A FOURIER-MUKAI APPROACH TO THE ENUMERATIVE GEOMETRY OF PRINCIPALLY POLARIZED ABELIAN SURFACES

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Abstract. We study twisted ideal sheaves of small length on an irreducible principally polarized abelian surface $(\mathbb{T}, \ell)$. Using Fourier-Mukai techniques we associate certain jumping schemes to such sheaves and completely classify such loci. We give examples of applications to the enumerative geometry of $\mathbb{T}$ and show that no smooth genus 5 curve on such a surface can contain a g^1_3. We also describe explicitly the singular divisors in the linear system $|2\ell|$.

Introduction

It is an old problem of algebraic geometry to describe the family of curves in a given linear system which go through a certain number of points. For projective spaces, this problem is essentially solved. The answer is given by the Plücker formulae and the reciprocity formulae (see [4, §2.4 and §5.4]). Our aim, in this paper, is to extend some of these results to a principally polarized Abelian surface $(\mathbb{T}, \ell)$ over the complex numbers. We shall find that, for a generic set of points, the family of curves has the dimension we would expect from a naïve dimension count. We aim to solve the problem of determining these families explicitly for divisors in translates of $|2\ell|$, this includes the non-generic cases. This will entail studying the case $|\ell|$ as well. The unique divisor D in $|\ell|$ and its translates will be viewed as "lines" on the abelian surface. There is a duality for these divisors in that any two intersect in two points (with multiplicity) while any two distinct points lie on exactly two of these "lines". We use the term "collinear" to mean that a 0-scheme lies on a "line".

Our main tools are sheaf theory and the Fourier-Mukai transform which provides exactly enough information to determine the incidence of points on divisors. We proceed in an inductive way: we relate the properties of the twisted ideal sheaf $L^2 \otimes \mathcal{I}_X$ to those of $L^2 \otimes \mathcal{I}_{X'}$ for $X' \subset X$. We then give an exhaustive treatment of $L^2 \otimes \mathcal{I}_X$ for each small value of $|X|$ in turn.

The paper is organized as follows. Section one sets up some convenient notation for zero-dimensional subschemes. The second section describes some of the basic properties of the linear systems $|\ell|$ and $|2\ell|$ including a description of the reducible divisors in $|2\ell|$. These are just the Θ and 2Θ linear systems and the reducible divisors are very well understood; forming the cornerstone of the theory of principally polarized abelian surfaces. But we

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recall some of the key facts translated into the language of sheaves. Section three is a brief overview of the Fourier-Mukai transform. In section four we shall define the basic objects of study: the cohomology jumping schemes associated to a zero-dimensional subscheme of $\mathbb{T}$. These describe the translates of $|2\ell|$ which contain extra divisors which go through X.

In sections five to nine we study the cases of length 1, 2, 3, 4 and 5 subschemes in detail. Each of these requires special treatment and a full analysis of each is required before proceeding to the next. In section 10 we can treat the general case. The conclusions are summarized in the two tables of the appendix to the paper. In section 11 we look briefly at stability questions for the Fourier transforms of the twisted ideal sheaves and in section 12 we look at two applications, one to answer the question of whether smooth divisors in $|2\ell|$ have a g^1_3 or not (they don not, as we shall see) and to answer a classical question to compute the locus of singular divisors in $|2\ell|$. The idea is to use the information about how non-reduced 0-schemes lie on divisors to detect singularities in the divisors of $|2\ell|$.

Our motivation for studying these questions come from the need to compute moduli spaces of stable and semistable sheaves over these tori. These moduli spaces provide an endless source of (relatively) easy to compute hyper-Kähler manifolds which lie in a good deformation family parametrized by the moduli space of flat hyper-Kähler tori. Some of these moduli spaces can be related to the space of 0-dimensional subschemes of the torus and the stability and local-freeness properties of the associated sheaves is determined by the incidence of the 0-subscheme on divisors from $|L^n|$. As an example of this, one can apply the results of sections eight, nine and ten to determine the moduli space of stable sheaves with Chern characters $(2, 0, -2)$ and $(2, 0, -3)$. It turns out that the standard algebraic compactification of the latter moduli space is isomorphic to $\operatorname{Hilb}^6 \mathbb{T} \times \mathbb{T}$. The former is isomorphic to another compactification of the bundle of Jacobians over the space of effective divisors in all translates of $|2\ell|$ (see [8]). It also provides us with a new irreducible hyperKähler manifold (see [12]). This paper forms the basis for a research programme which aims to give complete descriptions of a variety of moduli spaces of sheaves and more generally Bridgeland stable objects on a principally polarized abelian surface. The importance of moduli spaces arising from Hilbert schemes of points can be seen in [13] where it is shown that each fine moduli spaces of stable sheaves are birational to some $\operatorname{Hilb}^n \mathbb{T} \times \hat{\mathbb{T}}$.

The results we find in this paper are also useful to study moduli of Bridgeland stable objects with the same Chern character and in a subsequent work we compute the explicit wall crossing behaviour of such moduli spaces. In this paper, we operate at a more elementary level and avoid the use of Bridgeland stability but note that in section 11 a fuller treatment is best completed using the derived category. This has been done in [9] where the Bridgeland stable moduli spaces for Chern character $(1, 2\ell, n)$ are computed. It is also possible to give an account of n very ampleness for powers of L using these techniques. This will be the subject of a future article.

1. The Hilbert scheme of 0-subschemes

Let S be a smooth complex surface and let $\mathrm{Hilb}^n S$ denote the Hilbert scheme of length n 0-dimensional subschemes of S.

It is useful to introduce the following notation. A general 0-dimensional scheme will be denoted by X. Its length will be denoted $|X|$.

$$\begin{aligned} P &= \{p\} && \text{a single point,} \\ Q &= \{p,q\} && \text{a length 2 0-scheme,} \\ & && \text{ if } p=q \text{ then write } \{p,t\}, \text{ where } t\in \mathbb{P}T_pS \\ Y &= \{p,q,y\} && \text{a length 3 0-scheme,} \\ Z &= \{p,q,y,z\} && \text{a length 4 0-scheme,} \\ W &= \{p,q,y,z,w\} && \text{a length 5 0-scheme} \end{aligned}$$

The ideal sheaf associated to $X \subset S$ will be denoted $\mathcal{I}_X = \ker(\mathcal{O}_S \to \mathcal{O}_X)$. The notation P, Q, Y, Z, W is used to avoid double subscripts. It also reflects the fact that the properties of 0-schemes of different small lengths vary widely with respect to a given linear system whereas large length 0-schemes all behave similarly. As a mnemonic for remembering which are which note that the drawn letters have their length number of "points" on them (at the ends of lines or at acute angles).

Suppose that $D \subset S$ is a curve on a variety then we shall be concerned with questions of the following type: given a 0-dimensional subscheme $X \subset S$, which curves in a given linear system or systems containing D contain X? In other words, we want to understand $H^0(\mathcal{O}(D) \otimes \mathcal{I}_X)$.

Suppose X is non-reduced and contains precisely one closed point so that it is of the form $\operatorname{spec} A$ for some Artin local ring A. These will take the form $A = \mathbb{C}[\epsilon, \eta]/I$ for some ideal I.

Notation 1.1. We will have need to distinguish the three possibilities for the isomorphism class of Y (as a scheme) when $\operatorname{supp}(Y) = \{p\}$. We shall denote these as follows:

$$Y_c = \operatorname{spec} \frac{\mathbb{C}[\epsilon]}{(\epsilon^3)}, \qquad Y_d = \operatorname{spec} \frac{\mathbb{C}[\epsilon,\eta]}{(\epsilon-\eta^2, \epsilon\eta)} \quad \text{and} \quad Y_e = \operatorname{spec} \frac{\mathbb{C}[\epsilon,\eta]}{(\epsilon^2,\eta^2,\epsilon\eta)}$$

Note that the ideal giving rise to the first of these is regular while the ideals of the other two are not. This means, in particular, that the dimension of the fibre of $\mathcal{I}_Y$ at its support is 2 for Y_c and 3 for Y_d and Y_e.

Recall that a node is a transverse intersection, a cusp is a locally irreducible double point of the curve and a tacnode is a double point which is not transverse. In terms of the classification of surface singularities given in [1, §II.8]: a node has type A_1, cusps have types A_{2n} and tacnodes have types A_{2n+1}. By a *simple* tacnode we mean an A_3 singularity.

2. The linear systems $|\ell|$ and $|2\ell|$

We shall now suppose that $(\mathbb{T}, \ell)$ is a principally polarized Abelian surface and $(\hat{\mathbb{T}}, \hat{\ell})$ is the dual torus. We let $\mathbb{T}_2$ denote the set of points of order 2 in $\mathbb{T}$. We choose a symmetric line bundle L in the class ℓ. From now on, we will use the notation L^n to denote the nth tensor product of L with itself (rather than $L^{\otimes n}$). Note that $h^0(L) = 1$ and we write D_L for the zero set of the unique non-zero holomorphic section of L. In this and subsequent sections we adopt the following convention: D_u will denote the translation of D_L by $u \in \mathbb{T}$. This makes the notation more concise and also allows us to write $T_r D_u$ to mean the tangent space at r of the translate of D_L by u. Observe the irritating anomaly that $D_u \in |\tau_{-u}^* L|$, where $\tau_x : \mathbb{T} \to \mathbb{T}$ denotes translation by x.

If D_L is irreducible then $\mathbb{T} = \mathrm{Jac}(D_L)$ and L corresponds to the θ polarization and then D_L must be smooth (see e.g. [3, Cor 11.8.2]).

We shall introduce the following terminology.

Definition 2.1. If we have $X \in \mathrm{Hilb}^n \mathbb{T}$ and $X \subset D_u$ for some $u \in \mathbb{T}$ then we say that X is *collinear*.

This slightly unorthodox use of 'collinear' makes some sense because the divisors D_u play a similar role to lines on $\mathbb{C}P^2$. In fact, we shall see that any two "lines" intersect in exactly two points (up to multiplicity) and dually, that any two distinct points are contained in exactly two "lines".

Turning now to $|2\ell|$, observe that $h^0(L^2) = 4$ and recall that L^2 is base-point free and so, by Bertini, the generic element $D \in |2\ell|$ is smooth. The reducible divisors in $|2\ell|$ are given by the following.

Lemma 2.2. *If $D \in |2\ell|$ is reducible then $D = D_x + D_{-x}$.*

The proof is a straightforward exercise (see [3, Chapter 10]).

