On one class of holonomy groups in pseudo-Riemannian geometry

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1 Introduction and main result

Holonomy groups were introduced by Élie Cartan in the twenties [15, 16] for the study of Riemannian symmetric spaces and since then the classification of holonomy groups has remained one of the classical problems in differential geometry.

Definition 1. *Let M be a smooth manifold endowed with an affine symmetric connection ∇. The holonomy group of ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_xM)$ that consists of the linear operators $A : T_xM \to T_xM$ being "parallel transport transformations" along closed loops γ with $\gamma(0) = \gamma(1) = x$.*

Problem. Given a subgroup $H \subset GL(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on M^n?

The fundamental results in this direction are due to Marcel Berger [6] who initiated the programme of classification of Riemannian and irreducible holonomy groups which was completed by D. V. Alekseevskii [1], R. Bryant [12, 13], D. Joice [22, 23, 24], L. Schwahhöfer, S. Merkulov [34]. Very good historical surveys can be found in [14], [37].

In the pseudo-Riemannian case the complete description of holonomy groups is a very difficult problem which still remains open and even particular examples are of interest (see [18, 19, 7, 10, 21, 28, 29, 30]). We refer to [20] for more information on recent development in this field.

In our paper, we deal with Levi-Civita connections only. In algebraic terms this means that we consider only subgroups of the (pseudo-)orthogonal group $\mathrm{SO}(g)$:

$$H \subset \mathrm{SO}(g) = \{A \in \mathrm{GL}(V) \mid g(Au, Av) = g(u, v),\ u, v \in V\},$$

where g is a non-degenerate bilinear form on V.

The main result of our paper is as follows.

Theorem 1. *For every g-symmetric operator $L : V \to V$, the identity connected component of its centraliser in* $\mathrm{SO}(g)$

$$G_L = \{X \in \mathrm{SO}(g) \mid XL = LX\}$$

is a holonomy group for a certain (pseudo)-Riemannian metric g.

Notice that in the Riemannian case this theorem becomes trivial: L is diagonalisable and its centraliser G_L is isomorphic to the standard direct product $\mathrm{SO}(k_1) \oplus \cdots \oplus \mathrm{SO}(k_m) \subset \mathrm{SO}(n)$, $\sum k_i \le n$, which is, of course, a holonomy group. In the pseudo-Riemannian case, L may have non-trivial Jordan blocks and the structure of G_L becomes more complicated.

The structure of the paper is as follows. First we recall in Section 2 the classical approach by Berger to studying holonomy groups which we, like many other authors, are going to use in our paper. However, in our opinion, the most interesting part of the present paper consists in two explicit matrix formulas (5) and (12) that, in essence, almost immediately lead to the solution[1]. These formulas came to "holonomy groups" from "integrable systems on Lie algebras" via "projectively equivalent metrics" and we explain this way in Sections 3 and 4. The proof itself is given in Sections 5 (Berger test) and 6 (geometric realisation). The last section (Appendix) contains some details of the proof which, we believe, are more or less standard for experts on pseudo-Riemannian geometry.

2 Some basic facts about holonomy groups: Ambrose-Singer theorem and Berger test

The famous Ambrose-Singer theorem [2] gives the following description of the Lie algebra $\mathfrak{hol}(\nabla)$ of the holonomy group $\mathrm{Hol}(\nabla)$ in terms of the curvature tensor of the connection:

$\mathfrak{hol}(\nabla)$ *is generated (as a vector space) by the operators of the form $R(u \wedge v)$ where R is the curvature tensor taken, perhaps, at different points $x \in M$.*

This motivates the following construction.

Definition 2. A map $R : \Lambda^2 V \to \mathrm{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V. \tag{1}$$

This definition simply means that R as a tensor of type $(1, 3)$ satisfies all usual algebraic properties of curvature tensors:

$$R^m_{k\,ij} = R^m_{k\,ji} \quad \text{and} \quad R^m_{k\,ij} + R^m_{i\,jk} + R^m_{j\,ki} = 0.$$

[1] We think that this kind of formulas did not appear in the context of holonomy groups before. However, we are not experts in this area and we would really appreciate any comments on this matter.

Definition 3. *Let* $\mathfrak{h} \subset \mathrm{gl}(V)$ *be a Lie subalgebra. Consider the set of all formal curvature tensors* $R : \Lambda^2 V \to \mathrm{gl}(V)$ *such that* $\mathrm{Im}\, R \subset \mathfrak{h}$*:*

$$\mathcal{R}(\mathfrak{h}) = \{R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ *is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors* $R \in \mathcal{R}(\mathfrak{h})$*, i.e.,*

$$\mathfrak{h} = \mathrm{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}.$$

Berger's test (which is sometimes referred to as Berger's criterion) is the following result which can, in fact, be viewed as a version of the Ambrose–Singer theorem:

Let ∇ *be a symmetric affine connection on* TM*. Then the Lie algebra* $\mathfrak{hol}\,(\nabla)$ *of its holonomy group* $\mathrm{Hol}\,(\nabla)$ *is Berger.*

Usually the solution of the classification problem for holonomy groups consists of two parts. First, one tries to describe all Lie subalgebras $\mathfrak{h} \subset \mathrm{gl}(n, \mathbb{R})$ of a certain type satisfying Berger's test (i.e., Berger algebras). This part is purely algebraic. The second (geometric) part is to find a suitable connection ∇ for a given Berger algebra $\mathfrak{h}$ which realises $\mathfrak{h}$ as the holonomy Lie algebra, i.e., $\mathfrak{h} = \mathfrak{hol}\,(\nabla)$.

We follow the same scheme but will use, in addition, some ideas from two other areas of mathematics: projectively equivalent metrics and integrable systems on Lie algebras. These ideas are explained in the two next sections. The reader who is interested only in the proof itself may skip them and proceed directly to Sections 5 and 6 which are formally independent of the preliminary discussion below.

