TANGLE SUMS AND FACTORIZATION OF A-POLYNOMIALS

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Abstract. We show that there exist infinitely many examples of pairs of knots, K_1 and K_2, that have no epimorphism $\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$ preserving peripheral structure although their A-polynomials have the factorization $A_{K_2}(L, M) \mid A_{K_1}(L, M)$. Our construction accounts for most of the known factorizations of this form for knots with 10 or fewer crossings. In particular, we conclude that while an epimorphism will lead to a factorization of A-polynomials, the converse generally fails.

1. Introduction

Cooper et al. [5] introduced the A-polynomial as a knot invariant derived from the $SL(2, \mathbb{C})$-representations of the fundamental group of the knot's complement. It is a polynomial in the variables M and L, which correspond to the eigenvalues of the $SL(2, \mathbb{C})$-representations of the meridian and longitude respectively. We can obtain a lot of geometric information from A-polynomials including boundary slopes of incompressible surfaces in the knot complement and the non-existence of Dehn surgeries yielding 3-manifolds with cyclic or finite fundamental groups, see for instance [4, 5, 1].

It is natural to ask if there is a correspondence between epimorphisms among the fundamental groups of knot complements and their A-polynomials. Actually, Silver and Whitten [19] showed that if there exists an epimorphism, $\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$, between the fundamental groups of two knot complements, that preserves peripheral structure, then the A-polynomial of K_1 has a factor corresponding to the A-polynomial of K_2 under a suitable change of coordinates. Here we say an epimorphism preserves peripheral structure if the image of the subgroup generated by the meridian and longitude of K_1 is included in the subgroup generated by the meridian and longitude of K_2. Hoste and Shanahan [12] refined this by demonstrating that the A-polynomial of K_1 has a factor which corresponds to the A-polynomial of K_2 under the change of coordinates $(L, M) \mapsto (L^d, M)$ for some $d \in \mathbb{Z}$. Ohtsuki, Riley, and Sakuma [17] made a systematic study of epimorphisms between 2-bridge link groups.

In this paper, we study factorizations of A-polynomials of knots obtained by specific tangle sums and the existence of epimorphisms. In particular, we show that there are infinitely many knots whose A-polynomials have factorizations for which there is no corresponding epimorphism. Moreover, our factorization is realized without change of coordinates. We found 16 examples of such factorizations of A-polynomials among the knots with 10 or fewer crossings.

We now introduce the tangle sum, which will play a central role in this paper. A marked tangle is one whose four ends have specific orientations as shown on the left in Figure 1. The sum of two marked tangles S and T is a marked tangle obtained as shown on the right, denoted by $S \dot{+} T$. Let $N(T)$ and $D(T)$ denote the numerator and denominator closure of a marked tangle T respectively.

First we consider the factorization of the Alexander polynomial of a knot $N(S \dot{+} T)$. Let $\Delta_K(t)$ denote the Alexander polynomial of a knot K in S^3. Using his formulation of the Alexander polynomial, Conway

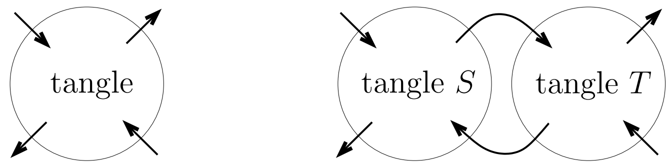

FIGURE 1. A marked tangle and the sum of marked tangles.

observed that

$$\Delta_{N(S\dot{+}T)}(t) = \Delta_{N(T)}(t)\Delta_{D(S)}(t) + \Delta_{D(T)}(t)\Delta_{N(S)}(t)$$

holds (cf. [7, Theorem 7.9.1]). In particular, if $N(S)$ is a split link then the Alexander polynomial has a factorization as

$$\Delta_{N(S\dot{+}T)}(t) = \Delta_{N(T)}(t)\Delta_{D(S)}(t) \tag{1}$$

since $\Delta_{N(S)}(t) = 0$.

If, for knots K_1 and K_2, $\pi_1(S^3 \setminus K_1)$ has an epimorphism onto $\pi_1(S^3 \setminus K_2)$, then $\Delta_{K_2}(t) \mid \Delta_{K_1}(t)$ (e.g., see [8]). It is known that converse does not hold in general. If we restrict our attention to epimorphisms which preserve peripheral structure, we can find an infinite family of counterexamples to the converse in 2-bridge knots, which is our first result.

Theorem 1. *Let $K = K(\beta/\alpha)$ be a 2-bridge knot, where α/β has continued fraction $[2, -n, k, n, -2]$, and $K_{2,k}$ be the $(2, k)$-torus knot, where $k > 2$ is odd and $n > 1$. Then $\pi_1(S^3 \setminus K)$ admits no epimorphism onto $\pi_1(S^3 \setminus K_{2,k})$ preserving peripheral structure, although $\Delta_{K_{2,k}}(t) \mid \Delta_K(t)$.*

Note that the $K(\beta/\alpha)$ has a diagram of the form $N(S\dot{+}T)$ with $N(T) = K_{2,k}$, see Figure 2. This is why we have the factorization $\Delta_{K_{2,k}}(t) \mid \Delta_K(t)$.

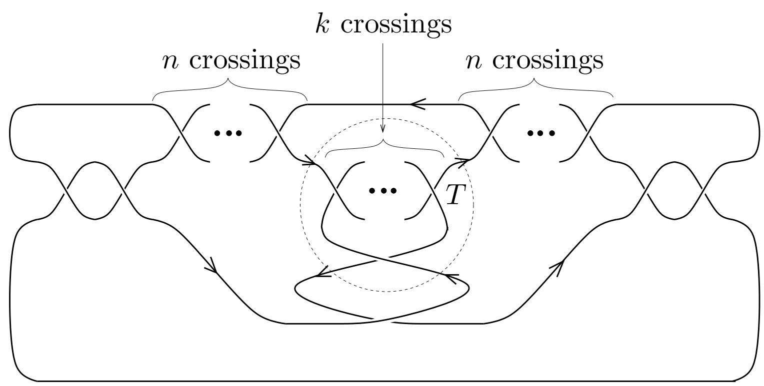

FIGURE 2. The 2-bridge knot $K(\beta/\alpha)$ where $\alpha/\beta = [2, -n, k, n, -2]$.

