Convergence in law for the branching random walk seen from its tip

by

Thomas Madaule

Université Paris XIII

Abstract. Considering a critical branching random walk on the real line. In a recent paper, Aidekon [3] developed a powerful method to obtain the convergence in law of its minimum after a log-factor normalization. By an adaptation of this method, we show that the point process formed by the branching random walk and its minimum converge in law to a Poisson point process colored by a certain point process. This result, confirming a conjecture of Brunet and Derrida [10], can be viewed as a discrete analog of the corresponding results for the branching brownian motion, previously established by Arguin et al. [5] [6] and Aidekon et al. [2].

1 Introduction

We consider a branching random walk on the real line $\mathbb{R}$. Initially, a single particle sits at the origin. Its children together with their displacements, form a point process L on $\mathbb{R}$ and the first generation of the branching random walk. These children have children of their own which form the second generation, and behave –relative to their respective birth positions– like independent copies of the same point process L. And so on.

Denote by $\mathbb{T}$ the genealogical tree of the particles in the branching random walk, then $\mathbb{T}$ is a Galton-Watson tree. We write $|z| = n$ if a particle z is in the n-th generation, and denote its position by $V(z)$. The collection of positions $(V(z), z \in \mathbb{T})$ is our branching random walk.

The study of the minimal position $M_n := \min_{|z|=n} V(z)$ has attracted many recent interests. The law of large numbers for the speed of the minimum goes back to the works of Hammersly [14], Kingman [18] and Biggins [7]. The second order was recently found separately by Hu and Shi [15], and Addario-Berry and Reed [1]. In [1], the authors computed the expectation of M_n up to $O(1)$, and showed, under suitable assumptions, that the sequence of the minimum is tight around its mean. Through recursive equations, Bramson and Zeitouni [9] obtained the tightness of M_n around its median, assuming some hypotheses on the decay of the tail distribution. A definite response was recently given by Aïdékon [3], where he proved the convergence of the minimum M_n centered around $\frac{3}{2}\log n$ for the general class of critical branching random walks.

One problem of great interest in the study of branching random walk is to characterize its behaviour seen from the minimal position, namely, the asymptotic of the point process formed by $\{V(z) - M_n, |z| = n\}$ as $n \to \infty$. The corresponding problem for the branching Brownian motion (the continuous analogue of branching random walk) was solved very recently by Arguin, Bovier, Kistler [5], [6] and paralleling by Aïdékon, Beresticky, Brunet, Shi [2].

The aim of this paper is to establish analogue results for branching random walk. Our main result, resumed by Theorem 1.1, will give the existence of the limiting point process together with a partial description, which also confirms the prediction in Brunet and Derrida [11]. Our method, largely inspired from Aïdékon [3], consists of an analysis of the Laplace transform of the point process.

Following [3], we assume

$$\mathbf{E}\left[\sum_{|z|=1} 1\right] > 1, \qquad \mathbf{E}\left[\sum_{|z|=1} \mathrm{e}^{-V(z)}\right] = 1, \qquad \mathbf{E}\left[\sum_{|z|=1} V(z)\mathrm{e}^{-V(z)}\right] = 0 \tag{1.1}$$

Every branching random walk satisfying mild assumptions can be reduced to this case by some renormalization. We refer to Appendix A in [16] for a precise discussion. Notice that we allow $\mathbf{E}\left[\sum_{|z|=1} 1\right] = \infty$, and even $\mathbf{P}\left(\sum_{|z|=1} 1 = \infty\right)$. The couple $(M_n, W_{n,\beta})$ is the most often encountered random variables in our work, with

$$M_n := \min\{V(x), |z| = n\}, \qquad W_{n,\beta} := \sum_{|z|=n} \mathrm{e}^{-\beta V(z)}, \qquad \beta > 1.$$

We also need the *derivative martingale*

$$Z_n := \sum_{|z|=n} V(z)\mathrm{e}^{-V(z)}, \qquad Z_\infty = \lim_{n\to\infty} Z_n. \tag{1.2}$$

By [8] and [3] we know that Z_∞ exists almost surely and is strictly positive on the set of non extinction of $\mathbb{T}$. As in the continuous case [2], we introduce the point process formed by the particles of the rescaled branching random walk:

$$\mu_n := \sum_{|z|=n} \delta_{\{V(z) - \frac{3}{2}\log n + \log Z_\infty\}}, \quad n \geq 1.$$

We will show the existence of a limiting point process as $n \to \infty$, then we deduce results on $\mu'_n := \sum_{|z|=n} \delta_{\{V(z)-M_n\}}$, $n \geq 1$. Writing for $y \in \mathbb{R} \cup \{\infty\}$, $y_+ := \max(y, 0)$, we introduce the random variables

$$X := \sum_{|z|=1} \mathrm{e}^{-V(z)}, \qquad \widetilde{X} := \sum_{|z|=1} V(z)_+ \mathrm{e}^{-V(z)}. \tag{1.3}$$

We finally assume that the distribution of L is non-lattice and

$$\mathbf{E}\left[\sum_{|z|=1} V(z)^2 \mathrm{e}^{-V(z)}\right] < \infty. \qquad \mathbf{E}\left[X(\log_+(X+\widetilde{X}))^3\right] < \infty \tag{1.4}$$

The main result of this paper is the following theorem:

Theorem 1.1 *As $n \to \infty$, on the set of non-extinction, the pair (μ_n, Z_n) converges jointly in distribution to (μ_∞, Z_∞) where μ_∞ and Z_∞ are independent and μ_∞ is obtained as follows.*

(i) Define $\mathcal{P}$ a Poisson process on $\mathbb{R}$, with intensity measure $\lambda \mathrm{e}^x dx$ for some (implicit) positive real constant λ.

(ii)For each atom x of $\mathcal{P}$, we attach a point process $x + \mathcal{D}^{(x)}$ where $\mathcal{D}^{(x)}$ are independent copies of a certain point process $\mathcal{D}$.

(iii) μ_∞ is the superposition of all the point processes $x + \mathcal{D}^{(x)}$, i.e, $\mu_\infty := \{x + y : x \in \mathcal{P}, y \in \mathcal{D}^{(x)}\}$.

Corollary 1.2 *Seen from the leftmost particle, the point process μ'_n formed by the particles $\{V(u) - M_n, |u| = n\}$ converges in distribution to the point process μ'_∞ obtained by replacing the Poisson point process $\mathcal{P}$ in step (i) above by $\mathcal{P}'$ described in step (i)' below:*

(i)'Let $\mathbf{e}$ be a standard exponential random variable. Conditionally on $\mathbf{e}$, Define $\mathcal{P}'$ to be a Poisson point process on $\mathbb{R}_+$, with intensity measure $\mathbf{e}\mathrm{e}^x \mathbb{1}_{\mathbb{R}_+} dx$ to which we add an atom in 0.

The decoration point process $\mathcal{D}$ remains the same.

These two results imitate the corresponding results for the branching Brownian motion, in particular Theorem 2.1 and Corollary 2.2 of Aïdékon, Beresticky, Brunet and Shi [2] (and also that of [5] and [6]). However, we do not adopt the same method as in [2] because, firstly the spine decomposition for the branching random walk leads to an use of Palm measures, which is much complicated than the case of branching brownian motion, and secondly, the path decomposition for a random walk is also less comfortable than the Brownian case. Instead, we shall imitate the fine analysis of Aïdékon [3] to analyse the Laplace transform of μ_n. More precisely, the main step in the proof of Theorem 1.1 is to establish the convergence in law of $(n^{\frac{3}{2}\beta_1} W_{n,\beta_1}, ..., n^{\frac{3}{2}\beta_1} W_{n,\beta_k})$ for any $k \geq 1$ and any $\beta_k > ... > \beta_1 > 1$. A crucial observation, inspired by [3], is that this convergence in law can be reduced to the study of its tail behaviour. From this analysis, we can prove the convergence in law stated in Theorem 1.1 , and as a by-product, we also get some expression for the Laplace transform of the limiting point process. The later might have some independent interest for further analysis of μ_∞.

The paper is organized as follows. The Section 2 contains the main estimates on the tail of distribution of $(n^{\frac{3}{2}\beta_1}W_{n,\beta_1},...,n^{\frac{3}{2}\beta_k}W_{n,\beta_k})$ for any any $k\geq 1$ and any $\beta_k>...>\beta_1>1$, from which we establish the convergence of some Laplace transforms of μ_n (Theorem 2.4) and give the proof of Theorem 1.1. The Section 3 is devoted to the proof Theorem 2.4 by admitting two preliminary estimates Proposition 2.1 and 2.2. Finally, we prove in Section 4 and 5 contain respectively Proposition 2.1 and 2.2.

2 Main steps of the proof of theorem 2.1

For shorten the statements we introduce some notations:

$$\widetilde{W}_{n,\beta}:=n^{\frac{3}{2}\beta}W_{n,\beta},\qquad \widehat{\mu}_n(\beta)=n^{\frac{3}{2}\beta}\sum_{|z|=n}\mathrm{e}^{-\beta(V(z)+\log Z_\infty)},$$
$$\widehat{\mu}_n^a(\beta)=n^{\frac{3}{2}\beta}\sum_{|z|=n}\mathrm{e}^{-\beta(a+V(z)+\log Z_\infty)}.$$

with $a\in\mathbb{R}$, $n\geq 1$, $\beta>1$. Remark that $\widehat{\mu}_n(\beta)$ is also equal to $\int_{\mathbb{R}}\mathrm{e}^{-\beta x}d\mu_n(x)$. In a general context many quantities with tilde are associated with the natural normalization $n^{\frac{3}{2}\beta}$ except for some obvious abuse of notation: For example in the sequel we will denote by simplification $\widetilde{W}_{n-|u|,\beta}:=n^{\frac{3}{2}\beta}W_{n-|u|,\beta}$. In a similar spirit we write $\widetilde{M}_n:=M_n-\frac{3}{2}\log n$ and $\widetilde{M}_{n-|u|}:=M_{n-|u|}-\frac{3}{2}\log n$ for some vertex $|u|\leq n$ (we shall recall these notations to avoid any confusion). At last we often encounter notations $\boldsymbol{\delta}$, $\boldsymbol{\beta}$ and $\boldsymbol{y}$ for respectively $(\delta_1,...,\delta_k)$, $(\beta_1,...,\beta_k)$ and $(y_1,...,y_k)$. The lengths of the vectors will be clear in the context.

2.1 Main preliminary results

In this section we state some technical results (deferring their proofs to the next sections) which will lead to the proof of Theorem 1.1.

Proposition 2.1 *There exists $c_1>0$, $\alpha>0$ and $N>0$ such that for any $n>N$, $j\geq 0$ and $x\in[1,\log\log n]$*

$$\mathbf{P}(\widetilde{W}_{n,\beta}\geq \mathrm{e}^{\beta x},\ \widetilde{M}_n\in[j-x-1,j-x])\leq c_1x\mathrm{e}^{-x}\mathrm{e}^{-\alpha j}. \tag{2.1}$$

In particular we see that $\mathbf{P}(\widetilde{W}_{n,\beta}\geq \mathrm{e}^{\beta x})\leq c_2x\mathrm{e}^{-x}$ for any $n>N$ (This Proposition, purely technical requires a proof very similar to the following. For this reason it will be found in the appendix.)