3 Motivation

The problem we are dealing with is closely related to the theory of projectively equivalent (pseudo)-Riemannian metrics [3, 4, 31, 33, 39, 35].

Definition 4. Two metrics g and $\bar{g}$ on the same manifold M are called *projectively equivalent*, if they have the same geodesics considered as unparametrized curves.

In the Riemannian case the local classification of projectively equivalent pairs g and $\bar{g}$ was obtained by Levi-Civita in 1896 [31]. For pseudo-Riemannian metrics, a complete description in reasonable terms of all possible projectively equivalent pairs is still an open problem although it has been intensively studied (see [3, 5, 8, 25, 26, 40]) and many particular examples and results in this direction have been obtained.

As a particular case of projectively equivalent metrics g and $\bar{g}$ one can distinguish the following.

Definition 5. Two metrics g and $\bar{g}$ are said to be *affinely equivalent* if their geodesics coincide as parametrized curves.

It is not hard to see that this condition simply means that the Levi-Chivita connections ∇ and $\bar{\nabla}$ related to g and $\bar{g}$ are the same, i.e., $\nabla = \bar{\nabla}$ or, equivalently,

$$\nabla \bar{g} = 0.$$

If instead of $\bar{g}$ we introduce a linear operator L (i.e. tensor field of type $(1,1)$) using the standard one-to-one correspondence $\bar{g} \leftrightarrow L$ between symmetric bilinear forms and g-symmetric operators:

$$\bar{g}(\xi, \eta) = g(L\xi, \eta),$$

then the classification of affinely equivalent pairs g and $\bar{g}$ is equivalent to the classification of pairs g and L, where L is covariantly constant w.r.t. the Levi-Civita connection ∇ related to g. This problem was partially solved by G. I. Kručkovič and A. S. Solodovnikov in [40] and this paper turned out to be extremely useful for us.

On the other hand, the existence of a covariantly constant $(1,1)$-tensor field L can be interpreted in terms of the holonomy group $\mathrm{Hol}(\nabla)$:

The connection ∇ *admits a covariantly constant* $(1,1)$*-tensor field if and only if* $\mathrm{Hol}(\nabla)$ *is a subgroup of the centralizer of* L *in* $\mathrm{SO}(g)$:

$$\mathrm{Hol}(\nabla) \subset G_L = \{X \in \mathrm{SO}(g) \mid XLX^{-1} = L\}.$$

In this formula, by L we understand the value of the desired $(1,1)$-tensor field at any fixed point $x_0 \in M$. Since L is supposed to be covariantly constant, the choice of $x_0 \in M$ does not play any role.

It is natural to conjecture that for a generic metric g satisfying $\nabla L = 0$, its holonomy group coincides with G_L exactly. Thus, we come to the following natural question:

Let g be a non-degenerate bilinear form on a vector space V, and $L : V \to V$ be a g-symmetric linear operator. Can the (identity connected component of the) centralizer of L in $\mathrm{SO}(g)$, i.e.

$$G_L = \{X \in \mathrm{SO}(g) \mid XLX^{-1} = L\}$$

be realised as the holonomy group of a suitable pseudo-Riemannian metric?

The relationship between this problem and the theory of projectively equivalent metrics, in fact, gives us not only the motivation, but also an approach to its solution: our proof is based on one unexpected and remarkable observation [9] concerning the algebraic structure of the curvature tensor of projectively equivalent metrics. This observation, in turn, opens possibility to use some technology and formulas from the theory of integrable Hamiltonian systems on Lie algebras. We briefly discuss these links in the next section.

4 Curvature tensor as a sectional operator

Our proof uses two "magic formulas" (5) and (12) which are closely related to the theory of integrable systems on Lie algebras. This section explains this relationship.

Definition 6. *We say that a linear map*

$$R : \mathrm{so}(n) \to \mathrm{so}(n)$$

is a sectional operator, if R is self-adjoint w.r.t. the Killing form and satisfies the algebraic identity:

$$[R(X), L] = [X, M] \quad \textit{for all } X \in \mathrm{so}(n), \tag{2}$$

where L and M are some fixed symmetric matrices.

These operators first appeared in the famous paper by S. Manakov [32] and then were studied by A. Mischenko and A. Fomenko in the framework of the argument shift method [36]. The role of such operators is explained by the following

Theorem 2 (Manakov, Mischenko, Fomenko)**.** *Let $R : \mathrm{so}(n) \to \mathrm{so}(n)$ be a sectional operator. Then the Euler equations on $\mathrm{so}(n)$ with the Hamiltonian $H = \frac{1}{2}(R(X), X)$:*

$$\frac{dX}{dt} = [R(X), X] \tag{3}$$

admit the following Lax representation with a spectral parameter

$$\frac{d}{dt}(X + \lambda L) = [R(X) + \lambda M, X + \lambda L]$$

and, therefore, possess first integrals of the form $\mathrm{Tr}\,(X + \lambda L)^k$. These integrals commute and, if L is regular, form a complete family in involution so that the Euler equations (3) *are completely integrable.*

Recall that (3) describe the dynamics of an n-dimensional rigid body. This construction was generalised by Mischenko and Fomenko [36] to the case of arbitrary semisimple Lie algebras. In particular, it follows from [36] that $\mathrm{so}(n)$ can be replaced by $\mathrm{so}(p, q)$ (or by $\mathrm{so}(g)$ in our notation) and the construction remains essentially the same. The terminology "sectional" was suggested by A. Fomenko and V. Trofimov [17] for a more general class of operators on Lie algebras with similar properties and originally was in no way related to "sectional curvature". However, such a relation exists and is, in fact, very close.