We use the notation $Ks(\mathbb{T}) = \mathbb{T}/\pm$ to denote the singular Kummer variety. This has sixteen singular points and they all have type A_2 (in terms of the classification of surface singularities given in [1, III.3]). There is a canonical family of divisors in $|\hat{L}^2 \otimes \mathcal{P}_\gamma|$ given by $D_u + D_y$, where $u + y = \gamma$. This family is parametrized by $Ks_\gamma(\hat{\mathbb{T}})$. We shall call the divisors *Kummer divisors*. The map $Ks_\gamma(\hat{\mathbb{T}}) \to |\hat{L}^2 \otimes \mathcal{P}_\gamma|$ is given by $[\alpha] \mapsto D_\alpha + D_{\gamma-\alpha}$, where $\alpha \sim \alpha'$ if and only if $\alpha = \gamma - \alpha$, so that $Ks_\gamma(\hat{\mathbb{T}}) = \hat{\mathbb{T}}/\sim \,\cong Ks(\hat{\mathbb{T}})$ isomorphic to the image of the above map.

Notation 2.3. We denote the Chern characters of sheaves on $\mathbb{T}$ by (r, c_1, χ), where $\chi(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$ and $r = \mathrm{rk}(E)$.

The following lemma (which works over any smooth surface) will be used several times. We let $\mathcal{L}(D)$ denote the line bundle associated to a divisor.

Lemma 2.4. *Suppose $X \subset D$ is a 0-subscheme of an effective divisor D on a surface S. Let R be a line bundle over S and $A = R\mathcal{I}_X/R\mathcal{L}(D)^*$. Then A contains no subscheme supported in codimension 2.*

Proof. If $T \subset A$ is supported in codimension 2 then $\mathrm{ch}_2(A/T) \leqslant \mathrm{ch}_2(A)$ and so $\mathrm{ch}_2(K) \geqslant \mathrm{ch}_2(R\mathcal{L}(D)^*)$, where $K = \ker(R\mathcal{I}_X \to A/T)$. But $K = R\mathcal{L}(D)^*\mathcal{I}_{X'}$ for some 0-scheme X' and so $\mathrm{ch}_2(K) = \mathrm{ch}_2(R\mathcal{L}(D)^*) - |X'|$. Hence $|X'| = 0$ and $T = 0$. □

Finally, we will need to describe torsion-free sheaves of rank 1 over reducible curves $D_1 + D_2 \subset \mathbb{T}$. If T is such a sheaf then we can consider its restrictions $T_i = T|_{D_i}$. Define the *degree* of T via the Riemann-Roch formula for embedded curves (see [1, II.3]) to be

$$\deg(T) = \chi(T) - \chi(\mathcal{L}(D_1 + D_2)).$$

Define the *restriction type* or *degree* of T to be $(\deg(T_1), \deg(T_2))$. If the restriction type of T is (n_1, n_2) and the singularity scheme X of T has length r then it is easy to see that $n_1 + n_2 = \deg(T) - r$. Note that $X \subset D_1 \cap D_2$ and so $r \leqslant |D_1 \cap D_2|$ if D_1 and D_2 have no common components. This generalizes in the obvious way to more than two components and to multiple curves.

3. The Fourier-Mukai Transform

One of the most useful tools in studying sheaves and bundles over tori is the Fourier-Mukai transform (see e.g. [10] or [11] for the original treatment and [6] or [2] for more up to date treatments). This takes the form of a functor $\Phi : D(\mathbb{T}) \to D(\hat{\mathbb{T}})$ between the derived categories of complexes of coherent sheaves. It is defined by $E \mapsto \mathbf{R}\hat{\pi}_*(\pi^*E \otimes \mathcal{P})$, where $\mathcal{P}$ denotes the Poincaré line bundle over $\mathbb{T} \times \hat{\mathbb{T}}$ and π and $\hat{\pi}$ are the projection maps to $\mathbb{T}$ and $\hat{\mathbb{T}}$ respectively. We denote the fibres of $\mathcal{P}$ over $\mathbb{T} \times \{\hat{x}\}$ by $\mathcal{P}_{\hat{x}}$ which gives rise to the isomorphism $\hat{\mathbb{T}} \cong \mathrm{Pic}^0\,\mathbb{T}$. A particularly important result concerning the Mukai transform is the fact that if $\Phi^j(E) = 0$ for all $j > m$ then the fibres of $\Phi^m(E)$ are given (canonically) by $H^m(E \otimes \mathcal{P}_{\hat{x}})$.

Definition 3.1. Following Mukai, we say that a sheaf E over $\mathbb{T}$ satisfies WIT_i if $\Phi^j(E) = 0$ for all $j \neq i$ and write $\hat{E}$ for $\Phi^i E$. We also say that E satisfies IT_i if $H^j(E \otimes \mathcal{P}_{\hat{x}}) = 0$ for all $\hat{x} \in \hat{\mathbb{T}}$ and $j \neq i$.

Note that ample line bundles satisfy IT_0. There is an unavoidable conflict of notation with $\hat{L}$. This will always denote the dual polarization which equals $(\Phi^0 L)^{-1}$. Notice that the fibres of the projective bundle $\mathbb{P}\widehat{L^2}$ are given canonically by the linear systems $|L^2 \otimes \mathcal{P}_{\hat{x}}|$ as $\hat{x}$ varies over $\hat{\mathbb{T}}$.

The inverse of Φ is $(-1)^*\hat{\Phi}[2]$ where $\hat{\Phi}(E) = \mathbf{R}\pi_*(\hat{\pi}^*E \otimes \mathcal{P})$. For practical purposes the fact that $(-1)^*\hat{\Phi}[2]$ is the quasi-inverse of Φ can be viewed as the following first-quadrant spectral sequence

$$E_2^{p,q} \Rightarrow \begin{cases} (-1_{\mathbb{T}})^*E & \text{for } p+q=2, \\ 0 & \text{otherwise,} \end{cases}$$

with $E_2^{p,q} = \hat{\Phi}^p(\Phi^q\, E)$. But $\Phi^2(E)$ is IT_0 while $\Phi^0(E)$ is WIT_2. Hence, the entire information content of this spectral sequence is contained in the exact sequences

$$0 \longrightarrow \mathcal{D} \longrightarrow (-1_{\mathbb{T}})^*E \longrightarrow \hat{\Phi}^0(\Phi^2(E)) \xrightarrow{d_2} \hat{\Phi}^2(\Phi^1(E)) \longrightarrow 0$$

and

$$0 \longrightarrow \hat{\Phi}^0(\Phi^1(E)) \xrightarrow{d_2} \hat{\Phi}^2\,\Phi^0(E)) \longrightarrow \mathcal{D} \longrightarrow \hat{\Phi}^1(\Phi^1(E)) \longrightarrow 0,$$

where $\mathcal{D}$ is an unknown. Moreover, if any one of $\hat{\Phi}^2(\Phi^0(E))$, $\hat{\Phi}^1(\Phi^1(E))$, $\hat{\Phi}^0(\Phi^2(E))$ is zero then we can eliminate $\mathcal{D}$.

We list some useful properties of Φ.

Proposition 3.2. *(see* [10]*)*

1. (1) $\mathcal{O}_X$ *satisfies* IT_0 *and* $\Phi^0(\mathcal{O}_X) = H_X$, *a homogeneous bundle and* $\Phi^0(\mathcal{O}_x) = \mathcal{P}_x$.
2. (2) $\mathcal{P}_{\hat{x}}$ *satisfies* WIT_2 *and* $\Phi^2(\mathcal{P}_{\hat{x}}) = \mathcal{O}_{-\hat{x}}$
3. (3) *If* E *satisfies* WIT *then so does* $\tau_x^* E$ *with transform* $\hat{E} \otimes \mathcal{P}_{-x}$.
4. (4) *If* $\mathrm{ch}(E) = (r, c, \chi)$ *then* $\mathrm{ch}(\Phi(E)) = (\chi, -\hat{c}, r)$.

Remark 3.3. Observe that $\mathbb{P}(\widehat{L^i})$ is flat as a projective bundle. Hence, $\widehat{L^i}$ admits an irreducible projectively flat connection and so $\widehat{L^i}$ is μ-stable. This is proved in a different way by Kempf (see [7]Thm 3). By μ-stable we mean the stability of Mumford-Takemoto: E is *μ-stable* if E is torsion-free and for all subsheaves $F \subset E$ with E/F torsion-free we have $d(F)/r(F) < d(E)/r(E)$. We obtain *μ-semistability* by replacing $<$ by $\leqslant$. Homogeneous bundles can be characterized as μ-semistable sheaves with $c_1 = 0$ and $c_2 = 0$.

Notation 3.4. As a useful shorthand, we shall drop the tensor product sign when no confusion will arise. We shall also write $L_{\hat{x}} = L\mathcal{P}_{\hat{x}} = L \otimes \mathcal{P}_{\hat{x}}$.

We will be interested in $L^i\mathcal{I}_X$ for $i > 0$. Since Φ is right exact we can apply it to short exact sequences to obtain a long exact sequence. For example,

$$\Phi(0 \longrightarrow L^i\mathcal{I}_X \longrightarrow L^i \longrightarrow \mathcal{O}_X \longrightarrow 0)$$

gives rise to

$$0 \to \Phi^0(L^i\mathcal{I}_X) \to \widehat{L^i} \to H_X \to \Phi^1(L^i\mathcal{I}_X) \to 0. \tag{3.1}$$

From this sequence we can immediately deduce:

Proposition 3.5. *For all* $i > 0$ *and 0-schemes* X, $\Phi^2(L^i\mathcal{I}_X) = 0$ *and hence* $H^2(L^i\mathcal{I}_X\mathcal{P}_{\hat{x}}) = 0$ *for all* $\hat{x}$.

In particular, $\Phi^0(L^i\mathcal{I}_X)$ is locally-free and, since $\widehat{L^i}$ is stable we have $\mu(\Phi^0(L^i\mathcal{I}_X)) < -1$. Observe that $\chi(L^i\mathcal{I}_X) = i^2 - |X|$.

Definition 3.6. Let $R_i^j(X) = \Phi^j(L^i\mathcal{I}_X)$ for $i > 0$ and $j = 0, 1$.

We can apply the Mukai spectral sequence to $L^i\mathcal{I}_X$ to obtain a long exact sequence

$$0 \to \hat{\Phi}^0(R_i^1(X)) \to \hat{\Phi}^2(R_i^0(X)) \to L^i\mathcal{I}_{-X} \to \hat{\Phi}^1(R_i^1(X)) \to 0. \tag{3.2}$$

Observe that it is impossible for the middle map to be zero unless $L^i\mathcal{I}_X$ satisfies WIT_1.