Proposition 2.2 *There exists $c_0 \in \mathbb{R}_+$ (see (4.9) for a precision) such that for any $k \geq 0$ there exists a function*

$$\chi : \begin{array}{l} (1,\infty)^k \times \mathbb{R}^k \to \mathbb{R}_+^* \\ (\boldsymbol{\beta}, \boldsymbol{\delta}) \mapsto \chi(\boldsymbol{\beta}, \boldsymbol{\delta}) \end{array} \tag{2.2}$$

which satisfies, $\forall K, \epsilon > 0$ there exists $A(K,\epsilon) > 0$ such that $\forall (\delta_1, ..., \delta_k) \in [-K,K]^k$ $\exists N(\epsilon, \boldsymbol{\delta})$ such that $\forall n > N$, $x \in [A, A + \log\log n]$

$$\left| \frac{\mathrm{e}^x}{x} \mathbf{P}\left(\bigcap_{j \leq k} \{ \widetilde{W}_{n,\beta_j} \geq \mathrm{e}^{\beta_j (x - \delta_j)} \} \right) - c_0 \chi(\boldsymbol{\beta}, \boldsymbol{\delta}) \right| \leq \epsilon.$$

Moreover function χ satisfies

(i) The restriction $\boldsymbol{\delta} \mapsto \chi(\boldsymbol{\beta}, \boldsymbol{\delta})$ is continuous,

(ii)For any $x \in \mathbb{R}$, $\boldsymbol{\delta} \in \mathbb{R}^k$, $\boldsymbol{\beta} \in (1,\infty)^k$, $\chi(\boldsymbol{\beta}, \boldsymbol{\delta} + x) = \mathrm{e}^x \chi(\boldsymbol{\beta}, \boldsymbol{\delta})$ with $\boldsymbol{\delta} + x := (\delta_1 + x, ..., \delta_k + x)$,

(iii)For any $\boldsymbol{\beta} \in (0,1)^k$ there exists $c_3 > 0$ such that $\chi(\boldsymbol{\beta}, \boldsymbol{\delta}) \leq c_3 \min_{i \in [1,k]} \mathrm{e}^{\delta_i}$, $\forall \boldsymbol{\delta} \in \mathbb{R}^k$.

The Proposition 2.2 yields an important consequence:

Corollary 2.3 *If $\widetilde{W}_{n,\beta}^{a,1}$ and $\widetilde{W}_{n,\beta}^{b,2}$ are the normalized partition functions of two independent branching random walks starting respectively from a and b real then, $\forall K, \epsilon > 0$ there exists $A(K,\epsilon) > 0$ such that $\forall (\Delta, \delta_1, ..., \delta_k) \in [-K,K]^{k+1}$ $\exists N(\epsilon, \Delta, \boldsymbol{\delta})$ such that $\forall n > N$, $x \in [A, A + \log\log n]$*

$$\left| \frac{\mathrm{e}^x}{x} \mathbf{P}\left(\bigcap_{j \leq k} \{ \widetilde{W}_{n,\beta}^{a,1} + \widetilde{W}_{n,\beta}^{b,2} \geq \mathrm{e}^{\beta_j (x - \delta_j)} \} \right) - c_0 (\mathrm{e}^a + \mathrm{e}^b) \chi(\boldsymbol{\beta}, \boldsymbol{\delta}) \right| \leq \epsilon.$$

The key step in the proof of Theorem 1.1 is the following result:

Theorem 2.4 *(i) $\forall l \in \mathbb{N}, \alpha \in \mathbb{R}_+$, there exists a function $F : (\beta_1, ..., \beta_l, \theta_1, ..., \theta_l) \in (1,\infty)^l \times \mathbb{R}_+ \to \mathbb{R}_+$ (see (3.2) for an explicit formula) such that $\lim_{\theta \to 0} F(\boldsymbol{\beta}, \boldsymbol{\theta}) = 0$, and*

$$\lim_{n\to\infty} \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l} \theta_i \widehat{\mu}_n(\beta_i)} \mathrm{e}^{-\alpha Z_\infty} \mathbb{1}_{\{Z_\infty > 0\}} \right) = \mathrm{e}^{-F(\boldsymbol{\beta}, \boldsymbol{\theta})} \mathbf{E}\left(\mathrm{e}^{-\alpha Z_\infty} \mathbb{1}_{\{Z_\infty > 0\}} \right). \tag{2.3}$$

In particular, $(\widehat{\mu}_n(\beta_1), ..., \widehat{\mu}_n(\beta_l))$ converge in law, when $n \to \infty$ to some random variable $(\widehat{\mu}_\infty(\beta_1), ..., \widehat{\mu}_\infty(\beta_l))$ independent of Z_∞ conditionally on $\{Z_\infty > 0\}$.

(ii) If $(a,b) \in \mathbb{R}^2$ respect $\mathrm{e}^a + \mathrm{e}^b = 1$. Let $\mathbb{T}^{a,b}$ the genealogical tree formed by two independents branching random walks starting respectively from a and b. Recalling that on

the set of non-extinction $Z_\infty := \lim_{n\to\infty} \sum_{|u|=n,\, u\in\mathbb{T}^{a,b}} V(u)\mathrm{e}^{-V(u)}$ *exists and is a.s positive. Let* $\mu_n^{a,b}$ *the point process formed by the particles* $\{V(u)-\log Z_\infty,\, u\in\mathbb{T}^{a,b},\, |u|=n\}$. *Define* $\mu_n^{a,b}(\beta)=\int_{\mathbb{R}}\mathrm{e}^{-\beta x}d\mu_n^{a,b}(x)$. *Then,*

$$\lim_{n\to\infty}\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\widehat{\mu}_n^{a,b}(\beta_i)}\mathbb{1}_{\{Z_\infty>0\}}\right)=\mathrm{e}^{-F(\boldsymbol{\beta},\boldsymbol{\theta})}\mathbf{P}(Z_\infty>0) \tag{2.4}$$

As a consequence, $(\widehat{\mu}_n^{a,b}(\beta_1),...,\widehat{\mu}_n^{a,b}(\beta_l))$ *converges in law to* $(\widehat{\mu}_\infty(\beta_1),...,\widehat{\mu}_\infty(\beta_l))$ *when* $n\to\infty$.

We are now in possession of sufficient tools to demonstrate the main theorem.

2.2 Proof of the main theorem

The main theorem follows from the subsequent lemma which is an easy consequence of Lemma 5.1 in [17].

Lemma 2.5 *Let* $(\mu_n)_{n\in\mathbb{N}}$ *a sequence of point process on* $\mathbb{R}$. *Suppose that*

(i) For any polynomial function Q *such that* $Q(0)=0$, $\int_{\mathbb{R}}Q(\mathrm{e}^{-x})d\mu_n(x)\overset{(d)}{\to}$ *some random variable*

(ii) For any $\epsilon>0$ *there exists* $A>0$ *such that* $\mathbf{P}\left(\int_{\mathbb{R}}\mathrm{e}^{-2x}d\mu_n(x)>A\right)\le\epsilon$ *for all* $n\ge 0$.

(iii) For any $\epsilon>0$ *there exists* $b>0$ *such that* $\mathbf{P}\left(\mu_n([-\infty,-b]>0\right)\le\epsilon$.

Then μ_n *converge in law to some point process.*

See appendix for the proof of Lemma 2.5.

To obtain the part of the existence of limit in theorem 1.1, it's enough to check that μ_n satisfies (i), (ii), (iii):

(i) is exactly (2.3), recalling that if $Q(X)=\sum_{i=1}^{l}\theta_iX^i$, then $\int_{\mathbb{R}}Q(\mathrm{e}^{-x})d\mu_n(x)=\sum_{i=1}^{l}\theta_i\widehat{\mu}_n(\beta_i)$.

(ii) follows from Proposition 2.1. Indeed

$$\begin{aligned}\mathbf{P}\left(\int_{\mathbb{R}}\mathrm{e}^{-2x}d\mu_n(x)>A;\,Z_\infty>0\right) &= \mathbf{P}\left(\widetilde{W}_{n,2}\ge\frac{A}{\mathrm{e}^{\beta\log Z_\infty}};\,Z_\infty>0\right)\\ &\le \mathbf{P}\left(\widetilde{W}_{n,2}\ge\frac{A}{\mathrm{e}^{\beta M}}\right)+\mathbf{P}\left(Z_\infty\ge M\right),\end{aligned}$$

which go to 0 when A then M go to ∞.

(iii) is a consequence of Theorem 1.1 [3].

The independence between μ_∞ and Z_∞ conditionally on $\{Z_\infty>0\}$ follows from (2.3), see theorem 2.4. It remains to describe μ_∞. To this end, we firstly recall some results on the superposable measures. Let $\mathcal{N}$ be the space of bounded finite counting measures on $\mathbb{R}$. If a random measure $\mathcal{L}$ takes values in $\mathcal{N}$, we call $\mathcal{L}$ a *point process*. For every $x \in \mathbb{R}$ define the translation operator $T_x : \mathcal{N} \to \mathcal{N}$, by $(T_x\mu)(A) = \mu(A - x)$ for every Borel set $A \subset \mathbb{R}$. Denote equality in law by $\overset{(d)}{=}$. Let $\mathcal{L}'$ be an independent copy on $\mathcal{L}$. We say that $\mathcal{L}$ is *superposable*, if

$$T_\alpha \mathcal{L} + T_\beta \mathcal{L} \overset{(d)}{=} \mathcal{L}, \text{ for every } \alpha, \beta \in \mathbb{R} \text{ such that } \mathrm{e}^\alpha + \mathrm{e}^\beta = 1$$

According to [23], $\mathcal{L}$ is a superposable process if and only $\mathcal{L}$ can be obtained as follows:

(a) Define $\mathcal{P}$ a Poisson process on $\mathbb{R}$, with intensity measure $\lambda \mathrm{e}^x dx$

(b) For each atom x of $\mathcal{P}$, we attach a point process $x + \mathcal{D}^{(x)}$ where $\mathcal{D}^{(x)}$ are independent copies of a certain point process $\mathcal{D}$ which respect

$$\int^\infty \mathbf{P}(\mathcal{D}(A - x) > 0)\mathrm{e}^{-x} dx < \infty.$$

(c) $\mathcal{L}$ is the superposition of all the point processes $x + \mathcal{D}^{(x)}$, i.e, $\mathcal{L} := \{x + y : x \in \mathcal{P}, y \in \mathcal{D}^{(x)}\}$.

In view of (a),(b), (c), the superposability of μ_∞ is a consequence of (2.4) in theorem 2.3. Theorem 1.1 follows. $\square$

3 Proof of theorem 2.4 by admitting Proposition 2.2

For $z \in \mathbb{T}$ we call trajectory of z all the positions of ancestor of z, i.e the vector $(V(z_1), ..., V(z_n) = V(z))$. Let $l \in \mathbb{N}$. We fix vectors $\boldsymbol{\beta} := (\beta_1, ..., \beta_l)$ and $\boldsymbol{\theta} := (\theta_1, ..., \theta_l)$. To simplify notations we denote $I := \{1, ..., t\}$ and for $I_k \subset I$, $B^{I_k} := \prod_{j \in I_k} \beta_j$ and $\Theta^{I_k} := \prod_{j \in I_k} \theta_j$. For $A \in \mathbb{N}$, let $\mathcal{Z}[A]$ denote the set of particles absorbed at level A, i.e.

$$\mathcal{Z}[A] := \{u \in T : V(u) \geq A, V(u_k) < A \ \forall k < |u|\},$$

and $Z_A := \sum_{u \in \mathcal{Z}[A]} V(u)\mathrm{e}^{-V(u)}$. By Proposition A.1 [3] we know that

$$\lim_{A \to \infty} Z_A = Z_\infty \qquad a.s. \tag{3.1}$$

Fix $x \in \mathbb{R}$ and let $\epsilon > 0$. For any $A > 0$, we have for n large enough

$$\mathbf{P}\left(\exists u \in \mathcal{Z}[A] : |u| \geq (\log n)^{10} \text{ or } V(u) \geq \frac{1}{2} \log\log n\right) \leq \epsilon,$$

Take $A, L > 0$. Let $\Xi_A(n, L) = \Xi_A := \{\max_{u \in \mathcal{Z}[A]} |u| \leq (\log n)^{10}, \max_{u \in \mathcal{Z}[A]} V(u) \leq A + \frac{1}{2} \log\log n, \log Z_A \in [-L, L]\}$ we observe that probability of Ξ_A increase to 1 when n then L go to infinity. On Ξ_A note that

$$\widetilde{W}_{n,\beta} := \sum_{u\in\mathcal{Z}[A]} \mathrm{e}^{-\beta V(u)}\widetilde{W}^u_{n,\beta}, \qquad \text{with} \quad W^u_{n,\beta} := \sum_{z>u,|z|=n} \mathrm{e}^{-\beta(V(z)-V(u))}.$$

Recall that $\widetilde{W}^u_{n,\beta}$ means $n^{\frac{3}{2}\beta}W^u_{n,\beta}$. Write $\mathbf{E}(Y;\Xi) := \mathbf{E}(Y1_\Xi)$ for any nonnegtaive r.v. Y and event Ξ. By Markov property, we have

$$\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i \log Z_A}\widetilde{W}_{n,\beta_i}}\mathrm{e}^{-\alpha Z_A}\mathbb{1}_{\{Z_A>0\}};\Xi_A\right) = \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\sum_{u\in\mathcal{Z}[A]}\mathrm{e}^{-\beta_i(V(u)+\log Z_A)}\widetilde{W}^u_{n,\beta_i}}\mathrm{e}^{-\alpha Z_A}\mathbb{1}_{\{Z_A>0\}};\Xi_A\right)$$

$$= \mathbf{E}\left(\prod_{u\in\mathcal{Z}[A]}\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i(V(u)+\log Z_A)}\widetilde{W}_{n-|u|,\beta_i}}\Big| u\in\mathcal{Z}[A], Z[A]\right)\mathrm{e}^{-\alpha Z_A}\mathbb{1}_{\{Z_A>0\}};\Xi_A\right)$$

$$= \mathbf{E}\left(\prod_{u\in\mathcal{Z}[A]}\Phi(n, V(u)+\log Z[A], |u|)\mathrm{e}^{-\alpha Z_A}\mathbb{1}_{\{Z_A>0\}};\Xi_A\right).$$

with

$$\Phi(n,t,p) := \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\mathrm{e}^{-\beta_i t}\widetilde{W}_{n-p,\beta_i}}\right), \qquad n\in\mathbb{N},\ t\in\mathbb{R},\ p\in[0,n].$$

We firstly establish a Proposition to estimate the amount under the product.