As in Figure 3 below (in Section 4), we can easily see that even if the tangle T is not marked, by applying Reidemeister move II, we can represent K as the sum of two marked tangles S' and T'. Moreover we have $N(T) = N(T')$ and $N(S) = N(S')$ (but usually $D(T) \neq D(T')$ and $D(S) \neq D(S')$). Therefore we always have $\Delta_{N(T)}(t) \mid \Delta_K(t)$ without assuming that the tangles are marked. We denote by $S + T$ the sum of non-marked tangles S and T. As in the marked tangle case, $N(T)$ and $D(T)$ denote the numerator and denominator closure, respectively, of a non-marked tangle T.

Next, we study the factorization of the A-polynomial of the knot $N(S+T)$. Let $A_K(L, M)$ denote the A-polynomial of a knot K in S^3 and $A^\circ_K(L, M)$ denote the product of the factors of $A_K(L, M)$ containing the variable L.

Theorem 2. *Suppose that $N(S + T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3. Then $A^\circ_{N(T)}(L, M) \mid A_{N(S+T)}(L, M)$.*

Note that $A_K(L,M)/A_K^\circ(L,M)$ is a polynomial with only one variable M. It is known that the roots of such a polynomial are on the unit circle, for instance see [5]. An example is the knot 9_{38} which, according to a calculation by Culler [3], has $(1-M^2)^2$ as a factor.

Several properties arising from the $SL(2,\mathbb{C})$-representations of the fundamental group of the complement of $N(T)$ are inherited by $N(S+T)$ as listed in the following corollary. The third statement of the corollary refers to r-curves, which we now define.

Definition 3. If $A_K(L,M)$ has a factor of the form $1 \pm L^b M^a$ (respectively, $L^b \pm M^a$), we say that the character variety of the knot K has an r-curve with $r = a/b$ (respectively, $r = -a/b$).

Remark 4. This corresponds to the definition of [2, Section 5].

Corollary 5. *Suppose that $N(S+T)$, $N(T)$, and $N(S)$ satisfy the conditions of Theorem 2.*

(1) *With the possible exception of $\frac{1}{0}$, the set of boundary slopes of $N(T)$ detected by its character variety is a subset of the boundary slopes of $N(S+T)$.*
(2) *If $A^\circ_{N(T)}(L,M)$ has a Newton polygon which does not admit cyclic/finite surgeries, then $N(S+T)$ does not have cyclic/finite surgeries.*
(3) *If the character variety of $N(T)$ has an r-curve with $r \neq \frac{1}{0}$, then that of $N(S+T)$ also has an r-curve with the same r.*

Our second corollary shows that the pairs of knots of Theorem 1 also constitute an infinite family where the A-polynomial of one factors that of the other even though there is no epimorphism between them.

Corollary 6. *Let K be the 2-bridge knot $K(\beta/\alpha)$ with $\alpha/\beta = [2,-n,k,n,-2]$ and $K_{2,k}$ the $(2,k)$-torus knot, where $k > 2$ is odd and $n > 1$. Then $\pi_1(S^3 \setminus K)$ admits no epimorphism onto $\pi_1(S^3 \setminus K_{2,k})$ preserving peripheral structure, although $A_{K_{2,k}}(L,M) \mid A_K(L,M)$.*

This corollary follows from Theorem 1, Theorem 2, and that ∞ is not a boundary slope of $K_{2,k}$.

In [18] Riley discusses three ways in which character varieties of 2-bridge knots and links may become reducible. The examples in Corollary 6 do not fall into any of those three categories.

This paper is organized as follows. We prove Theorems 1 and 2 in Sections 2 and 3 respectively. The 16 examples of factorizations of A-polynomials are listed in Table 1 of Section 4 where we also pose a few questions. In an appendix we explain the factorization of the Alexander polynomials $\Delta_{N(T)}(t) \mid \Delta_{N(S+T)}(t)$ from the viewpoint of $SL(2,\mathbb{C})$-representations.

We would like to express our gratitude to Makoto Sakuma for his precious comments and for informing us of Riley's result on the reducibility of the character variety of 2-bridge knots. We would like to thank Fumikazu Nagasato for telling us some important details about A-polynomials as relates to Theorem 2. The first author is supported by MEXT, Grant-in-Aid for Young Scientists (B) (No. 22740032). The third author is supported by the MEXT, Grant-in-Aid for Scientific Research (C) (No. 22540066).

In this study, we often referred to the list of A-polynomials computed by Hoste and Culler and other knot invariants in the database `KnotInfo` [3]. We also used the program `Knotscape` of Hoste and Thistlethwaite for checking the knot types of given knot diagrams. We thank them for these useful computer programs and their database.

2. Proof of Theorem 1

Proof of Theorem 1. We follow the notation of [10] so that

$$[2,-n,k,n,-2] = \frac{k(2n-1)^2}{kn(2n-1)+1}.$$

Then, $\alpha = k(2n-1)^2$ and $\beta = kn(2n-1)+1$, both are positive odd integers, and $0 < \beta < \alpha$.

In [11, Theorem 16], the authors observe that a knot K admits an epimorphism (preserving peripheral structure) onto the group of a torus knot if and only if K has property Q. In [10], they present an algorithm that will determine whether or not a given fraction for a 2–bridge knot will result in a knot with Property Q. We will use that algorithm to show that K does not have Property Q.

We must argue that $\gcd(\alpha,\beta)=1$. To this end, note that we can write 2 as a combination of α and β:

$$2=(kn-1)\alpha+(2(1-kn)+k)\beta.$$

Since α is odd, we can write $\alpha=1+2m$. Then

$$\begin{aligned}1&=\alpha-2m\\&=\alpha-m((kn-1)\alpha+(2(1-kn)+k)\beta)\\&=(1-m(kn-1))\alpha-m(2(1-kn)+k)\beta\end{aligned}$$

whence $\gcd(\alpha,\beta)=1$, as required.

Following the algorithm of [10, Remark (2) on p.452], in Step 0, we set $d=\alpha$, $q=\alpha/\beta$. Since $\gcd(\alpha,\beta)=1$, q is an integer only if $\beta=1$. However, $\beta=kn(2n-1)+1>1$. So we pass on to step 2. Notice that $\alpha<2\beta$. Then $\lfloor q\rfloor=1$, where $\lfloor x\rfloor=\max\{y\in\mathbb{Z}\mid y\le x\}$ for $x\in\mathbb{R}$. So, $\gcd(\lfloor q\rfloor,d)=1$. On the other hand, since $d=\alpha$ is odd, $\gcd(\lfloor q\rfloor+1,d)=1$, too. Thus $d':=\max\{\gcd(\lfloor q\rfloor,d),\gcd(\lfloor q\rfloor+1,d)\}=1$ and K does not have property Q, as we wished to show.