If $X' \subset X$ with $X \setminus X' = X''$ then the sequence

$$0 \longrightarrow L^i\mathcal{I}_X \longrightarrow L^i\mathcal{I}_{X'} \longrightarrow \mathcal{O}_{X''} \longrightarrow 0$$

gives rise to

$$0 \to R_i^0(X) \to R_i^0(X') \to H_{X''} \to R_i^1(X) \to R_i^1(X') \to 0. \tag{3.3}$$

This is particularly useful if $|X''| = 1$.

4. Cohomology Jumping Schemes

Let $p : U \to S$ be a flat morphism of projective varieties. Let F be a sheaf on U and let F_r denote its fibre over $r \in S$. Suppose that, for some sheaf F, $R^i p_* F = 0$ for $i > 1$. Then the fibres of $R^1 p_* F$ over $r \in S$ are canonically isomorphic to $H^1(F_r)$ since $H^i(F_r) = 0$ for all $i > 1$. We want to consider the points of S where the dimension of $H^1(F_r)$ jumps (up, by semicontinuity). We will consider the situation where $F = p_1^*(L^i \mathcal{I}_X) \otimes \mathcal{P}$ over $\mathbb{T} \times \hat{\mathbb{T}}$ and $S = \hat{\mathbb{T}}$. Since $\chi(F_r)$ does not depend on r we see that $\dim(H^0(L^i \mathcal{I}_X \mathcal{P}_{\hat{x}})) = \dim(H^1(L^i \mathcal{I}_X \mathcal{P}_{\hat{x}})) = \dim(\Phi^1(L^i \mathcal{I}_X) \otimes \mathcal{O}_{\hat{x}})$. From sequence 3.1 we know that

$$\Phi^1(L^i \mathcal{I}_X) = \operatorname{coker}(\Phi^0(L^i \to \mathcal{O}_X) : \widehat{L}^i \to H_X).$$

We can therefore make the following definition.

Definition 4.1. The *cohomology jumping scheme* $S_i(X)$ associated to $L^i \mathcal{I}_X$ is defined to be the determinantal locus of $\Phi^0(L^i \to \mathcal{O}_X)$.

More often than not, we shall only be interested in the support of $S_i(X)$.

Definition 4.2. Define $\Phi_i(X) \subset \mathbb{P}\widehat{L}^i$ to be $\{D \mid X \subset D\}$. Note that $\mathbb{P}\widehat{L}^i$ is a projective bundle over $\mathbb{T}$ and we denote the intersection of $\Phi(X)$ with the fibre over $\hat{x}$ by $\Phi_i(X)_{\hat{x}}$.

Observe that

$$\Phi_i(X)_{\hat{x}} \cong \mathbb{P}H^0(L^i \mathcal{P}_{\hat{x}} \mathcal{I}_X).$$

By semicontinuity of cohomology, for generic $\hat{x}$, $\Phi_i(X)_{\hat{x}} \cong \mathbb{C}P^r$ for some r. Then, for such $\hat{x}$,

$$h^1(L^i \mathcal{I}_X \mathcal{P}_{\hat{x}}) = |X| - i^2 + r.$$

In any case, $r \leqslant \dim \Phi_i(X)_{\hat{x}} \leqslant i^2$. It also follows that the support of $S_i(X)$ is just $\{\hat{x} \in \hat{\mathbb{T}} \mid \dim \Phi_i(X)_{\hat{x}} \geqslant r + 1\}$.

Our aim will be to compute $\Phi_1(X)$ and $\Phi_2(X)$ for any X. We do this by considering separately the cases $|X| = 1, 2, 3, 4$ and then extrapolating to the general case.

5. $|X| = 1$

Consider $S_1(P)$ first. From sequence 3.1 we see that $L\mathcal{I}_P$ satisfies WIT_1 and its transform is given by $\mathcal{P}_p / \hat{L}^{-1} = \mathcal{O}_{D_{-p}} \mathcal{P}_p$. Hence, $S_1(P) = D_{-p}$. In fact, any degree 0 line bundle over D_u satisfies WIT_1 with transform $L_\beta \mathcal{I}_{-u}$.

Turning now to $S_2(P)$, observe that $\chi(L^2 \mathcal{I}_P) = 3$ and hence $\dim H^0(L^2 \mathcal{I}_P) \geqslant 3$. Thus the fibres of $\Phi_2(P)$ contain $\mathbb{C}P^2$. But $|L^2 \mathcal{P}_{\hat{x}}|$ are all base-point free and hence the fibres of $\Phi_2(P)$ must all equal $\mathbb{C}P^2$. In other words, $S_2(P) = \varnothing$ and $L^2 \mathcal{I}_P$ satisfies IT_0.

Proposition 5.1. *For all P, $R_2^0(P) = \Phi^0(L^2 \mathcal{I}_p)$ are μ-stable.*

Proof. This follows because $\Phi^0(L^2)$ is μ-stable and $R^0_1(P)$ is a sub-bundle of $\Phi^0(L^2)$ of slope $-4/3$. Then there are no integers a and b such that $-4/3 \leqslant 2a/b < -1$ with $0 < b < 3$. □

6. $|X| = 2$

Part of the following is well known (see [3]).

Proposition 6.1. *For all $Q \in \operatorname{Hilb}^2 \mathbb{T}$, $L\mathcal{I}_Q$ satisfies WIT_1 and $R^1_1(Q) = \hat{L}_{p+q}\mathcal{I}_{S_1(Q)}$, where $S_1(Q) \in \operatorname{Hilb}^2 \hat{\mathbb{T}}$.*

Proof. Use sequence 3.1 with $i = 1$ and $X = Q$. Then $R^0_1(Q) = 0$ or the sequence splits. But H_Q satisfies WIT_2 whereas $\hat{\Phi}^2(R^1_1(Q)) = 0$ from the Mukai spectral sequence. Hence, $L\mathcal{I}_Q$ satisfies WIT_1.

Now sequence 3.3 with $\{p\} \subset Q$ gives

$$0 \longrightarrow \mathcal{P}_q \longrightarrow R^1_1(Q) \longrightarrow \mathcal{O}_{D_{-p}}\mathcal{P}_p \longrightarrow 0. \tag{6.1}$$

This cannot split as $\hat{\Phi}^2(R^1_1(Q)) = 0$. Hence, $R^1_1(Q)$ is torsion-free and so takes the form $\hat{L}_x\mathcal{I}_{Q'}$ with $Q' \in \operatorname{Hilb}^2 \hat{\mathbb{T}}$. From 6.1, we see that $x = p + q$. □

As an easy corollary of this and sequence 3.3 to give the answer for $S_1(X)$ for any X.

Corollary 6.2. *If X is a 0-scheme and length at least 2 then $L\mathcal{I}_X$ satisfies WIT_1 and $R^1_1(X)$ is torsion-free.*

We can view Proposition 6.1 more explicitly as follows. Using the fact that $S_1(P) = D_{-p}$ we see that $S_1(Q) = D_{-p} \cap D_{-q}$. If Q is reduced we can deduce that $S_1(Q) = \{l - p, -l' - p\}$, where $p - q = l - l'$ for $l, l' \in D_L$. Furthermore, $S_1(Q)$ is reduced if and only if $p - q \notin 2D_L$.

Suppose Q is not reduced and given by $t \in \mathbb{P}T_p\mathbb{T}$. We have a degree 2 map $D_p \xrightarrow{\phi} \mathbb{C}P^1$ given by $u \mapsto \mathbb{P}(T_pD_u)$. This is just the Gauss map. Then $S_1(Q) = \phi^{-1}[t]$. So, if $-p + l \in \phi^{-1}[t]$ then $S_1(Q) = \{-p + l, -p - l\}$. By Hurwitz ([5, Cor. IV.2.4]) ϕ has a ramification divisor R of degree 6. In particular, as D_L is irreducible, the reducible divisors in $|2\ell|$ which have a tacnode are precisely $D_{-l} + D_l$, where $l \in D_L \setminus \mathbb{T}_2$. A similar argument can be found in [11, Lemma 5.1].

Let us turn now to $S_2(Q)$.

Proposition 6.3. *For all $Q \in \operatorname{Hilb}^2 \mathbb{T}$, with $Q = \{p, q\}$ (possibly $p = q$), $R^0_2(Q)$ is a rank 2 vector bundle and $R^1_2(Q)$ is a torsion sheaf. Then $R^1_2(Q) = \mathcal{O}_{-p-q}$ and $R^0_2(Q) = H_{S_1(Q)}\hat{L}^{-1}$. In particular, $R^0_2(Q)$ is μ-semistable.*

Proof. Consider $\{p\} \subset Q$ and use sequence 3.3 with $i = 2$. In this case, $R^1_2(P) = 0$ and so if $R^1_2(Q)$ had non-zero rank it would be isomorphic to $\mathcal{P}_q$ which is impossible. Hence $R^1_2(Q)$ has rank 0 and $R^0_2(Q)$ has rank 2 since $\chi(L^2\mathcal{I}_Q) = 2$. This proves the first statement of the proposition.

Let $c_1(R^1_2(Q)) = b$ and factor the middle map of sequence 3.3 via $B\mathcal{I}_X$, where X is a 0-scheme and B is a line bundle with $c_1(B) = -b$. Then $c_1(R^0_2(Q)) = -2\ell + b$. But

Proposition 3.5 implies that

$$\mu(R^0_2(Q)) = -2 + b\cdot\ell/2 < -1.$$

In other words, $b\cdot\ell < 2$. On the other hand, $b\cdot\ell \geqslant 0$ from the definition of b. Notice also that $B\mathcal{I}_X \subset \mathcal{P}_q$ so that

$$\hat{\Phi}^2(B) \cong \hat{\Phi}^2(B\mathcal{I}_X) \neq 0.$$

This implies that $b^2 \geqslant 0$. Then there are two cases to consider:

(1) $b\cdot\ell = 1$**.** Then the Hodge Index Theorem implies that $b^2 < 1/2$ and so $b^2 = 0$. Hence, $B^* = \hat{L}_k\mathcal{P}_x$, say and the torus is a product which we do not allow.

