangle) is stochastically complete; see for instance the consequence given in Corollary 20

below). In particular, using again Remark 2, one gets that on a gradient Ricci soliton we have always the validity of the full Omori–Yau maximum principle for the operator Δ . This last fact was, very recently, established in [2]. We are grateful to S. Pigola for pointing out this to us. See also [5] for the shrinking gradient case.

Proof. (of Theorem 1). First of all we observe that since $(M, \langle \cdot, \cdot \rangle, X)$ is a soliton structure on M ,

$$\text{Ric}_X(\nabla r, \nabla r) = \lambda.$$

Since by assumption X satisfies (3), Lemma 14 holds and the function $\gamma(x) = r(x)^2$ satisfies assumptions (36), (37), (38) of Lemma 11. We therefore can apply Lemma 13. On the other hand, by (24) of Lemma 9, we have

$$(64) \quad \frac{1}{2}\Delta_X S = \lambda S - |\text{Ric}|^2 = \lambda S - \frac{S^2}{m} - \left| \text{Ric} - \frac{S}{m} \langle \cdot, \cdot \rangle \right|^2$$

from which, setting $u = -S$ we immediately deduce the differential inequality

$$(65) \quad \frac{1}{2}\Delta_X u \geq \lambda u + \frac{u^2}{m}.$$

We apply Lemma 13 to (65) with the choices $F(t) = t^2$,

$$\rho(u, |\nabla u|) = \lambda u + \frac{u^2}{m}.$$

Then $u^* < +\infty$ and

$$(66) \quad \lambda u^* + \frac{(u^*)^2}{m} \leq 0.$$

But $u^* = -S_*$ so that the claimed bounds on S_* in the statement of Theorem 1 follow immediately from (66). The rest of the proof is as in the proof of Theorem 1.4 of [12]. \square

Before proceeding with the proof of Theorem 4 we recall the next estimate due to Huisken, see [8], Lemma 3.4:

$$(67) \quad |\text{T}_{ik} \text{T}_{jk} W_{ijkl}| \leq \frac{\sqrt{2}}{2} \sqrt{\frac{m-2}{m-1}} |W| |T|^2.$$

Proof. (of Theorem 4). From (29), Okumura’s lemma [14] and Huisken’s estimate (67) we obtain

$$\frac{1}{2}\Delta_X |T|^2 \geq |\nabla T|^2 + 2 \left(\lambda - \frac{m-2}{m(m-1)} S - \sqrt{\frac{m-2}{2(m-1)}} |W| \right) |T|^2 - \frac{4}{\sqrt{m(m-1)}} |T|^3.$$

We set $u = |T|^2$ and from the above we deduce

$$(68) \quad \frac{1}{2}\Delta_X u \geq 2 \left(\lambda - \frac{m-2}{m(m-1)} S^* - \sqrt{\frac{m-2}{2(m-1)}} |W|^* \right) u - \frac{4}{\sqrt{m(m-1)}} u^{\frac{3}{2}}.$$

We note that if $|T|^* = +\infty$ then (9) is obviously satisfied otherwise $u^* < +\infty$ and in the assumptions of Theorem 4 we have the validity of Lemma 11. It follows that

$$u^* \left[\frac{1}{2} \left(\lambda \sqrt{m(m-1)} - \frac{m-2}{\sqrt{m(m-1)}} S^* - \sqrt{\frac{m-2}{2(m-1)}} |W|^* \right) - \sqrt{u^*} \right] \leq 0,$$

from which we deduce that either $u^* = 0$, that is, $T \equiv 0$ on M , or $|T|^*$ satisfies (9). In the first case (M, \langle, \rangle) is Einstein. \square

Remark 17. As observed (in the gradient case) in [3], if (M, \langle, \rangle) is Einstein and in addition it is a shrinking soliton which is not Ricci flat, by Theorem 1, S is a positive constant and thus (M, \langle, \rangle) is compact by Myers' Theorem. In this latter case, if $|W|$ is sufficiently small, precisely if $|W|^2 \leq \frac{4}{(m+1)m(m-1)(m-2)} S^2$, then (M, \langle, \rangle) has positive curvature operator ([8], Corollary 2.5). Thus, from Tachibana [27], (M, \langle, \rangle) has positive constant sectional curvature and therefore is a finite quotient of S^m .

The next Proposition is a consequence of Lemma 14.

Proposition 18. *Let (M, \langle, \rangle) be a complete manifold and $X \in \mathfrak{X}(M)$ a vector field satisfying*

$$|X| \leq \sqrt{G(r)}$$

for some nonnegative nondecreasing function $G \in C^0([0, +\infty))$. Assume

$$\text{Ric}_X(\nabla r, \nabla r) \geq -(m-1)\overline{G}(r)$$

for some smooth positive function $\overline{G}(r)$ on $[0, +\infty)$ such that

$$\inf_{[0, +\infty)} \frac{\overline{G}'}{\overline{G}^{3/2}} > -\infty.$$

Then

$$(69) \quad \text{vol}(\partial B_r) \leq C e^{B \int_0^r (\sqrt{\overline{G}(s)} + \sqrt{G(s)}) ds}$$

for almost every r , and, as a consequence,

$$(70) \quad \text{vol}(B_r) \leq C \int_0^r \left(e^{B \int_0^t (\sqrt{\overline{G}(s)} + \sqrt{G(s)}) ds} \right) dt + D,$$

with C, B, D sufficiently large positive constants.

