LARGE DEVIATIONS FOR TRUNCATED HEAVY-TAILED RANDOM VARIABLES: A BOUNDARY CASE

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Abstract. This paper investigates the decay rate of the probability that the row sum of a triangular array of truncated heavy tailed random variables is larger than an integer (k) times the truncating threshold, as both - the number of summands and the threshold go to infinity. The method of attack for this problem is significantly different from the one where k is not an integer, and requires much sharper estimates.

1. Introduction

Suppose $Y, Y_1, Y_2, \ldots$ are i.i.d. random variables with $P(Y > \cdot)$ regularly varying with index $-\alpha$, for some $\alpha > 0$, and let (M_n) be a sequence satisfying

$$\lim_{n\to\infty} nP(Y > M_n) = 0 .$$

Define the triangular array $\{X_{nj} : 1 \le j \le n\}$ by

$$X_{nj} := Y_j \mathbf{1}(|Y_j| \le M_n) , \tag{1.1}$$

and denote the row sum by

$$S_n := \sum_{j=1}^{n} X_{nj} . \tag{1.2}$$

This paper studies the decay rate of

$$P\left(S_n > kM_n\right) \tag{1.3}$$

as $n \to \infty$, for a fixed positive integer k, under some additional assumptions. For the motivation behind studying such a truncated heavy-tailed model, the reader is referred to Chakrabarty and Samorodnitsky (2009).

The case when k is an integer is more challenging than the case when k is a fraction because of the following simple reason. When k is not an integer, a result similar to Theorem 2.2 in Chakrabarty (2011) holds, which

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essentially shows that the probability in (1.3) is asymptotically equivalent to

$$\binom{n}{\lceil k \rceil} P\left(\sum_{j=1}^{\lceil k \rceil} X_{nj} > kM_n\right),$$

and therefore argues that the probability is of the order $\{nP(Y > M_n)\}^{\lceil k \rceil}$. When k is an integer, the natural guess is that the same equivalence holds, with $\lceil k \rceil$ replaced by $k+1$, that is, S_n will be larger than kM_n, "if and only if", $\sum_{i=1}^{k+1} X_{nj_i}$ is larger than kM_n for some $1 \le j_1 < \ldots < j_{k+1} \le n$. However, as is shown in Theorem 2.1 below, this is not always the case. This disagreement with intuition is the primary reason behind the boundary case, that is the case when k is an integer, being harder. The author could solve this boundary problem only when $\alpha \ge k$ or $\alpha < k/(k+2)$. Surprisingly, it turns out that the asymptotic magnitude of the probability is different for these two regimes of α, a phenomenon one wouldn't guess a priori. The behavior in these two regimes are studied in sections 2 and 3 respectively. In Section 4, we consider a couple of examples.

We conclude this section by pointing out some classical works in the theory of large deviations for (untruncated) heavy tailed random variables. The study started in late sixties with Heyde (1968), Nagaev (1969a) and Nagaev (1969b), significant contributions being made later by Nagaev (1979) and Cline and Hsing (1991), among others. Recently, Hult et al. (2005) studied the functional version of the large deviations principle.

2. The case $\alpha \ge k$

Suppose that $Y, Y_1, Y_2, \ldots$ are i.i.d. random variables such that $P(|Y| > \cdot)$ is regularly varying with index $-\alpha$, for some $\alpha > 1$, and

$$p := \lim_{x\to\infty} \frac{P(Y > x)}{P(|Y| > x)} \text{ exists, and is positive}\,. \tag{2.1}$$

We assume that

$$E(Y) = 0\,. \tag{2.2}$$

If $\alpha = 2$, we assume that

$$E(Y^2) < \infty\,. \tag{2.3}$$

Furthermore, we assume that given $\delta > 0$, there exist $T_0 > 0$ and $u_0 \in (0,1)$ such that for all $T \ge T_0$ and $1 - u_0 \le a \le b \le 1$,

$$\left| \frac{P(Y > aT) - P(Y > bT)}{P(Y > T)} - (a^{-\alpha} - b^{-\alpha}) \right| \le \delta(b - a)\,. \tag{2.4}$$

In Section 4, we shall see an example of a random variable with regularly varying tail, for which this holds.

Define the quantile sequence (b_n) as

$$b_n := \inf\left\{x > 0 : P(Y > x) \le \frac{1}{n}\right\}. \tag{2.5}$$

Let (M_n) be a sequence of positive numbers so that

$$M_n \gg b_n,\ \text{if } \alpha < 2\,, \tag{2.6}$$

and

$$M_n \gg n^{1/2+\gamma} \text{ for some } \gamma > 0,\ \text{if } \alpha \ge 2\,. \tag{2.7}$$

The triangular array $\{X_{nj} : 1 \le j \le n\}$ and their row sum S_n are as defined in (1.1) and (1.2) respectively.

The first result of this paper, Theorem 2.1 below, describes the decay rate of $P(S_n > kM_n)$ when k is a positive integer and $\alpha \ge k$, under some additional assumptions. The result below refers to stable distributions; an overview of this topic can be found in the first chapter of Samorodnitsky and Taqqu (1994).

Theorem 2.1. *Suppose that $\alpha \ge k$, where k is a positive integer (we still assume that $\alpha > 1$). Assume furthermore that if $\alpha = k \ge 3$, then*

$$\int_0^\infty y^\alpha P(Y \in dy) < \infty\,. \tag{2.8}$$

Then

$$P(S_n > kM_n) \sim c_n^k n^k M_n^{-k} P(Y > M_n)^k \frac{\alpha^k}{(k!)^2} \int_0^\infty s^k P(Z_\alpha \in ds)\,,$$

as $n \to \infty$, where

$$c_n := \begin{cases} b_n, & \alpha < 2\,, \\ n^{1/2}, & \alpha \ge 2\,, \end{cases}$$

and Z_α follows an α-stable distribution with scale, location and skewness parameters as 1, 0 and $2p-1$ respectively if $\alpha < 2$, and a normal distribution with mean zero and variance same as that of Y if $\alpha \ge 2$.

For proving the result, we need some lemmas. First, let us fix some notations. For $l \ge 1$, let C_{nl} be the set of l-tuples $j = (j_1, \ldots, j_l)$ such that $1 \le j_1 < \ldots < j_l \le n$. For any $j \in C_{nl}$, denote

$$j^c := \{1, \ldots, n\} \setminus \{j_1, \ldots, j_l\}\,.$$

Lemma 2.1. *Suppose that (x_n) is a sequence satisfying*

$$M_n \gg x_n \gg b_n,\ \text{if } \alpha < 2\,, \tag{2.9}$$

and

$$M_n \gg x_n \gg n^{1/2+\gamma},\ \text{if } \alpha \ge 2\,, \tag{2.10}$$

where γ is same as that in (2.7)*. Then,*

$$P\left(\bigcap_{j\in C_{nk}}\left\{\sum_{i\in j^c} X_{ni} > x_n, S_n > kM_n\right\}\right) = O\left(n^{k+1}P(Y > x_n)^{k+1}\right),$$

for all fixed $k \geq 1$, as $n \to \infty$.

Proof. The proof is by induction on k. The proof for $k = 1$ is very similar to the induction step, and hence we do not show the former separately. As the induction hypothesis, assume that the result is true for $1, \ldots, k-1$, and we shall show it for k. Observe that if $\alpha \geq 2$, then

$$b_n = O\left(n^{1/2+\gamma}\right),$$

and hence by (2.9) and (2.10), it follows that

$$b_n \ll x_n, \tag{2.11}$$

for all α. Since, p as defined in (2.1), is positive, it follows that $P(Y > \cdot)$ is regularly varying with index $-\alpha$. It immediately follows from (2.5) that

$$\lim_{n\to\infty} nP(Y > b_n) = 1\,. \tag{2.12}$$

Fix

$$\varepsilon \in \left(0, \frac{1}{k+2}\right),$$

and let u be such that

$$0 < u < \frac{1}{\alpha}\left(\frac{1}{k+2} - \varepsilon\right),$$

and

$$\frac{1-u}{2-\alpha u} < \frac{1}{2} + \gamma\,. \tag{2.13}$$

Notice that by (2.12),

$$n^{-u}b_n^{1/(k+2)-\varepsilon} \sim P(Y > b_n)^u b_n^{1/(k+2)-\varepsilon} \ll P(Y > x_n)^u x_n^{1/(k+2)-\varepsilon},$$

the inequality following from (2.11) and the fact that the function x going to $P(Y > x)^u x^{1/(k+2)-\varepsilon}$, is regularly varying with a positive index, namely $1/(k+2) - \varepsilon - u\alpha$. Thus,

$$b_n^{1/(k+2)-\varepsilon} x_n^{(k+1)/(k+2)+\varepsilon} \ll n^u P(Y > x_n)^u x_n\,. \tag{2.14}$$

Let $\delta \in (0, 2 - \alpha u)$ be such that

$$\frac{1-u}{2-\alpha u-\delta} \leq \frac{1}{2} + \gamma\,;$$

such a δ exists because of (2.13). When $\alpha \geq 2$, observe that

$$x_n^2 P(Y > x_n)^u \gg x_n^{2-\alpha u-\delta} \gg n^{(1/2+\gamma)(2-\alpha u-\delta)} \geq n^{1-u}\,.$$

In view of (2.14) and the above inequality, there exists a sequence (z_n) satisfying

$$b_n^{\frac{1}{k+2}-\varepsilon} x_n^{\frac{k+1}{k+2}+\varepsilon} \ll z_n \ll n^u P(Y > x_n)^u x_n, \text{ if } \alpha < 2\,, \tag{2.15}$$

and

$$b_n^{\frac{1}{k+2}-\varepsilon} x_n^{\frac{k+1}{k+2}+\varepsilon} + \frac{n}{x_n} \ll z_n \ll n^u P(Y > x_n)^u x_n, \text{ if } \alpha \geq 2\,. \tag{2.16}$$

Fix such a (z_n).