(2) $b\cdot\ell = 0$**.** Since $b^2 \geqslant 0$ we must have $b^2 = 0$. Hence, B is flat. So $R^1_2(Q) = \mathcal{O}_X$. Since $H^0(L^2\mathcal{I}_Q) \neq 0$ we can pick $L \hookrightarrow L^2\mathcal{I}_Q$. Let the quotient be A. Then A is supported on D_L and

$$R^1_2(Q) \cong \Phi^1(A) \text{ and } \Phi^0(A) = R^0_2(Q)/\hat{L}^*.$$

If $T \subset A$ is the torsion subsheaf of A then T is supported in codimension 0. On the other hand, Lemma 2.4 implies that $T = 0$. Hence, $A = \mathcal{O}_D L_{\hat{x}}$, for some $\hat{x}$. The short exact sequence $\mathcal{P}_{\hat{x}} \to L_{\hat{x}} \to A$ implies that

$$\Phi^0(A) = \hat{L}^*\mathcal{P}_{-\hat{x}} \text{ and } \Phi^1(A) = \mathcal{O}_{-\hat{x}} (= \mathcal{O}_X).$$

Now apply det to sequence 3.1 to get $\mathcal{P}_{\hat{x}} = \det H_Q$ so that $\hat{x} = p + q$. This deals with the two cases.

Observe that $\mathrm{ch}\big(R^0_2(Q) \otimes \hat{L}\big) = (2,0,0)$ and since $R^0_2(Q)$ is locally-free we must have $R^0_2(Q) \otimes \hat{L} = H_{\tilde{Q}}$ for some $\tilde{Q} \in \mathrm{Hilb}^2\hat{\mathbb{T}}$. The Mukai spectral sequence now gives us a surjection $LH_{-\tilde{Q}} \to L^2\mathcal{I}_Q$ and so we see that $\tilde{Q} = S_1(Q)$. □

7. $|X| = 3$

We have dealt with $S_1(Y)$ in the preceding section. We shall first treat the collinear case.

Proposition 7.1. *If $Y \subset D_v$ then $S_2(Y) = D_{v-\sum Y}$, $R^1_2(Y)$ is a degree 1, rank 1 torsion-free sheaf over $S_2(Y)$ and $R^0_2(Y) = \hat{L}^*\mathcal{P}_{-v}$. Conversely, if $R^1_2(Y)$ is supported on a translate of D_L and torsion-free on its support then Y is collinear.*

Proof. If $Y \subset D_v$ then we have a short exact sequence $L_v \to L^2\mathcal{I}_Y \to A$, where A is supported on D_v. Apply Φ to obtain

$$0 \to \hat{L}^*\mathcal{P}_{-v} \to R^0_2(Y) \to \Phi^0(A) \to 0 \qquad \text{and} \qquad R^1_2(Y) \cong \Phi^1(A).$$

We know from Lemma 2.4 that the torsion of A is supported in dimension 1 and so A is torsion-free over D_v. Since $\chi(A) = 0$, Riemann-Roch implies that A has degree 1 over D_v.

Conversely, if $R^1_2(Y)$ is supported on D_u then $R^0_1(Y) \cong \hat{L}^*\mathcal{P}_a$, say and so the degree of $R^1_2(Y)$ is 1. On the other hand, sequence 3.2 shows that $\hat{\Phi}^0(R^1_2(Y)) = 0$ and Y is collinear. □

Corollary 7.2. *If $X \in \operatorname{Hilb}^n \mathbb{T}$ with $n > 3$ then $R^0_2(X)$ has rank 1 if and only if X is collinear. Hence, $L^2\mathcal{I}_X$ satisfies WIT_1 if and only if X is not collinear.*

Proof. Suppose that X is collinear. We must have $\operatorname{rk}(R^0_2(X)) < 2$ by Proposition 6.3. On the other hand, if $X \subset D_u$ then $X \subset D_u + D_v$ for any v and so $\Phi_2(X)_x \neq \varnothing$ for all x. Hence, $R^0_2(X)$ is a line bundle. Conversely, if $R^0_2(X)$ is a line bundle then we induct on n. If $X' \subset X$ has length $n-1$ and then sequence 3.3 implies that $R^0_2(X) \cong R^0_2(X')$ and $\hat{\Phi}^2(R^0_2(X)) \to L^2\mathcal{I}_{-X}$ implies that X is collinear (by induction). This holds also if $n = 4$ and so the induction starts. $\square$

Theorem 7.3. *Suppose that $Y \in Hilb^3\mathbb{T}$ is not collinear. Then $S_2(Y) \cong Y$ as schemes and $R^0_2(Y) \cong \hat{L}^{-2}\mathcal{P}_{-\Sigma Y}$.*

Proof. Let $Q \subset Y$ be a length 2 subscheme of Y. Then we can analyze sequence 3.3:

$$0 \longrightarrow R^0_2(Y) \xrightarrow{\phi} R^0_2(Q) \longrightarrow \mathcal{P}_y \longrightarrow R^1_2(Y) \xrightarrow{\psi} \mathcal{O}_{-p-q} \longrightarrow 0.$$

Let $A = \operatorname{coker}(\phi)$ and $B = \ker(\psi)$. We know from Proposition 3.5 and the sequence 3.3 for $Q \subset Y$ that $R^0_2(Y)$ is a line bundle. Let $r = c_1(R^0_2(Y))$. Then $\operatorname{ch}(A) = (1, -2\ell - r, 2 - r^2/2)$. Since A is torsion-free it takes the form $L^{-2}R^{-2}\mathcal{I}_X$ modulo $\operatorname{Pic}^0\mathbb{T}$ for some 0-scheme X. Then $2 - r^2/2 = 4 + 2r\cdot\ell + r^2/2 - |X|$ and so $|X| = 2 + 2r\cdot\ell + r^2 \geqslant 0$. From the μ-semistability of $R^0_2(Q)$ (see Proposition 6.3) the fact that and $\mu(R^0_2(Q)) = -2$ we must have $r\cdot\ell < -1$. On the other hand, $c_1(B)\cdot\ell \geqslant 0$. Hence, $r\cdot\ell \geqslant -4$. We now treat the values of $r\cdot\ell$ separately. **(1)** $r\cdot\ell = -2$**.** The Hodge Index Theorem implies that $r^2 \leqslant 2$ but $-2 + r^2 = |X| \geqslant 0$ and so $r^2 = 2$ with $|X| = 0$. This implies that $-r$ is a principal polarization and, since $-r\cdot\ell = 2$, $r = \ell$. Then B is locally-free over a translate of D_L. But then $c_1(A) = \ell$ and so $c_1(R^1_2(Y)) = \ell$. But, since $r(R^1_2(Y)) = 0$ we have that $R^1_2(Y)$ is supported on a translate of D_L and so by Theorem 7.1, Y must be collinear, a contradiction.

(2) $r\cdot\ell = -4$**.** In this case, $r^2 \geqslant 6$ and the Hodge Index Theorem implies that $r^2 = 6$. Then the degree of A is 0 and so, since $A \to \mathcal{P}_y$ is non-zero, A^{**} must be flat. Hence, $R = L^{-2}$ and $X \in \operatorname{Hilb}^2\hat{\mathbb{T}}$. Then $B = \mathcal{O}_X$ and

$$R^1_2(Y) = \mathcal{O}_{S_2(Y)} \quad \text{with} \quad S_2(Y) \in \operatorname{Hilb}^3\hat{\mathbb{T}}.$$

If we apply det to sequence 3.1 then we see that $R^0_2(Y) = L^{-2}\mathcal{P}_{-p-q-y}$. If Y is reduced then by repeating the argument above with each $Q \subset Y$ we obtain

$$S_2(Y) = \{-p-q, -q-y, -y-p\}$$

and so is isomorphic to Y. To discuss the others observe that by applying det to sequence 3.1 we see that

$$\sum S_2(Y) = -2\sum Y = -2p - 2q - 2y.$$

This deals with the case where the support of Y contains at least 2 points. Observe that the argument used to eliminate case (2) also implies that $|S_2(Y)| \subset \bigcup_{Q\subset Y} |S_2(Q)|$. If Y is of type c, d or e then the union of the $|S_2(Q)|$'s is a single point and therefore $S_2(Y)$ is of type c, d or e. Using the continuity of

$$(p,q,y) \mapsto (-p-q, -q-y, -y-p)$$

and $Y \to S_2(Y)$ we see that $Y \cong S_2(Y)$. In fact, we can use the sequences

$$\begin{aligned} \{(0,0),\ (-x_n,0),\ (x_n,0)\} &\longrightarrow Y_c \\ \{(0,0),\ (x_n,0),\ (x_n,x_n^2)\} &\longrightarrow Y_d \\ \text{and}\quad \{(0,0),\ (0,x_n),\ (x_n,0)\} &\longrightarrow Y_e \end{aligned}$$

as $x_n \to 0$. Since the torus is an abelian Lie group the addition of points corresponds to addition of these tangent vectors. Then we see that $S_2(Y)$ for each of these is in the same configuration as $x_n \to 0$. □

Notes 7.4.

(1) This theorem is the geometrical reason why there are no rank 2 stable bundles with Chern character $(2,0,-1)$. Any such bundle E must fit into a short exact sequence $L_{\hat{x}}^{-1} \to E \to L_{\hat{x}+\hat{d}} \otimes \mathcal{I}_Y$ for some $Y \in \operatorname{Hilb}^3 \mathbb{T}$. But only collinear Y's give rise to locally-free extensions and for these $H^0(E \otimes \mathcal{P}_{\hat{z}}) \neq 0$ when $Y \subset \tau_{\hat{x}+\hat{d}+\hat{z}} D_L$. This contradicts the stability of E.

(2) The case when Y is collinear provides us with a good example of a sheaf where we know the Fourier transform cohomology sheaves R^i and even the boundary maps in the Mukai spectral sequence but we cannot reconstruct the original sheaf from this alone. Whatever Y is, the spectral sequence degenerates at the E_2 level. When Y is not collinear then $R_2^1(Y) = \mathcal{O}_{\tilde{Y}}$ and $R_2^0(Y) = L^{-2}\mathcal{P}_{-\tau}$ and from τ together with $\tilde{Y}$ we can easily reconstruct Y. But when Y is collinear R^1 is a degree 1 line bundle over D_u and $R_2^0(Y) = L^{-1}\mathcal{P}_{-\tau-u}$. Then R^1 is determined by its cohomology jumping divisor in $\mathbb{T}$ which must be D_v. But applying H^* to the long exact sequence $R_2^0(Y) \to \widehat{L^2} \to \hat{H}_Y \to R_2^1(Y)$ we see that $Y \subset D_v$ and so $v = \tau + u$. Hence, we only have the parameters u and τ free which are not enough to determine Y.