Proposition 3.1 *$\forall\epsilon>0, L>0$ there exists $N(\epsilon), A(\epsilon)>0$ such that for any $n\geq N(\epsilon)$, $t\in[A,\frac{1}{2}\log\log n]$, $s\in[-L,L]$, $p\leq(\log n)^{10}$,*

$$\lim_{n\to\infty}\left|\Phi(n,t+s,p) - \left(1-F(\boldsymbol{\beta},\boldsymbol{\theta})t\mathrm{e}^{-(t+s)}\right)\right| \leq \epsilon t\mathrm{e}^{-t},$$

with

$$F(\boldsymbol{\beta},\boldsymbol{\theta}) := \sum_{k=1}^{l}(-1)^{k+1}g_k(\boldsymbol{\beta},\boldsymbol{\theta}), \tag{3.2}$$

$$g_k(\boldsymbol{\beta},\boldsymbol{\theta}) := \sum_{i_1<\ldots<i_k}\theta_{i_1}\ldots\theta_{i_k}\int_{\mathbb{R}^k}\mathrm{e}^{\sum_{j=1}^{k}-\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}}c_0\chi(\boldsymbol{\beta},-\boldsymbol{y})d\boldsymbol{y}. \tag{3.3}$$

The functions g_k are continuous at 0.

Before the proof of Proposition 3.1, we begin with a technical Lemma:

Lemma 3.2 *The functions g_k are well defined, function F is non negative. There exists a constants c_4 such that for all $x\geq 1$, $k\leq l$, $g_k(\boldsymbol{\beta},\boldsymbol{\theta})\leq c_4\sum_{i_1<\ldots<i_k}\min_{j\leq k}\theta^{1/\beta_{i_j}}$.*

Proof of Lemma 3.2. The first assertion is an easy consequence of Proposition 2.2 (iii) and the inequality

$$\sum_{i_1<\ldots<i_k} \theta_{i_1}\ldots\theta_{i_k} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}} \min_{j\in[1,k]} \mathrm{e}^{-y_{i_j}}\, d\boldsymbol{y} < +\infty.$$

The second is also simple because F is a sum of decreasing alternating terms with the first which is non negative. It remains to show the continuity at 0. Observe that

$$\begin{aligned}
\theta_{i_1}\ldots\theta_{i_k} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}} c_0\chi(\boldsymbol{\beta},-\boldsymbol{y})d\boldsymbol{y} &\leq \theta_{i_1}\ldots\theta_{i_k} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}} c_3 \min_{j\in[1,k]} \mathrm{e}^{-y_{i_j}}\, d\boldsymbol{y} \\
&\leq \theta_{i_1}\ldots\theta_{i_k} \min_{j\in[1,k]} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}} c_3 \mathrm{e}^{-y_i}\, d\boldsymbol{y} \\
&= c_4 \min_{j\in[1,k]} \theta_{i_j}^{1/\beta_{i_j}},
\end{aligned}$$

which goes to 0 as $\theta \to 0$. □

Proof of Proposition 3.1. The function F appears immediately with:

$$\begin{aligned}
&\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i(t+s)}\widetilde{W}_{n-p,\beta_i}}\right) = \mathbf{E}\left(\prod_{i=1}^{l}\left(1-\theta_i\int_0^\infty \mathrm{e}^{-\theta_i y}\mathbb{1}_{\{\widetilde{W}_{n-p,\beta_i}\geq \mathrm{e}^{\beta_i(t+s)}y\}}dy\right)\right) \\
&= \mathbf{E}\Bigg[1-\sum_{i=1}^{l}\theta_i\int_0^\infty \mathrm{e}^{-\theta_i y}\mathbb{1}_{\{\widetilde{W}_{n-p,\beta_i}\geq \mathrm{e}^{\beta_i(t+s)}y\}}dy + \ldots + (-1)^k \sum_{I_k\subset I}\Theta^{I_k} \\
&\times \int_{\mathbb{R}_+^k}\prod_{i_j\in I}\mathrm{e}^{-\theta_{i_j}y_{i_j}}\mathbb{1}_{\{\widetilde{W}_{n-p,\beta_{i_j}}\geq \mathrm{e}^{\beta_{i_j}(t+s)}y\}}d\boldsymbol{y} + \ldots + (-1)^t\Theta^I\int_{\mathbb{R}_+^k}\prod_{j\in I}\mathrm{e}^{-\theta_i y_i}\mathbb{1}_{\{\widetilde{W}_{n-p,\beta_i}\geq \mathrm{e}^{\beta_i(t+s)}y\}}d\boldsymbol{y}\Bigg].
\end{aligned}$$

Then, Fubini's Theorem and the simple change of variable $y_{i_j} = \mathrm{e}^{\beta_{i_j}y_{i_j}}$ provide that

$$\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i(t+s)}\widetilde{W}_{n-p,\beta_i}}\right) = \mathbf{E}\left(1-\sum_{k=1}^{l}(-1)^{k+1}h_k^n(\boldsymbol{\beta},\boldsymbol{\theta},x)\right),$$

with

$$h_k^n(\boldsymbol{\theta},x) := \sum_{I_k\subset I} B^{I_k}\Theta^{I_k}\int_{\mathbb{R}^k}\prod_{i_j\in I_k}\mathrm{e}^{-\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}}\mathbb{1}_{\{\widetilde{W}_{n-p,\beta_{i_j}}\geq \mathrm{e}^{\beta_{i_j}(t+s+y_{i_j})}\}}d\boldsymbol{y}. \tag{3.4}$$

To conclude it remains to prove that *for any* $\epsilon > 0$ *there exist* $A(\epsilon)$ *such that for any* $t \in [A(\epsilon), \frac{1}{2}\log\log n]$, $s \in [-L, L]$ *and* $p \leq (\log n)^{10}$, $\lim_{n\to\infty} |\mathbf{E}(h_k^n|\mathcal{F}_A) - t\mathrm{e}^{-(t+s)} g_k| \leq \epsilon t\mathrm{e}^{-t}$.

By Proposition 2.2 (iii), it is possible to define

$$c_5 := c_0 \max_{k\leq t} \max_{I_k \subset I} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j} \mathrm{e}^{\beta_{i_j} y_{i_j}} + \beta_{i_j} y_{i_j}} \chi(\boldsymbol{\beta}, -\boldsymbol{y}) d\boldsymbol{y} < \infty, \tag{3.5}$$

and $K > 0$ sufficiently large such that for all $(i_1, ..., i_k)$,

$$\int_{([-K,K]^k)^c} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j} \mathrm{e}^{\beta_{i_j} y_{i_j}} + \beta_{i_j} y_{i_j}} [c_1 \min_{j\leq k} \max(1, \mathrm{e}^{-y_{i_j}}) + c_0 \chi(\boldsymbol{\beta}, -\boldsymbol{y} - s)] d\boldsymbol{y} \leq \frac{\epsilon}{c_6 2^t \mathrm{e}^L} \tag{3.6}$$

with $c_6 := \max_{I_k \subset I,\, k\leq t} \Theta^{I_k} B^{I_k} = \max_{I_k \subset I,\, k\leq t} \Theta^{I_k} B < \infty$.

Now we can use Proposition 2.2. The idea is to cut the integral into two parts, one on the hypercube and the other on its complement:

$$h_k^n(\boldsymbol{\theta}, x) = \sum_{I_k \subset I} B^{I_k} \Theta^{I_k} \left[\int_{[-K,K]^k} ... + \int_{([-K,K]^k)^c} ... \right].$$

There exists $A = A(K, \frac{\epsilon}{4M_1 2^t \kappa}) > 0$ such that for any $t \in [A, \frac{1}{2}\log\log n]$, $s \in [-L, L]$, $I_k \subset I$, $k \leq l$, $(y_{i_1} ... y_{i_k}) \in [-K, K]^k$, $\exists N(\frac{\epsilon}{4M_1 2^t \alpha}, \boldsymbol{y})$ such that $\forall n > N$

$$\begin{aligned} &\left| \frac{\mathrm{e}^t}{t} \mathbf{P} \left(\bigcap_{j\leq k} \{ \widetilde{W}_{n-p,\beta_{i_j}} \geq \mathrm{e}^{\beta_{i_j}(t+s+y_{i_j})} \} \right) - c_0 \chi(\boldsymbol{\beta}, -\boldsymbol{y} - s) \right| = \\ &\left| \frac{\mathrm{e}^t}{t} \mathbf{P} \left(\bigcap_{j\leq k} \{ \widetilde{W}_{n-p,\beta_{i_j}} \geq \mathrm{e}^{\beta_{i_j}(t+s+y_{i_j})} \} \right) - c_0 \mathrm{e}^{-s} \chi(\boldsymbol{\beta}, -\boldsymbol{y}) \right| \leq \frac{\epsilon}{c_5 c_6 2^l}. \end{aligned}$$

Thus,

$$\begin{aligned} &\limsup_{n\to\infty} \left| \int_{[-K,K]^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j} \mathrm{e}^{\beta_{i_j} y_{i_j}} + \beta_{i_j} y_{i_j}} \left[\mathbf{P} \left(\bigcap_{j\leq k} ... \right) - t\mathrm{e}^{-(t+s)} c_0 \chi(\boldsymbol{\beta}, -\boldsymbol{y}) \right] d\boldsymbol{y} \right| \\ \leq\;& \frac{\epsilon}{c_5 c_6 2^l} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} -\theta_{i_j} \mathrm{e}^{\beta_{i_j} y_{i_j}} + \beta_{i_j} y_{i_j}} t\mathrm{e}^{-t} c_0 \chi(\boldsymbol{\beta}, -\boldsymbol{y}) d\boldsymbol{y} \\ =\;& \frac{\epsilon}{c_5 c_6 2^l} t\mathrm{e}^{-t} \int_{\mathbb{R}^k} \mathrm{e}^{\sum_{j=1}^{k} \theta_{i_j} \mathrm{e}^{\beta_{i_j} y_{i_j}} + \beta_{i_j} y_{i_j}} c_0 \chi(\boldsymbol{\beta}, -\boldsymbol{y}) d\boldsymbol{y} \\ \leq\;& \frac{\epsilon}{2^l c_6} t\mathrm{e}^{-t}. \end{aligned}$$