As mentioned in the introduction, since $K=N(S\dot{+}T)$ with $N(S)$ a trivial link of two components and $N(T)=K_{2,k}$, then $\Delta_{K_{2,k}}(t)\mid\Delta_K(t)$. □

3. Proof of Theorem 2

We prove Theorem 2 in this section. Let F_2 denote the free group of rank 2. We first introduce a lemma that allows us a specific choice for the generators of F_2.

Lemma 7. *Let $\langle a,b\rangle$ be generators of F_2 and $\hat{a}$ be an element in F_2 conjugate to a. Then there exists $\hat{b}\in F_2$ conjugate to b such that $\hat{a}$ and $\hat{b}$ generate $\langle a,b\rangle=F_2$.*

Proof. By hypothesis, there exists $c\in F_2$ such that $\hat{a}=cac^{-1}$. Set $\hat{b}=cbc^{-1}$. Let $\varphi:F_2\to\langle\hat{a},\hat{b}\rangle\subset\langle a,b\rangle=F_2$ be a homomorphism defined by $\varphi(x)=cxc^{-1}$. If $\varphi(x_1)=\varphi(x_2)$ then $x_1=x_2$, hence φ is injective. Since $\varphi(c^{-1}yc)=c(c^{-1}yc)c^{-1}=y$ for any $y\in\langle a,b\rangle$, φ is a map onto $\langle a,b\rangle=F_2$. Therefore $\varphi:F_2\to F_2$ is an isomorphism and $\langle\hat{a},\hat{b}\rangle$ generate F_2. □

Let $N(S+T)$, $N(S)$, and $N(T)$ be as in Theorem 2. Since $N(S+T)$ is a knot, the split link $N(S)$ consists of two link components, say S_1 and S_2. Since $\pi_1(S^3\setminus N(S))\cong\pi_1(S^3\setminus S_1)*\pi_1(S^3\setminus S_2)$, the abelianizations $\pi_1(S^3\setminus S_i)\to H_1(S^3\setminus S_i)\cong\mathbb{Z}$, $i=1,2$, define a quotient map $q:\pi_1(S^3\setminus N(S))\to F_2$ that sends meridians of the two different components to the two generators a and b of F_2. Set $\hat{a},b'$ to be the elements in $F_2=\langle a,b\rangle$ corresponding to the meridional loops around the two strands of the numerator closure of the tangle T. By replacing a (resp. b) by its inverse element if necessary, we may assume that a and $\hat{a}$ (resp. b and b') are conjugate. By Lemma 7, there exists an element $\hat{b}$ conjugate to b such that $\hat{a}$ and $\hat{b}$ generate $F_2=\langle a,b\rangle$. Since b' is conjugate to b, there exists $c\in\langle\hat{a},\hat{b}\rangle$ such that $b'=c\hat{b}c^{-1}$. We further assume that the elements in $\pi_1(S^3\setminus N(T))$ corresponding to $\hat{a}$ and b' are conjugate by replacing one of them by its inverse element if necessary.

Let ρ_0 be a representation in $\mathrm{Hom}(\pi_1(S^3\setminus N(T)),SL(2,\mathbb{C}))$.

Lemma 8. *Suppose that $\rho_0(\hat{a})=\begin{pmatrix}M&0\\0&M^{-1}\end{pmatrix}$ and that $\rho_0(b')=\begin{pmatrix}b'_{11}&b'_{12}\\b'_{21}&b'_{22}\end{pmatrix}$ satisfies $b'_{11}\neq M^{\pm1}$. Then there exists a representation $\rho\in\mathrm{Hom}(\langle\hat{a},\hat{b}\rangle,SL(2,\mathbb{C}))$ such that $\rho(\hat{a})=\rho_0(\hat{a})$ and $\rho(b')=\rho_0(b')$.*

Proof. Set $\rho(\hat{a})=\rho_0(\hat{a})$. We will find a ρ such that $\rho(b')=\rho_0(b')$. Set $\rho(\hat{b})=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}\in SL(2,\mathbb{C})$ and let $f_{11},f_{12},f_{21},f_{22}$ be the polynomial functions, in the variables M and the b_{ij}'s, given by

$$\begin{pmatrix}f_{11}&f_{12}\\f_{21}&f_{22}\end{pmatrix}=\rho(c)\rho(\hat{b})\rho(c)^{-1},$$

where $f_{11}f_{22}-f_{12}f_{21}=1$. We eliminate the variables b_{22} and b_{21} by substituting $b_{21}=\frac{1}{b_{12}}(b_{11}b_{22}-1)$ and $b_{22}=M+\frac{1}{M}-b_{11}$, where the second equation holds since $\hat{a}$ and b' are conjugate. The remaining variables are M, b_{11}, and b_{12}.

We first prove that f_{11} depends on the variables b_{11} and b_{12}. Assume it does not, i.e., f_{11} is constant for each, fixed, choice of M. Setting $\rho(\hat{b}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ (resp. $\rho(\hat{b}) = \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix}$) we have $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ (resp. $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix}$). Therefore we have $M = M^{-1}$, i.e., $M = \pm 1$ since f_{11} is constant. However, in the case $M = \pm 1$, since $\rho_0(\hat{a}) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, the equality

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \rho(\hat{b}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

is satisfied for any choice of the b_{ij}'s, which contradicts the assumption that f_{11} does not depend on b_{11}.