For a set A and $m \geq 1$, let $S_m(A)$ denote the family of all subsets of A that have cardinality m. Fix $1 \leq l \leq k-1$. For $j \in C_{nl}$, define

$$D_j := \bigcap_{i \in S_{n-k}(j^c)} \left\{ \sum_{u=1}^{n-k} X_{ni_u} > x_n, \sum_{v \in j^c} X_{nv} > (k-l)M_n \right\};$$

recall the definitions of C_{nl} and j^c from the text preceding the statement of the current lemma. Define the events

$$\begin{aligned}
E_n &:= \{|X_{nj}| > z_n \text{ for at least } (k+2) \text{ many } j\text{'s } \leq n\}\,, \\
F_n &:= \left\{ \sum_{j=1}^n X_{nj} \mathbf{1}(|X_{nj}| \leq z_n) > M_n - \frac{x_n}{2} \right\}, \\
G_n &:= \left\{ X_{nj} > \frac{x_n}{2(k+1)} \text{ for at least } (k+1) \text{ many } j\text{'s } \leq n \right\}, \\
H_n &:= \bigcup_{j \in C_{nk}} \Bigg\{ X_{nj_i} > \frac{x_n}{2(k+1)} \text{ for } 1 \leq i \leq k \\
&\qquad \text{and } \sum_{i \in j^c} X_{ni} \mathbf{1}(|X_{ni}| \leq z_n) > \frac{x_n}{2} \Bigg\}, \\
I_n &:= \bigcup_{l=1}^{k-1} \bigcup_{j \in C_{nl}} \left[\left\{ X_{nj_i} > \frac{x_n}{2(k+1)} \text{ for } 1 \leq i \leq l \right\} \cap D_j \right].
\end{aligned}$$

Our claim is that

$$\bigcap_{j \in C_{nk}} \left\{ \sum_{i \in j^c} X_{ni} > x_n, S_n > kM_n \right\} \subset E_n \cup F_n \cup G_n \cup H_n \cup I_n\,. \tag{2.17}$$

To see this, if possible, fix a sample point in the left hand side, which is in neither of E_n, F_n, G_n, H_n or I_n. Let

$$l := \# \left\{ u \in \{1, \ldots, n\} : X_{nu} > \frac{x_n}{2(k+1)} \right\}.$$

Case 1: $l = 0$**.** Write

$$S_n := \sum_{j=1}^{n} X_{nj}\mathbf{1}(|X_{nj}| \le z_n) + \sum_{j=1}^{n} X_{nj}\mathbf{1}(|X_{nj}| > z_n)\,.$$

By the assumption that the sample point does not belong to F_n, it follows that the first sum on the right is at most $M_n - x_n/2$. Since E_n does not hold, the number of surviving summands in the second sum is at most $(k+1)$. As $l = 0$, it follows that each summand is at most $x_n/2(k+1)$. Therefore, the second sum is at most $x_n/2$. Thus, $S_n \le M_n$, which is a contradiction.
Case 2: $1 \le l \le k-1$**.** Let $1 \le j_1 < \ldots < j_l \le n$ be such that

$$X_{nj_i} > \frac{x_n}{2(k+1)} \text{ for all } 1 \le i \le l\,.$$

Denote $j := (j_1, \ldots, j_l) \in C_{nl}$. Clearly D_j is a superset of the left hand side of (2.17), and hence trivially the sample point is in D_j. Thus, the sample point is in I_n, which is a contradiction.
Case 3: $l = k$**.** Once again, let $j_1 < \ldots < j_k$ denote the indices i for which $X_{ni} > x_n/2(k+1)$, and set $j := (j_1, \ldots, j_k)$. Write

$$\sum_{i \in j^c} X_{ni} = \sum_{i \in j^c} X_{ni}\mathbf{1}(|X_{ni}| \le z_n) + \sum_{i \in j^c} X_{ni}\mathbf{1}(|X_{ni}| > z_n)\,.$$

Since H_n does not hold, the first sum on the right is at most $x_n/2$. As E_n does not hold, in the second sum, at most $(k+1)$ terms survive. Also, each summand in that sum is at most $x_n/2(k+1)$. Thus, the second sum is at most $x_n/2$. This shows that the left hand side is at most x_n, which clearly is a contradiction.
Case 4: $l \ge k+1$**.** This case cannot arise because G_n does not hold. Thus, the inclusion (2.17) is true.

In view of (2.17), all that needs to be shown is that

$$P(E_n)+P(F_n)+P(G_n)+P(H_n)+P(I_n) = O\left[\{nP(Y > x_n)\}^{k+1}\right]\,. \tag{2.18}$$

To that end, notice that

$$\begin{aligned} P(E_n) &\le n^{k+2} P\left(|Y| > z_n\right)^{k+2} \\ &= O\left[n^{k+2} P\left(Y > z_n\right)^{k+2}\right]\,, \end{aligned} \tag{2.19}$$

the second step following from the assumption that p, as defined in (2.1), is positive. By (2.11) and (2.12), it follows that

$$b_n^{1/(k+2)-\varepsilon} x_n^{(k+1)/(k+2)+\varepsilon} \gg b_n\,,$$

and

$$\lim_{n\to\infty} nP(Y > x_n) = 0\,. \tag{2.20}$$

These, in view of (2.15) and (2.16), show that

$$b_n \ll z_n \ll x_n\,. \tag{2.21}$$

Set

$$\theta := \alpha(k+2)\varepsilon\,,$$

and observe that as $n \to \infty$,

$$\begin{aligned}
\frac{n^{k+2}P(Y>z_n)^{k+2}}{n^{k+1}P(Y>x_n)^{k+1}} &\sim \left\{\frac{P(Y>z_n)}{P(Y>x_n)}\right\}^{k+1}\frac{P(Y>z_n)}{P(Y>b_n)}\\
&\le 2\left\{\left(\frac{z_n}{x_n}\right)^{-\alpha-\theta/(k+1)}\right\}^{k+1}\left(\frac{z_n}{b_n}\right)^{-\alpha+\theta}\\
&= 2\left(\frac{z_n}{b_n^{1/(k+2)-\varepsilon}x_n^{(k+1)/(k+2)+\varepsilon}}\right)^{-\alpha(k+2)}\\
&\to 0\,,
\end{aligned}$$

the inequality in the second line holding for large n, and following by the Potter bounds (Proposition 2.6 in Resnick (2007)) and (2.21). This, in view of (2.19), shows that

$$P(E_n) = o\left(n^{k+1}P(Y>x_n)^{k+1}\right)\,.$$

It is obvious that

$$P(G_n) = O\left(n^{k+1}P(Y>x_n)^{k+1}\right)\,.$$

Next we proceed to show that

$$P(H_n) = O\left(n^{k+1}P(Y>x_n)^{k+1}\right)\,. \tag{2.22}$$

To that end, we shall use a result from Prokhorov (1959), which states that if $T_1, T_2, \ldots, T_N$ are independent zero mean random variables such that $|T_j| \le C$ for all $1 \le j \le n$, then

$$P\left(\sum_{j=1}^N T_j > \lambda\right) \le \exp\left\{-\frac{\lambda}{2C}\sinh^{-1}\frac{\lambda C}{2\mathrm{Var}(\sum_{j=1}^N T_j)}\right\}\,. \tag{2.23}$$