It remains to bound the integral on $[-K,K]^c$, i.e to control the following $\limsup$:

$$(3.7)\qquad \limsup_{n\to\infty}\left|\int_{([-K,K]^k)^c} \mathrm{e}^{\sum_{j=1}^{k}-\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}}\left[\mathbf{P}\left(\bigcap_{j\le k}...\right)+t\mathrm{e}^{-(t+s)}c_0\chi(\boldsymbol{\beta},-\boldsymbol{y})\right]d\boldsymbol{y}\right|.$$

According to Proposition 2.2, for n large enough, we have

$$\begin{aligned}\mathbf{P}(\bigcap_{j\le k}...) &\le \min_{j\le k}\mathbf{P}\left(\widetilde{W}_{n-p,\beta_{i_j}}\ge \mathrm{e}^{\beta_{i_j}(t+s+y_{i_j})}\right)\\ &\le \min_{j\le k}\begin{cases} c_1t\mathrm{e}^{-(t+s)} & \text{if}\quad y_{i_j}\ge 0\\ \min(c_1t\mathrm{e}^{-(t+s)-y_{i_j}},1) & \text{if}\quad y_{i_j}\le 0\end{cases}.\end{aligned}$$

As a consequence,

$$\begin{aligned}(3.7) &\le \left(\int_{([-K,K]^k)^c}\mathrm{e}^{\sum_{j=1}^{k}-\theta_{i_j}\mathrm{e}^{\beta_{i_j}y_{i_j}}+\beta_{i_j}y_{i_j}}\left[c_1\min_{j\le k}\max(1,\mathrm{e}^{-y_{i_j}})+c_0\chi(\boldsymbol{\beta},-\boldsymbol{y})\right]d\boldsymbol{y}\right)t\mathrm{e}^{-(t+s)}\\ &\le \frac{\mathrm{e}^{-s}\epsilon}{\mathrm{e}^{L}c_6 2^l}t\mathrm{e}^{-t}.\end{aligned}$$

Finally we have demonstrated that:

$$(3.8)\qquad \limsup_{n\to\infty}\left|\mathbf{E}(h_k^n(\boldsymbol{\theta},x))-t\mathrm{e}^{-(t+s)}g_k(\boldsymbol{\theta},x)\right|\le\frac{\epsilon}{l}t\mathrm{e}^{-t},$$

then

$$\left|\mathbf{E}\left(\sum_{k=1}^{l}h_k^n(\boldsymbol{\theta},x)\right)-(t\mathrm{e}^{-(t+s)}\sum_{k=1}^{l}g_k(\boldsymbol{\theta},x))\right|\le\epsilon t\mathrm{e}^{-t},$$

and the Proposition is proved. □

We are now able to obtain

Proposition 3.3 *For all* $(\theta_1,...,\theta_t,\beta_1,...,\beta_t)\in\mathbb{R}_+^t\times(1,\infty)^k$ *and* $\alpha\in\mathbb{R}_+$

$$(3.9)\qquad \lim_{A\to\infty}\lim_{n\to\infty}\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\mathrm{e}^{-\beta_i\log Z_A}\widetilde{W}_{n,\beta_i}}\mathbb{1}_{\{Z_A>0\}}\mathrm{e}^{-\alpha Z_A}\right)=\mathrm{e}^{-F(\boldsymbol{\beta},\boldsymbol{\theta})}\mathbf{P}(\mathrm{e}^{-\alpha Z_\infty};\,Z_\infty>0).$$

Proof of proposition 3.3. Let $\epsilon > 0$, for L large enough such that $\mathbf{P}\left(\log Z_A \notin [-L, L],\ \forall A \geq 0\right) \leq \epsilon$, we have

$$\limsup_{A\to\infty} \limsup_{n\to\infty} \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l} \theta_i \mathrm{e}^{-\beta_i \log Z_A} \widetilde{W}_{n,\beta_i}} \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}} \right) \leq$$

$$\limsup_{A\to\infty} \limsup_{n\to\infty} \mathbf{E}\left(\prod_{u\in\mathcal{Z}[A]} \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l} \theta_i \mathrm{e}^{-\beta_i (V(u)+\log Z_A)} \widetilde{W}_{n-|u|,\beta_i}} \Big| u \in \mathcal{Z}[A], Z_A \right) \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}},\ \Xi_A \right) + \epsilon.$$

By the dominated convergence theorem, we deduce from Proposition 3.1 that

$$\begin{aligned} &\limsup_{A\to\infty} \limsup_{n\to\infty} \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l} \theta_i \mathrm{e}^{-\beta_i \log Z_A} \widetilde{W}_{n,\beta_i}} \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}} \right) \\ &\leq \mathbf{E}\left(\lim_{A\to\infty} \prod_{u\in\mathcal{Z}[A]} (1 - F(\boldsymbol{\beta}, \boldsymbol{\theta}) V(u) \mathrm{e}^{-(V(u)+\log Z_A)} - \epsilon V(u) \mathrm{e}^{-V(u)}) \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}} \right) + \epsilon. \end{aligned}$$

By (3.1),

$$\begin{aligned} & \lim_{A\to\infty} \sum_{u\in\mathcal{Z}[A]} \log\left(1 - F(\boldsymbol{\beta}, \boldsymbol{\theta}) V(u) \mathrm{e}^{-(V(u)+\log Z_A)} - \epsilon V(u) \mathrm{e}^{-V(u)}\right) \mathbb{1}_{\{Z_A>0\}} \\ = & \lim_{A\to\infty} \left(-F(\boldsymbol{\beta}, \boldsymbol{\theta}) \frac{\sum_{u\in\mathcal{Z}[A]} V(u) \mathrm{e}^{-V(u)}}{Z_A} \mathbb{1}_{\{Z_A>0\}} + \epsilon \sum_{u\in\mathcal{Z}[A]} V(u) \mathrm{e}^{-V(u)} \mathbb{1}_{\{Z_A>0\}} \right) \\ = & (-F(\boldsymbol{\beta}, \boldsymbol{\theta}) + \epsilon Z_\infty) \mathbb{1}_{\{Z_\infty>0\}}, \end{aligned}$$

which is equivalent to say that

$$\begin{aligned} & \lim_{A\to\infty} \prod_{u\in\mathcal{Z}[A]} (1 - F(\boldsymbol{\beta}, \boldsymbol{\theta}) V(u) \mathrm{e}^{-(V(u)+\log Z_A)} - \epsilon V(u) \mathrm{e}^{-V(u)}) \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}} \\ & = \mathrm{e}^{-F(\boldsymbol{\beta}, \boldsymbol{\theta}) + \epsilon Z_\infty} \mathrm{e}^{-\alpha Z_\infty} \mathbb{1}_{\{Z_\infty>0\}}. \end{aligned}$$

It follows that

$$\limsup_{A\to\infty} \limsup_{n\to\infty} \mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l} \theta_i \mathrm{e}^{-\beta_i \log Z_A} \widetilde{W}_{n,\beta_i}} \mathrm{e}^{-\alpha Z_A} \mathbb{1}_{\{Z_A>0\}} \right) \leq \mathrm{e}^{-F(\boldsymbol{\beta}, \boldsymbol{\theta})} \mathbf{E}\left(\mathrm{e}^{(\epsilon-\alpha) Z_\infty} \mathbb{1}_{\{Z_\infty>0\}} \right) + \epsilon.$$

Letting $\epsilon \to 0$ gives the upper bound. The lower bound follows from the same way. $\square$

Proof of Theorem 2.4. For (i) it suffices to show

$$\lim_{A\to\infty}\lim_{n\to\infty}\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i \log Z_A}\widetilde{W}_{n,\beta_i}}\mathrm{e}^{-\alpha Z_A}\mathbb{1}_{\{Z_A>0\}}\right)=\lim_{n\to\infty}\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\widehat{\mu}_n(\beta_i)}\mathrm{e}^{-\alpha Z_\infty}\mathbb{1}_{\{Z_\infty>0\}}\right).$$

Let $\epsilon \geq 0$. For any $A>0$ and $m>0$ sufficiently large

$$\begin{aligned}&\left|\mathbf{E}\left(\mathrm{e}^{-\sum_{i=1}^{l}\theta_i \mathrm{e}^{-\beta_i \log Z_A}\widetilde{W}_{n,\beta_i}}-\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\widehat{\mu}_n(\beta_i)}; Z_\infty>0\right)\right| \leq \mathbf{P}\left(\max_{1\leq i\leq l}\left|\frac{1}{(Z_\infty)^{\beta_i}}-\frac{1}{(Z_A)^{\beta_i}}\right|\geq \frac{\epsilon}{m},\, Z_\infty>0\right)\\ &+P(\widetilde{W}_{n,\beta}>m)+\mathbf{E}\left(\left[\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\frac{\widetilde{W}_{n,\beta_i}}{(Z_A)^{\beta_i}}}-\mathrm{e}^{-\sum_{i=1}^{l}\theta_i\frac{\widetilde{W}_{n,\beta_i}}{(Z_\infty)^{\beta_i}}}\right]\mathbb{1}_{\left\{Z_\infty>0,\ \max_{1\leq i\leq l}\left|\frac{1}{(Z_\infty)^{\beta_i}}-\frac{1}{(Z_A)^{\beta_i}}\right|\leq\frac{\epsilon}{m},\widetilde{W}_{n,\beta}\leq m\right\}}\right)\\ &\leq (l+2)\epsilon,\end{aligned}$$

by the Proposition 2.1. It remains to show(ii). With the notations $Z^a_\infty := \lim_{n\to\infty}\sum_{|u|,u\in\mathbb{T}^a}V(u)\mathrm{e}^{-V(u)}$ and $Z^b_\infty := \lim_{n\to\infty}\sum_{|u|,u\in\mathbb{T}^b}V(u)\mathrm{e}^{-V(u)}$, it's clear that

$$\mathbb{1}_{\{Z_\infty>0\}}=\mathbb{1}_{\{Z^a_\infty>0\}}\mathbb{1}_{\{Z^b_\infty=0\}}+\mathbb{1}_{\{Z^a_\infty=0\}}\mathbb{1}_{\{Z^b_\infty>0\}}+\mathbb{1}_{\{Z^a_\infty>0\}}\mathbb{1}_{\{Z^b_\infty>0\}}.$$

On the two first events there is nothing to prove. On the third we can repeat exactly the same proof as (2.3) by keeping in mind Corollary 2.3. The proof of Theorem 2.4 is complete. □

4 Results for the killed Branching Random walk

4.1 The many-to-one formula and Lyons' change of measure

Under (1.1), there exists a centered random walk $(S_n, n\geq 0)$ such that for any $n\geq 1$ and any measurable function $g :\in \mathbb{R}^n \to [0,\infty)$,

$$\mathbf{E}\left\{\sum_{|z|=n} g(V(z_1),...,V(z_n))\right\}=\mathbf{E}\left\{\mathrm{e}^{S_n}g(S_1,...,S_n)\right\}. \tag{4.1}$$

In particular, under (1.3), S_1 has a finite variance $\sigma^2 := \mathbf{E}[S_1^2]=\mathbf{E}\left[\sum_{|z|=1}V(z)^2\mathrm{e}^{-V(z)}\right]$. We can see the so-called many-to-one formula (4.1) as a consequence of Proposition 4.1 below. We introduce the *additive martingale*

$$W_n := \sum_{|z|=n}\mathrm{e}^{-V(z)}, \tag{4.2}$$

and define a probability measure $\mathbf{Q}$ such that for any $n \geq 0$,

$$\mathbf{Q}\big|_{\mathcal{F}_n} := W_n \bullet \mathbf{P}\big|_{\mathcal{F}_n}, \tag{4.3}$$

where $\mathcal{F}_n$ denotes the sigma-algebra generated by the positions $(V(z), |z| \leq n)$ up to time n. To give the description of the branching random walk under $\mathbf{Q}$, we introduce the point process $\hat{L}$ with Radon-Nykodim derivative $\int \mathrm{e}^{-x}\mathcal{L}(dx)$ with respect to the law of L, and we define the following process. At time 0, the population is composed of one particle w_0 located at $V(w_0) = 0$. Then, at each step n, particles of generation n die and give birth to independent point processes distributed as L except for the particle w_n which generates a point process distributed as $\hat{L}$. The particle w_{n+1} is chosen among the children z of w_n with probability proportional to $\mathrm{e}^{-V(z)}$. We denote by $\mathcal{B} := (V(z))$ the family of the positions of this system. We still call $\mathbb{T}$ the genealogical tree of the process, so that $(w_n)_{n\geq 0}$ is a ray of $\mathbb{T}$, which we will call the *spine*. This change of probability was used in [21], see also [15]. We refer to [22] for the case of the Galton-Watson tree, to [12] for the analog for the branching Brownian motion, and to [8] for the spine decomposition in various types of branching.