Now f_{11} does depend on at least one of the variables b_{11} and b_{12}, so we solve the equation $f_{11} = b'_{11}$ in terms of one of these variables. The inequality $f_{11} = b'_{11} \neq M^{\pm 1}$ implies $f_{12} \neq 0$ and $f_{21} \neq 0$, otherwise we cannot have $f_{11}f_{22} - f_{12}f_{21} = 1$. For the same reason, we have $b'_{12} \neq 0$ and $b'_{21} \neq 0$. The conjugation of ρ by the matrix

$$P = \begin{pmatrix} \sqrt{b'_{12}/f_{12}} & 0 \\ 0 & \sqrt{f_{12}/b'_{12}} \end{pmatrix}$$

satisfies

$$P\rho(\hat{a})P^{-1} = \rho(\hat{a}) \quad \text{and} \quad P \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} P^{-1} = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix},$$

where the bottom two equalities in the second matrix equation are automatically satisfied by the equation $f_{11} + f_{22} = b'_{11} + b'_{22}$ and the fact that these matrices are in $SL(2,\mathbb{C})$. Hence we obtain the representation required. □

Let $f^+(M)$ be the rational function of one variable M that appears as the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ when

$$\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\hat{b}) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}.$$

Similarly, we define $f^-(M)$ to be the rational function of one variable M that is the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ when

$$\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\hat{b}) = \begin{pmatrix} M^{-1} & 1 \\ 0 & M \end{pmatrix}.$$

Lemma 9. *$f^+(M)$ and $f^-(M)$ are not constant.*

Proof. We can set $\rho(c) = \begin{pmatrix} M^k & c_{12} \\ 0 & M^{-k} \end{pmatrix}$, where c_{12} is a rational function in one variable, M, whose denominator, if any, is a power of M, and $k \in \mathbb{Z}$. Then

$$\begin{aligned} \rho(c)\rho(\hat{b})\rho(c)^{-1} &= \begin{pmatrix} M^k & c_{12} \\ 0 & M^{-k} \end{pmatrix} \begin{pmatrix} M^{\pm 1} & 1 \\ 0 & M^{\mp 1} \end{pmatrix} \begin{pmatrix} M^{-k} & -c_{12} \\ 0 & M^k \end{pmatrix} \\ &= \begin{pmatrix} M^{\pm 1} & (M^{k\mp 1} - M^{k\pm 1})c_{12} + M^{2k} \\ 0 & M^{\mp 1} \end{pmatrix}, \end{aligned}$$

i.e.,

$$f^{\pm}(M) = (M^{k\mp 1} - M^{k\pm 1})c_{12} + M^{2k}.$$

This cannot be constant since, even if c_{12} has a denominator, it is only a power of M. □

Lemma 10. *Suppose that $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ and $\rho_0(b') = \begin{pmatrix} M & b'_{12} \\ b'_{21} & M^{-1} \end{pmatrix}$ with $b'_{12}b'_{21} = 0$. Suppose further that $f^+(M) \neq 0$. Then there exists a reducible representation $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b} \rangle, SL(2,\mathbb{C}))$ such that $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$.*

Proof. Set $\rho(\hat{a}) = \rho_0(\hat{a})$. We will find a reducible representation ρ such that $\rho(b') = \rho_0(b')$. Consider the case where $b'_{21} = 0$. As above, we have $b' = c\hat{b}c^{-1}$. Set $\rho(\hat{b}) = \begin{pmatrix} M & b_{12} \\ 0 & M^{-1} \end{pmatrix}$; then the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ becomes $f^+(M)b_{12}$. Since $f^+(M) \neq 0$, $b_{12} = b'_{12}/f^+(M)$ gives the required reducible representation. The proof for the case $b'_{12} = 0$ is similar. □

Lemma 11. *Suppose that* $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ *and* $\rho_0(b') = \begin{pmatrix} M^{-1} & b'_{12} \\ b'_{21} & M \end{pmatrix}$ *with* $b'_{12}b'_{21} = 0$. *Suppose further that* $f^-(M) \neq 0$. *Then there exists a reducible representation* $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b}\rangle, SL(2,\mathbb{C}))$ *such that* $\rho(\hat{a}) = \rho_0(\hat{a})$ *and* $\rho(b') = \rho_0(b')$.

Proof. Similar to the proof of Lemma 10. □

Proof of Theorem 2. Let $\mathcal{R}(K)$ denote the representation variety $\mathrm{Hom}(\pi_1(S^3 \setminus K), SL(2,\mathbb{C}))$ of a knot K in S^3.

Let M and M^{-1} be the eigenvalues of $\rho_0(\hat{a})$. Assume that $f^\pm(M) \neq 0$ and $M \neq \pm 1$. Lemma 9 ensures that, except for a finite number of values, every $M \in \mathbb{R}$ satisfies these conditions. Since $M \neq \pm 1$, $\rho_0(\hat{a})$ is diagonalizable and hence we can set $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ by conjugation. Then by Lemma 8, Lemma 10, and Lemma 11, for each representation $\rho_0 \in \mathcal{R}(N(T))$, there exists $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b}\rangle, SL(2,\mathbb{C}))$ such that $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. The quotient map $q : \pi_1(S^3 \setminus N(S)) \to \langle \hat{a}, \hat{b}\rangle$ induces a representation $\rho \in \mathcal{R}(N(S))$ which satisfies $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. Let $D_{N(S+T)}$ be a knot diagram of $N(S+T)$ such that we can see the tangle decomposition into $N(S)$ and $N(T)$ on that diagram. Fix a Wirtinger presentation of $\pi_1(S^3 \setminus N(S+T))$ on $D_{N(S+T)}$. Clearly, ρ_0 satisfies the relations of the Wirtinger presentation in the tangle T and ρ also satisfies the relations in the tangle S. Therefore these representations satisfy all the relations of the Wirtinger presentation, in other words, we obtain an $SL(2,\mathbb{C})$-representation of $\pi_1(S^3 \setminus N(S+T))$.

Each irreducible component of $A^\circ_{N(T)}(L,M) = 0$ corresponds to an irreducible component Y of $\mathcal{R}(N(T))$ on which M varies. Since each representation $\rho_0 \in Y$ corresponds to a representation $\rho_1 \in \mathcal{R}(N(S+T))$, except for a finite number of M values, there always exists a subvariety Z in $\mathcal{R}(N(S+T))$ which corresponds to Y.

Let Z_Δ be the algebraic subset of Z consisting of all $\rho_1 \in Z$ such that $\rho_1(\ell_1)$ and $\rho_1(m_1)$ are upper triangular, where (m_1, ℓ_1) is the meridian-longitude pair of $N(S+T)$. Let $\xi : Z_\Delta \to \mathbb{C}^2$ be the eigenvalue map $\rho_1 \mapsto (L_1, M_1)$, where L_1 and M_1 are the top-left entries of $\rho_1(\ell_1)$ and $\rho_1(m_1)$ respectively. It is known by [6, Corollary 10.1] that $\dim \xi(Z_\Delta) \leq 1$. Since M varies on $\xi(Z_\Delta)$, we have $\dim \xi(Z_\Delta) = 1$. This means that there exists a factor of the A-polynomial $A_{N(S+T)}(L,M)$ which vanishes at $(L,M) = (L_1, M_1)$.