To explain this, we first notice that $\Lambda^2 V$ can be naturally identified with $\mathrm{so}(g)$. Therefore, in the (pseudo)-Riemannian case, a curvature tensor can be understood as a linear map

$$R : \mathrm{so}(g) \to \mathrm{so}(g).$$

In this setting, by the way, the symmetry $R_{ij,kl} = R_{kl,ij}$ of the curvature tensor amounts to the fact that R is self-adjoint w.r.t. the Killing form, and "constant curvature" means that $R = a \cdot \mathrm{Id}$, $a = \mathrm{const}$. So this point of view on curvature tensors is quite natural.

The following observation was made in [9].

Theorem 3. *If g and $\bar{g}$ are projectively equivalent, then the curvature tensor of g considered as a linear map*

$$R : \mathrm{so}(g) \to \mathrm{so}(g)$$

is a sectional operator, i.e., satisfies the identity

$$[R(X), L] = [X, M] \quad \text{for all } X \in \mathrm{so}(g)$$

with L defined by $\bar{g}^{-1}g = \det L \cdot L$ and M being the Hessian of $2\mathrm{tr}\, L$, i.e. $M^i_j = 2\nabla^i \nabla_j \mathrm{tr}\, L$.

This result is, in fact, a reformulation of the so-called second Sinjukov equation [39] for projectively equivalent metrics.

What is important for us is an explicit formula that represents $R(X)$ in terms of L and M. To get this formula, one first needs to notice that (2) immediately implies that M belongs to the center of the centralizer of L and, therefore, can be presented as $M = p(L)$ where $p(t)$ is a plynomial. Then, we have:

$$R(X) = \frac{d}{dt}|_{t=0} p(L + tX). \tag{4}$$

The proof that this R indeed satisfies (2) is obvious. It is sufficient to differentiate the identity

$$[p(L + tX), L + tX] = 0$$

with respect to t to get

$$[\frac{d}{dt}|_{t=0} p(L + tX), L] + [p(L), X] = 0$$

i.e., $[R(X), L] + [M, X] = 0$ as needed.

As was already remarked (see Theorem 3), in the theory of projectively equivalent metrics, M is the Hessian of $2\mathrm{tr}\, L$. However, if g and $\bar{g}$ are affinely equivalent (we are going to deal with this case only!), then L is automatically covariantly constant and $M = 0$. Thus, the curvature tensor R satisfies a simpler equation

$$[R(X), L] = 0,$$

which, of course, directly follows from $\nabla L = 0$ and seems to make all the discussion above not relevant to our very particular situation. However, formula (4) still defines a non-trivial operator, if $p(t)$ is a non-trivial polynomial satisfying $p(L) = M = 0$, for example, the minimal polynomial for L.

We want to "transfer" this algebraic formula (4) from "integrable systems" to "Riemannian geometry", but first it would be natural to ask ourselves the following question:

Let $R : \mathrm{so}(g) \to \mathrm{so}(g)$ be an abstract sectional operator (i.e., satisfying (2)). Is R a formal curvature tensor? In other words, does R always satisfy the Bianchi identity?

Apriori this property does not seem to be obvious at all. But in the generic situation this is indeed the case. In particular, every operator R defined by (4) satisfies the Bianchi identity and, therefore, can be considered as a formal curvature tensor (see Lemma 2 below).

This discussion gives us a very good candidate for the role of a formal curvature tensor in our construction, namely, the operator defined by (4) with $p(t)$ being the minimal polynomial of L. As we shall show below, this operator satisfies all required conditions and this fact almost immediately leads to the proof of Theorem 1.

5 Step one. Berger's test

We consider a non-degenerate bilinear form g on a finite-dimensional real vector space V, and a g-symmetric linear operator $L : V \to V$, i.e.,

$$g(Lv, u) = g(v, Lu), \quad \text{for all } u, v \in V.$$

By $\mathrm{so}(g)$ we denote the Lie algebra of the orthogonal group associated with g. Recall that this Lie algebra consists of g-skew-symmetric operators:

$$\mathrm{so}(g) = \{X : V \to V \mid g(Xv, u) = -g(v, Xu), \quad u, v \in V\}.$$

We are going to verify that the centralizer of L in $\mathrm{so}(g)$, i.e.,

$$\mathfrak{g}_L = \{X \in \mathrm{so}(g) \mid XL - LX = 0\}$$

is a Berger algebra.

It is important to notice from the very beginning that $\mathfrak{g}_L$ is trivial for regular operators L. Regularity condition in this context is equivalent to one of the following:

- the centralizer of L in $\mathrm{gl}(V)$ has minimal possible dimension and is generated by powers of L, i.e., operators $\mathrm{Id}, L, L^2, \dots, L^{n-1}$, $n = \dim V$;
- the minimal polynomial of L coincides with the characteristic polynomial;
- to each eigenvalue of L corresponds exactly one Jordan block.

The first of these properties shows that a regular g-symmetric operator L may commute with g-symmetric operators only. This means, in particular, that our

Lie algebra $\mathfrak{g}_L$ (which consists of g-skew-symmetric operators!) is trivial. Thus, we are interested in singular operators L only.

The next remark is that without loss of generality we may assume that L has a single eigenvalue λ and, moreover, this eigenvalue is zero, so that L is nilpotent. This follows immediately from the standard procedure, namely, decomposition of L into λ-blocks (see Appendix for details). The construction presented below is, however, very general and works for any L.

Following Definition 3, to verify that $\mathfrak{g}_L$ is a Berger algebra we need to describe formal curvature tensors $R : \Lambda^2 V \to \mathfrak{g}_L$ and analyse the subspace in $\mathfrak{g}_L$ spanned by their images. In particular, if we can find just one single formal curvature tensor R such that $\operatorname{Im} R = \mathfrak{g}_L$, then our problem is solved.