Proposition 4.1 *(i)Under $\mathbf{Q}$, the branching random walk has the distribution of $\mathcal{B}$.*

(ii)For any $|z| = n$, we have

$$\mathbf{Q}\left\{w_n = z \Big| \mathcal{F}_n\right\} = \frac{\mathrm{e}^{-V(z)}}{W_n}. \tag{4.4}$$

(iii) The spine process $(V(w_n), n \geq 0)$ has distribution of the centered random walk $(S_n, n \geq 0)$ under $\mathbf{Q}$ satisfying (4.1).

Before closing this subsection, we collect some elementary facts about the centered random walks with finite variance:

Lemma 4.2 *(i) There exists a constant $\alpha_1 > 0$ such that for any $x \geq 0$ and $n \geq 1$,*

$$\mathbf{P}_x\left(\min_{j\leq n} S_j \geq 0\right) \leq \alpha_1(1+x)n^{-\frac{1}{2}}. \tag{4.5}$$

(ii) There exists a constant $\alpha_2 > 0$ such that for any $b \geq a, x \geq 0$ and $n \geq 1$,

$$\mathbf{P}_x\left(S_n \in [a,b], \min_{j\leq n} S_j \geq 0\right) \leq \alpha_2(1+x)(1+b-a)(1+b)n^{-\frac{3}{2}}. \tag{4.6}$$

(iii) Let $0 < \Lambda < 1$. There exists a constant $\alpha_3 = \alpha_3(\Lambda) > 0$ such that for any $b \geq a, x \geq 0, y \in \mathbb{R}$

$$\mathbf{P}_x\left(S_n \in [y+a, y+b], \min_{j\leq n} S_j \geq 0, \min_{\Lambda n\leq j\leq n} S_j \geq y\right) \tag{4.7}$$

$$\leq \alpha_3(1+x)(1+b-a)(1+b)n^{-\frac{3}{2}}.$$

See [19] for (4.5). The estimates (4.6) and (4.7) are for example Lemmas A.1 and A.3 in [4]. In our case (S_n) is the centered random walk under $\mathbf{P}$, with finite variance $\mathbf{E}[S_1^2] = \sigma$ which appears in the many-to-one Lemma. We introduce its renewal function $R(x)$ which is zero if $x < 0$, 1 if $x = 0$, and $x > 0$

$$R(x) := \sum_{k\geq 0} \mathbf{P}\left(S_k \geq -x,\, S_k < \min_{0\leq j\leq k-1} S_j\right). \tag{4.8}$$

It is known that there exists $c_0 > 0$ (the constant of Proposition 2.2) such that

$$\lim_{x\to\infty} \frac{R(x)}{x} = c_0. \tag{4.9}$$

4.2 Definition of M_n^{kill} and $W_{n,\beta}^{kill}$

Following Aïdékon [3], to determinate the tail of distribution of the partition function of the branching random walk, we study the same amount for the killed branching random walk:

$$W_{n,\beta}^{kill} := \sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{k\leq n} V(z_k)\geq 0\}}.$$

Let us adopt some notation of [3]. We denote the minimum of the killed branching random walk

$$M_n^{kill} := \inf\{V(z), |z| = n, V(z_k) \geq 0,\ \forall 0 \leq k \leq n\}.$$

and $m^{kill,(n)}$ a vertex chosen uniformly in the set

$$\{V(z), |z| = n, V(z) = M_n^{kill},\ \forall 0 \leq k \leq n\}.$$

For $|z| = n$, we say that $z \in Z^{x,L}$ if

$$V(z) \in I_n(x),\ \min_{k\leq n} V(z_k) \geq 0,\ \min_{\frac{n}{2}<k\leq n} V(z_k) \geq a_n(x+L).$$

As the typical order of M_n^{kill} is $\frac{3}{2}\log n$, it will be convenient to use the following notation, for $x \geq 0$:

$$a_n(x) := \frac{3}{2}\log n - x, \tag{4.10}$$

$$I_n(x) := [a_n(x) - 1, a_n(x)]. \tag{4.11}$$

The choice of an interval of length 1 is arbitrary and could be change. Our goal is the following result:

Proposition 4.3 *For any $k \geq 0$ there exists*

$$c : \begin{array}{l} (1,+\infty)^k, \mathbb{R}^{k+1} \to \mathbb{R}_+ \\ (\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \mapsto c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \end{array} \tag{4.12}$$

which satisfies

(i) For any $K > 0$ there exists $c_K, \alpha_K > 0$ such that for any $j \geq 0$ $\boldsymbol{\delta} \in [-K, K]^k$

$$c(\boldsymbol{\beta}, \boldsymbol{\delta}, -j) \leq c_K \mathrm{e}^{-\alpha_K j}, \qquad c(\boldsymbol{\beta}, \boldsymbol{\delta}, j) \leq c_K \mathrm{e}^{-\alpha_K j}. \tag{4.13}$$

(ii)The restriction $(\boldsymbol{\delta}, \Delta) \mapsto c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta))$ is continuous for any $\boldsymbol{\beta}$.

(iii)$\forall \boldsymbol{\beta}, \boldsymbol{\delta}$ and Δ, $c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) = \mathrm{e}^{\Delta} c(\boldsymbol{\beta}, \boldsymbol{\delta} - \Delta, 0)$.

(iv)For all $K, \epsilon > 0$, there exists $A(K, \epsilon) > 0$ such that for any $(\delta_1, ..., \delta_k, \Delta) \in [-K, K]^{k+1}$, $\exists N(\epsilon, \boldsymbol{\delta}, \Delta)$ such that $\forall n > N, x \in [A, A + \frac{1}{2} \log n)]$

$$\left| \mathrm{e}^x \mathbf{P} \left(\bigcap_{j \leq k} \{ \widetilde{W}^{kill}_{n,\beta_j} \geq \mathrm{e}^{\beta_j (x - \delta_j)} \};\ M_n^{kill} \in I_n(x - \Delta) \right) - c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon. \tag{4.14}$$

(v) We deduce immediately that on the same conditions

$$\left| \mathrm{e}^x \mathbf{P} \left(\bigcap_{j \leq k} \{ \widetilde{W}^{kill}_{n,\beta_j} \geq \mathrm{e}^{\beta_j (x - \delta_j)} \};\ M_n^{kill} \leq \frac{3}{2} \log n - (x - \Delta) \right) - \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon, \tag{4.15}$$

with $\chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) := \sum_{j \geq 0} \mathrm{e}^{\Delta - j} c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta - j)$.

Remark: Obviously the reader expects $\chi(\boldsymbol{\beta}, \boldsymbol{\delta}, -\infty) = \chi(\boldsymbol{\beta}, \boldsymbol{\delta})$, and will see that it's true.

(i) (ii) and (iii) are necessary for prove (iv) which is the heart of the Proposition. Reader will note the great similarity between this result and the Proposition 3.1 [3]. It is not surprising, indeed first we note that $\{\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x}\} = \{-\log W^{kill}_{n,\beta} \leq a_n(x)\}$ and two, the most important term of $\widetilde{W}^{kill}_{n,\beta}$ is one provided by M_n^{kill}. Our prove consist then to verify that the method of [3] run in our case.

Finally we mention here a very useful Lemma stated in [3].

Lemma 3.5, Aïdékon *There exists $c_{aid} > 0$ such that for any $n \geq 1$ and $x \geq 0$*

$$\mathrm{e}^x \mathbf{P} \left(M_n^{kill} \leq \frac{3}{2} \log n - x \right) \leq c_{aid}$$

4.3 First result about $W^{kill}_{n,\beta}$

Two subsequent Lemma show that on the set $\{\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x}\}$, the non-negligible contribution to $\widetilde{W}_{n,\beta}$ are provided by the particles z whose

(i)the path satisfies some conditions,

(ii)the final position $V(z)$ isn't to large.

Lemma 4.4 *There exists $c_7, c_8 > 0$ such that $\forall x \in \mathbb{R}$, $y \geq 0$, $n \geq 1$,*

$$\text{(4.16)} \quad \mathbf{P}_y\left(\sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{k\geq n} V(z_k)\geq 0,\ \min_{\frac{n}{2}<k\geq n} V(z_k)\leq a_n(x+L)\}} \geq \frac{\mathrm{e}^{\beta x}}{n^{\frac{3}{2}\beta}}\right) \leq c_7(1+y)\mathrm{e}^{-c_8 L}\mathrm{e}^{-y}\mathrm{e}^{-x}.$$

Remark: The conditions on the paths are those proposed by Aïdékon [3], moreover the proof requires Lemma 3.3 of [3].
Proof of lemma 4.4. We denote $\mathbf{P}_{(4.16)}(y)$ the probability of (4.16) and by $\mathbf{P}_{(4.17)}(y)$ the probability

$$\text{(4.17)} \qquad \mathbf{P}_y\left(\exists|z| = n; V(z) \leq a_n(x), \min_{k\in\{0,\dots,n\}} V(z_k) \geq 0, \min_{k\in\{n/2,\dots,n\}} V(z_k) \leq a_n(x+L)\right)$$

which appears in Lemma 3.3 [3]. Lemma 3.3 says that $\mathbf{P}_{(4.17)}(y) \leq c_9(1+y)\mathrm{e}^{-c_{10}}\mathrm{e}^{-y-x}$. We need some surgery on the path. For $|z| = n$, $j \geq 0$, $\frac{n}{2} < k \leq n$ and $L' \geq L$ we define the event

$$\text{(4.18)} \quad E^j_{k,L'}(z) := \{\min_{l\leq n} V(z_l) \geq 0,\ V(z_k) = \min_{\frac{n}{2}<l\leq n} V(z_l) \in I_n(x+L'),\ V(z_n) \in I_n(x)+j\}.$$

For any $a \geq 0$ define also

$$\text{(4.19)} \qquad F_{a,L'}(z) := \bigcup_{k\in[\frac{n}{2},n-a]} \bigcup_{j\geq 0} E^j_{k,L'}(z), \qquad F^a_{L'}(z) := \bigcup_{k\in[n-a,n]} \bigcup_{j\geq 0} E^j_{k,L'}(z),$$

and similarly (for the centered random walk $(S_n)_{n\geq 0}$)

$$\text{(4.20)} \qquad E^j_{k,L'}(S) := \{\min_{l\leq n} S_l \geq 0,\ S_k = \min_{\frac{n}{2}<l\leq n} S_l \in I_n(x+L'),\ S_n \in I_n(x)+j\},$$

$$\text{(4.21)} \qquad F_{a,L'}(S) := \bigcup_{k\in[\frac{n}{2},n-a]} \bigcup_{j\geq 0} E^j_{k,L'}(S), \qquad F^a_{L'}(S) := \bigcup_{k\in[n-a,n]} \bigcup_{j\geq 0} E^j_{k,L'}(S).$$

We need to estimate $\mathbf{P}_y(E^j_k(S))$ for $\frac{n}{2} < k \leq n-a$. By the Markov property at time k,

$$\mathbf{P}_y(E^j_{k,L'}(S)) \leq \mathbf{P}_y\left(\min_{l\geq k} S_l \geq 0,\ \min_{\frac{n}{2}<l\leq k} S_l \geq a_n(x+L'),\ S_k \in I_n(x+L')\right) \times$$
$$\mathbf{P}\left(S_{n-k} \in [L'-1+j, L'+1+j], \min_{k\leq n-k} S_l \geq 0\right).$$