(3) There are precisely six 0-schemes Y which are fat points of type c (so $Y \cong \operatorname{spec} \mathbb{C}[\epsilon]/(\epsilon^3)$) supported at $e \in \mathbb{T}$. These points are the ramification divisor of the Gauss map ϕ discussed after Corollary 6.2. To see this observe that $R_2^1(Y_c)$ is properly torsion-free precisely when $L\mathcal{I}_{S_1(Q)}$ satisfies the Cayley Bacharach condition, where $Q \subset Y_c$ is the unique length 2 subscheme. But this happens precisely when when $S_1(Q)$ is fat. So $\operatorname{supp} S_1(Q) \in D_L \cap \mathbb{T}_2$. This tell us that the inflection points of D_L are the points of $D_L \cap \mathbb{T}_2$.

8. $|X| = 4$

From Corollary 7.2 we can deduce the following.

Proposition 8.1. *$L^2\mathcal{I}_Z$ satisfies WIT_1 if and only if Z is not collinear. If Z is collinear then the rank of $R_2^0(Z)$ is 1.*

Proposition 8.2. *If $Z \subset D_v$ then $R^0_2(Z) = \hat{L}^{-1}\mathcal{P}_{-v}$ and $R^1_2(Z) = \hat{L}\mathcal{P}_{\sigma-v}\mathcal{I}_{2v-\sigma}$, where $\sigma = \sum Z$.*

Proof. Suppose first that Z contains some Y. Then sequence 3.3 implies that $R^0_2(Z) \cong R^0_2(Y) = \hat{L}^{-1}\mathcal{P}_{-v}$. We also have a sequence $0 \to \mathcal{P}_z \to R^1_2(Z) \to \mathcal{O}_{D_{v-p-q-y}}(1) \to 0$, where (1) denotes degree 1. This sequence cannot split so any torsion in $R^1_2(Z)$ is supported on a proper dimension 1 subset of $D_{v-p-q-y}$. Sequence 3.3 shows that $R^1_2(Z)$ satisfies WIT_1. If $Y \subset Z$ then $R^1_2(Z) = \hat{L}\mathcal{P}_x\mathcal{I}_a$. To compute x we apply det to sequence 3.1. This gives $x = \sigma - v$. To compute a we apply $\hat{\Phi}$ to give $L_{-v} \to L^2\mathcal{I}_{-Z} \to \mathcal{O}_{D_{-x-a}}\mathcal{P}_a$ and so $-x - a = -v$, so $a = 2v - \sigma$.

□

Theorem 8.3. *If $Z \in \mathrm{Hilb}^4\mathbb{T}$ is not collinear then $T = R^1_2(Z)$ is a degree 3 torsion-free sheaf of rank 1 of a divisor D' in $|\hat{L}^2\mathcal{P}_\sigma|$, where $\sigma = \sum Z$. Moreover, D' is a Kummer divisor if and only if Z contains a collinear length 3 subscheme. The restriction type of D' is $(2,1)$ precisely when there is exactly one such collinear length 3 subscheme and is $(1,1)$ if there are two.*

Proof. We know that if Z is not collinear then $L^2\mathcal{I}_Z$ satisfies WIT_1. T has no torsion supported in dimension 0 since T satisfies WIT_1. If $Y \subset Z$ is not collinear then sequence 3.3 gives

$$0 \longrightarrow \hat{L}^{-2}\mathcal{P}_{-\tau} \longrightarrow \mathcal{P}_z \longrightarrow T \longrightarrow \mathcal{O}_{S_2(Y)} \longrightarrow 0.$$

The middle map factors through a degree 0 line bundle over a divisor in $|\hat{L}^2\mathcal{P}_\sigma|$.

Suppose that $Y \subset Z$ satisfies $Y \subset D_v$. Then sequence 3.3 gives

$$0 \longrightarrow \hat{L}^*\mathcal{P}_{-v} \longrightarrow \mathcal{P}_z \longrightarrow T \longrightarrow \mathcal{O}_{D_{v-p-q-y}}(1) \longrightarrow 0.$$

The middle map factors through $\mathcal{O}_{D_{-v-z}}\mathcal{P}_z$ and so, T is supported on a Kummer divisor in $|\hat{L}^2\mathcal{P}_{v+z-v+p+q+y}| = |\hat{L}^2\mathcal{P}_\sigma|$. Conversely, suppose that T is supported on a Kummer divisor $D_\alpha + D_\beta$. Notice that it is not possible for T to be reducible since $\Phi^1(T) = L^2\mathcal{I}_{-Z}$. We need to understand the possible restriction types of T to D_i. Since T does not satisfy IT_1 there must be some $X \in \mathrm{Hilb}^5\hat{\mathbb{T}}$ and $x \in \mathbb{T}$ such that $T = L^2\mathcal{P}_x\mathcal{I}_X/\mathcal{O}$. Let T_i be the restriction of T to D_i with degree n_i and let $T'_i = \ker(T \to T_i)$. Then $\chi(T'_i) = -n_i$. Since $\Phi^0(T) = 0$ and $\Phi^1(T)$ is torsion-free we must have $\chi(T'_i) < 0$ and so $n_i \geqslant 1$. If T is locally-free over $D_1 + D_2$ then $n_{2-i} = 3 - n_i$ and so $n_i \leqslant 2$. Proposition 7.1 implies that, if T is not locally-free then the restriction type $(n_1, n_2) = (1,1)$. This implies that the only possible restriction types are $(2,1)$, $(1,2)$ and $(1,1)$. The first two cases correspond to a single collinear $Y \subset Z$ and the latter to two different collinear $Y \subset Z$. □

Remark 8.4. We can also see part of Theorem 8.3 by observing that $\chi(L^2\mathcal{I}_Z) = 0$ and so $S_2(Z)$ is supported on a divisor from $\widehat{L^2}$. We can be more precise about this by applying Φ to the structure sequence of Z. This implies that $\det\Phi(L^2\mathcal{I}_Z)^* \cong \hat{L}^2\mathcal{P}_\sigma$.

9. $|X| = 5$

The case $|X| = 5$ is a curious one because $R^1_2(X)$ can have torsion reflecting the structure of X.

Remark 9.1. Since $D \cdot D_x = 4$ for $D \in |2\ell|$ we see that if X is collinear then $S(X) = \varnothing$.

Theorem 9.2. *Suppose that $W \in \operatorname{Hilb}^5 \mathbb{T}$ is not collinear. Let S be the torsion subsheaf of $R^1_2(W)$ and let $F = R^1_2(W)/S$. Then S satisfies one of the following.*

1. *If $S = 0$ then $R^1_2(W) = \widehat{L^2 \mathcal{P}_a} \mathcal{I}_{W'}$ for some $W' \in \operatorname{Hilb}^5 \hat{\mathbb{T}}$. If S is not zero then it must be a torsion-free sheaf of rank 1 supported on a divisor in $|\hat{L}\mathcal{P}_x|$ for some x.*
2. *If S is supported on a translate of D_L then it must have degree 0 or -1.*
3. *S is a degree 0 line bundle on a translate of D_L if and only if W contains a length 4 collinear subscheme.*
4. *S is a degree -1 line bundle on a translate of D_L if and only if every length 4 subscheme $Z \subset W$ contains a unique collinear length three subscheme but is not itself collinear.*

Proof. Recall from Corollary 7.2 that $L^2 \mathcal{I}_W$ satisfies WIT_1. If $R^1_2(W)$ is torsion-free then from $\operatorname{ch}(R^1_2(W)) = (1, 2L, -1)$ we must have $R^1_2(W) \cong \hat{L}^2 \mathcal{P}_a \mathcal{I}_{W'}$ for some $W' \in \operatorname{Hilb}^5 \hat{\mathbb{T}}$. If $\mathcal{O}_x \subset R^1_2(W)$ then this contradicts the fact that $\hat{\Phi}^0(R^1_2(W)) = 0$. This implies that the torsion of $R^1_2(W)$ is supported on a divisor and is torsion-free over that divisor. This proves part (1).

If we apply sequence 3.3 to $Z \subset X$ then we obtain

$$0 \longrightarrow R^0_2(Z) \longrightarrow \mathcal{P}_w \longrightarrow R^1_2(W) \longrightarrow R^1_2(Z) \longrightarrow 0.$$

If Z is collinear then $R^0_2(Z) \cong \hat{L}^* \mathcal{P}_v$ and $R^1_2(Z)$ is torsion-free. This implies that $S = \mathcal{P}_w / L^* \mathcal{P}_v = \mathcal{O}_{D_{v-w}} \mathcal{P}_w$. If Z is not collinear we have a commuting diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S & = & S & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_w & \longrightarrow & R^1_2(W) & \longrightarrow & R^1_2(Z) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_w & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{9.1}$$

The bottom sequence implies that B is torsion-free supported on a divisor. In the case we are looking at, we have $R^1_2(Z) = T$, a torsion-free sheaf of rank 1 supported on $D \in |L^2 \mathcal{P}_a|$

for some a. We cannot have $S = T$ since that would imply that $A = \mathcal{P}_w$ contradicting the fact that $R^1_2(W)$ satisfies WIT_1. Since B is torsion-free on a divisor we must have that S is supported on $D_x \subset D$. This implies that D is reducible and so Z contains a collinear Y. Moreover, the restriction type of T is $(2,1)$ or $(1,1)$ and so the degree of S is either 0 or -1. This proves (2) and part of (3).

To complete (3) suppose that S is degree 0 on D_x. Then $\hat{S} \cong L\mathcal{P}_y\mathcal{I}_c$ for some x and c and $R^1_2(W)/S \cong \hat{L}\mathcal{P}_a\mathcal{I}_b$. So if we apply $\hat{\Phi}$ to

$$0 \longrightarrow S \longrightarrow R^0_2(W) \longrightarrow \hat{L}\mathcal{P}_a\mathcal{I}_b \longrightarrow 0$$

we see that $L\mathcal{P}_y\mathcal{I}_c \subset L^2\mathcal{I}_{-W}$. This implies that $W \setminus \{p\} \in \mathrm{Hilb}^4 \mathbb{T}$ is collinear.