In general, this is a rather difficult problem because the Bianchi identity represents a highly non-trivial system of linear relations. However, as was explained in Section 4, in our case we have a very good candidate for this role.

In what follows, we use the following natural identification of $\Lambda^2 V$ and $\mathrm{so}(g)$:

$$\Lambda^2 V \longleftrightarrow so(g), \quad v \wedge u = v \otimes g(u) - u \otimes g(v).$$

Here the bilinear form g is understood as an isomorphism $g : V \to V^*$ between "vectors" and "covectors". Taking into account this identification, we define a linear mapping $R : \mathrm{so}(g) \simeq \Lambda^2 V \to \mathrm{gl}(V)$ by:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX), \tag{5}$$

where $p_{\min}(\lambda)$ is the minimal polynomial of L.

Proposition 1. *Let $L : V \to V$ be a g-symmetric operator. Then* (5) *defines a formal curvature tensor $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ for the Lie algebra $\mathfrak{g}_L$. In other words, R satisfies the Bianchi identity and its image is contained in $\mathfrak{g}_L$.*

Proof consists of two lemmas.

Lemma 1. *The image of R is contained in $\mathfrak{g}_L$.*

Proof. First we check that $R(X) \in \mathrm{so}(g)$, i.e., $R(X)^* = -R(X)$, where $*$ denotes "g–adjoint":

$$g(A^* u, v) = g(u, Av), \qquad u, v \in V.$$

Since $L^* = L$, $X^* = -X$, $(p_{\min}(L + tX))^* = p_{\min}(L^* + tX^*)$ and "$\frac{d}{dt}$" and " $*$ " commute, we have

$$R(X)^* = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX)^* = \frac{d}{dt}\Big|_{t=0} p_{\min}(L^* + tX^*) = \\ = \frac{d}{dt}\Big|_{t=0} p_{\min}(L - tX) = -\frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX) = -R(X),$$

as needed. Thus, $R(X) \in \mathrm{so}(g)$. Notice that this fact holds true for any polynomial $p(\lambda)$, not necessarily minimal.

To prove that $R(X)$ commutes with L, we consider the obvious identity

$$[p_{\min}(L+tX), L+tX] = 0.$$

and differentiate it at $t=0$:

$$[\frac{d}{dt}\Big|_{t=0} p_{\min}(L+tX), L] + [p_{\min}(L), X] = 0.$$

Clearly, $p_{\min}(L) = 0$ as it is a minimal polynomial, whence $[R(X), L] = 0$, as required. Thus, $R(X) \in \mathfrak{g}_L$. □

Lemma 2. *R satisfies the Bianchi identity, i.e.*

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \quad \text{for all } u, v, w \in V.$$

Proof. It is easy to see that our operator $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathrm{gl}(V)$ can be written as $R(X) = \sum_k C_k X D_k$, where C_k and D_k are some g-symmetric operators (in our case these operators are some powers of L). Thus, it is sufficient to check the Bianchi identity for operators of the form $X \mapsto CXD$.

For $X = u \wedge v$ we have

$$C(u \wedge v)Dw = Cu \cdot g(v, Dw) - Cv \cdot g(u, Dw)$$

Similarly, if we cyclically permute u, v and w:

$$C(v \wedge w)Du = Cv \cdot g(w, Du) - Cw \cdot g(v, Du)$$

and

$$C(w \wedge u)Dw = Cw \cdot g(u, Dv) - Cu \cdot g(w, Dv).$$

Adding these three expressions and taking into account that C and D are g-symmetric, we obtain zero, as required. □

For given L and g the image of (5) can be described explicitly. In some cases it coincides with the whole Lie algebra $\mathfrak{g}_L$ which immediately implies that $\mathfrak{g}_L$ is Berger. In particular, straightforward computation leads to (see details in Appendix)

Proposition 2. *Let $L : V \to V$ be a g-symmetric nilpotent operator that consists of two Jordan blocks. Then the image of the formal curvature tensor $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ defined by (5) coincides with $\mathfrak{g}_L$. In particular, $\mathfrak{g}_L$ is Berger.*

In the general case, the proof can be obtained by a kind of "block-wise" modification of formula (5). Roughly speaking, we decompose L into Jordan blocks $L_1, \dots, L_k$, then for each pair of blocks L_i and L_j we define an operator $\hat{R}_{ij} : \mathrm{so}(n) \to \mathfrak{g}_L$ by means of (5) on the subspace spanned by these blocks and extending it trivially onto all other blocks, and finally we set:

$$R_{\text{formal}} = \sum_{i,j} \hat{R}_{ij} \tag{6}$$

This operator solves our problem (see details in Appendix). Namely, we have

Theorem 4. *Let $L : V \to V$ be a g-symmetric operator. Then $R_{\text{formal}} : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ given by* (6) *is a formal curvature tensor such that* $\operatorname{Im} R = \mathfrak{g}_L$. *In particular, $\mathfrak{g}_L$ is a Berger algebra.*

6 Step two: Realisation

Now for a given operator $L : T_{x_0}M \to T_{x_0}M$, we need to find a (pseudo)-Riemannian metric g on M and a $(1,1)$-tensor field $L(x)$ (with the initial condition $L(x_0) = L$) such that

1. $\nabla L(x) = 0$;
2. $\mathfrak{hol}\,(\nabla) = \mathfrak{g}_L$.

Notice that the first condition guarantees that $\mathfrak{hol}\,(\nabla) \subset \mathfrak{g}_L$. On the other hand, $\operatorname{Im} R(x_0) \subset \mathfrak{hol}\,(\nabla)$, where $x_0 \in M$ is a fixed point and R is the curvature tensor of g. Thus, taking into account Theorem 4, the second condition can be replaced by

$2'$. $R(x_0)$ coincides with the formal curvature tensor R_{formal} from Theorem 4.