We know from (4.6) that there exists a constant c_{10} such that

$$\mathbf{P}\left(S_{n-k}\in[L'-1+j,L'+1+j],\ \min_{k\le n}S_l\ge 0\right)\le c_{10}(n-k+1)^{-\frac{3}{2}}(1+L'+j).$$

For the first term, we have to discuss on the value of k. Suppose that $\frac{3}{4}n\le k\le n$, then by (4.7)

$$\mathbf{P}_y\left(\min_{l\ge k}S_l\ge 0,\ \min_{\frac{n}{2}<l\le k}S_l\ge a_n(x+L'),\ S_k\in I_n(x+L')\right)\le c_{11}\frac{1+y}{n^{\frac{3}{2}}}.$$

If $\frac{1}{2}n\le k\le\frac{3}{4}n$ we simply write

$$\begin{aligned}\mathbf{P}_y\left(\min_{l\ge k}S_l\ge 0,\ \min_{\frac{n}{2}<l\le k}S_l\ge a_n(x+L'),\ S_k\in I_n(x+L')\right) &\le \mathbf{P}\left(S_k\in I_n(x+L'),\min_{l\le k}S_l\ge 0\right)\\ &\le c_{12}(1+y)n^{-\frac{3}{2}}\log n.\end{aligned}$$

To resume we have obtained

$$\mathbf{P}_y(E^j_{k,L'}(S))\le\begin{cases} c_{13}\frac{(1+y)\log n}{n^{\frac{3}{2}}(n-k+1)^{\frac{3}{2}}}(1+L'+j) & \text{if}\quad \frac{n}{2}<k\le\frac{3}{4}n\\ c_{13}\frac{1+y}{n^{\frac{3}{2}}(n-k+1)^{\frac{3}{2}}}(1+L'+j) & \text{if}\quad \frac{3}{4}n<k\le n-a\end{cases}. \tag{4.22}$$

Now we can tackle the proof, observe that

$$\mathbf{P}_{(4.16)}(y)\le\mathbf{P}_y\left(\sum_{|z|=n}\mathrm{e}^{-\beta V(z)}(\mathbb{1}_{\{F^a_{L'}(z)\}}+\mathbb{1}_{\{F_{a,L'}(z)\}})\ge\frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}};\ M_n\ge a_n(x)\right)+\mathbf{P}_{(4.17)}(y).$$

By Lyons' change of measure,

$$\begin{aligned}\frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathbf{E}_y\left(\sum_{|z|=n}\mathrm{e}^{-\beta V(z)}\mathbb{1}_{\{F_{a,L'}(z)\}}\right) &= \frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathrm{e}^{-y}\mathbf{E}_y\left[\mathrm{e}^{(1-\beta)S_n}\mathbb{1}_{\{F_{a,L'}(S)\}}\right] && (4.23)\\ &\le \frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathrm{e}^{-y}\sum_{k=n/2}^{n-a}\sum_{j\ge 0}\mathrm{e}^{(1-\beta)(a_n(x)+j)}\mathbf{P}_y(E^j_{k,L'}(S)) && (4.24)\\ &\le c_{14}(1+y)(1+L')\mathrm{e}^{-x-y}a^{-\frac{1}{2}}, && (4.25)\end{aligned}$$

for any $a\ge 1$. We also get

$$\mathbf{P}\left(\sum_{|z|=n}\mathbb{1}_{\{F^a_{L'}(z)\}}\geq 1\right)\leq\sum_{k\in[n-a,n]}\sum_{j\geq 0}\mathbf{P}_y\left(\sum_{|u|=n}\mathbb{1}_{\{E^j_{k,L'}(z)\}}\geq 1\right)$$
$$\leq\sum_{k\in[n-a,n]}\mathbf{P}_y\left(\exists |z|=k:\min_{l\geq k}V(z_l)\geq 0,\ \min_{\frac{n}{2}<l\leq k}V(z_l)\geq a_n(x+L'),\ V(z_k)\in I_n(x+L')\right),$$

which is again by an application of Lyons' change of measure smaller than

$$\sum_{k\in[n-a,n]}c_{15}(1+y)\mathrm{e}^{-x-y-L'}=c_{15}(1+a)(1+y)\mathrm{e}^{-x-y-L'}. \tag{4.26}$$

Now let $a(L+p)_{p\geq 0}=\mathrm{e}^{\alpha(L+p)}$, in combining (4.25) and (4.26) we obtain

$$\mathbf{P}_{(4.16)}(y)\leq\mathbf{P}_{(4.17)}(y)+$$
$$\mathbf{P}_y\left(\sum_{p\geq 0}\sum_{|z|=n}\mathrm{e}^{-\beta(V(z)+y)}\mathbb{1}_{\{F_{a(p),L+p}(z)\}}\geq\frac{\mathrm{e}^{\beta x}}{2n^{\frac{3}{2}\beta}}\right)+\mathbf{P}\left(\sum_{p\geq 0}\sum_{|z|=n}\mathbb{1}_{\{F^{a(p)}_{L+p}(z)\}}\geq 1\right).$$

The two last terms are smaller than

$$\leq\sum_{p\geq 0}\left(\frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathbf{E}\left(\sum_{|z|=n}\mathrm{e}^{-\beta V(z)}\mathbb{1}_{\{F_{a(p),L+p}(z)\}}\right)+\mathbf{P}\left(\sum_{|z|=n}\mathbb{1}_{\{F^{a(p)}_{L+p}(z)\}}\geq 1\right)\right)$$
$$\leq\sum_{p\geq 0}\left(c_{15}(1+a(L+p))(1+y)\mathrm{e}^{-x-y-L+p}+c_{14}(1+y)(1+L+p)\mathrm{e}^{-x-y}a(L+p)^{-\frac{1}{2}}\right)$$
$$\leq c_{16}(1+y)\mathrm{e}^{-c_4L}\mathrm{e}^{-y-z}.$$

The Lemma is proved. □

We have shown that the main contributions to the partition function, are given by the particles whose paths stay above $a_n(x+L)$, after the generation $\frac{n}{2}$. The following natural Lemma says that for A and L large enough

$$W^{A,L}_{n,\beta}(x):=W^{A,L}_{n,\beta}=\sum_{|z|=n}\mathrm{e}^{-\beta V(z)}\mathbb{1}_{\{\min_{k\leq n}V(z_k)+y\geq 0,\ \min_{\frac{n}{2}<k\leq n}V(z_k)+y\geq a_n(x+L),\, V(z)+y\leq a_n(x)+A\}}. \tag{4.27}$$

and $W^{kill}_{n,\beta}$ are almost equal. Hence only particles whose the positions at generation n are less than $a_n(x)+A$ give a non-negligible contribution.

Lemma 4.5 *There exists $c_{17} \geq 0$ such that for all $n \geq 0$:*

$$\mathbf{P}_y \left(\sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{k\leq n} V(z_k)\geq 0,\, \min_{\frac{n}{2}<k\leq n} V(z_k)\geq a_n(x+L),\, V(z)\geq a_n(x)+A\}} \geq \frac{\mathrm{e}^{\beta x}}{n^{\frac{3}{2}\beta}} \right) \leq c_{17}(1+y)\mathrm{e}^{-x-y} L \mathrm{e}^{-A(\beta-1)}.$$

Proof of Lemma 4.5. The trivial inequality $\mathbf{P}(X \geq 1) \leq \mathbf{E}(X)$, for X positive gives:

$$\mathbf{P}_y \left(\sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{k\leq n} V(z_k)\geq 0,\, \min_{\frac{n}{2}<k\leq n} V(z_k)\geq a_n(x+L),\, V(z)\geq a_n(x)+A\}} \geq \frac{\mathrm{e}^{\beta x}}{n^{\frac{3}{2}\beta}} \right) \leq$$
$$\frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}} \mathbf{E}_y \left(\sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{k\leq n} V(z_k)\geq 0,\, \min_{\frac{n}{2}<k\leq n} V(z_k)\geq a_n(x+L),\, V(z)\geq a_n(x)+A\}} \right).$$

By the Lyons' change of measure this is equal to

$$\begin{aligned}
&= \frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}} \mathrm{e}^{-y} \sum_{k\in\mathbb{N}} \mathbf{E}_y \left(\mathrm{e}^{(1-\beta)S_n} \mathbb{1}_{\{\min_{k\leq n} S_k\geq 0,\, \min_{\frac{n}{2}<k\leq n} S_k\geq a_n(x+L),\, S_n\in I_n(x-A-k)\}} \right) \\
&\leq \mathrm{e}^{-x-A(\beta-1)} \sum_{k\in\mathbb{N}} \mathrm{e}^{(1-\beta)k} n^{\frac{3}{2}} \mathbf{P}_y \left(\min_{k\leq n} S_k \geq 0,\ \min_{\frac{n}{2}<k\leq n} S_k \geq a_n(x+L),\ S_n \in I_n(x-A-k) \right) \\
&\leq c_{17}(1+y)\mathrm{e}^{-x-y} L \mathrm{e}^{-A(\beta-1)}
\end{aligned}$$

by (4.7).

□

The following Lemma shows the tension exponential for the partition function of the killed branching random walk. This is the analogue of Lemma 3.3 in [3].

Lemma 4.6 *There exists $c_{18} > 0$, c_{19}, $c_{20} > 0$ such that $\forall x \in \mathbb{R}$, $y \geq 0$, $n \geq 1$, $j \in \mathbb{Z}$*

$$\mathrm{e}^{x+y} \mathbf{P}_y \left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x}, M^{kill}_n \in I_n(x-j) \right) \leq c_{18}(1+y) j \mathrm{e}^{-c_{19} j}. \tag{4.28}$$

$$\mathrm{e}^{x+y} \mathbf{P}_y \left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x} \right) \leq (1+y) c_{20}. \tag{4.29}$$

Proof of Lemma 4.6. We note that for any $L > 0$

$$\begin{aligned}
& \mathbf{P}_y\left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x}, M^{kill}_n \in I_n(x-j)\right) \\
\leq\; & c_7(1+y)\mathrm{e}^{-y-x}\mathrm{e}^{-c_8 L} + \mathbf{P}_y\left(\sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \mathbb{1}_{\{\min_{j\leq n} V(z_k)\geq 0,\ \min_{j\in[n/2,n]} V(z_k)\geq a_n(x+L), V(z)\geq a_n(x-j)\}} \geq \frac{\mathrm{e}^{\beta x}}{n^{\frac{3}{2}\beta}}\right) \\
\leq\; & c_7(1+y)\mathrm{e}^{-y-x}\mathrm{e}^{-c_8 L} + c_{17}(1+y)\mathrm{e}^{-x-y} L \mathrm{e}^{-j(\beta-1)},
\end{aligned}$$

by Lemma 4.4 and 4.5. Set $L = j$ and $c_{19} = \min(\beta-1, \alpha)/2$ to obtain (4.28). (4.29) follows easily from

$$\mathbf{P}_y\left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x}\right) = \mathbf{P}_y\left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x},\ M^{kill}_n \leq a_n(x)\right) + \mathbf{P}_y\left(\widetilde{W}^{kill}_{n,\beta} \geq \mathrm{e}^{\beta x},\ M^{kill}_n \geq a_n(x)\right),$$

(4.28) and Lemma 3.5 of [3] $\square$

4.4 Proof of Proposition 4.3

The proof is divided in three parts. First we suppose that the subsequent Lemma below holds and we demonstrate the point (iv) of the Proposition, two we prove the Lemma and three we collect all our work to show that we get also the other points. Recall that $\widetilde{W}^{A,L}_{n,\beta} := n^{\frac{3}{2}\beta} W^{A,L}_{n,\beta}$.