By the above, it follows that

$$\begin{aligned}
&P(H_n)\\
\le\;& n^k P\left(Y > \frac{x_n}{2(k+1)}\right)^k P\left(\sum_{j=k+1}^n X_{nj}\mathbf{1}(|X_{nj}|\le z_n) > \frac{1}{2}x_n\right)\\
=\;& O\left[P\left(\sum_{j=k+1}^n X_{nj}\mathbf{1}(|X_{nj}|\le z_n) > \frac{1}{2}x_n\right)\right]\,,
\end{aligned}$$

the last step following from (2.20). The assumption (2.2) implies that for n large,

$$\begin{aligned} nE\left[X_{n1}\mathbf{1}(|X_{n1}|\le z_n)\right] &= -nE\left[Y\mathbf{1}(|Y|>z_n)\right] \\ &= O\left(nz_nP(Y>z_n)\right) \\ &= o\left(nx_nP(Y>x_n)\right) \\ &= o(x_n)\,, \end{aligned} \tag{2.24}$$

where the second step follows from Karamata's theorem; see Theorem 2.1 in Resnick (2007). Thus, for n large enough,

$$\begin{aligned} &P\left(\sum_{j=k+1}^n X_{nj}\mathbf{1}(|X_{nj}|\le z_n) > \frac{1}{2}x_n\right) \\ \le\ & P\left(\sum_{j=k+1}^n X_{nj}\mathbf{1}(|X_{nj}|\le z_n) - (n-k)E\left[X_{n1}\mathbf{1}(|X_{n1}|\le z_n)\right] > \frac{1}{4}x_n\right) \\ \le\ & \exp\left\{-K_1\frac{x_n}{z_n}\sinh^{-1}K_2\frac{z_nx_n}{(n-k)\mathrm{Var}(X_{n1}\mathbf{1}(|X_{n1}|\le z_n))}\right\}, \end{aligned} \tag{2.25}$$

for some finite positive constants K_1 and K_2. Thus, for (2.22), it suffices to show that

$$x_nz_n \gg n\mathrm{Var}(X_{n1}\mathbf{1}(|X_{n1}|\le z_n))\,, \tag{2.26}$$

$$\text{and } \frac{x_n}{z_n} \gg (nP(Y>x_n))^{-u}\,. \tag{2.27}$$

This is because if the above hold, then by (2.25) and the fact that

$$\lim_{x\to\infty}\sinh^{-1}x=\infty\,,$$

it follows that,

$$P(F_n)=O\left(\exp\left\{-K_1(nP(Y>x_n))^{-u}\right\}\right)=o\left[(nP(Y>x_n))^{k+1}\right].$$

Notice that (2.27) is a restatement of the second inequalities in (2.15) and (2.16). We shall show (2.26) separately for the cases $\alpha<2$ and $\alpha\ge 2$.

Case $\alpha<2$**:** By the Karamata's theorem, it follows that

$$\begin{aligned} n\mathrm{Var}(X_{n1}\mathbf{1}(|X_{n1}|\le z_n)) &= O\left(nz_n^2P(Y>z_n)\right) \\ &= o(z_n^2) \\ &= o(z_nx_n)\,. \end{aligned}$$

Case $\alpha\ge 2$**:** The assumption (2.3) implies that

$$\begin{aligned} n\mathrm{Var}(X_{n1}\mathbf{1}(|X_{n1}|\le z_n)) &= O(n) \\ &= o(x_nz_n)\,, \end{aligned}$$

the last step following from the left inequality in (2.16). Thus, (2.26) holds, and hence so does (2.22).

By similar arguments as above, the fact that

$$P(F_n) = O\left(n^{k+1}P(Y > x_n)^{k+1}\right) \tag{2.28}$$

will follow once it can be shown that

$$nE\left[X_{n1}\mathbf{1}(|X_{n1}| \le z_n)\right] = o(\hat{M}_n)\,,$$

$$\hat{M}_n z_n \gg n\mathrm{Var}(X_{n1}\mathbf{1}(|X_{n1}| \le z_n))\,,$$

and

$$\frac{\hat{M}_n}{z_n} \gg (nP(Y > x_n))^{-u}\ ,$$

where

$$\hat{M}_n := M_n - \frac{1}{2}x_n\,.$$

These, follow immediately from (2.24), (2.26) and (2.27) respectively, along with the fact that $x_n \ll \hat{M}_n$. This establishes (2.28).

To show (2.18), and thus complete the proof of the lemma, all that needs to be shown is

$$P(I_n) = O\left(n^{k+1}P(Y > x_n)^{k+1}\right)\ . \tag{2.29}$$

This is the only place where we shall use the induction hypothesis; note that when $k = 1$, I_n is the null event, and hence the above arguments give a complete proof for that case. To the end of proving this for general k, observe that

$$\begin{aligned} P(I_n) &\le \sum_{l=1}^{k-1}\sum_{j\in C_{nl}} P\left(\left\{X_{nj_i} > \frac{x_n}{2(k+1)} \text{ for } 1 \le i \le l\right\} \cap D_j\right) \\ &\le \sum_{l=1}^{k-1}\binom{n}{l} P\left(Y > \frac{x_n}{2(k+1)}\right)^l P\left(D_{(1,\ldots,l)}\right) \\ &= \sum_{l=1}^{k-1} O\left[n^l P\left(Y > x_n\right)^l P\left(D_{(1,\ldots,l)}\right)\right]\ . \end{aligned}$$

Notice that,

$$P\left(D_{(1,\ldots,l)}\right) = O\left(n^{k-l+1}P(Y > x_n)^{k-l+1}\right)$$

for all fixed $l = 1, \ldots, k-1$, since by the induction hypothesis, the claim of the theorem is true for k replaced by $k - l$. This shows (2.29), and consequently the result for k, that is, the induction step, which in turn completes the proof. □

Lemma 2.2. *For $k \ge 1$, there exists a function $\varepsilon_k : (0,1) \times (0,\infty) \to \mathbb{R}$ such that*

$$\lim_{T\to\infty, u\downarrow 0} u^{-k}\varepsilon_k(u,T) = 0\,, \tag{2.30}$$

and

$$P\left(\sum_{i=1}^k X_{ni} > (k-u)M_n\right) \over P(Y>M_n)^k \tag{2.31}$$

$$\frac{P\left(\sum_{i=1}^k X_{ni} > (k-u)M_n\right)}{P(Y>M_n)^k} = \frac{\alpha^k}{k!}u^k + \varepsilon_k(u, M_n)\,, \tag{2.31}$$

for all $n \geq 1$, $0 < u < 1$.

Proof. The proof is by induction on k. First let us show the result for $k = 1$. To that end, define

$$\varepsilon_1(u,T) := \frac{P(Y > (1-u)T) - P(Y > T)}{P(Y > T)} - \alpha u\,,$$

and notice that (2.31) holds with this $\varepsilon_1(\cdot,\cdot)$. Also,

$$|\varepsilon_1(u,T)| \leq \left|\frac{P(Y > (1-u)T) - P(Y > T)}{P(Y > T)} - \left\{(1-u)^{-\alpha} - 1\right\}\right| + \left|\left\{(1-u)^{-\alpha} - 1\right\} - \alpha u\right|\,.$$

By (2.4), the first term on the right hand side is $o(u)$ as $T \to \infty$ and $u \downarrow 0$. The second term is $o(u)$ by Taylor's theorem. This shows the result for $k = 1$. Suppose now that the claim is true for k; we shall show the same for $k+1$. Clearly, for $n \geq 1$ and $0 < u < 1$,

$$\begin{aligned} & P\left(\sum_{i=1}^{k+1} X_{ni} > (k+1-u)M_n\right) \\ = & \int_{(1-u,1]} P\left(\sum_{i=1}^{k} X_{ni} > \{k - (u+x-1)\}\, M_n\right) P\left(M_n^{-1}Y \in dx\right) \\ = & P(Y > M_n)^k \int_{(1-u,1]} \left[\frac{\alpha^k}{k!}(u+x-1)^k + \varepsilon_k(u+x-1, M_n)\right] P\left(M_n^{-1}Y \in dx\right)\,. \end{aligned}$$

Define

$$\varepsilon_{k+1}(u,T) := \int_{(1-u,1]} \left[\frac{\alpha^k}{k!}(u+x-1)^k + \varepsilon_k(u+x-1, T)\right] \frac{P\left(T^{-1}Y \in dx\right)}{P(Y>T)} - \frac{\alpha^{k+1}}{(k+1)!}u^{k+1}\,.$$

To complete the proof, all that needs to be shown is

$$\lim_{T\to\infty, u\downarrow 0} u^{-(k+1)}\varepsilon_{k+1}(u,T) = 0\,. \tag{2.32}$$

To that end, we shall first show

$$\lim_{T\to\infty, u\downarrow 0} u^{-(k+1)} \int_{(1-u,1]} (u+x-1)^k \frac{P\left(T^{-1}Y \in dx\right)}{P(Y>T)} = \frac{\alpha}{k+1}\,. \tag{2.33}$$

Fix $\alpha > \delta > 0$, and $T_0 > 0$, $u_0 \in (0,1)$ such that (2.4) holds. Then, for $T \geq T_0$ and $1-u_0 \leq a < b \leq 1$,

$$\alpha - \delta \leq \frac{P(Y \in (aT, bT])}{(b-a)P(Y>T)} \leq \alpha(1-u_0)^{-\alpha-1} + \delta\,.$$