Thus, our goal in this section is to construct (at least one example of) $L(x)$ and $g(x)$ satisfying conditions 1 and $2'$. Apart formula (5) (whose modification (12) leads to the desired example), the construction below is based on two well known geometric facts.

The first one allows us to use a nice coordinate system in which all computations at a fixed point become much simpler. Roughly speaking, linear terms of g as a function of x can be ignored.

Proposition 3. *For every metric g there exists a local coordinate system such that $\frac{\partial g_{ij}}{\partial x^\alpha}(0) = 0$ for all i, j, α. In particular, in this coordinate system we have $\Gamma^k_{ij}(0) = 0$ and the components of the curvature tensor at $x_0 = 0$ are defined as some combinations of the second derivatives of g.*

The second result states that covariantly constant $(1,1)$-tensor fields L are actually very simple. To the best of our knowledge, this theorem was first proved by A. P. Shirokov [38] (see also [11, 27, 41]).

Theorem 5. *If L satisfies $\nabla L = 0$ for a symmetric connection ∇, then there exists a local coordinate system $x^1, \dots, x^n$ in which L is constant.*

In this coordinate system, the equation $\nabla L = 0$ can be rewritten in a very simple way:

$$\left(\frac{\partial g_{ip}}{\partial x^\beta} - \frac{\partial g_{i\beta}}{\partial x^p}\right) L^\beta_k = \left(\frac{\partial g_{i\beta}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\beta}\right) L^\beta_p \tag{7}$$

This equation is linear and if we represent g as a power series in x, then (7) must hold for each term of this expansion. Moreover, if we consider the constant and second order terms only, then they give us a particular (local) solution.

This suggest the idea to set $L(x) = \text{const}$ and then to try to find the desired metric $g(x)$ in the form:

constant + quadratic

i.e.,

$$g_{ij}(x) = g^0_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q \tag{8}$$

where $\mathcal{B}$ satisfies obvious symmetry relations, namely, $\mathcal{B}_{ij,pq} = \mathcal{B}_{ji,pq}$ and $\mathcal{B}_{ij,pq} = \mathcal{B}_{ij,qp}$.

Before discussing an explicit formula for $\mathcal{B}$, we give some general remarks about "quadratic" metrics (8).

- The condition $\nabla L = 0$ amounts to the following equation for $\mathcal{B}$:

$$(\mathcal{B}_{ip,\beta q} - \mathcal{B}_{i\beta,pq}) L^\beta_k = (\mathcal{B}_{\beta i,kq} - \mathcal{B}_{ik,\beta q}) L^\beta_p \tag{9}$$

- The condition that L is g symmetric reads:

$$\mathcal{B}_{ij,pq} L^i_l = \mathcal{B}_{il,pq} L^i_j \tag{10}$$

- The curvature tensor of g at the origin $x = 0$ takes the following form:

$$R^i_{k\,\alpha\beta} = g^{is}(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}), \tag{11}$$

  and, in particular, R (at the origin) depends on $\mathcal{B}$ linearly:

$$R_{\lambda_1 \mathcal{B}_1 + \lambda_2 \mathcal{B}_2} = \lambda_1 R_{\mathcal{B}_1} + \lambda_2 R_{\mathcal{B}_2}$$

Thus, the realisation problem admits the following purely algebraic version: find $\mathcal{B}$ satisfying (9), (10) and such that (11) coincides with R_{formal} from Theorem 4. From the formal viewpoint, this is a system of linear equations on $\mathcal{B}$ which we need to solve or just to guess a particular solution.

Example. Consider the simplest case when

$$g = g_0 + \mathcal{B}(x,x), \quad \mathcal{B}_{ij}(x,x) = \sum \mathcal{B}_{ij,pq} x^p x^q \quad \text{with } \mathcal{B} = \mathcal{C} \otimes \mathcal{D},$$

where $\mathcal{C}$ and $\mathcal{D}$ are the bilinear forms associated with g_0-symmetric operators C and D, i.e., $\mathcal{B}_{ij,pq} = \mathcal{C}_{ij} \cdot \mathcal{D}_{pq}$, $\mathcal{C}_{ij} = (g_0)_{i\alpha} C^\alpha_j$, $\mathcal{D}_{pq} = (g_0)_{p\alpha} D^\alpha_q$, then the conditions (9), (10), (11) can respectively be rewritten (in terms of C and D) as

$$[CXD, L] + [CXD, L]^* = 0 \quad \text{for any } X \in \mathrm{gl}(V), \tag{9$'$}$$

$$CL = LC \tag{10$'$}$$

$$R(X) = -CXD + (CXD)^*, \qquad X \in \mathrm{so}(g_0) \tag{11$'$}$$

Similarly, if $\mathcal{B} = \sum_\alpha \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha$, then the corresponding conditions on $\mathcal{B}$ are obtained from (9′), (10′), (11′) by summing over α.

These simple observations leads us to the following conclusion. Let $B = \sum C_\alpha \otimes D_\alpha$ where C_α and D_α are g_0-symmetric operators. Consider B as a linear map

$$B : \mathrm{gl}(V) \to \mathrm{gl}(V) \quad \text{defined by } B(X) = \sum C_\alpha X D_\alpha,$$

In other words, $B(X)$ is obtained from B by "replacing" $\otimes$ by X. Then for the corresponding quadratic metric $g = g_0 + \mathcal{B}(x, x)$, the conditions (9), (10), (11) can be rewritten as

$$[B(X), L] + [B(X), L]^* = 0 \quad \text{for any } X \in \mathrm{gl}(V), \tag{9''}$$

$$[C_\alpha, L] = 0 \tag{10''}$$

$$R(X) = -B(X) + B(X)^*, \qquad X \in \mathrm{so}(g_0) \tag{11''}$$

As the reader may notice, we prefer to work with "operators" rather than "forms". We used the same idea before when we replaced $\Lambda^2(V)$ by $\mathrm{so}(g)$. The reason is easy to explain: operators form an associative algebra, i.e., one can multiply them, and we use this property throughout the paper.