Proof. We let

$$h(r) = e^{\frac{B}{m-1} \int_0^r (\sqrt{\overline{G}(s)} + \sqrt{G(s)}) ds} - 1,$$

with $B \geq \max\{A, 2\}$. Since $\Delta r = \Delta_X r + \langle X, \nabla r \rangle$, from (57), (3) and the choice of h we have

$$\Delta r \leq (m-1) \frac{h'(r)}{h(r)}$$

pointwise on $M \setminus (\{0\} \cup \text{cut}(o))$ and weakly on all of M . We now follow the argument of the proof of Theorem 2.14 of [22]. Thus for each $\varphi \in \text{Lip}_0(M)$, $\varphi \geq 0$,

$$(71) \quad - \int \langle \nabla r, \nabla \varphi \rangle \leq (m-1) \int \frac{h'(r(x))}{h(r(x))} \varphi.$$

Next, we fix $0 < r < R$ and $\varepsilon > 0$ and we let ρ_ε be the piecewise linear function

$$\rho_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0, r) \\ \frac{t-r}{\varepsilon} & \text{if } t \in [r, r+\varepsilon) \\ 1 & \text{if } t \in [r+\varepsilon, R-\varepsilon) \\ \frac{R-t}{\varepsilon} & \text{if } t \in [R-\varepsilon, R) \\ 0 & \text{if } t \in [R, +\infty), \end{cases}$$

and we define the radial cut-off function

$$\varphi_\varepsilon(x) = \rho_\varepsilon(r(x))h(r(x))^{1-m}.$$

Indicating with $\chi_{s,t}$, $s < t$, the characteristic function of the annulus $B_t \setminus B_s$, we have

$$\nabla \varphi_\varepsilon = \left\{ \frac{1}{\varepsilon} \chi_{r, r+\varepsilon} - \frac{1}{\varepsilon} \chi_{R-\varepsilon, R} - (m-1) \frac{h'(r)}{h(r)} \rho_\varepsilon \right\} h(r)^{1-m} \nabla r$$

for a.e. $x \in M$. Therefore, using φ_ε into (71) and simplifying we get

$$\frac{1}{\varepsilon} \int_{B_R \setminus B_{R-\varepsilon}} h(r)^{1-m} \leq \frac{1}{\varepsilon} \int_{B_{r+\varepsilon} \setminus B_r} h(r)^{1-m}.$$

Using the co-area formula

$$\frac{1}{\varepsilon} \int_{R-\varepsilon}^R h(t)^{1-m} \text{vol}(\partial B_t) dt \leq \frac{1}{\varepsilon} \int_r^{r+\varepsilon} h(t)^{1-m} \text{vol}(\partial B_t) dt$$

and letting $\varepsilon \searrow 0^+$

$$\frac{\text{vol}(\partial B_R)}{h(R)^{m-1}} \leq \frac{\text{vol}(\partial B_r)}{h(r)^{m-1}},$$

for a.e. $0 < r < R$. Letting $r \rightarrow 0$, recalling that $\text{vol}(\partial B_r) \sim c_m r^{m-1}$ and $h(r) \sim r$, we obtain that

$$\text{vol}(\partial B_r) \leq C e^{\text{B} \int_0^R (\sqrt{\overline{G}(s)} + \sqrt{G(s)}) ds}$$

for some constant $C > 0$ and a.e. R . Using the co-area formula we then also deduce (70). \square

With the aid of the above estimates we can again conclude that

Corollary 19. *In the assumptions of Proposition 18 if $G(r)$ and $\overline{G}(r)$ are of the form $r^2 \prod_{j=1}^N (\log^{(j)}(r))^2$ for $r \gg 1$, then (M, \langle, \rangle) is stochastically complete.*

Proof. From Proposition 18 and the assumption on $G(r)$ and $\overline{G}(r)$

$$\frac{r}{\log \text{vol}(B_r)} \notin L^1(+\infty)$$

and we can apply Grigor'yan sufficient condition, [6], for stochastic completeness. \square

An immediate consequence of Corollary 19 is, for instance, the following

Corollary 20. *If $(M, \langle \cdot, \cdot \rangle)$ is a complete manifold supporting a generic Ricci soliton structure satisfying (3) then it can not be minimally immersed into a non-degenerate cone of the Euclidean space.*

Proof. By (5) and the above Corollary 19, or by Remark 16, $(M, \langle \cdot, \cdot \rangle)$ is stochastically complete. Then the result follows from Theorem 1.4 in [11]. \square

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, I-20133 MILANO, ITALY

E-mail address: `paolo.mastrolia@gmail.com`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, I-20133 MILANO, ITALY

E-mail address: `marco.rigoli@unimi.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, I-20133 MILANO, ITALY

E-mail address: `michele.rimoldi@unimi.it`