We turn now to (4). If the degree of S is -1 then no $Z \subset W$ is collinear and the restriction type of $R^1_2(Z)$ is $(2,1)$. So each length 4 subscheme $Z \subset W$ has a unique collinear length 3 subscheme by Theorem 8.3. Conversely, suppose $L^2\mathcal{I}_W$ it WIT_1 and its transform $R^1_2(W)$ is contained in a short exact sequence

$$0 \longrightarrow S \longrightarrow R^1_2(W) \longrightarrow \hat{L}_x \longrightarrow 0.$$

Note that $\dim \mathrm{Ext}^1(\hat{L}_x, S) = 4$ and so there is a 9 dimensional family of such extensions R. These extensions all satisfy WIT_1 and the Fourier transform is generically torsion-free as both $\hat{S}$ and $\hat{\Phi}^0(\hat{L}_x)$ are locally-free, and any torsion in $\hat{\Phi}(R)$ must be a quotient $\mathcal{P}_{\hat{y}}/L_x^{-1}$ coming from a lift of $L_x^{-1} \to \hat{S}$. But the generic such map does not lift. □

Notes 9.3.

(1) It is easy to see that if $R^1_2(W)$ is torsion-free then the factor of $\mathcal{P}_x$ in it is given by $\mathcal{P}_{\Sigma W}$.

(2) One might conjecture that $S_2(W) \cong W$ as schemes when $R^1_2(W)$ is torsion-free. In fact, this need not be the case. As an example consider a generic divisor $D \in |2\ell|$ and $D_x + D_{-x} \in |2\ell|$ also generic. Then we can arrange that D intersects D_x and D_{-x} transversely and that $D \cap D_x \cap D_{-x} = \varnothing$. Now pick $Y \subset D \cap D_x$ and $Q \subset D \cap D_{-x}$ and let $W = Y \cup Q$. Note that W is not collinear because Y is collinear and $Q \not\subset D_x$. Then the line spanned by D and $D_x + D_{-x}$ in $|2\ell|$ is contained in $\Phi(W)_0$. Then the dimension of the fibre of $R^1_2(W) \cong \hat{L}\mathcal{P}_x\mathcal{I}_{W'}$ at 0 is 3. This implies that W' is not reduced at 0 and so cannot be isomorphic to W which is simple by construction.

10. General X

For $|X| > 5$ we can prove that when X is not collinear then it has a torsion-free Fourier-Mukai transform if and only if it does not contain a collinear colength 1 subscheme. The hard work goes into proving the $|X| = 6$ case and then the general case follows by induction.

Theorem 10.1. *Let $X \subset \mathbb{T}$ be a 0-dimensional subscheme of an irreducible ppas $(\mathbb{T}, \ell)$ of length $|X| \geqslant 6$. Suppose that X is not collinear. Then $R^1_2(X)$ admits torsion S if and only if X contains X', a collinear length $|X| - 1$ subscheme. Moreover, in that case S must be* *a degree 0 line bundle over some translate of D_L. In particular, for a non-collinear length 5 subscheme $W \subset X$ we have that the torsion subsheaf of $R^1_2(W)$ has degree 0.*

Proof. Observe that $L^2\mathcal{I}_X$ satisfies WIT_1 as X is not collinear. This follows from Corollary 7.2. If X contains a collinear colength 1 subscheme X' then we have an inclusion $L_{\hat{x}}\mathcal{I}_p \subset L^2\mathcal{I}_X$. This implies that $R^1_1(\{p\}) \subset R^1_2(X)$ and so $R^1_2(X)$ admits torsion.

Conversely, suppose that $R^1_2(X)$ contains torsion S. Arguing as in the first paragraph of the proof of Theorem 9.2, we can deduce that S must be supported on a divisor and be torsion-free with rank one over that divisor. We now induct on $|X|$. Consider first the case $|X| = 6$. Suppose that we have a non-collinear length 5 subscheme $W \subset X$. Then from sequence 3.3 we see that $S \subset R^1_2(W)$. Then Theorem 9.2 implies that $\chi(S) \leqslant -1$. Let $F = R^1_2(X)/S$. Then

$$R^1_2(W)/S = F/\mathcal{P}_x,$$

where $\mathcal{P}_x = \ker(R^1_2(X) \to R^1_2(W))$. Let G be the torsion subsheaf of $R^1_2(W)/S$ (regarded as a sheaf on its support so that G is supported at points). Then $G \subset F/\mathcal{P}_x$. But $\mathrm{Ext}^1(G, \mathcal{P}_x) = 0$ and so this map lifts to a map $G \to F$. But F is torsion-free (as an $\mathcal{O}_{\mathbb{T}}$-module) and this implies that $G = 0$. In other words, S is the torsion subsheaf of $R^1_2(W)$ as well. Then Theorem 9.2 implies that $\chi(S) = -1$ or -2.

Suppose that $\chi(S) = -2$. Then F is a rank two vector bundle and $F/\mathcal{P}_x = R^1_2(W)/S = \hat{L}_y$ for some y. This implies that $\hat{\Phi}^2(F) = \mathcal{O}_{-x}$ — apply $\hat{\Phi}$ to the short exact sequence

$$0 \longrightarrow \mathcal{P}_x \longrightarrow F \longrightarrow \hat{L}_y \longrightarrow 0.$$

But this contradicts the fact that $R^1_2(W)$ satisfies WIT_1.

Then $\chi(S) = -1$ and we have a short exact sequence

$$0 \longrightarrow \mathcal{P}_x \longrightarrow F \longrightarrow \hat{L}_y\mathcal{I}_p \longrightarrow 0.$$

This implies that $\hat{\Phi}^0(F) = 0$ and so F satisfies WIT_1. We then have a short exact sequence

$$0 \longrightarrow \hat{S} \longrightarrow L^2\mathcal{I}_{-X} \longrightarrow \hat{F} \longrightarrow 0.$$

But $\hat{S} \cong L_{\hat{x}}\mathcal{I}_p$ for some $\hat{x}$ and p and so $X \setminus \{-p\}$ is collinear. This proves the theorem for the case $|X| = 6$.

Now suppose that $|X| > 6$ and we have proved the theorem for all 0-schemes of length $|X| - 1 \geqslant 6$. We suppose that $R^1_2(X)$ contains torsion S and suppose that a length $|X| - 1$ subscheme $X' \subset X$ is not collinear. Then $S \subset R^1_2(X')$ and the induction hypothesis implies that there is a length $|X'| - 1$ subscheme $X'' \subset X'$ which is collinear. Let $Q = X \setminus X''$. then we have a short exact sequence

$$0 \longrightarrow L_{\hat{x}}\mathcal{I}_Q \longrightarrow L^2\mathcal{I}_X \longrightarrow A \longrightarrow 0 \tag{10.1}$$

for some $\hat{x}$ and line bundle A supported on $D_{\hat{x}}$. Note that $\chi(A) = 5 - |X|$.

Suppose, for a contradiction, that A is locally-free on its support. Then, since $\deg(A) < 0$ we must have that A satisfies IT_1. Then applying Φ to 10.1 we have a short exact sequence

$$0 \longrightarrow \hat{L}_x\mathcal{I}_{Q'} \longrightarrow R^1_2(X) \longrightarrow \hat{A} \longrightarrow 0,$$

for some x. But $R^1_2(X)$ admits torsion. This is a contradiction.

So $\mathcal{O}_{\tilde{X}} \subset A$. Let $\tilde{A} = A/\mathcal{O}_{\tilde{X}}$ be locally-free on its support. Then $\chi(\tilde{A}) < \chi(A)$. But there is a surjection $L^2\mathcal{I}_X \to \tilde{A}$ and its kernel is torsion-free of rank 1 with singularity set strictly contained in Q. This implies that it is the ideal sheaf of a single point $\{p\}$ and we have $L_x\mathcal{I}_p \subset L^2\mathcal{I}_X$.

We now see that S is either 0 or has degree 0 over a translate of D_L. This means that if $W \subset X$ is non-collinear then $S \subset R^1_2(W)$ and so the degree of the torsion subsheaf of $R^1_2(W)$ must be greater or equal to 0. □

We know that $D \cdot D' = 8$ for $D, D' \in |2\ell|$. In particular, this means that there must exist $X \in \operatorname{Hilb}^8 \mathbb{T}$ such that $S_2(X) \in \operatorname{Hilb}^2 \hat{\mathbb{T}}$. We know from Theorem 9.2 that when $R^1_2(W)$ is torsion-free then $S_2(W) \in \operatorname{Hilb}^5 \hat{\mathbb{T}}$. We would now like to compute the length of $S_2(X)$ for $|X| \geqslant 6$. This is answered in the following theorem.

Theorem 10.2. *Let $X \in \operatorname{Hilb}^n \mathbb{T}$ with $n \geqslant 6$. Suppose that X is not collinear and that $R^1_2(X)$ is torsion-free. Then*

$$\operatorname{length}\big(\operatorname{sing}(R^1_2(X))\big) = \operatorname{length}(S_2(X)) \leqslant 3.$$

Furthermore, if $|X| \geqslant 7$ then $\operatorname{length}(S_2(X)) \leqslant 2$.

Proof. Observe first that

$$\hat{\Phi}^2(R^1_2(X)^{**}) = 0. \tag{10.2}$$

This follows by applying $\hat{\Phi}$ to

$$0 \longrightarrow R^1_2(X) \longrightarrow R^1_2(X)^{**} \longrightarrow \mathcal{O}_{S_2(X)} \longrightarrow 0.$$

Let $|X| = 6$. Consider $W \subset X$. By Theorem 10.1, we know that W cannot be collinear. Suppose first that $R^1_2(W)$ is torsion-free. Without loss of generality, assume that $R^1_2(W) = \hat{L}^2\mathcal{I}_{W'}$. Then we have the following 3 by 3 diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_v & \longrightarrow & R^1_2(X) & \longrightarrow & L^2\mathcal{I}_{W'} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_v & \longrightarrow & R^1_2(X)^{**} & \longrightarrow & L^2\mathcal{I}_{W'\setminus S_2(X)} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{S_2(X)} & = & \mathcal{O}_{S_2(X)} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{10.3}$$

The middle horizontal sequence cannot split by 10.2. But

$$H^1(L^2\mathcal{I}_{W'\setminus S_2(X)}) = 0 \quad \text{if} \quad |W \setminus S_2(X)| \leqslant 1.$$

So $5 - \operatorname{length}(S_2(X)) \geqslant 2$.

Now suppose that $R^1_2(W)$ has non-zero torsion subsheaf S. By Theorem 9.2 and the last part of Theorem 10.1 we have

$$R^1_2(W)/S \cong \hat{L}_x \mathcal{I}_p,$$

for some x and p. Then

$$\ker(R^1_2(X) \to \hat{L}_x \mathcal{I}_p) = \hat{L}_y \mathcal{I}_Q, \quad Q \in \operatorname{Hilb}^2 \hat{\mathbb{T}}$$

and so we have $Q \subset S_2(X) \subset Q \cup \{p\}$ and we conclude that

$$2 \leqslant \operatorname{length}(S_2(X)) \leqslant 3.$$

This completes the proof of the first part of the theorem.