In summary, for each generic point $(L_0, M_0) \in \{A^\circ_{N(T)}(L,M) = 0\}$, there is a representation $\rho_0 \in \mathcal{R}(N(T))$ such that the top-left entries of $\rho_0(\ell_0)$ and $\rho_0(m_0)$ are L_0 and M_0 respectively, where (m_0, ℓ_0) is the meridian-longitude pair of $N(T)$, and there exists a representation $\rho_1 \in \mathcal{R}(N(S+T))$ corresponding to ρ_0 such that the image (L_1, M_1) satisfies $A_{N(S+T)}(L_1, M_1) = 0$. Thus if we have $\rho_0(m_0) = \rho_1(m_1)$ and $\rho_0(\ell_0) = \rho_1(\ell_1)$ then $M_0 = M_1$ and $L_0 = L_1$, and hence we have $A_{N(S+T)}(L_0, M_0) = 0$. This means that the factor $A^\circ_{N(T)}(L,M)$ appears in $A_{N(S+T)}(L,M)$. Since $m_0 = m_1$ from the construction, we have $\rho_0(m_0) = \rho_1(m_1)$. Hence, it is enough to show that $\rho_0(\ell_0) = \rho_1(\ell_1)$.

Let Σ be the Seifert surface of $N(S+T)$ described on the diagram $D_{N(S+T)}$ by using Seifert's algorithm. The boundary of Σ determines ℓ_1. Using the Wirtinger presentation of $\pi_1(S^3 \setminus N(S+T))$ on $D_{N(S+T)}$, the longitude ℓ_1 in $\pi_1(S^3 \setminus N(S+T))$ is represented as a product of words of the generators in the Wirtinger presentation by reading the words along the boundary of Σ. This word presentation of ℓ_1 has the form

$$\ell_1 = \ell_{T,1}\ell_{S,1}\ell_{T,2}\ell_{S,2},$$

where, for $i = 1, 2$, $\ell_{T,i}$ is a product of generators in the tangle T and $\ell_{S,i}$ is a product of generators in the tangle S. Since each $\ell_{S,i}$ represents one of the boundary components of a Seifert surface of the split link $N(S)$ and the representation ρ_1 is defined via the quotient map $q : \pi_1(S^3 \setminus N(S)) \to F_2$, $\rho_1(\ell_{S,i})$ is the identity matrix. Therefore we have $\rho_1(\ell_1) = \rho_1(\ell_{T,1})\rho_1(\ell_{T,2}) = \rho_0(\ell_0)$. □

TABLE 1. Factorizations of RTЯ knots

	RTЯ	type	A-poly. fac.	epi.	Alex. poly.
8_{10}	$1/3, 3/2, -1/3$	A	3_1	$8_{10} \to 3_1$	$(3_1)^3$
8_{11}	$[2, -2, 3, 2, -2]$	B	3_1	No	$(3_1)(6_1)$
9_{24}	$1/3, 5/2, -1/3$	A	4_1	$9_{24} \to 3_1$	$(3_1)^2(4_1)$
9_{37}	$1/3, 5/3, -1/3$	B	4_1	$9_{37} \to 4_1$	$(4_1)(6_1)$
10_{21}	$[2, -2, 5, 2, -2]$	B	5_1	No	$(5_1)(6_1)$
10_{40}	$[2, 2, 3, -2, -2]$	B	3_1	$10_{40} \to 3_1$	$(3_1)(8_8)$
10_{59}	$2/5, 3/2, -2/5$	A	3_1	$10_{59} \to 4_1$	$(3_1)(4_1)^2$
10_{62}	$1/3, 5/4, -1/3$	A	5_1	$10_{62} \to 3_1$	$(3_1)^2(5_1)$
10_{65}	$1/3, 7/4, -1/3$	A	5_2	$10_{65} \to 3_1$	$(3_1)^2(5_2)$
10_{67}	$1/3, 7/5, -1/3$	B	5_2	No	$(5_2)(6_1)$
10_{74}	$1/3, 7/3, -1/3$	B	5_2	$10_{74} \to 5_2$	$(5_2)(6_1)$
10_{77}	$1/3, 7/2, -1/3$	A	5_2	$10_{77} \to 3_1$	$(3_1)^2(5_2)$
10_{98}	$1/3,\ T_0,\ -1/3$	B	$3_1\#3_1$	$10_{98} \to 3_1$	$(3_1)^2(6_1)$
10_{99}	$1/3,\ T_1,\ -1/3$	A	$3_1\#3_1^{\text{mir}}$	$10_{99} \to 3_1$	$(3_1)^4$
10_{143}	$1/3, 3/4, -1/3$	A	3_1	$10_{143} \to 3_1$	$(3_1)^3$
10_{147}	$1/3, 3/5, -1/3$	B	3_1	No	$(3_1)(6_1)$

4. EXAMPLES AND QUESTIONS

4.1. **RTЯ examples of 10 or fewer crossings.**

Definition 12. A knot K in S^3 is said to be an RTЯ *knot* if it satisfies the following:

1. (1) K is of the form $N(R + T + Я)$, where R is rational, Я is the mirror reflection of R, and T is some tangle.
2. (2) K is not isotopic to $N(T)$.

The second condition is added to exclude trivialities, for example the case where R consists of two horizontal arcs. Since $N(R + Я)$ is always a trivial link of two components, $N(R + T + Я)$ satisfies the conditions of Theorem 2 with $S = R + Я$.

Here are two simple families of RTЯ knots:

- The 2-bridge knots of the form $[a_1, a_2, a_3, \cdots, a_k, \cdots, a_{2n-1}]$ with $a_i = -a_{2n-i}$ for $i = 1, \cdots, n-1$ and a_n odd.
- Three-tangle Montesinos knots of the form $(p/q, r/s, -p/q)$.

Note that the infinite collection of pairs of 2-bridge knots of Theorem 1 and Corollary 6 are included in the first of these families.