Thus, for $T \geq T_0$ and $0 < u \leq u_0$,

$$(\alpha-\delta)\int_{1-u}^{1} (u+x-1)^k dx \leq \int_{(1-u,1]} (u+x-1)^k \frac{P\left(T^{-1}Y \in dx\right)}{P(Y>T)}$$

$$\leq \left\{\alpha(1-u_0)^{-\alpha-1} + \delta\right\} \int_{1-u}^{1} (u+x-1)^k dx\,. \tag{2.34}$$

Since δ and u_0 can be chosen to be arbitrarily small, it follows that as $T \to \infty$ and $u \downarrow 0$,

$$\begin{aligned}\int_{(1-u,1]} (u+x-1)^k \frac{P\left(T^{-1}Y \in dx\right)}{P(Y>T)} \quad &\sim \quad \alpha \int_{1-u}^{1} (u+x-1)^k dx \\ &= \quad \frac{\alpha}{k+1} u^{k+1}\,.\end{aligned}$$

This shows (2.33). Now using the induction hypothesis that

$$\lim_{T\to\infty, u\downarrow 0} u^{-k}\varepsilon_k(u,T) = 0\,,$$

and the above computations, (2.32) follows. This establishes the induction step, and thus completes the proof of the lemma. □

Lemma 2.3. *Under the assumptions of Theorem 2.1,*

$$\lim_{n\to\infty} \int_0^\infty s^k P\left(c_n^{-1} S_n \in ds\right) = \int_0^\infty s^k P(Z_\alpha \in ds) < \infty\,, \tag{2.35}$$

where (c_n) and Z_α are as defined in the statement of that theorem.

Proof. Since α is always assumed to be larger than 1, notice that either $k < \alpha < 2$ or $\alpha \geq 2$ always holds. In the former case, Z_α has an α-stable distribution, and hence the integral on the right hand side of (2.35) is finite because $k < \alpha$. In the latter case Z_α is a normal random variable, and hence the integral is finite for all k.

Notice that the assumption (2.2) implies that as $n \to \infty$,

$$c_n^{-1} \sum_{j=1}^n Y_j \Longrightarrow Z_\alpha\,.$$

Recall that S_n, defined in (1.2), is the row sum of the triangular array $\{X_{nj} : 1 \leq j \leq n\}$. Notice that

$$\begin{aligned}P\left(S_n \neq \sum_{j=1}^n Y_j\right) \quad &\leq \quad nP(|Y| > M_n) \\ &= \quad o(1)\,,\end{aligned}$$

the last equality being an immediate consequence of (2.6) and (2.7). Thus, it follows that

$$c_n^{-1} S_n \Longrightarrow Z_\alpha \,. \tag{2.36}$$

For the proof, we shall use Theorem 3.2 in de Acosta and Giné (1979), which states the following:
Let $\{V_{nj} : 1 \le j \le n\}$ be a triangular array satisfying

$$\lim_{n\to\infty} \max_{1\le j\le n} P\left(|V_{nj}| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0\,, \tag{2.37}$$

and the row sum $\tilde{S}_n := \sum_{j=1}^n V_{nj}$ converges weakly to a probability measure ν. If ϕ is a continuous function from $\mathbb{R}$ to $[0,\infty)$ such that there exists $a \in (0,\infty)$ with

$$\phi(x+y) \le a\phi(x)\phi(y) \text{ for all } x, y\,, \tag{2.38}$$

and

$$\lim_{T\to\infty} \sup_{n\ge 1} \sum_{j=1}^n E\left[\phi(V_{nj}) \mathbf{1}(|V_{nj}| > T)\right] = 0\,, \tag{2.39}$$

then

$$\lim_{n\to\infty} E\phi(\tilde{S}_n) = \int_{\mathbb{R}} \phi d\nu\,.$$

The plan is to use the above result with ϕ defined by

$$\phi(x) := 2 + |x|^k \mathbf{1}(x > 0)\,.$$

A quick inspection will reveal that ϕ thus defined, satisfies (2.38) with $a = 2^k$. Let $\{X_{nj}\}$ be as defined in (1.1), and set

$$V_{nj} := c_n^{-1} X_{nj},\ 1 \le j \le n\,.$$

Thus, checking (2.37) and (2.39) suffices for the proof of the lemma. Verifying (2.37) is trivial. A quick argument will yield that instead of showing (2.39), it is enough to show

$$\lim_{T\to\infty} \limsup_{n\to\infty} \sum_{j=1}^n E\left[\phi(V_{nj}) \mathbf{1}(|V_{nj}| > T)\right] = 0\,. \tag{2.40}$$

For (2.40), observe that

$$\begin{aligned} & \sum_{j=1}^n E\left[\phi(V_{nj}) \mathbf{1}(|V_{nj}| > T)\right] \\ = \;& 2nP\left(|X_{n1}| > c_n T\right) + n c_n^{-k} E\left[X_{n1}^k \mathbf{1}\left(X_{n1} > c_n T\right)\right] \\ \le \;& 2nP\left(|Y| > c_n T\right) + n c_n^{-k} \int_{[c_n T, \infty)} y^k P(Y \in dy)\,. \end{aligned}$$

By (2.3), it follows that when $\alpha \ge 2$,

$$P(Y > x) = o(x^{-2}) \text{ as } x \to \infty\,. \tag{2.41}$$

This, along with (2.12), shows that

$$\limsup_{n\to\infty} nP(|Y| > c_n) < \infty\,, \tag{2.42}$$

for all α. In view of this, it follows that there is a finite constant C, independent of T and n, satisfying

$$\limsup_{n\to\infty} nP\left(|Y| > c_nT\right) = CT^{-\alpha}\,.$$

Thus, it follows that

$$\lim_{T\to\infty}\limsup_{n\to\infty} nP\left(|Y| > c_nT\right) = 0\,. \tag{2.43}$$

For (2.40), it remains to show that

$$\lim_{T\to\infty}\limsup_{n\to\infty} nc_n^{-k}\int_{[c_nT,\infty)} y^kP(Y\in dy) = 0\,. \tag{2.44}$$

We shall prove (2.44) separately for the cases $\alpha > k$ and $\alpha = k$.
Case $\alpha > k$**:** Observe that for fixed $T > 0$, by the Karamata's theorem,

$$\begin{aligned}\limsup_{n\to\infty} nc_n^{-k}\int_{[c_nT,\infty)} y^kP(Y\in dy) &= \frac{\alpha}{\alpha-k}T^k\limsup_{n\to\infty} nP(Y > c_nT)\\ &\le C\frac{\alpha}{\alpha-k}T^{k-\alpha}\,.\end{aligned}$$

This completes the proof of (2.44).
Case $\alpha = k$**:** Since the assumption $\alpha > 1$ is in force, for this case it is necessarily true that $\alpha = k \ge 2$. Thus, for $T \ge 1$,

$$\begin{aligned}nc_n^{-k}\int_{[c_nT,\infty)} y^kP(Y\in dy) &\le nc_n^{-k}\int_{[c_n,\infty)} y^kP(Y\in dy)\\ &= n^{1-k/2}\int_{[n^{1/2},\infty)} y^\alpha P(Y\in dy)\\ &\le \int_{[n^{1/2},\infty)} y^\alpha P(Y\in dy)\,.\end{aligned}$$

By the assumptions (2.3) and (2.8), it follows that rightmost quantity goes to zero as $n\to\infty$. This shows (2.44). Equations (2.43) and (2.44) establish (2.40), which completes the proof. □

Proof of Theorem 2.1. We start with proving the upper bound, that is,

$$\limsup_{n\to\infty}\frac{P(S_n > kM_n)}{n^kP(Y > M_n)^kc_n^kM_n^{-k}} \le \frac{\alpha^k}{(k!)^2}\int_0^\infty s^kP(Z_\alpha\in ds)\,. \tag{2.45}$$

To that end, observe that

$$M_n \gg c_n\,, \tag{2.46}$$

which is a consequence of (2.6) and (2.7). Our first claim is that

$$M_n^kP(Y > M_n) \ll c_n^kn^{-1}\,. \tag{2.47}$$

We shall show this separately for the cases $\alpha > k$ and $\alpha = k$.
Case $\alpha > k$**:** By (2.46) and the fact that $\alpha > k$, it follows that

$$M_n^k P(Y > M_n) \ll c_n^k P(Y > c_n) = O\left(c_n^k/n\right) ,$$

the rightmost equality following from (2.42).
Case $\alpha = k$**:** It is necessary that in this case $\alpha \geq 2$. By (2.41) and the assumption (2.8), it follows that

$$M_n^k P(Y > M_n) \ll 1 \leq n^{k/2-1} = c_n^k n^{-1} .$$

This completes the proof of (2.47). Thus, it follows that

$$P(Y > M_n)^{k+1} \ll c_n^k n^{-1} M_n^{-k} P(Y > M_n)^k . \tag{2.48}$$

Set

$$\beta := \begin{cases} 0, & \alpha < 2 , \\ \gamma, & \alpha \geq 2 , \end{cases}$$

where γ is same as that in (2.7). It follows that

$$M_n \gg n^\beta c_n .$$

This, along with (2.48), shows that

$$P(Y > M_n)^{k+1} \ll \min\left\{c_n^k n^{-1} M_n^{-k} P(Y > M_n)^k, P(Y > n^\beta c_n)^{k+1}\right\} .$$