Lemma 4.7 $\forall K, \eta > 0$, $\exists A_0(\eta)$, $L_0(\eta)$ *such that for all* $(\delta_1, ..., \delta_k, \Delta) \in [-K, K]^{k+1}$, $L \geq L_0$, $A \geq A_0$ *there exists* $D(A, L, \eta, K) > 0$ *and* $N(A, L, D, \eta, \boldsymbol{\delta}, \Delta) \geq 0$ *such that* $\forall n > N$ *and* $\forall x \in [D, \log n]$

(4.30)
$$\left| \mathrm{e}^x \mathbf{P}\left(\bigcap_{j\leq k}\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j) \geq \mathrm{e}^{\beta_j(x-\delta_j)}\};\ M^{kill}_n \in I_n(x), m^{kill,(n)} \in Z^{x-\Delta,L}\right) - c(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta)\right| \leq \eta.$$

4.4.1 Part 1, Proof of Proposition 4.3 (iv) in admitting Lemma 4.30

Observe that the only difference between (4.30) and (4.14) is that $\widetilde{W}^{kill}_{n,\beta}$ is replaced by $\widetilde{W}^{A,L}_{n,\beta}$. An easy consequence of Lemma 3.3 [3] is that for any $\epsilon > 0$ there exists $L_0 > 0$ such that for any $L \geq L_0$, $x \geq 0, n \in \mathbb{N}$,

$$\left| \mathbf{E}\left(\prod_{j\leq k} \mathbb{1}_{\{\widetilde{W}^{kill}_{n,\beta_j} \geq \mathrm{e}^{\beta_j(x-\delta_j)}\}} \left(\mathbb{1}_{\{M^{kill}_n \in I_n(x)\}} - \mathbb{1}_{\{M^{kill}_n \in I_n(x), m^{kill,(n)} \in Z^{x,L}\}}\right)\right)\right| \leq \epsilon \mathrm{e}^{-x}. \tag{4.31}$$

On other hand, for $A > A_0$, $L \geq L_0$, $x, n \geq 0$, we set

$$er_{A,L}(\boldsymbol{\delta}, x, n) := \left| \mathbf{E} \left(\prod_{j \leq k} \mathbb{1}_{\{\widetilde{W}^{kill}_{n,\beta_j} \geq \mathrm{e}^{\beta_j(x-\delta_j)}\}} - \prod_{j \leq k} \mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j) \geq \mathrm{e}^{\beta_j(x-\delta_j)}\}} \right) \right|.$$

It is not difficult to check that for any $\alpha \in [0,1]$:

$$er_{A,L}(\boldsymbol{\delta}, x, n) \leq \sum_{j \leq k} \mathbf{P}\left(\widetilde{W}^{kill}_{n,\beta_j} - \widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j) \geq (1-\alpha)\mathrm{e}^{\beta(x-\delta_j)} \right) + \mathbf{P}\left(\alpha \leq \frac{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)}{\mathrm{e}^{\beta_1(x-\delta_j)}} \leq 1 \right).$$

We will bound the terms both. Suppose the following assertion.

For any $\epsilon > 0$, there exists α_0 , near enough to 1 such that for any $\alpha < \alpha_0$ there exists $A_0(\epsilon), L_0(\epsilon)$ such that for any $A > A_0, L > L_0$ there exists D and N large enough such that for any $n > N$, $x \in [D, \log n]$ and $j \in [0,k]$,

$$\mathbf{P}\left(\alpha \leq \frac{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)}{\mathrm{e}^{\beta_j(x-\delta_j)}} \leq 1 \right) \leq \epsilon \mathrm{e}^{-x};$$

true. Then for $\alpha < \alpha_0$,

$$\mathbf{P}\left(\widetilde{W}^{kill}_{n,\beta_j} - \widetilde{W}^{A,L}_{n,\beta_j} \geq (1-\alpha)\mathrm{e}^{\beta_j(x-\delta_j)} \right) \leq (A) + (B),$$

with

$$\begin{aligned}
(A) \quad &= \mathbf{P}\left(\sum_{|z|=n} \mathrm{e}^{-\beta_j V(z)} \mathbb{1}_{\{\min_{k \leq n} V(z_k) \geq 0,\ \min_{\frac{n}{2} \leq k \leq n} V(z_k) \leq a_n(x-\delta+L)\}} \geq \frac{(1-\alpha)\mathrm{e}^{\beta_j - \delta_j}}{2n^{\frac{3}{2}\beta_j}} \right), \\
(B) \quad &= \mathbf{P}\left(\sum_{|z|=n} \mathrm{e}^{-\beta_j V(z)} \mathbb{1}_{\{\min_{k \leq n} V(z_k) \geq 0,\ \min_{\frac{n}{2} \leq k \leq n} V(z_k) \geq a_n(x-\delta_j+L),\, V(z) \geq a_n(x-\delta_j)+A\}} \geq \frac{(1-\alpha)\mathrm{e}^{\beta_j - \delta}}{2n^{\frac{3}{2}\beta_j}} \right).
\end{aligned}$$

Both terms (A) and (B) are small. Indeed we recognize the terms of the Lemmas 4.4 and 4.5, with $x = x + \frac{1}{\beta_j}\log(\frac{1-\alpha}{2})$ and $L = L - \frac{1}{\beta_j}\log(\frac{1-\alpha}{2})$, and $A = A + \frac{1}{\beta_j}\log(\frac{1-\alpha}{2})$. Thus we can fix A, L, N and D large enough to conclude $er_{A,L}(\boldsymbol{\delta}, x, n) \leq 2\epsilon \mathrm{e}^{-x}$. In combining with (4.30) we obtain (iv). It remains thus to show our assertion in italic. We need a Lemma,

Lemma 4.8 *For all $\epsilon > 0$ there exists $L_*(\epsilon)$ and $A_*(\epsilon)$ such that $\forall A > A_*, L > L_*$ there exists $D > 0$ and N large enough such that for any $n \geq N$ and $x \in [D, \log n]$,*

$$\left| \mathrm{e}^x \mathbf{P}\left(\widetilde{W}^{A,L}_{n,\beta}(x) \geq \mathrm{e}^{\beta x} \right) - \left(C_1 + \sum_{j \geq 1} \mathrm{e}^j c(\beta, 0, -j) \right) \right| \leq \epsilon. \tag{4.32}$$

with C_1 the constant which appear in Proposition 1.2 [3].

Proof of Lemma 4.8. Let $\epsilon > 0$. For j_0 large enough, $\sum_{j \geq j_0} c_{12} j \mathrm{e}^{-c_{(4.6)} j} \leq \epsilon$ (with $c_{(4.6)}$ the constant which appears in Lemma 4.6) and it implies that

$$\sum_{j \geq j_0} \mathrm{e}^j c(\beta, 0, -j) + \sum_{j \geq j_0} \mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}, M_n^{kill} \in I_n(x-j)\right) \leq 2\epsilon \qquad \forall A,\ L,\ n,\ x.$$

Now we fix $L_*(\epsilon), A_*(\epsilon) \geq 0$ such that:

by Lemma 4.7 (in an uni-dimensional case), there exists $D(A_*, L_*, D, \frac{\epsilon}{j_0}, j_0)$ such that for all $A \geq A_0$ and $L \geq L_0$, $j \leq j_0$, $\exists N(A, L, D, \frac{\epsilon}{j_0}, j_0)$, such that $\forall n > N$, $x \in [D, \log n]$

$$\left| \mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}, M_n^{kill} \in I_n(x-j)\right) - \mathrm{e}^j c(\beta, 0, -j) \right| \leq \frac{\epsilon}{j_0}. \tag{4.33}$$

by Lemma 6.1 (see appendix) $\forall L > L_*,\ A, x \geq 1,\ n \in \mathbb{N}^*$,

$$\left| \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}, M_n^{kill} \leq a_n(x)\right) - C_1 \right| \leq \epsilon \mathrm{e}^{-x}. \tag{4.34}$$

Hence, with

$$\mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}\right) = \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x},\ M_n^{kill} \leq a_n(x)\right) + \sum_{j \geq 0} \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}, M_n^{kill} \in I_n(x-j)\right),$$

for all $A \geq A_0$ et $L \geq L_0$, $n > N$ and $x \in [D, \log n]$

$$\begin{aligned} \left| \mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}\right) - (C_1 + \sum_{j \geq 1} \mathrm{e}^j c(\beta, -j)) \right| &\leq \epsilon + \epsilon + 2 \sum_{j \geq j_0} c_{12} j \mathrm{e}^{-\alpha j} + \sum_{j \leq j_0} \frac{\epsilon}{j_0}, \\ &\leq 4\epsilon. \end{aligned}$$

by (4.33) and (4.34). The Lemma is proved. $\square$

Proof of the assertion in italics. It's still rigorous to suppose $\delta = 0$. Let $\epsilon > 0$. We choose α near enough to one such that

$$\left(C_1 + \sum_{j \geq 1} \mathrm{e}^j c(\beta, 0, -j)\right) \left| \frac{1}{\alpha^{\frac{1}{\beta}}} - 1 \right| \leq \epsilon.$$

Let $A_1(\epsilon) = A_*(\epsilon) - \frac{1}{\beta} \log \alpha$ and $L_1(\epsilon) = L_*(\epsilon)$ (A_*, L_* are the constants defined by the previous Lemma). With

$$\mathbf{P}\left(\alpha \leq \frac{\widetilde{W}_{n,\beta}^{A,L}(x)}{\mathrm{e}^{\beta x}} \leq 1\right) = \mathbf{P}\left(\frac{\widetilde{W}_{n,\beta}^{A,L}(x)}{\mathrm{e}^{\beta x}} \geq \alpha\right) - \mathbf{P}\left(\frac{\widetilde{W}_{n,\beta}^{A,L}(x)}{\mathrm{e}^{\beta x}} \geq 1\right) \qquad \text{and}$$

$$\mathbf{P}\left(\frac{\widetilde{W}_{n,\beta}^{A,L}(x)}{\mathrm{e}^{\beta x}} \geq \alpha\right) = \mathbf{P}\left(\frac{\widetilde{W}_{n,\beta}^{A+\frac{1}{\beta}log(\alpha), L-\frac{1}{\beta}log(\alpha)}(x + \frac{1}{\beta}log(\alpha))}{\mathrm{e}^{\beta(x+\frac{1}{\beta}log(\alpha))}} \geq 1\right),$$

we may affirm that $\forall A \geq A_1$, $L \geq L_1$, there exists D, N such that for any $n > N$ and $x \in [D, \log n]$.

$$\left| \mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x} \right) - \left(C_1 + \sum_{j \geq 1} \mathrm{e}^j c(\beta, 0, -j) \right) \right| \leq \epsilon, \tag{4.35}$$

$$\left| \mathrm{e}^x \mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L} \geq \alpha \mathrm{e}^{\beta x} \right) - \left(C_1 + \sum_{j \geq 1} \mathrm{e}^j c(\beta, 0, -j) \right) \frac{1}{\alpha^{\frac{1}{\beta}}} \right| \leq \epsilon. \tag{4.36}$$

The assertion in italics follows. □

Finally, admitting Lemma 4.30 the Proposition 4.3 is true.

4.4.2 Part 2, Proof of the Lemma 4.7

Proof of the Lemma is inspired of [3]. We use the same tools, ideas and several results are very similar. Thus some lemma will be stated without proof, deferring it to the appendix.

Definition 4.9 *For b integer, we define the event ξ_n by*

$$\xi_n := \xi_n(x, b, A) := \{\forall k \leq n - b, \forall v \in \Omega(w_k), \min_{u \geq v, |u| = n} V(u) > a_n(x) + A\}, \tag{4.37}$$

where $\Omega(w_k)$ denotes the set of brothers of w_k. On the event $\xi_n \cap \{M_n^{kill} \in I_n(x)\}$ we are sure that any particle located at the minimum separated from the spine after the time n-b.