The last formula (11″), in fact, shows how to reconstruct B from R: we need to "replace" X by $\otimes$, i.e., $B = -\frac{1}{2}R(\otimes)$.

Thus, consider the following formal expression:

$$B = -\frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot \otimes), \tag{12}$$

where $p_{\min}(\lambda)$ is the minimal polynomial of L. This formula looks a bit strange, but, in fact, it defines a tensor B of type $(2, 2)$ whose meaning is very simple. If $p_{min}(t) = \sum_{m=0}^n a_m t^m$ is the minimal polynomial of L, then

$$B = -\frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} L^{m-1-j} \otimes L^j. \tag{13}$$

This formula is obtained from the right hand side of (5), i.e.,

$$\frac{d}{dt}\Big|_{t=0} \left(\sum_{m=0}^{n} a_m (L + t \cdot X)^m \right) = \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} L^{m-1-j} X L^j,$$

by substituting $\otimes$ instead X.

Proposition 4. *Consider the quadratic metric* $g(x) = g^0 + \mathcal{B}(x,x)$ *with* $\mathcal{B}_{ij,pq} = g_{i\alpha} g_{p\beta} B^{\alpha,\beta}_{j,q}$ *where where* B *is defined by* (12) (*or, equivalently, by* (13)). *Then*

1) L *is* g*-symmetric;*

2) $\nabla L = 0$*, where* ∇ *is the Levi-Civita connection for* g*;*

3) *The curvature tensor for* g *at the origin is defined by* (5)*, i.e.,*

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX).$$

Proof. Since B is of the form $\sum_\alpha C_\alpha \otimes D_\alpha$, where C_α and D_α are some powers of L, we can use formulas (9″), (10″), (11″) (see Example above).

Item 1) is equivalent to (10″) and hence is obvious.

Next, to check 2) it suffices, according to (9″), to show that

$$[B(X), L] = 0, \quad \text{where } B = -\frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot X)$$

but this has been done in Lemma 1.

Finally, we compute the curvature tensor R at the origin by using (11″):

$$R(X) = -B(X) + B(X)^* = -2B(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX).$$

as stated. Here we again use Lemma 1 which says, in particular, that our $B(X)$ belongs to $\mathrm{so}(g_0)$, i.e., $B(X) = -B(X)^*$. ☐

This proposition together with Proposition 2 solve the realisation problem in the most important "two Jordan blocks" case. To get the realisation for the general case, we proceed just in the same way as we did for the algebraic part. Namely, we split L into Jordan blocks and define for each pair L_i, L_j of Jordan blocks a formal curvature tensor $\hat R_{ij}$ (see Appendix for details). Then by using (12) we can realise this formal curvature tensor by an appropriate quadratic metric $g(x) = g^0 + \hat B_{ij}(x,x)$ satisfying $\nabla L = 0$. We omit the details because this construction is absolutely straightforward and just repeats its algebraic counterpart. Now if we set

$$g(x) = g^0 + \mathcal{B}(x,x), \quad \text{with } B = \sum_{i,j} \hat B_{ij},$$

(or, shortly $B = -\frac{1}{2} R_{\mathrm{formal}}(\otimes)$ if we accept this "strange" notation) then, by linearity, this metric still satisfies $\nabla L = 0$ and its curvature tensor coincides with R_{formal} from Theorem 4. This completes the realisation part of the proof.

7 Appendix: Details of the proof

7.1 Reduction to the nilpotent case

First of all, we notice that it is sufficient to prove this result for two special cases only:

- either L has a single real eigenvalue,
- or L has a pair of complex conjugate eigenvalues.

The reduction from the general case to one of these is more or less standard. If L has several eigenvalues then V can be decomposed into L-invariant subspaces

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where V_i is either a generalised eigensubspace corresponding to a real eigenvalue λ, or a similar subspace corresponding to a pair of complex conjugate eigenvalues $\lambda_j, \bar{\lambda}_j$.

This decomposition is orthogonal w.r.t. g (due to the fact that L is g-symmetric) and the algebra $\mathfrak{g}_L$ is compatible with this decomposition in the sense that $\mathfrak{g}_L$ is the direct sum of Lie algebras $\mathfrak{g}_{L_i}$ each of which is naturally associated with V_i and is the centraliser of $L_i = L|_{V_i}$ in $\mathrm{so}(g|_{V_i})$. In other words, in the basis adapted to this decomposition, all of our objects, in turn, split into independent blocks:

$$g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_s \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_s \end{pmatrix}, \quad \mathfrak{g} = \begin{pmatrix} \mathfrak{g}_{L_1} & & & \\ & \mathfrak{g}_{L_2} & & \\ & & \ddots & \\ & & & \mathfrak{g}_{L_s} \end{pmatrix},$$

where $g_i = g|_{V_i}$, $\mathfrak{g}_{L_i} = \{X \in \mathrm{so}(g_i) \mid XL_i - L_iX = 0\}$.

It is an obvious fact that $\mathfrak{g}$ is a Berger algebra if and only if so is each of $\mathfrak{g}_{L_i}$. But L_i, by construction, has either a single real eigenvalue, or two complex conjugate eigenvalues.

Below we consider in detail the case when L has a single real eigenvalue $\lambda \in \mathbb{R}$ [2]. Moreover, without loss of generality, we shall assume from now on that this eigenvalue is zero, i.e., L is nilpotent (if not, we simply replace L by $L - \lambda \cdot \mathrm{Id}$).