The right hand side, clearly, goes to zero. Thus, there exists a sequence (T_n) satisfying

$$\begin{aligned} P(Y > M_n)^{k+1} &\ll P(Y > T_n c_n)^{k+1} \\ &\ll \min\left\{c_n^k n^{-1} M_n^{-k} P(Y > M_n)^k, P(Y > n^\beta c_n)^{k+1}\right\} . \end{aligned} \tag{2.49}$$

Let us record some quick consequences of the above. One immediate observation is that

$$n^\beta c_n \ll c_n T_n \ll M_n . \tag{2.50}$$

In particular,

$$\lim_{n\to\infty} T_n = \infty . \tag{2.51}$$

Notice that

$$\{S_n > kM_n\} \subset \left(\bigcup_{j\in C_{nk}} \left\{\sum_{u=1}^k X_{nj_u} > kM_n - T_n c_n, S_n > kM_n\right\}\right)$$
$$\cup \left(\bigcap_{j\in C_{nk}} \left\{\sum_{i\in j^c} X_{ni} > T_n c_n, S_n > kM_n\right\}\right) ,$$

where C_{nk}, as defined earlier, denotes the set of k-tuples $j = (j_1, \ldots, j_k)$ such that $1 \le j_1 < \ldots < j_k \le n$. Thus,

$$\begin{aligned}
& P(S_n > kM_n) \\
\le\ & \binom{n}{k} P\left(\sum_{i=1}^{k} X_{ni} > kM_n - T_n c_n, S_n > kM_n\right) + \\
& P\left(\bigcap_{j \in C_{nk}} \left\{\sum_{i \in j^c} X_{ni} > T_n c_n, S_n > kM_n\right\}\right) \\
=:\ & Q_1 + Q_2 \,.
\end{aligned}$$

Clearly,

$$\begin{aligned}
& \binom{n}{k}^{-1} Q_1 \\
=\ & \int_{(0,T_n]} P\left(\sum_{i=1}^{k} X_{ni} > kM_n - sc_n\right) P\left(c_n^{-1} \sum_{i=k+1}^{n} X_{ni} \in ds\right) \\
& + P\left(\sum_{i=1}^{k} X_{ni} > kM_n - T_n c_n\right) P\left(\sum_{i=k+1}^{n} X_{ni} > T_n c_n\right) \\
=:\ & Q_{11} + Q_{12} \,.
\end{aligned}$$

Denote

$$P_n(ds) := P\left(c_n^{-1} \sum_{i=k+1}^{n} X_{ni} \in ds\right) . \tag{2.52}$$

By Lemma 2.2, it follows that

$$\begin{aligned}
Q_{11} \ =\ & P(Y > M_n)^k \frac{\alpha^k}{k!} c_n^k M_n^{-k} \int_{(0,T_n]} s^k P_n(ds) \\
& + P(Y > M_n)^k \int_{(0,T_n]} \varepsilon_k \left(sc_n M_n^{-1}, M_n\right) P_n(ds) \\
=:\ & Q_{111} + Q_{112} \,,
\end{aligned}$$

where $\varepsilon_k(\cdot, \cdot)$ satisfies (2.30). Clearly,

$$\begin{aligned}
Q_{111} \ \le\ & P(Y > M_n)^k \frac{\alpha^k}{k!} c_n^k M_n^{-k} \int_0^\infty s^k P_n(ds) \\
\sim\ & P(Y > M_n)^k \frac{\alpha^k}{k!} c_n^k M_n^{-k} \int_0^\infty s^k P(Z_\alpha \in ds) \,,
\end{aligned}$$

the equivalence following from Lemma 2.3. Also,

$$\begin{aligned} |Q_{112}| &\le P(Y>M_n)^k c_n^k M_n^{-k} \int_0^\infty s^k P_n(ds) \sup_{0<t\le c_n T_n M_n^{-1}} \frac{|\varepsilon_k(t,M_n)|}{t^k} \\ &= o\left(P(Y>M_n)^k c_n^k M_n^{-k}\right), \end{aligned}$$

the second equality following by (2.30), (2.50) and Lemma 2.3. Hence,

$$\limsup_{n\to\infty} \frac{Q_{11}}{P(Y>M_n)^k c_n^k M_n^{-k}} \le \frac{\alpha^k}{k!} \int_0^\infty s^k P(Z_\alpha \in ds). \tag{2.53}$$

Next, we proceed to show that

$$Q_{12} = o\left(P(Y>M_n)^k c_n^k M_n^{-k}\right). \tag{2.54}$$

To that end, notice that

$$P\left(\sum_{i=k+1}^n X_{ni} > T_n c_n\right) \le P\left(\sum_{i=k+1}^n Y_i > T_n c_n\right) + nP(|Y|>M_n).$$

It is well known that

$$P\left(\sum_{i=k+1}^n Y_i > T_n c_n\right) = O\left(nP(Y>T_n c_n)\right);$$

see, for example, Lemma 2.1 in Hult et al. (2005). This, in view of (2.50), shows that

$$P\left(\sum_{i=k+1}^n X_{ni} > T_n c_n\right) = O\left(nP(Y>T_n c_n)\right).$$

Using Lemma 2.2 once again, it follows that

$$P\left(\sum_{i=1}^k X_{ni} > kM_n - T_n c_n\right) = O\left(P(Y>M_n)^k (T_n c_n M_n^{-1})^k\right).$$

Thus,

$$Q_{12} = O\left(nP(Y>T_n c_n) P(Y>M_n)^k (T_n c_n M_n^{-1})^k\right).$$

In view of this, (2.54) will follow if it can be shown that

$$\lim_{n\to\infty} nP(Y>T_n c_n) T_n^k = 0. \tag{2.55}$$

Once again, (2.55) will be shown separately for the cases $\alpha > k$ and $\alpha = k$.
Case $\alpha > k$: Fix $\delta \in (0, \alpha - k)$. By (2.42), it follows that there exists $K < \infty$, independent of n, such that

$$\begin{aligned} nP(Y>T_n c_n) T_n^k &\le K \frac{P(Y>T_n c_n)}{P(Y>c_n)} T_n^k \\ &\le 2K T_n^{k-\alpha+\delta}, \end{aligned}$$

for n large enough, the second inequality following by the Potter bounds and (2.51). This shows (2.55).

Case $\alpha = k$: By (2.41), and the assumption (2.8), it is ensured that

$$nP(Y > T_n c_n) T_n^k \ll n c_n^{-k} = n^{1-k/2} \leq 1 \, .$$

Thus, (2.55) holds, and hence so does (2.54). By (2.53) and (2.54), it follows that

$$\limsup_{n\to\infty} \frac{Q_1}{n^k P(Y > M_n)^k c_n^k M_n^{-k}} \leq \frac{\alpha^k}{(k!)^2} \int_0^\infty s^k P(Z_\alpha \in ds) \, . \tag{2.56}$$

By Lemma 2.1 and (2.50), it follows that

$$\begin{aligned} Q_2 &= O\left(n^{k+1} P(Y > T_n c_n)^{k+1}\right) \\ &= o\left(n^k P(Y > M_n)^k c_n^k M_n^{-k}\right) , \end{aligned}$$

the second inequality following from (2.49). This, along with (2.56), completes the proof of the upper bound (2.45).

Next we proceed to establish the lower bound, that is

$$\liminf_{n\to\infty} \frac{P(S_n > kM_n)}{n^k P(Y > M_n)^k c_n^k M_n^{-k}} \geq \frac{\alpha^k}{(k!)^2} \int_0^\infty s^k P(Z_\alpha \in ds) \, . \tag{2.57}$$

To that end, fix $T > 0$ and define the event

$$B_j := \left\{ \sum_{i=1}^k X_{nj_i} > kM_n - Tc_n, S_n > kM_n \right\}, \; j = (j_1, \ldots, j_k) \in C_{nk} \, .$$

Clearly,

$$\begin{aligned} P(S_n > kM_n) &\geq P\left(\bigcup_{j \in C_{nk}} B_j \right) \\ &\geq \sum_{j \in C_{nk}} P(B_j) - \sum\nolimits_{j^1, j^2 \in C_{nk}, j^1 \neq j^2} P\left(B_{j^1} \cap B_{j^2}\right) \\ &= \binom{n}{k} P\left(B_{(1,\ldots,k)}\right) - \sum_{j^1, j^2 \in C_{nk}, j^1 \neq j^2} P\left(B_{j^1} \cap B_{j^2}\right) \\ &=: Q_3 - Q_4 \, . \end{aligned}$$