In the following, we represent the rational tangle corresponding to the rational number p/q by $R(p/q)$. For example, the Montesinos knot of the form $(p/q, r/s, -p/q)$ is represented as $N(R(p/q) + R(r/s) + R(-p/q))$.

Table 1 lists the RTЯ knots of 10 or fewer crossings of which we know. In the table, T_0 is the tangle obtained as the $+\pi/2$-rotation of the tangle sum $R(-1/1) + R(1/3) + R(1/3)$ and T_1 is obtained as the $+\pi/2$-rotation of the tangle sum $R(1/3) + R(-1/3)$. We use 3_1^{mir} to denote the mirror image of 3_1 and use # for the connected sum of two knots. In the table, we include information of epimorphisms among the knot groups and Alexander polynomials for convenience. The epimorphism data is from [13]. In the column "Alex. poly.," we represent a knot's Alexander polynomial by enclosing the knot's symbol in parenthesis.

There are two types of RTЯ knots depending on how the strands enter and leave the tangle T. We say that the RTЯ knot $N(R + T + Я)$ is *of type A* if the tangle T is a marked tangle. Otherwise we say it is *of type B*.

Lemma 13. *Let* $K = N(R + T + Я)$ *be an* RTЯ *knot with* $R = R(p/q)$ *and* $q > 0$. *Then*

(i) $q > 1$.
(ii) *If K is of type A then* $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{D(R)}(t)^2$.
(iii) *If K is of type B then* $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{N(\mathrm{R}+\mathrm{R}(1/1)+\text{Я})}(t)$.
(iv) *The knot determinant of K is divisible by q^2.*

Proof. If $q = 1$ then we have $N(R + T + \text{Я}) = N(T)$. Such a knot is not RTЯ by definition. Thus we have assertion (i). Assertion (ii) follows from equation (1) and the equations

$$\Delta_{D(S)}(t) = \Delta_{D(R)\#D(\text{Я})}(t) = \Delta_{D(R)}(t)^2.$$

Next we prove assertion (iii). Since K is of type B, we need to modify the diagram of $N(R + T + \text{Я})$ as shown in Figure 3 such that it becomes the sum of marked tangles. We denote the marked tangle obtained from T by T' and the complementary tangle of T' by S'. From the figure, we can see that $D(S') = N(R + R(1/1) + \text{Я})$. Thus assertion (iii) follows from equation (1).

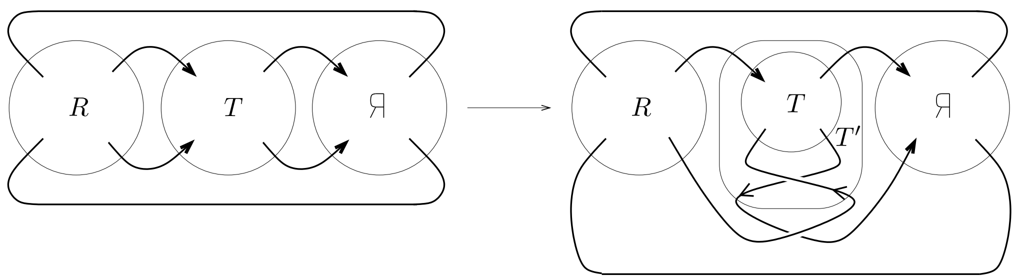

FIGURE 3. Changing an RTЯ knot of type B into the sum of two marked tangles.

Finally, we check the last assertion. It is known that the knot determinant of a knot is equal to the absolute value of its Alexander polynomial evaluated at $t = -1$ (see for instance [16, Proposition 6.1.5]). We also know that the knot determinant of $D(R(p/q))$ is q. Thus, if K is of type A then assertion (iv) follows immediately from the factorization in (ii). Suppose K is of type B. Then, from

$$N(R(p/q) + R(1/1) + R(-p/q)) = N(R(p/q) + R((q-p)/q)),$$

the knot determinant of $D(S') = N(R + R(1/1) + \text{Я})$ is calculated as $|pq + (q - p)q| = q^2$, see [9] and also [16, Theorem 9.3.5]. Thus, from the factorization in (iii), we again have assertion (iv). ☐

Proposition 14. *Let K be a prime knot of 10 or fewer crossings. Suppose that K is not 8_{18}, 9_{40}, 10_{82}, 10_{87}, or 10_{103}. Then K is RTЯ with $N(T)$ a non-trivial knot of 10 or fewer crossings if and only if it is in Table 1.*

We were unable to determine if the remaining five knots are RTЯ or not. In the proof below, we show further that if 8_{18}, 9_{40}, or 10_{82} is RTЯ then it is of type A and if 10_{103} is RTЯ then it is of type B.

Proof. We first consider the case of type A. Set $R = R(p/q)$ with $q > 0$. By Lemma 13 (i), we have $q > 1$. We check if the Alexander polynomial of a knot, up to 10 crossings, has a factorization of the form in Lemma 13 (ii). Since the knot determinant of $D(R)$ is equal to $\Delta_{D(R)}(-1)$, $q > 1$ implies that the polynomial $\Delta_{D(R)}(t)^2$ in Lemma 13 (ii) is non-trivial. Now we check if the Alexander polynomial of a knot has such a non-trivial, multiple factor corresponding to a knot up to 10 crossings. The candidate knots K are

$$8_{10}, 8_{18}, 8_{20}, 9_{24}, 9_{40}, 10_{59}, 10_{62}, 10_{65}, 10_{77}, 10_{82}, 10_{87}, 10_{98}, 10_{99}, 10_{123}, 10_{137}, 10_{140}, 10_{143}.$$

If $N(T)$ is one of 8_{20}, 10_{123}, 10_{137}, and 10_{140} then we have $\Delta_{N(T)}(t) = 1$, i.e., $N(T)$ is trivial since it is assumed to be of 10 or fewer crossings. Such a case is excluded by assumption. Since 8_{10}, 9_{24}, 10_{59}, 10_{62}, 10_{65}, 10_{77}, 10_{99}, and 10_{143} are in Table 1, the remaining knots are only 8_{18}, 9_{40}, 10_{82}, 10_{87}, and 10_{98}.