7.2 Explicit matrix description for L, g, $\mathrm{so}(g)$ and $\mathfrak{g}_L$

We shall use the following well known analog of the Jordan normal form theorem for g-symmetric operators in the case when g is pseudo-Euclidean. We recall this result for the nilpotent case only.

Proposition 5. *Let $L : V \to V$ be a g-symmetric nilpotent operator. Then by an appropriate choice of a basis in V, we can simultaneously reduce L and g to*

[2] A pair of complex eigenvalues does not represent essentially different situation. The point is that our approach is based on Proposition 2 which still holds true for two complex Jordan blocks.

the following block diagonal matrix form:

$$L = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{pmatrix} \tag{14}$$

where

$$L_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad g_i = \pm \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

are square matrices of the same size $n_i \times n_i$*, and* $n_1 \le n_2 \le \cdots \le n_k$*. As a particular case, we admit* 1×1 *matrices* $L_i = 0$ *and* $g_i = \pm 1$*.*

In this proposition and below, we use the same notation L for the operator and its matrix. This does not lead to any confusion because from now on we can choose and fix a canonical basis. The same convention is applied to the bilinear form g and its matrix.

In what follows, we shall assume that g_i has $+1$ on the antidiagonal. This assumption is not very important, but allows us to simplify the formulae below.

The next step is the explicit description of the algebra $\mathfrak{g}_L$ which can be easily obtained by the straightforward computation. We also give the explicit matrix representation of $\mathrm{so}(g)$ in the canonical basis from Proposition 5.

Proposition 6. *In the canonical basis from Proposition 5, the orthogonal Lie algebra* $\mathrm{so}(g)$ *consists of block matrices of the form*

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & \cdots & \cdots & X_{kk} \end{pmatrix} \tag{15}$$

where X_{ij} *is an* $n_i \times n_j$ *block. The diagonal blocks* X_{ii}*'s are skew-symmetric with respect to their antidiagonal. The off-diagonal blocks* X_{ij} *and* X_{ji} *are related by*

$$X_{ji} = -g_j X_{ij}^\top g_i.$$

More explicitly:

$$X_{ij} = \begin{pmatrix} x_{11} & \cdots & x_{1n_j} \\ \vdots & \ddots & \vdots \\ x_{n_i 1} & \cdots & x_{n_i n_j} \end{pmatrix}, \quad X_{ji} = \begin{pmatrix} -x_{n_i n_j} & \cdots & -x_{1 n_j} \\ \vdots & \ddots & \vdots \\ -x_{n_i 1} & \cdots & -x_{11} \end{pmatrix} \tag{16}$$

The Lie algebra $\mathfrak{g}_L$ consists of block matrices of the form:

$$\begin{pmatrix} 0 & M_{12} & \cdots & M_{1k} \\ M_{21} & 0 & & \vdots \\ \vdots & & \ddots & M_{k-1,k} \\ M_{k1} & \cdots & M_{k,k-1} & 0 \end{pmatrix} \tag{17}$$

where M_{ij}'s for $i < j$ are $n_i \times n_j$ matrices of the form

$$M_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \mu_1 & \mu_2 & \cdots & \mu_{n_i} \\ 0 & \cdots & 0 & 0 & \mu_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \mu_2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mu_1 \end{pmatrix}, \ \mu_i \in \mathbb{R}. \tag{18}$$

If $n_i = n_j$, then M_{ij} is a square matrix and the first zero columns are absent. The blocks M_{ij} and M_{ji} are related in the same way as X_{ij} and X_{ji}, i.e., $M_{ji} = -g_j M_{ij}^\top g_i$.

The subspace $\mathfrak{m}_{ij} \subset \mathfrak{g}_L$ $(i < j)$ that consists of two blocks M_{ij} and M_{ji} is a commutative subalgebra of dimension n_i. As a vector space, $\mathfrak{g}_L$ is the direct sum $\sum_{i<j} \mathfrak{m}_{ij}$. In particular, $\dim \mathfrak{g}_L = \sum_{i=1}^{k} (k-i) n_i$.

7.3 The special case of two blocks

We consider this special case for two reasons. Firstly, the proof can be viewed as a handy algorithm for an explicit computation of (5). Secondly, and more importantly, our main result strongly relies upon it. The following statement coincides with Proposition 2.

Lemma 3. *Let L be nilpotent and consist of two Jordan blocks of sizes $m \times m$ and $n \times n$. Then the image of the formal curvature operator R defined by* (5) *coincides with $\mathfrak{g}_L$ and, therefore, $\mathfrak{g}_L$ is a Berger algebra.*

Proof. We will get this result by straightforward computation in the canonical basis described above in Proposition (5). We consider $L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, where L_1 and L_2 are standard nilpotent Jordan blocks of size m and n respectively, $m \le n$. The minimal polynomial for L is $p_{\min}(t) = t^n$.

If we represent $X \in \mathrm{so}(g)$ as a block matrix (the sizes of blocks are naturally related to m and n, of course)

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

then we see immediately that our operator $R = \frac{d}{dt}\big|_{t=0} (L + tX)^n = L^{n-1}X + L^{n-2}XL + \cdots + XL^{n-1}$ acts independently of each block of X, i.e.

$$R(X) = \begin{pmatrix} R_{11}(X_{11}) & R_{12}(X_{12}) \\ R_{21}(X_{21}) & R_{22}(X_{22}) \end{pmatrix} \tag{19}$$

The blocks $R_{11}(X_{11})$, $R_{12}(X_{12})$, $R_{21}(X_{21})$ and $R_{22}(X_{22})$ can be explicitly computed, and we shall see that the image of R is exactly our Lie algebra $\mathfrak{g}_L$.