Note that

$$\begin{aligned} Q_3 &\sim \frac{n^k}{k!} P\left(B_{(1,\ldots,k)}\right) \\ &\geq \frac{n^k}{k!} \int_{(0,T)} P\left(\sum_{i=1}^k X_{ni} > kM_n - sc_n \right) P_n(ds) \, , \end{aligned}$$

where $P_n(\cdot)$ is as defined in (2.52). By Lemma 2.2, (2.36) and arguments similar to those leading to (2.53), it follows that as $n \to \infty$,

$$\begin{aligned} & \int_{(0,T)} P\left(\sum_{i=1}^k X_{ni} > kM_n - sc_n\right) \\ \sim \; & \frac{\alpha^k}{k!} M_n^{-k} P(Y > M_n)^k c_n^k \int_{(0,T)} s^k P(Z_\alpha \in ds)\,. \end{aligned}$$

This shows that

$$\liminf_{n\to\infty} \frac{Q_3}{M_n^{-k} P(Y > M_n)^k c_n^k n^k} \ge \frac{\alpha^k}{(k!)^2} \int_{(0,T)} s^k P(Z_\alpha \in ds)\,.$$

If it can be shown that

$$Q_4 = o\left(M_n^{-k} P(Y > M_n)^k c_n^k n^k\right)\,, \tag{2.58}$$

then (2.57) will follow by letting $T \to \infty$. To that end, write

$$Q_4 = \sum_{l=0}^{k-1} \sum_{j^1, j^2 \in C_{nk} : \#(j^1 \cap j^2) = l} P\left(B_{j^1} \cap B_{j^2}\right)\,.$$

Fix $l \in \{0, \ldots, k-1\}$ and $j^1, j^2 \in C_{nk}$ such that $\#(j^1 \cap j^2) = l$. Notice that for n so large that $c_n^{-1} M_n \ge 2T$,

$$\begin{aligned} & P\left(B_{j^1} \cap B_{j^2}\right) \\ \le \; & P\left(\sum_{i=1}^k X_{ni} > (k-1/2)M_n, \sum_{i=k-l+1}^{2k-l} X_{ni} > (k-1/2)M_n\right) \\ \le \; & P\left(X_{ni} > M_n/2,\, 1 \le i \le 2k-l\right) \\ \le \; & K_l P(Y > M_n)^{2k-l}\,, \end{aligned}$$

for some K_l independent of j^1 and j^2. Thus,

$$\begin{aligned} Q_4 &= \sum_{l=0}^{k-1} O\left(n^{2k-l} P(Y > M_n)^{2k-l}\right) \\ &= O\left(n^{k+1} P(Y > M_n)^{k+1}\right) \\ &= o\left(M_n^{-k} P(Y > M_n)^k c_n^k n^k\right)\,, \end{aligned}$$

the equality in the last line following from (2.47). This shows (2.58), which in turn establishes the lower bound (2.57), and thus completes the proof. □

3. The case $\alpha < k/(k+2)$

Suppose that the random variables $Y_1, Y_2, \ldots$ are as defined in the beginning of Section 2, **except that now we assume** $0 < \alpha < 1$. The assumption (2.4) is still in force. Let (M_n) be a real sequence, the assumption on which will be stated in the main result, namely Theorem 3.1 below. Suppose that $\{X_{nj} : 1 \le j \le n\}$ and S_n are as defined in (1.1) and (1.2) respectively. The above mentioned result studies the behavior of $P(S_n > kM_n)$ when k is a positive integer satisfying $\alpha < k/(k+2)$. For stating the result, we need some more notations. For $k \ge 1$, define the function c_k from $[k-1, k]$ to $[0, \infty]$ recursively, by

$$\begin{aligned} c_1(t) &= t^{-\alpha} - 1 \text{ for } 0 \le t \le 1\,, \\ c_{k+1}(t) &= \alpha \int_{t-k}^{1} c_k(t-z) z^{-\alpha-1} dz \text{ for } k \le t \le k+1,\ k \ge 1. \end{aligned}$$

In the preceding definition, we have followed the convention that $0^{-\alpha} = \infty$.

Theorem 3.1. *For a positive integer k,*

$$c_{k+1}(k) < \infty\,. \tag{3.1}$$

Suppose that $0 < \alpha < k/(k+2)$ and

$$M_n^{1-\frac{\alpha(k+2)}{k}-\gamma} \gg n^{\frac{1}{\alpha}} \tag{3.2}$$

for some $0 < \gamma < 1 - \frac{\alpha(k+2)}{k}$. Then, as $n \to \infty$,

$$P(S_n > kM_n) \sim \frac{c_{k+1}(k)}{(k+1)!} n^{k+1} P(Y > M_n)^{k+1}\,. \tag{3.3}$$

Notice that the assumption (3.2) implies that

$$\lim_{n\to\infty} nP(Y > M_n) = 0\,, \tag{3.4}$$

which in particular shows that this is stronger than the corresponding assumption (2.6), for the case $1 < \alpha < 2$ in Theorem 2.1. Before starting the proof of Theorem 3.1, we shall prove a couple of lemmas, which will be used in the proof of the former.

Lemma 3.1. *For $k \ge 1$, and $x \in (k-1, k]$,*

$$P\left(\sum_{j=1}^{k} X_{nj} > xM_n\right) \sim c_k(x) P(Y > M_n)^k\,, \tag{3.5}$$

as $n \to \infty$, and (3.1) is true.

Proof. Since $k > \alpha$ by assumption, the proof of the lemma will be complete if we can show that for all $k \ge 1$, (3.5) holds, and

$$c_k(k-u) = O\left(u^k\right)\,, \tag{3.6}$$

as $u \downarrow 0$.

The proof is by induction on k. For $k = 1$, (3.5) and (3.6) are trivial to check. As the induction hypothesis, we assume them to be true for k. We proceed to show them to be true for $k+1$. To that end, fix $x \in (k, k+1]$, and write

$$\frac{P\left(\sum_{j=1}^{k+1} X_{nj} > xM_n\right)}{P(Y > M_n)^{k+1}}$$

$$= \int_{(x-k,1]} \frac{P\left(\sum_{j=1}^{k} X_{nj} > (x-s)M_n\right)}{P(Y > M_n)^k} \frac{P(X_{n1} \in M_n ds)}{P(Y > M_n)} . \tag{3.7}$$

Using the induction hypothesis (3.5) for k, and the fact that the function $s \mapsto \frac{P\left(\sum_{j=1}^{k} X_{nj} > (x-s)M_n\right)}{P(Y>M_n)^k}$ is non-decreasing, it follows that the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{s\in[x-k,1]} \left| \frac{P\left(\sum_{j=1}^{k} X_{nj} > (x-s)M_n\right)}{P(Y > M_n)^k} - c_k(x-s) \right| = 0 .$$

Thus, as $n \to \infty$,

$$\int_{(x-k,1]} \frac{P\left(\sum_{j=1}^{k} X_{nj} > (x-s)M_n\right)}{P(Y > M_n)^k} \frac{P(X_{n1} \in M_n ds)}{P(Y > M_n)}$$

$$= \int_{(x-k,1]} c_k(x-s) \frac{P(X_{n1} \in M_n ds)}{P(Y > M_n)} + o(1) . \tag{3.8}$$

Clearly,

$$\lim_{n\to\infty} \int_{(x-k,1]} c_k(x-s) \frac{P(X_{n1} \in M_n ds)}{P(Y > M_n)} = \int_{(x-k,1]} c_k(x-s)\alpha s^{-\alpha-1} ds = c_{k+1}(x) .$$

This, in view of (3.7) and (3.8), shows that (3.5) holds for $k+1$. That (3.6) holds with $k+1$, follows trivially from the hypothesis that the same holds with k, and the definition of $c_{k+1}(\cdot)$. This completes the induction step, and thus completes the proof of the lemma. □

Lemma 3.2. *If k is a positive integer, and (ε_n) is a sequence of positive numbers such that*

$$\varepsilon_n = o\left(P(Y > M_n)^{1/k}\right) , \tag{3.9}$$

then

$$P\left(\sum_{j=1}^{k+1} X_{nj} > (k-\varepsilon_n)M_n\right) \sim c_{k+1}(k) P(Y > M_n)^{k+1} , \tag{3.10}$$

as $n \to \infty$.