The Alexander polynomial of 10_{98} is $\Delta_{10_{98}}(t) = (2t-1)(t-2)(t^2-t+1)^2$ and hence we know that $N(T)$ is either 6_1 or 9_{46}. In either case, $g_4(N(T)) = 0$, where $g_4(\cdot)$ represents the 4-genus of a knot. On the other hand, $g_4(10_{98}) = 2$. Now, let $(F, \partial F) \subset (B^4, \partial B^4)$ be an orientable surface in the 4-ball B^4

with $\partial F = N(T)$ such that the genus of F is 0, where ∂B^4 is the 3-sphere bounding B^4. Since $N(R+Я)$ bounds two disjoint disks $D_1 \sqcup D_2$ embedded in ∂B^4, by gluing $F \subset B^4$ and $D_1 \sqcup D_2 \subset \partial B^4$ suitably, we can obtain an orientable surface in B^4 bounded by $N(R+T+Я) = 10_{98}$ and of genus 0. However we have $g_4(10_{98}) = 2$ and this is a contradiction. The remaining four knots are excluded by hypothesis.

Next we consider RTЯ knots of type B. It is shown in the proof of Lemma 13 (iv) that the knot determinant of $D(S)$ is $q^2 > 1$. Hence $D(R)$ is a 2-bridge knot with denominator $q > 1$. Moreover, since $N(T)$ is assumed to be a non-trivial knot of 10 or fewer crossings, $\Delta_{N(T)}(t)$ is non-trivial. Hence we know that the factorization of $\Delta_K(t)$ in Lemma 13 (iii) is non-trivial.

Now we make a list of knots K, up to 10 crossings that satisfy

- the Alexander polynomial of K factors into two non-trivial Alexander polynomials, and
- the knot determinant of K is divisible by q^2 for some integer $q > 1$.

The following knots satisfy these conditions:

$$8_{10}, 8_{11}, 8_{18}, 8_{20}, 9_1, 9_6, 9_{23}, 9_{24}, 9_{37}, 9_{40}, 10_{21}, 10_{40}, 10_{59}, 10_{62}, 10_{65}, 10_{66}, 10_{67},$$

$$10_{74}, 10_{77}, 10_{82}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, 10_{123}, 10_{137}, 10_{140}, 10_{143}, 10_{147}.$$

Among them, the knots 8_{11}, 9_{37}, 10_{21}, 10_{40}, 10_{67}, 10_{74}, 10_{98}, and 10_{147} are in Table 1.

If K is one of $8_{10}, 8_{18}, 8_{20}, 9_1, 9_6, 9_{23}, 9_{24}, 10_{62}, 10_{65}, 10_{77}, 10_{82}, 10_{140}$, and 10_{143}, we have $q^2 = 9$, i.e., $q = 3$. The only knot of the form $N(R+1+Я)$ with $q = 3$ is 6_1. Since $\Delta_{6_1}(t) = (2t-1)(t-2)$, we have $(2t-1)(t-2) \mid \Delta_K(t)$. However, none of the above knots satisfy this property. Hence they are not RTЯ knots.

If K is one of $9_{40}, 10_{59}, 10_{66}, 10_{103}, 10_{106}$, and 10_{137} we have $q^2 = 25$, i.e., $q = 5$. The only knots of the form $N(R+1+Я)$ with $q = 5$ are 8_8 and 10_3. Since $\Delta_{8_8}(t) = (2t^2-2t+1)(t^2-2t+2)$ and $\Delta_{10_3}(t) = (3t-2)(2t-3)$ one of theses is a factor of $\Delta_K(t)$. However, none of the above knots other than 10_{103} satisfy this property. Hence they are not RTЯ knots. As for 10_{103}, it is excluded by hypothesis.

This leaves only 10_{87}, 10_{99}, and 10_{123} of which 10_{87} is excluded by hypothesis. If K is 10_{99} then $q = 3$ or 9. The case $q = 3$ can not happen since $(2t-1)(t-2)$ is not a factor of $\Delta_{10_{99}}(t)$ as before. If $q = 9$ then, since the knot determinant of 10_{99} is 81, the factorization in Lemma 13 (iii) implies $\Delta_{N(T)}(t) = 1$, i.e., $N(T)$ is trivial. Such a case is excluded by assumption. If K is 10_{123} then, since the knot determinant of 10_{123} is 121, we have $q = 11$ and $\Delta_{N(T)}(t) = 1$, i.e., $N(T)$ is trivial. This is again excluded. This completes the proof. □

Remark 15. There is no direct relationship between the RTЯ construction and the list of epimorphisms in [13]. First of all, we can see from Table 1 that the following 9 knots

$$8_{11}, 9_{24}, 10_{21}, 10_{59}, 10_{62}, 10_{65}, 10_{67}, 10_{77}, 10_{147}$$

have the factorization of the A-polynomials but have no epimorphisms to the corresponding knot groups.

Even for the other knots in Table 1, we believe that there is no relationship for the following reason: In [14] it is written that there is an epimorphism $\pi_1(S^3 \setminus 8_{10}) \to \pi_1(S^3 \setminus 3_1)$ which maps the longitude of 8_{10} to $1 \in \pi_1(S^3 \setminus 3_1)$, see Figure 4, while Theorem 2 shows that the longitude of 8_{10} corresponds to that of 3_1 in our construction. In this example, the epimorphism is given by the tangle R and the factorization of the A-polynomial is given by the tangle T. In general, for any RTЯ knot of type A, there is an epimorphism from $\pi_1(S^3 \setminus N(R+T+Я))$ to $\pi_1(S^3 \setminus D(R))$ such that the image of the longitude of this RTЯ knot is $1 \in \pi_1(S^3 \setminus D(R))$; however, the longitude of $N(R+T+Я)$ corresponds to that of $D(R)$ when we compare their A-polynomials. This shows that the type A examples do not correspond to the epimorphisms. We remark that there may exist other epimorphisms from $\pi_1(S^3 \setminus N(R+T+Я))$ to $\pi_1(S^3 \setminus D(R))$ preserving peripheral structure. This is why we cannot exclude the possibility that there is a relationship between the factorization of A-polynomials and epimorphisms for these examples.