Suppose now that $|X| = 7$. We may assume as before that X contains a non-collinear length 6 subscheme X'. By Theorem 10.1 we have two cases to consider: either $R^1_2(X')$ is torsion-free or it contains a torsion sheaf S with $E = R^1_2(X')/S$ a torsion-free sheaf with Chern character $(2, \ell, 0)$.

If $R^1_2(X')$ is torsion-free then we can argue as before. Let $F = R^1_2(X)^{**}/\mathcal{P}_x$, where $\mathcal{P}_x = \ker\big(R^1_2(X) \to R^1_2(X')\big)$. Then $\operatorname{ch}(F \otimes L^*) = (2, 0, 0)$. On the other hand, F is locally-free and so F must be a homogeneous bundle. Therefore, $\hat{\Phi}^2(R^1_2(X)^{**}) \neq 0$. This implies that $\hat{\Phi}^2(R^1_2(X)) \neq 0$; a contradiction.

On the other hand, suppose that $R^1_2(X')$ admits torsion. We now need a lemma:

Lemma 10.3. *If $X' \in \operatorname{Hilb}^6 \mathbb{T}$ is not-collinear and $R^1_2(X')$ admits torsion S then $E = R^1_2(X')/S$ is locally-free.*

Proof. As above, if $W \subset X'$ is not collinear then $R^1_2(W)/\text{torsion} = \hat{L}_x \mathcal{I}_p$ and we have a short exact sequence

$$0 \longrightarrow \mathcal{P}_x \longrightarrow E \longrightarrow \hat{L}_x \mathcal{I}_p \longrightarrow 0. \tag{10.4}$$

Suppose that E is not locally-free then we have the following 3 by 3 diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_y & \longrightarrow & E & \longrightarrow & \hat{L}_x \mathcal{I}_p & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_y & \longrightarrow & E^{**} & \longrightarrow & L_x & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_p & = & \mathcal{O}_p & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle horizontal sequence is split because $\hat{L}^{-1}$ satisfies IT_2 but this implies that there is a non-zero map $E \to \mathcal{P}_x$ which contradicts the fact that $\hat{\Phi}^2(E) = 0$. Alternatively, sequence 10.4 cannot be split and so $\{p\}$ satisfied the Cayley-Bacharach condition with respect to $|\hat{L}|$.

On the other hand, if all $W \subset X'$ are collinear then we have $E \cong R^1_2(W)$ and so E is locally-free by Corollary 7.2. □

Returning to the proof of the theorem, we now have a short exact sequence

$$0 \longrightarrow \hat{L}_z \mathcal{I}_Q \longrightarrow R^1_2(X) \longrightarrow E \longrightarrow 0$$

for some z and $Q \in \operatorname{Hilb}^2 \hat{\mathbb{T}}$ and locally-free sheaf E. This implies that

$$\operatorname{length}\big(\operatorname{sing}(R^1_2(X))\big) = 2.$$

This completes the proof of the theorem. □

Corollary 10.4. *If $R^1_2(X)$ is not torsion-free for some $X \in \operatorname{Hilb}^n \mathbb{T}$, $n \geqslant 6$, then $\Phi_2(X)_{\hat{x}} \subset Ks_{\hat{x}}(\mathbb{T})$ for all $\hat{x} \in S_2(X)$.*

Proof. By Theorem 10.1 we know that X contains a collinear length $n-1$ subscheme $X' \subset D_y$. Let $X = X' \cup \{x\}$. Then $X \subset D_y + D_z$ for $z \in S_1(\{x\})$. Let the torsion subsheaf of $R^1_2(X)$ be S as usual. Note that $S_2(X)$ equals $D_{\hat{y}}$ as a scheme for some $\hat{y}$. This is because $R^1_2(X)/S$ is locally-free. This means that

$$\Phi_2(X) = \big\{D_y + D_{-z} \mid z \in S_1(\{x\})\big\}$$

as required □

11. Stability Properties of $R^1_i(X)$

We now turn to stability properties of $R^1_i(X)$ for $i = 1$ and $i = 2$. The following has been proved before (Mukai [11, Thm 0.3] and Yoshioka [13, Prop 3.5]). We give a direct and brief proof in the spirit of this paper.

Theorem 11.1. *If $|X| \geqslant 2$ then $R^1_1(X)$ is μ-stable.*

Proof. We induct on the length $|X|$. Observe that the $|X| = 2$ case is trivial because $R^1_1(X)$ is rank 1 torsion-free and so μ-stable. Now assume that $|X| > 2$. Pick $X' \subset X$ of length $|X| - 1$ and let $X = X' \cup \{x\}$ as schemes. Then we have a short exact sequence,

$$0 \to \mathcal{P}_x \to R^1_1(X) \to R^1_1(X') \to 0.$$

Now that slope of $R^1_1(X)$ is $2/(|X| - 1)$ and $\mu(R^1_1(X')) = 2/(|X| - 2)$. Suppose M is maximally destabilizing. Then the map $M \to R^1_1(X)$ cannot lift to $\mathcal{P}_x$ and so we have a non-trivial map $M \to R^1_1(X')$. If $r(M) < |X| - 2$ then the induction hypothesis implies that $d(M) < 2$ which is impossible. On the other hand, if $r(M) = |X| - 2$, we have that the map $M \to R^1_1(X')$ must inject and so $\chi(M) \leqslant -1$. But $\operatorname{ch}(R^1_1(X)/M) = (1, 0, -1 - \chi(M))$ and, by the maximality of the destabilizing sheaf, $R^1_1(X)/M$ is torsion-free and so $\chi(M) = -1$. But that is a contradiction as M would then split the original short exact sequence. □

Conversely, given a μ-stable sheaf E of Chern character $(r, \ell, -1)$ with $r \geqslant 2$ we see that it cannot be IT_1 and it must admit an injection from a homogeneous bundle of rank $r-1$. The quotient must take the form $L_x \mathfrak{I}_Q$ and so E must be WIT_1 with transform given by $L_y \mathfrak{I}_X$ for some 0-scheme X of length $r+1$. So we recover:

Corollary 11.2. [11, (Thm 0.3 and Cor 5.5)] *The moduli space $\mathcal{M}(r, \ell, -1)$ of μ-stable sheaves is compact and is biholomorphic to $\mathrm{Hilb}^{r+1} \mathbb{T} \times \hat{\mathbb{T}}$ for $r \geqslant 2$. The non-locally free boundary has transform given by a collinear 0-scheme.*

We now turn to the case $i = 2$. The interesting cases are $|X| \geqslant 6$. Observe first that if X is collinear then $R_2^0(X) = L^{-1}\mathcal{P}_y$ and $R_2^1(X)$ has rank $|X| - 3$ and degree 2. It is an easy induction to show that $R_2^1(X)$ must always be μ-stable and locally-free.

We now look in detail at $|X| = 6$. This is also treated, from the point of view of moduli spaces, in moduli spaces, in [8] (see, in particular, Theorem 4.10).

Theorem 11.3. *Let $X \in \mathrm{Hilb}^6 \mathbb{T}$. Suppose that X is not collinear.*

(1) *If $R_2^1(X)$ is locally-free then $R_2^1(X)$ is μ-stable.*
(2) *If $R_2^1(X)$ is torsion-free then it is Gieseker stable.*
  (a) *X contains a collinear length 4 subscheme if and only if $R_2^1(X)$ is μ-destabilized by $L_x \mathfrak{I}_Q$ for some x and Q. Then $\mathrm{sing}(R_2^1(X)| = 2$.*
  (b) *Otherwise $R_2^1(X)$ may be destabilized by $L_x \mathfrak{I}_Y$, for some x and Y. Then $|\mathrm{sing}(R_2^1(X)| = 3$.*
(3) *If $R_2^1(X)$ admits torsion S then $R_2^1(X)/S$ is μ-stable.*

Proof. Suppose first that $R_2^1(X)$ is torsion-free. Then by Theorem 10.1 we have a non-collinear length 5 subscheme $W \subset X$ and a short exact sequence from sequence 3.3

$$0 \longrightarrow \mathcal{P}_x \longrightarrow R_2^1(X) \longrightarrow R_2^1(W) \longrightarrow 0$$

Note that $\mu(R_2^1(X)) = 2$. Suppose that $M \subset R_2^1(X)$ is a rank 1 destabilizing subsheaf. Then M takes the form $L_y^n \mathfrak{I}_{X'}$ for some $n > 0$ and 0-scheme X'. Then the composite to $R_2^1(W) = L_z^2 \mathfrak{I}_{W'}$ must inject. Hence, $\mathcal{P}_x \to R_2^1(X)/M$ also injects and so $n \leqslant 2$. If $n = 2$ then $R_2^1(X)/M \cong \mathcal{P}_x$ and so $|X'| = 3$. But $W \subseteq X'$ which is a contradiction. So we are left with the case $n = 1$. We may assume the quotient $R_2^1(X)/M$ is torsion-free and so takes the form $L_w \mathfrak{I}_{X''}$. The Euler characters now tell us that $|X'| + |X''| = 3$. If $|X'| = 0$ then M is IT_0 and this contradicts the fact that $R_2^1(X)$ is WIT_1. Otherwise, $R_2^1(X)$ must have a non-zero singularity set (containing X'). This establishes the first part.

If $|X'| = 1$ then the transform of $M \to R_2^1(X)$ gives a non-zero map $\hat{M} \to L^2 \mathfrak{I}_X$. But this is impossible as $\hat{M}$ is a torsion sheaf. Hence $|X'| > 1$ and so $R_2^1(X)$ is Gieseker stable (as $\chi(M)/1 \leqslant -1$ while $\chi(R_2^1(X)) = -1/2$).

Observe that X has a collinear length 4 subscheme if and only if we have $L_y \mathfrak{I}_{Q'} \to L^2 \mathfrak{I}_X$ for some y and Q'. This happens if and only if there is an injection $\widehat{L_y \mathfrak{I}_{Q'}} \to R_2^1(X)$. But $\widehat{L_y \mathfrak{I}_{Q'}} = L_x \mathfrak{I}_Q$ for some x and Q.