Proof. The proof is again by induction on k. Our induction hypothesis is a bit stronger than the statement of the result, namely the following. For all $k \geq 1$,

$$P\left(\sum_{j=1}^{k+1} X_{nj} > (k-u)M_n\right) = O\left(u^k P(Y > M_n)^k + P(Y > M_n)^{k+1}\right), \tag{3.11}$$

as $u \downarrow 0$ and $n \to \infty$. In addition, if (ε_n) is a sequence of positive numbers satisfying (3.9), then (3.10) holds. We first verify this hypothesis for $k = 1$. To that end, fix $0 < u < \delta < 1/2$, and notice that

$$\begin{aligned}
& P\left(X_{n1} + X_{n2} > (1-u)M_n\right) \\
= & \int_{(-u,1]} P\left(X_{n1} > (1-u-z)M_n\right) P\left(X_{n1} \in M_n dz\right) \\
= & \int_{(-u,0]} + \int_{(0,\delta]} + \int_{(\delta,1-\delta]} + \int_{(1-\delta,1-u]} + \int_{(1-u,1]} \\
=: & \; I_1 + I_2 + I_3 + I_4 + I_5 \,.
\end{aligned}$$

Clearly,

$$\begin{aligned}
I_1 & \leq P\left(X_{n1} > (1-u)M_n\right) \\
& = O\left(uP(Y > M_n)\right),
\end{aligned}$$

as $u \downarrow 0$ and $n \to \infty$, the last step following from Lemma 2.2. Using that result once again, it follows that there is $C < \infty$ independent of n, u and δ, satisfying

$$\begin{aligned}
I_2 & \leq CP(Y > M_n) \int_{(0,\delta]} (u+z) P\left(X_{n1} \in M_n dz\right) \\
& \leq CP(Y > M_n) \left[u + M_n^{-1} \int_{(0,\delta M_n]} zP(Y \in dz)\right].
\end{aligned}$$

By the Karamata's theorem, it follows that as $n \to \infty$,

$$M_n^{-1} \int_{(0,\delta M_n]} zP(Y \in dz) \sim \frac{\alpha}{1-\alpha} \delta^{1-\alpha} P(Y > M_n)\,.$$

This shows that

$$I_2 = O\left[uP(Y > M_n) + P(Y > M_n)^2\right].$$

A restatement of Lemma 3.1 is that

$$P(Y > M_n)^{-1} P\left(X_{n1} > \cdot\right) \to c_1(\cdot)\,,$$

uniformly on $[\delta/2, 1]$, from which it follows that for $u \leq \delta/2$,

$$\lim_{n\to\infty} \sup_{\delta \leq z \leq 1-\delta} \left| \frac{P\left(X_{n1} > (1-u-z)M_n\right)}{P(Y > M_n)} - c_1(1-u-z) \right| = 0\,.$$

Thus,

$$\begin{aligned} I_3 &\sim P(Y > M_n) \int_{(\delta, 1-\delta]} c_1(1-u-z) P(X_{n1} \in M_n dz) \\ &\sim P(Y > M_n)^2 \alpha \int_\delta^{1-\delta} c_1(1-u-z) z^{-\alpha-1} dz \,, \end{aligned}$$

as $n \to \infty$. In other words,

$$I_3 = O\left(P(Y > M_n)^2\right) .$$

By arguments similar to those leading to (2.34), it follows that for δ small enough and n large enough,

$$\begin{aligned} I_4 &\le 2\alpha P(Y > M_n) \int_{1-\delta}^{1-u} P\left(X_{n1} > (1-u-z)M_n\right) dz \\ &= 2\alpha P(Y > M_n) M_n^{-1} \int_0^{(\delta-u)M_n} P(Y > v) dv \\ &\le 2\alpha P(Y > M_n) M_n^{-1} \int_0^{\delta M_n} P(Y > v) dv \\ &\sim 2\frac{\alpha}{1-\alpha} \delta^{1-\alpha} P(Y > M_n)^2 \,, \end{aligned}$$

as $n \to \infty$, the last step following from Karamata's theorem. Consequently,

$$I_4 = O\left(P(Y > M_n)^2\right) .$$

Finally, the same arguments show that,

$$\begin{aligned} I_5 &= O\left(P(Y > M_n) \int_{1-u}^1 P\left(X_{n1} > (1-u-z)M_n\right) dz \right) \\ &= O\left(uP(Y > M_n)\right) . \end{aligned}$$

The above calculations show that (3.11) holds for $k = 1$. Now suppose that (ε_n) is a sequence of positive numbers satisfying

$$\varepsilon_n = o\left(P(Y > M_n)\right) .$$

Define I_1 to I_5 as above, with u replaced by ε_n. The same calculations as above, will reveal that

$$I_3 \sim \alpha \int_\delta^{1-\delta} c_1(1-z) z^{-\alpha-1} dz$$

as $n \to \infty$, and

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \left[I_1 + I_2 + I_4 + I_5\right] = 0 .$$

Thus, (3.10) holds, again with $k = 1$.

Next, we proceed to prove the induction step, that is, assuming that the hypothesis is true for $k - 1$, we shall show that to be true for k. Let

$0 < u < \delta < 1/2$, and write

$$\begin{aligned}
& P\left(\sum_{j=1}^{k+1} X_{nj} > (k-u)M_n\right) \\
= \quad & \int_{(-u,1]} P\left(\sum_{j=1}^{k} X_{nj} > (k-u-z)M_n\right) P(X_{n1} \in M_n dz) \\
= \quad & \int_{(-u,0]} + \int_{(0,\delta]} + \int_{(\delta,1-\delta]} + \int_{(1-\delta,1-u]} + \int_{(1-u,1]} \\
=: \quad & J_1 + J_2 + J_3 + J_4 + J_5 \,.
\end{aligned}$$

By similar arguments as above, it follows that

$$J_1 + J_2 = O\left(u^k P(Y > M_n)^k + P(Y > M_n)^{k+1}\right) \,,$$

as $u \downarrow 0$ as $n \to \infty$, and

$$J_3 \sim P(Y > M_n)^{k+1} \int_{\delta}^{1-\delta} c_k(k-u-z)\alpha z^{-\alpha-1} dz \,,$$

as $n \to \infty$. Once again, by arguments similar to those leading to (2.34), it follows that for δ small enough and n large enough,

$$\begin{aligned}
J_4 \quad & \le \quad 2\alpha P(Y > M_n) \int_{1-\delta}^{1-u} P\left(\sum_{j=1}^{k} X_{nj} > (k-u-z)M_n\right) dz \\
& \le \quad 2\alpha\delta P(Y > M_n) P\left(\sum_{j=1}^{k} X_{nj} > (k-1)M_n\right) \,.
\end{aligned}$$

Using the induction hypothesis (3.11) for $k-1$, it follows that

$$P\left(\sum_{j=1}^{k} X_{nj} > (k-1)M_n\right) = O\left(P(Y > M_n)^k\right) \,,$$

which in turn, shows that there exists $C < \infty$ (independent of δ, u and n) satisfying

$$J_4 \le C\delta P(Y > M_n)^{k+1} \,,$$

for n large and δ small enough. Finally, the same arguments show that,

$$J_5 \quad = \quad O\left(P(Y > M_n) \int_{1-u}^{1} P\left(\sum_{j=1}^{k} X_{nj} > (k-u-z)M_n\right) dz\right) \,.$$

Using the induction hypothesis, we claim that there is $C < \infty$ such that

$$P\left(\sum_{j=1}^{k} X_{nj} > (k-u-z)M_n\right) \le$$

$$C\left[(u+z-1)^{k-1}P(Y>M_n)^{k-1}+P(Y>M_n)^k\right],$$

whenever $1-u\le z\le 1$, and u small enough. Thus, for such an u,

$$\begin{aligned}&\int_{1-u}^1 P\left(\sum_{j=1}^k X_{nj}>(k-u-z)M_n\right)dz\\ \le\ & C\left[P(Y>M_n)^{k-1}\int_{1-u}^1(u+z-1)^{k-1}dz+P(Y>M_n)^k\right]\\ =\ & C\left[P(Y>M_n)^{k-1}\frac{u^k}{k}+P(Y>M_n)^k\right].\end{aligned}$$

This shows that

$$J_5=O\left(u^kP(Y>M_n))^k+P(Y>M_n))^{k+1}\right),$$

and thus concludes the proof of (3.11) for k. Once again, a close inspection of the calculations above will reveal that if (ε_n) is a sequence of positive numbers satisfying (3.9), then (3.10) holds. This proves the induction step, and thus completes the proof of the lemma □

Proof of Theorem 3.1. In view of Lemma 3.1, (3.3) is what remains to show. To that end, we start with the proof of the upper bound, that is the lim sup of the left hand side divided by the right one is at most 1. Define

$$u_n:=P(Y>M_n)^{1/k}M_n,\ n\ge 1.$$

By (3.2), it follows that

$$P(Y>M_n)^{-\frac{k+1}{k+2}}P(Y>u_n)\ll n^{-\frac{1}{k+2}},$$

which we restate as

$$P(Y>u_n)\ll n^{-\frac{1}{k+2}}P(Y>M_n)^{\frac{k+1}{k+2}}.$$

Clearly, the right hand side goes to zero as $n\to\infty$, and hence there exists a sequence (z_n) satisfying

$$P(Y>u_n)\ll P(Y>z_n)\ll n^{-\frac{1}{k+2}}P(Y>M_n)^{\frac{k+1}{k+2}}. \tag{3.12}$$