This computation can, however, be essentially simplified, if we take into account the inclusion $\operatorname{Im} R \subset \mathfrak{g}_L$ (Lemma 1) and the fact that $\mathfrak{g}_L$ consists of the block matrices of the form $\begin{pmatrix} 0 & M \\ \widetilde{M} & 0 \end{pmatrix}$, where

$$M = \begin{pmatrix} 0 & \cdots & 0 & \mu_1 & \mu_2 & \cdots & \mu_m \\ 0 & \cdots & 0 & 0 & \mu_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \mu_2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mu_1 \end{pmatrix} \tag{20}$$

is $m \times n$ matrix, and $\widetilde{M} = -g_2 M^\top g_1$, then without any computation we can conclude that $R_{11}(X_{11}) = 0$, $R_{22}(X_{22}) = 0$ and $R_{21}(X_{21}) = -g_2^\top R_{12}(X_{12}) g_1$.

Thus, we should only explain how the parameters $\mu_1, \dots, \mu_m$ of the block $M = R_{12}(X_{12})$ depend on the entries of

$$X_{12} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{k2} & \dots & x_{mn} \end{pmatrix}$$

We have

$$R_{12}(X_{12}) = L_1^{n-1} X_{12} + L_1^{n-2} X_{12} L_2 + \cdots + X_{12} L_2^{n-1} \tag{21}$$

and an easy computation gives

$$\begin{aligned} \mu_1 &= x_{m1}, \\ \mu_2 &= x_{m-1,1} + x_{m2}, \\ \mu_3 &= x_{m-2,1} + x_{m-1,2} + x_{m3}, \\ &\vdots \\ \mu_k &= x_{11} + x_{22} + x_{33} + \cdots + x_{mm}. \end{aligned}$$

Thus, there are no relations between μ_i's and therefore the image of R coincides with $\mathfrak{g}_L$, which completes the proof.

□

7.4 General case (end of the proof)

Let us now consider the general case, i.e., L and g of the form (14). We start with the following obvious and well known remark.

Let $V' \subset V$ be a subspace of V such that $g' = g|_{V'}$ is non-degenerate. Let $\mathfrak{h} \subset \mathrm{so}(g')$ be a Berger subalgebra. Then $\mathfrak{h}$ will be a Berger subalgebra of $\mathrm{so}(g)$ too, if we consider the standard embedding $\mathrm{so}(g') \to \mathrm{so}(g)$ induced by the inclusion $V' \to V$.

Moreover, if $R' : \mathrm{so}(g') \to \mathrm{so}(g')$ is a formal curvature tensor, then its trivial extension $R : \mathrm{so}(g) \to \mathrm{so}(g)$ defined by

$$R \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} R'(X) & 0 \\ 0 & 0 \end{pmatrix}$$

is a formal curvature tensor too.

This remark allows to reformulate Lemma 3 in the following way. Consider the operator $\widehat{R}_{12} : \mathrm{so}(g) \to \mathrm{so}(g)$ defined by:

$$\widehat{R}_{12} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & 0 \\ R_{21}(X_{21}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{22}$$

where $R_{12}(X_{12})$ and $R_{21}(X_{21})$ are defined as in Lemma 3 and all the other blocks in the right hand side vanish. Then $\widehat{R}_{12}$ is a formal curvature tensor and its image coincides with the Abelian subalgebra $\mathfrak{m}_{12} \subset \mathfrak{g}_L$ (see Lemma 3). In particular, $\mathfrak{m}_{12} \subset \mathrm{so}(g)$ is a Berger algebra.

To construct the curvature operator $R : \mathrm{so}(g) \to \mathfrak{g}_L$ in the general case we need to slightly modify formula (5) taking into account the fact that the blocks of L and X may have different sizes.

We define R as follows:

$$R \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & R_{1k}(X_{1k}) \\ R_{21}(X_{21}) & 0 & \cdots & R_{2k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{k1}(X_{k1}) & R_{k2}(X_{k2}) & \cdots & 0 \end{pmatrix} \tag{23}$$

In other words, R acts on each block X_{ij} independently (compare with the proof of Lemma 3). Moreover, each of its components

$$R_{ij} : X_{ij} \mapsto R_{ij}(X_{ij})$$

is defined in the exactly same way as in Lemma 3, if we ignore all the blocks of L except for L_i and L_j. More precisely,

$$R_{ij}(X_{ij}) = L_i^{n_{ij}-1} X_{ij} + L_i^{n_{ij}-2} X_{ij} L_j + \cdots + X_{ij} L_j^{n_{ij}-1}, \tag{24}$$

where $n_{ij} = \max\{n_i, n_j\}$, and n_i, n_j are sizes of the nilpotent Jordan blocks L_i and L_j.

If we introduce the operators $\widehat{R}_{ij} : \mathrm{so}(g) \to \mathrm{so}(g)$ by generalising (22) for arbitrary indices $i < j$, then we can rewrite (23) and (24) as

$$R = \sum_{i<j} \widehat{R}_{ij}.$$

The following statement completes the proof of Theorem 4.

Proposition 7. *The operator R defined by* (23) *and* (24) *is a formal curvature tensor. Moreover,* $\operatorname{Im} R = \mathfrak{g}_L$ *and, therefore,* $\mathfrak{g}_L$ *is a Berger algebra.*

Proof. Since each $\widehat{R}_{ij}$ is a formal curvature tensor, so is R by linearity. The image of $\widehat{R}_{ij}$ is the subalgebra $\mathfrak{m}_{ij}$. From (23) it is easily seen that each $\widehat{R}_{ij}$ acts only the blocks X_{ij} and X_{ji} and does not interact with other blocks at all. This (together with Proposition 6) immediately implies that

$$\operatorname{Im} R = \sum_{i<j} \operatorname{Im} \widehat{R}_{ij} = \sum_{i<j} \mathfrak{m}_{ij} = \mathfrak{g}_L,$$

as required. □

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