Now assume $R^1_2(X)$ has torsion sheaf T and let $E = R^1_2(X)/T$. We have already seen that E must be locally-free. Then we have a short exact sequence

$$0 \to \mathcal{P}_y \to E \to L_x \mathcal{I}_p \to 0$$

for points x, y and p. But now any μ-destabilising bundle would have to have rank 1 and slope at least 2. But then it cannot map to either $L_x\mathcal{I}_p$ or to $\mathcal{P}_y$. This establishes the final part. □

12. Applications

We can deduce the following result (which can also be deduced from classical projective geometry)

Theorem 12.1. *Suppose D' is a genus five curve in an irreducible principally polarized abelian variety $(\mathbb{T}, \ell)$. If D' is smooth then D' has no g^1_3. There are (singular) irreducible divisors $D' \in |2\ell|$ which admit a 1-dimensional family of g^1_3's.*

Proof. Observe first that if T is a degree 3 line bundle over a smooth divisor $D' \in |2\ell|$ and if T is the restriction of $L_{\hat{x}}\mathcal{I}_p$ then T has no g^1_3. This is because the exact sequence

$$0 \longrightarrow L^*\mathcal{P}_{\hat{y}} \longrightarrow L_{\hat{x}}\mathcal{I}_p \longrightarrow T \longrightarrow 0$$

shows that $H^0(T) \cong H^0(L_{\hat{x}}\mathcal{I}_p)$ which has dimension 0 or 1. This means that if D' is not reducible and admits a g^1_3 then that g^1_3 must satisfy WIT_1 with transform $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Z$ for some $Z \in \mathrm{Hilb}^4 \mathbb{T}$.

Observe that

$$\dim(H^0(T)) = \dim\big((L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Z)_{(0)}\big),$$

where $A_{(x)}$ denotes the fibre of a sheaf A at x. Then the dimension of $H^0(T)$ is greater than 1 only if $(\mathcal{I}_Z)_0$ is not a regular ideal. This can only happen if Z contains either Y_d or Y_e. But by Theorem 7.3 we see that $S_2(Y)$ has type d or e. On the other hand, $S_2(Y) \subset S_2(Z)$ and so $S_2(Z)$ must be singular. If we take the example

$$Z \cong \mathbb{C}[\epsilon, \eta]/(\epsilon^3, \eta^2, \epsilon\eta)$$

then we see that $Z \subset S_2(Z)$ by Theorem 7.3. The subset of $\mathrm{Hilb}^4 \mathbb{T} \times \hat{\mathbb{T}}$ of such Z is 6-dimensional, whereas the space of singular divisors in $\mathbb{P}\widehat{L^2}$ is at most 4-dimensional. This implies that $\dim(W^1_3) \geqslant 1$ for such $S_2(Z)$. On the other hand, such $S_2(Z)$ cannot be generically reducible since then every $Y_c \subset \mathbb{T}$ would be collinear which is impossible because it would tell us that Y_c is, generically, contained in two different translates of D_L. This contradicts Proposition 7.1. □

We can also use our results to understand the set of singular divisors in $|2\ell|$. Recall that $\mathbb{T}_2$ denotes the 2-torsion points of $\mathbb{T}$.

Theorem 12.2. *Let Σ_x denote the linear system of divisors in $|2\ell|$ which pass through x.*

(1) *If $x \notin \mathbb{T}_2$ then the only singular divisors in Σ_x are Kummer divisors.*

(2) *If* $x \in \mathbb{T}_2$ *then any divisor* D *in* Σ_x *is singular. If* D *is not a Kummer divisor then its singularity set consists of at most 3 nodes. There are* 120 *(skew) lines of divisors with 2 nodes and each line contains fourteen points whose corresponding divisors have exactly 3 nodes.*

In particular, the locus of singular divisors consists of $Ks(\mathbb{T})$ *union all sixteen singular planes in* $\mathbb{P}^3 = |2\ell|$.

Proof. Observe that if Q is a length 2 subscheme supported at x then $S_2(Q) = \{-2x\}$. (1) If $x \notin \mathbb{T}_2$ then $h^0(L^2\mathfrak{I}_Q) = 1$. Let Q_1 and Q_2 be two distinct length 2 subschemes supported at x and suppose that $\Phi_2(Q_1)_0 \cap \Phi_2(Q_2)_0$ is 1 dimensional. Then this holds for all pairs Q_1 and Q_2 since each divisor in the intersection has a singularity at x. But then such singular divisors must also contain $Y = Q_1 \cup_x Q_2$ for some $Q_1 \neq Q_2$. But $S_2(Y) = \{-2x\}$ and so $\Phi_2(Y)_0$ consists of a single divisor. This is a contradiction and shows that there is at most 1 singular divisor in Σ_x with a singularity at x. But for any x we can find y such that $D_y + D_{-y}$ has a node at x.

(2) If $x \in \mathbb{T}_2$ then $\Phi_2(Q)_0 = \Sigma_x$ and so every divisor in Σ_x is singular at x. Note that there is a line of Kummer divisors in Σ_x which have a tacnode or are multiple. The sets Σ_x correspond to singular planes in $\mathbb{P}^3$. These intersect along lines and any three of them intersect in a single point. In fact the non-Kummer points correspond to Göpel tetrahedra (see [3]). Since there are no degenerate Göpel tetrahedra, we see that these points are all distinct (so the divisors cannot have more than 4 singular points unless they are multiple Kummer divisors.

It remains to prove that if a divisor $D \in \Sigma_x$ is not a Kummer divisor then it has only nodes for singularities. Observe now that $\Phi_2(Y)_0$ has dimension 1 for all length 3 Y supported at x. Pick a regular Y and $D \in \Sigma_x \setminus \Phi_2(Y)_0$. Let Y' be a length 3 subscheme supported at x and contained in D. Then $\Phi_2(Y)_0 \cap \Phi_2(Y')_0 = \{D\}$. The space of pairs of regular length 3 subschemes supported at x is given by $S^2\mathbb{P}^1 \cong \mathbb{P}^2$ and the above observation furnishes us with a map $\mathbb{P}^2 \setminus \Delta \to \Sigma_x \cong \mathbb{P}^2$, where Δ denotes the diagonal. This must be finite since otherwise there would be divisor containing all regular length 3 subschemes supported at x and this is impossible. On the other hand the map has degree one since it is given by the intersection of hypersurfaces and so it must be the identity. The extension over the diagonal maps to the Kummers and so we see that the non-Kummer divisors have a simple node at x. □

Remark 12.3. We can also see directly that there is at most one divisor in $|2\ell|$ which is singular at three given distinct points because for a generic choice of length 6 subscheme X supported at the three points $h^1(L^2\mathfrak{I}_X) \leqslant 1$ because X does not contain collinear length 5 subschemes and is not, itself, collinear.

Appendix

We will now summarize the computations of the Fourier-Mukai transforms derived in section 5–10.

X	$S_1(X)$	$S_2(X)$	R^1_1	R^0_2	R^1_2	where
P	D_p	$\varnothing$	$\mathcal{P}_p\mathcal{O}_{D_{-p}}$	(Note 1)	0	
Q	$\{-p+l,-p-l'\}$	$\{-p-q\}$	$\mathcal{I}_{S_1}\hat{L}$		$\mathcal{O}_{S_2}$	$p-q=l-l'$
Y	$\varnothing$	$\cong Y$		$\hat{L}^{-2}\mathcal{P}_{-\tau}$	$\mathcal{O}_{S_2}$	$\tau=\sum Y$
	$\{v\}$	D_u		$\hat{L}^{-1}_{-v}$	(Note 2)	$u=v-\tau$
Z	$\varnothing$	D'		0	(Note 3)	$D'\in\lvert\hat{L}^2\mathcal{P}_\sigma\rvert$
	$\{v\}$	$\{2v-\sigma\}$		$\hat{L}^{-1}_{-v}$	$\hat{L}_{\sigma-v}\mathcal{I}_{2v-\sigma}$	$\sigma=\sum Z$

(1) $R^1_2(P)$ is a rank 3 μ-stable vector bundle.
(2) $R^1_2(Y)$ is a degree 1 line bundle over D_u.
(3) $R^1_2(Z)$ is a degree 3 line bundle over D'

X	$S_1(X)$	$S_2(X)$	R^0_2	R^1_2	where
W	$\varnothing$	W'	0	$\hat{L}^2\mathcal{P}_\alpha\mathcal{I}_{W'}$	$W'\in\operatorname{Hilb}^5\hat{\mathbb{T}}$
	$\varnothing$	D_u	0	$T\ltimes\hat{L}_{\hat{x}}\mathcal{I}_y$	collinear $Z\subset W$, $\deg(T)=0$ on $D_{\hat{u}}$
	$\varnothing$	D_u	0	$T\ltimes\hat{L}_{\hat{x}}$	$\exists x$, $\forall Z\subset W$, $\exists Y\subset Z$, $Y\subset D_x$
					$\deg(T)=-1$ on $D_{\hat{u}}$
	$\{v\}$	$\varnothing$	$\hat{L}^{-1}_v$	(Note 1)	
X	$\varnothing$	$\varnothing$	0	(Note 2)	$\lvert X\rvert\geqslant 6$
	$\varnothing$	$\{u\}$	0	(Note 3)	$\lvert X\rvert\geqslant 6$
	$\varnothing$	$\{u,v\}$	0	(Note 3)	$6\leqslant\lvert X\rvert\leqslant 8$
	$\varnothing$	$\{u,v,z\}$	0	(Note 3)	$\lvert X\rvert=6$
	$\varnothing$	D_u	0	$T\ltimes E$	$\lvert X\rvert\geqslant 6$, collinear $X'\subset X$
					colength 1, $\deg(T)=0$ on D_u
	$\{v\}$	$\varnothing$	$\hat{L}^{-1}_v$	(Note 4)	$\lvert X\rvert>5$

(1) $R^1_2(W)$ is a rank 2 μ-stable vector bundle over $\hat{\mathbb{T}}$ with Chern character $(2,\ell,0)$.
(2) $R^1_2(X)$ is a vector bundle over $\hat{\mathbb{T}}$ with Chern character $(|X|-4,2\ell,-1)$.
(3) $R^1_2(X)$ is a torsion-free sheaf over $\hat{\mathbb{T}}$ with Chern character $(|X|-4,2\ell,-1)$ and singularity set equal to $S_2(X)$. This is μ-semistable when $|X|=6$.
(4) $R^1_2(X)$ is a vector bundle over $\hat{\mathbb{T}}$ with Chern character $(|X|-3,\ell,0)$.

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