Fix such a (z_n). An immediate consequence of the left inequality above is that

$$z_n\ll P(Y>M_n)^{1/k}M_n.$$

Fix a sequence (ε_n) satisfying

$$\frac{z_n}{M_n}\ll\varepsilon_n\ll P(Y>M_n)^{1/k}. \tag{3.13}$$

Define the events

$$\begin{aligned}
E_n &:= \left\{ \sum_{i=1}^{k+1} X_{nj_i} > (k-\varepsilon_n)M_n \text{ for some } 1 \le j_1 \le \ldots \le j_{k+1} \le n \right\} , \\
F_n &:= \{ |X_{nj}| > z_n \text{ for at least } (k+2) \text{ many } j\text{'s } \le n \} , \\
G_n &:= \left\{ \sum_{j=1}^{n} X_{nj} \mathbf{1}\left(|X_{nj}| \le z_n\right) > \varepsilon_n M_n \right\} , \\
H_n &:= \left\{ \sum_{i=1}^{k} X_{nj_i} > (k-\varepsilon_n)M_n \text{ for some } 1 \le j_1 \le \ldots \le j_k \le n \right\} .
\end{aligned}$$

Clearly, for n large enough so that $\varepsilon_n \le 1$, it holds that

$$\{S_n > kM_n\} \subset E_n \cup F_n \cup G_n \cup H_n \, .$$

Note that

$$\begin{aligned}
P(E_n) &\le \frac{n^{k+1}}{(k+1)!} P\left(\sum_{j=1}^{k+1} X_{nj} > (k-\varepsilon_n)M_n \right) \\
&\sim \frac{n^{k+1}}{(k+1)!} c_{k+1}(k) P(Y > M_n)^{k+1} ,
\end{aligned}$$

as $n \to \infty$, the equivalence following from Lemma 3.2 and the right inequality in (3.13). In order to complete the proof of the upper bound, we shall next show that

$$P(F_n) + P(G_n) + P(H_n) = o\left(n^{k+1} P(Y > M_n)^{k+1} \right) . \tag{3.14}$$

To that end, notice that the right inequality in (3.12) implies that

$$P(F_n) = o\left(n^{k+1} P(Y > M_n)^{k+1} \right) .$$

By Lemma 2.2, it follows that

$$\begin{aligned}
P(H_n) &= O\left(n^k \varepsilon_n^k P(Y > M_n)^k \right) \\
&= o\left(n^k P(Y > M_n)^{k+1} \right) \\
&= o\left(n^{k+1} P(Y > M_n)^{k+1} \right) ,
\end{aligned}$$

the equality in the second line following from the right inequality in (3.13). For estimating $P(G_n)$, we shall appeal once more to the result in Prokhorov

(1959); see (2.23). First notice that

$$\begin{aligned} E\left|\sum_{j=1}^n X_{nj}\mathbf{1}\left(|X_{nj}|\le z_n\right)\right| &\le n\int_{[0,z_n]} xP(|Y|\in dx) \\ &= O\left(nz_nP(Y>z_n)\right) \\ &= o\left[z_n\left\{nP(Y>M_n)\right\}^{\frac{k+1}{k+2}}\right] \\ &= o(z_n) \\ &= o(\varepsilon_n M_n)\,, \end{aligned}$$

the last three steps following from the right inequality in (3.12), the limit in (3.4) and the left inequality in (3.13) respectively. Thus, for n large enough, (2.23) implies that

$$P(G_n)\le \exp\left\{-K_1\frac{\varepsilon_n M_n}{z_n}\sinh^{-1}K_2\frac{\varepsilon_n M_n z_n}{n\mathrm{Var}[X_{n1}\mathbf{1}(|X_{n1}|\le z_n)]}\right\},$$

for some finite and positive constants K_1 and K_2. Note that

$$\begin{aligned} \frac{\varepsilon_n M_n z_n}{n\mathrm{Var}[X_{n1}\mathbf{1}(|X_{n1}|\le z_n)]} &\ge \frac{\varepsilon_n M_n z_n}{nE[X_{n1}^2\mathbf{1}(|X_{n1}|\le z_n)]} \\ &\sim K\frac{\varepsilon_n M_n}{nz_nP(Y>z_n)} \\ &\gg [nP(Y>z_n)]^{-1} \\ &\gg [nP(Y>M_n)]^{-(k+1)/(k+2)}\,, \end{aligned}$$

where K is the constant from Karamata's theorem. In view of this, the fact that $\sinh^{-1}x\ge\log x$ for $x>0$, and (3.13), it follows that

$$-K_1\frac{\varepsilon_n M_n}{z_n}\sinh^{-1}K_2\frac{\varepsilon_n M_n z_n}{n\mathrm{Var}[X_{n1}\mathbf{1}(|X_{n1}|\le z_n)]}\gg -\log\left[nP(Y>M_n)\right]\,,$$

which shows that

$$P(G_n)=o\left(n^{k+1}P(Y>M_n)^{k+1}\right)\,.$$

This completes the proof of (3.14), and thus shows that

$$\limsup_{n\to\infty}\frac{P(S_n>kM_n)}{n^{k+1}P(Y>M_n)^{k+1}}\le\frac{c_{k+1}(k)}{(k+1)!}\,. \tag{3.15}$$

For the reverse inequality with lim inf, notice that for all $\varepsilon>0$,

$$\{S_n>kM_n\}\supset\bigcup_{j\in C_{nk}}\left\{\sum_{i=1}^{k+1}X_{nj_i}>(k+\varepsilon)M_n,\left|\sum_{i\in j^c}X_{ni}\right|\le\varepsilon M_n\right\}\,,$$

where C_{nk} and j^c are as defined just before Lemma 2.1. By arguments similar to those proving (2.58), it follows that

$$\begin{aligned}
&P\left(\bigcup_{j\in C_{nk}}\left\{\sum_{i=1}^{k+1}X_{nj_i}>(k+\varepsilon)M_n,\left|\sum_{i\in j^c}X_{ni}\right|\le\varepsilon M_n\right\}\right)\\
\sim\ &\sum_{j\in C_{nk}}P\left(\sum_{i=1}^{k+1}X_{nj_i}>(k+\varepsilon)M_n,\left|\sum_{i\in j^c}X_{ni}\right|\le\varepsilon M_n\right)\\
=\ &\binom{n}{k+1}P\left(\sum_{i=1}^{k+1}X_{ni}>(k+\varepsilon)M_n\right)P\left(\left|\sum_{i=k+2}^{n}X_{ni}\right|\le\varepsilon M_n\right).
\end{aligned}$$

By Lemma 3.1 and the fact that $M_n^{-1}\sum_{i=k+2}^n X_{ni}$ goes to zero in probability, it follows that the right hand side is asymptotically equivalent to

$$\frac{n^{k+1}}{(k+1)!}P(Y>M_n)^{k+1}c_{k+1}(k+\varepsilon)\,.$$

The above calculations put together show that

$$\liminf_{n\to\infty}\frac{P(S_n>kM_n)}{n^{k+1}P(Y>M_n)^{k+1}}\ge\frac{c_{k+1}(k+\varepsilon)}{(k+1)!}\,.$$

By letting $\varepsilon\downarrow 0$, the reverse inequality of (3.15) with lim inf, follows. This completes the proof. □

4. Examples

We end the paper with a couple of examples. The first example is of a random variable with a regularly varying tail, for which (2.4) does not hold. Let $\alpha>0$ and suppose Y is a random variable whose tail probability is given by

$$P(Y>x)=\frac{1}{2}\left(1+\frac{1}{\lfloor x\rfloor}\right)x^{-\alpha},\ x\ge 1\,.$$

Set

$$a_n:=1-n^{-2},\ n\ge 1\,,$$

and observe that

$$\lim_{n\to\infty}\frac{1}{1-a_n}\left[\frac{P(Y>na_n)}{P(Y>n)}-a_n^{-\alpha}\right]=1\,.$$

Clearly, (2.4) cannot hold for this Y, although the tail of Y is regularly varying with index $-\alpha$.

Next, we point out that the assumption (2.4) is automatically satisfied if the tail of Y is "normalized slowly varying", that is, there exist $c,T\in(0,\infty)$ and a function ε from $[0,\infty)$ to $\mathbb{R}$ such that

$$\lim_{x\to\infty}\varepsilon(x)=-\alpha\,,$$

and

$$P(Y > x) = c \int_T^x \frac{\varepsilon(t)}{t} dt, \text{ for all } x > T\,.$$

The class of normalized slowly varying functions is known to coincide with the Zygmund class of slowly varying functions; see Chapter 1.5.3 of Bingham et al. (1987) for a definition of the latter and a proof of this fact. For example, if Y is a random variable with c.d.f.

$$F(x) := \begin{cases} \max\left\{1/2, 1 - x^{-\alpha}(\log x)^{-2}\right\}, & x \geq 0\,, \\ \min\left\{1/2, |x|^{-\alpha}(\log|x|)^{-2}\right\}, & x < 0\,, \end{cases}$$

then Y satisfies the hypotheses of theorems 2.1 and 3.1, when $\alpha \geq k$ and $\alpha < k/(k+2)$ respectively.

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