4.2. r**–curves.** The idea of an r–curve in the character variety of a knot was introduced by Boyer and Zhang [2]. A natural question to ask is, which r can occur? Taking advantage of mirror reflections, we can assume $r \geq 0$. If we take $N(T)$ to be a (p,q)-torus knot, then any $N(S+T)$ with $N(S)$ a split link will have a pq-curve in its character variety. It is also known that the $(-2,3,-3)$-pretzel knot admits a 0-curve, see [15], and this argument can be generalized to show that $(-2,p,-p)$-pretzel knots (p odd) all

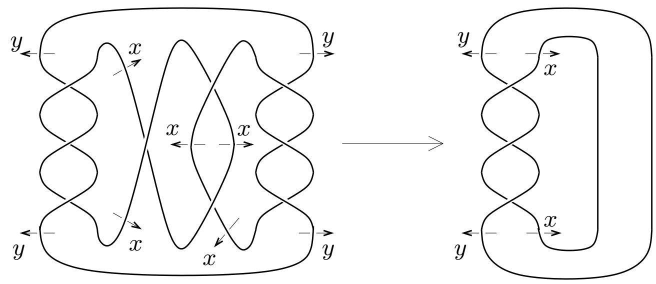

FIGURE 4. An epimorphism $\pi_1(S^3 \setminus 8_{10}) \to \pi_1(S^3 \setminus 3_1)$. Construct a Seifert surface and observe that the longitude of 8_{10} vanishes in $\pi_1(S^3 \setminus 3_1)$.

have 0-curves. Together, this covers all integers except for prime powers. Calculations by Culler [3] do provide a few examples of such r-curves: 8_{21} has a 2–curve, 9_{37} has a 4–curve, 10_{143} has an 8–curve, and 10_{152} has an 11–curve. Still, we can ask, are there other examples, and more generally:

Question 16. Are there families of knots with r–curves such that r is a prime power?

The following related question was suggested by Thang Lê:

Question 17. Is there an example of a 2–bridge knot with a 0–curve?

According to a calculation by Culler [3], the knot 9_{38} has a $\frac{1}{0}$–curve. Aside from this, there are no known examples of a knot with non-integral r–curve. It is known that a small knot admits no such r-curve, see [2]. This leads to our final question.

Question 18. Other than 9_{38}, are there examples of knots with r–curves such that r is not integral?

Appendix A. On the roots of Alexander polynomials

In this appendix we prove Theorem 19, stated below, using reducible representations in $SL(2,\mathbb{C})$. The assertion is a weaker version of the factorization of Alexander polynomials explained in the introduction and the proof is somehow analogous to the proof of Theorem 2.

Theorem 19. *Any root of $\Delta_{N(T)}(t) = 0$ is also a root of $\Delta_{N(S+T)}(t) = 0$.*

Set $\hat{a}, b'$ to be elements in $\langle a, b\rangle = F_2$ corresponding to the two strands of the numerator closure of the tangle T as before. We may assume that a and $\hat{a}$ (resp. b and b') are conjugate. Furthermore, by replacing b and b' by b^{-1} and $(b')^{-1}$ if necessary, we assume that $\hat{a}$ and b' are conjugate in $\pi_1(S^3 \setminus N(T))$. By Lemma 7, there exists an element $\hat{b}$ conjugate to b such that $\langle \hat{a}, \hat{b}\rangle = \langle a, b\rangle$. Since b' is conjugate to b, there exists $c \in \langle \hat{a}, \hat{b}\rangle$ such that $b' = c\hat{b}c^{-1}$. Let $\mathcal{R}(K)$ denote the representation variety $\mathrm{Hom}(\pi_1(S^3 \setminus K), SL(2,\mathbb{C}))$ of a knot K.

It is shown in Lemma 10 that if $f^+(M) \neq 0$ then a reducible representation $\rho_0 \in \mathcal{R}(N(T))$ extends to a reducible representation of $N(S)$. In the case $f^+(M) = 0$, we do not know if the same extension is possible. Nevertheless, we can prove Theorem 19 by showing the existence of a representation of $N(S+T)$ for $M \in \mathbb{R}$ with $f^+(M) = 0$ directly.

Lemma 20. *Suppose that M is a root of $f^+(M) = 0$. Then there exists a reducible representation $\rho \in \mathcal{R}(N(S+T))$ with non-abelian image such that $\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$.*

Proof. Suppose that M is a root of $f^+(M) = 0$. Set $\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ and $\rho(\hat{b}) = \begin{pmatrix} M & b_{12} \\ 0 & M^{-1} \end{pmatrix}$; then ρ satisfies

$$\rho(\hat{a}) = \rho(b') = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}.$$

If we set the representation of each generator of the Wirtinger presentation of $\pi_1(S^3 \setminus N(S+T))$ in the tangle T to be $\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$, then the relations in the Wirtinger presentation on T are automatically satisfied. The quotient map $q : \pi_1(S^3 \setminus N(S)) \to \langle \hat{a}, \hat{b} \rangle$ induces a reducible representation $\rho \in \mathcal{R}(N(S))$ that satisfies $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. Such a ρ is an element in $\mathcal{R}(N(S+T))$ by construction. Choosing $b_{12} \neq 0$, we have a reducible representation with non-abelian image. □

Proof of Theorem 19. Let M^2 be a root of $\Delta_{N(T)}(t) = 0$. Then there exists a reducible representation $\rho_0 \in \mathcal{R}(N(T))$ which has non-abelian image and sends the meridian to an element with eigenvalues M and M^{-1} by [5, Section 6.1]. Note that $M^2 \neq 1$ since $\Delta_{N(T)}(1) = 1$ and that $M = 0$ is excluded because the Alexander polynomial is defined up to multiplication by powers of t. Assume $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ by conjugation of ρ_0.

If $f^+(M) \neq 0$ then Lemma 10 ensures that there exists a reducible representation ρ of $\langle \hat{a}, \hat{b} \rangle$ that satisfies $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. The quotient map $q : \pi_1(S^3 \setminus N(S)) \to \langle \hat{a}, \hat{b} \rangle$ induces a reducible representation $\rho \in \mathcal{R}(N(S))$ that satisfies $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. Such a ρ is an element in $\mathcal{R}(N(S+T))$ by construction. This ρ has non-abelian image because this is already true of its restriction to $\pi_1(S^3 \setminus N(T))$. Moreover, it obviously sends the meridian to an element with eigenvalues M and M^{-1}. Hence, by [5, Section 6.1], M^2 is a root of $\Delta_{N(S+T)}(t) = 0$.

If $f^+(M) = 0$ then Lemma 20 ensures the existence of such a ρ. Hence M^2 is a root of $\Delta_{N(S+T)}(t) = 0$ in any case. □

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