Definition 4.10 *Let for x, L, $A > 0$ and $b \in \mathbb{N}^*$ we define*

(i) the event

$$\diamondsuit_{A,L,b}(\beta_j, \delta_j, y) := \mathbb{1}_{\left\{ \mathrm{e}^{-\beta_j(\delta_j + L)} \leq \sum_{|z| = b} \mathrm{e}^{-\beta_j (V(z) + y)} \mathbb{1}_{\{V(z) + y \leq \delta_j + L + A,\, \min_{k \leq b} V(z_k) + y \geq \delta_j\}} \right\}}.$$

(ii)The function $F_{A,L,b}$ by

$$F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y) := \mathbf{E}_{\mathbf{Q}_y}\left[\frac{\mathrm{e}^{V(\omega_b) - L} \mathbb{1}_{\{V(\omega_b) = M_b\}}}{\sum_{|u| = b} \mathbb{1}_{\{V(u) = M_b\}}} \mathbb{1}_{\{V(\omega_b) \in [\Delta + L - 1, \Delta + L], \min_{k \leq b} V(\omega_b) \geq \Delta\}} \prod_{j \leq k} \diamondsuit_{A,L,b}(\beta_j, \delta_j, 0) \right]. \tag{4.38}$$

We stress that M_b which appears in the definition of $F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y)$ is the minimum at time b of the ***non killed*** *branching random walk.*

(iii) $c_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) := \frac{C_- C_+ \sqrt{\pi}}{\sigma \sqrt{\pi}} \int_{x \geq 0} F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y) R_-(y) dy$*, where C_-, C_+ and $R_-(x)$ are defined in introduction.*

By adding

$$\begin{aligned}
&\mathbf{P}\left(\bigcap_{j\leq k}\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\leq \mathrm{e}^{\beta_j(x-\delta_j)}\};\ M_n^{kill}\in I_n(x), m^{kill,(n)}\in Z^{x-\Delta,L}\right)\\
=\ &\mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{V(\omega_n)}\mathbb{1}_{\{V(\omega_n)=M_n^{kill},\omega_n\in Z^{x-\Delta,L}\}}}{\sum\limits_{|u|=n}\mathbb{1}_{\{V(u)=M_n^{kill}\}}}\prod_{j\leq k}\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\}}\right],
\end{aligned}$$

(which is true by lyons'change of measure) to the three Lemma (see Appendix for the proofs)

Lemma 4.11 *$\forall K,\eta,L,A>0$ $\exists D(A,\eta)>0$ and $B(A,L,K,\eta)\geq 1$ such that $\forall b\geq B$, $n\geq b$, $x\geq D$ and $\Delta\in[-K,K]$*

$$\mathbf{Q}((\xi_n)^c,\omega_n\in Z_n^{x-\Delta,L})\leq \eta n^{-\frac{3}{2}}. \tag{4.39}$$

Lemma 4.12 *$y\mapsto F_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,y)$ is Riemann integrable and there exists a non-increasing function $\bar{F}:\mathbb{R}_+\to\mathbb{R}$ such that $|F(x)|\leq \bar{F}(x)$ for any $x\geq 0$ and $\int_{x\geq 0}x\bar{F}(x)<\infty$.*

Lemma 4.13 *Let $L,A>0$ and $K,\eta>0$. Let D and B be as in Lemma 4.11 then $\forall b\geq B,(\delta_1,...,\delta_k)\in[-K,K]^k$ $\exists N(b,L,\delta_1,...,\delta_k,\eta)>0$ such that $\forall n>N$ and $\forall x\in[D,\log n]$*

$$\left|\mathrm{e}^x\mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{V(\omega_n)}\mathbb{1}_{\{V(\omega_n)=M_n^{kill},\omega_n\in Z^{x-\Delta,L}\}}}{\sum\limits_{|u|=n}\mathbb{1}_{\{V(u)=M_n^{kill}\}}}\prod_{j\leq k}\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\}},\xi_n\right]-c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right|\leq(2+\mathrm{e}^{K+1})\eta. \tag{4.40}$$

we are nearly to the Proposition 4.3. Indeed by combining the Lemma 4.13 and 4.11 we can drop ξ_n in the expectation of (4.40), so we obtain that the probability of Lemma 4.3 almost behaves like a constant factor e^{-x} as $x\to\infty$. "Almost" because the factor depends to A,L and b. With the following we will can drop "almost".

Lemma 4.14 *(i) For all $\boldsymbol{\beta}>1,\boldsymbol{\delta}\in K$, $\Delta\in\mathbb{R}$, $c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta):=\lim_{b\to\infty}c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ exists.*

(ii) $\lim_{A,L\to\infty}c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ converge in increasing and we denote $c(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ the limit.

(iii)$(\boldsymbol{\delta},\Delta)\mapsto c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ and $(\boldsymbol{\delta},\Delta)\mapsto c(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ are continuous and thus $c_{A,L}(\boldsymbol{\beta}.,.)$ converges uniformly on compact subsets to $c(\boldsymbol{\beta},.,.)$ by Dini Lemma.

Proof of lemma 4.14. Let $\eta>0$. We call $\mathbf{Q}_{(4.40)}$ the expectation in the left-hand side of (4.40), we introduce

$$c^-_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta):=\liminf_{x\to\infty}\ \liminf_{n\to\infty}\mathrm{e}^x\mathbf{Q}_{(4.40)}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta),$$

$$c^+_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) := \limsup_{x\to\infty}\limsup_{n\to\infty} \mathrm{e}^x \mathbf{Q}_{(4.40)}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta).$$

In particular, taking $n\to\infty$ then $x\to\infty$ in (4.40), for all $A,L\in\mathbb{R}^+$ there exists $B(A,L,K,\eta)$ such that for any $b\geq B(A,L,K,\eta)$, $\Delta\in[-K,K]$

$$c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)-\eta \leq c^-_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) \leq c^+_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) \leq c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)+\eta.$$

Notice that ξ_n (hence $\mathbf{Q}_{(4.40)}$) is increasing with b. It implies that $c^-_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ and $c^+_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ are both increasing in b. Let $c^-_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ and $c^+_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ the respectively limits when $b\to\infty$. By Lemma 4.6 both are bounded uniformly in A and L. We have then

$$\limsup_{b\to\infty} c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)-\eta \leq c^-_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) \leq c^+_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) \leq \liminf_{b\to\infty} c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)+\eta.$$

Letting η go to 0, it yields that $c_{A,L,b}$ has a limit as $b\to\infty$, that we denote by $c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) = c^+_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) = c^-_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$. We stress that this equality is valid for all A, $L>0$. Similarly we see that $\mathbf{Q}_{(4.40)}$ is increasing with L, thus $c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ is increasing with L. Same $c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ is increasing with A and thus $c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ is increasing with A. Finally $c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ is bounded and increasing with A and L. This prove (i) and (ii), it remains (iii). Here are two useful lemmas (proved in the appendix):

Lemma 4.15 *For all $L>0$, there exists C_L such that*

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathrm{e}^x \mathbf{P}(M_n^{kill}\in I_n(x), m^{kill,(n)}\in Z^{x,L}) = C_L. \tag{4.41}$$

Lemma 4.16 *For all $A,L>0$, $(\delta,\Delta)\in\mathbb{R}^2$*

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathrm{e}^x \mathbf{P}\left(\widetilde{W}^{A,L}_{n,\beta}(x-\delta)\geq \mathrm{e}^{\beta(x-\delta)}, M_n^{kill}\in I_n(x), m^{kill,(n)}\in Z^{x-\Delta,L}\right) = c_{A,L}(\beta,\delta,\Delta). \tag{4.42}$$

For $(\Delta,\Delta')\in\mathbb{R}^2$, $(\boldsymbol{\delta},\boldsymbol{\delta'})\in K^2$, by Lyons' change of measure

$$\begin{aligned}
&\mathrm{e}^x|\mathbf{Q}_{(4.40)}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)-\mathbf{Q}_{(4.40)}(\boldsymbol{\beta},\boldsymbol{\delta'},\Delta')| \leq \\
&\sum_{j\leq k}\mathrm{e}^x \mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{V(\omega_n)}\mathbb{1}_{\{V(\omega_n)=M_n^{kill}\}}}{\sum_{|u|=n}\mathbb{1}_{\{V(u)=M_n^{kill}\}}}\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)},\,\omega_n\in Z^{x-\Delta,L}\}} - \mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta'_j)\geq \mathrm{e}^{\beta_j(x-\delta'_j)},\,\omega_n\in Z^{x-\Delta',L}\}}\right] \\
&\leq \sum_{j\leq k}\mathrm{e}^x\mathbf{E}\left[\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)},\,m^{kill,(n)}\in Z^{x-\Delta,L}\}} - \mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta'_j)\geq \mathrm{e}^{\beta_j(x-\delta'_j)},\,m^{kill,(n)}\in Z^{x-\Delta',L}\}}\right].
\end{aligned}$$

Thus it's enough to control

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathrm{e}^x \mathbf{E}\left(\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\}} - \mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta'_j)\geq \mathrm{e}^{\beta_1(x-\delta'_j)}\}}; m^{kill,(n)} \in Z^{x-\Delta,L}\right), \tag{4.43}$$

and

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathbf{E}\left(\mathbb{1}_{\{m^{kill,(n)}\in Z^{x-\Delta,L}\}} - \mathbb{1}_{\{m^{kill,(n)}\in Z^{x-(\Delta'),L}\}}\right). \tag{4.44}$$

By (4.41) and a change of variable $\lim_{x\to\infty}\lim_{n\to\infty}(4.44) \leq C_L \mathrm{e}^{\Delta}|\mathrm{e}^{\Delta'-\Delta}-1|$. Other hand

$$\begin{aligned}(4.44) \leq \mathrm{e}^x \mathbf{E}&\left(\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\, m^{kill,(n)}\in Z^{x-\Delta,L}\}} - \mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta'_j)\geq \mathrm{e}^{\beta_1(x-\delta'_j)},\, m^{kill,(n)}\in Z^{(x-\delta'+\delta)-\Delta,L}\}};\right) \\ &+\mathbf{E}\left(\mathbb{1}_{\{m^{kill,(n)}\in Z^{x-\Delta,L}\}} - \mathbb{1}_{\{m^{kill,(n)}\in Z^{x-(\Delta-\delta+\delta'),L}\}}\right)\end{aligned}$$

and by (6.3) and again a change of variable

$$\lim_{x\to\infty}\lim_{n\to\infty}(4.44) \leq c_{A,L}(\beta,\delta,\Delta)|\mathrm{e}^{\delta_j}-\mathrm{e}^{\delta'_j}| + C_L \mathrm{e}^{\delta_j}|\mathrm{e}^{\delta'-\delta}-1|.$$

We conclude by

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathrm{e}^x |Q_{(4.40)}(\boldsymbol{\delta},\Delta) - Q_{(4.40)}(\widetilde{\delta'},\Delta')| \leq c_{21}\sum_{j\leq k}|\mathrm{e}^{\delta_j}-\mathrm{e}^{\delta'_j}| + k\mathrm{e}^{\Delta}|\mathrm{e}^{\Delta'-\Delta}-1|.$$

for some $c_{21}>0$. As it's a bound uniform in A, L, b, it implies the continuity of $(\boldsymbol{\delta},\Delta)\mapsto c_{A,L}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$ and $(\boldsymbol{\delta},\Delta)\mapsto c(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$. $\square$

End of proof of Lemma 4.7. Let $K>0$, $\eta>0$. Let $A_0, L_0>0$ such that for any $A>A_0$, $L>L_0$ there exists D such that for any $(\boldsymbol{\delta},\Delta)\in[-K,K]^{k+1}$ there exists B, large enough such that for any $b\geq B_0$ $\exists N(b,A,L,\boldsymbol{\delta},\Delta,\eta)$ such that for any $n>N$ and $x\in[D,\log n]$,

$$\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(\omega_n)}\mathbb{1}_{\{V(\omega_n)=M_n^{kill},\,\omega_n\in Z^{x-\Delta,L}\}}}{\sum_{|u|=n}\mathbb{1}_{\{V(u)=M_n^{kill}\}}}\prod_{j\leq k}\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\}}; \xi_n^c\right] \leq \eta \mathrm{e}^{-x},$$

$$|c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta) - c(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)| \leq \eta$$

and

$$\left|\mathrm{e}^x\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(\omega_n)}1_{\{V(\omega_n)=M_n^{kill},\,\omega_n\in Z^{x-\Delta,L}\}}}{\sum_{|u|=n}1_{\{V(u)=M_n^{kill}\}}}\prod_{j\leq k}\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\geq \mathrm{e}^{\beta_j(x-\delta_j)}\}}, \xi_n\right] - c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right| \leq 3\eta.$$

The combination of this three inequalities implies

$$\left|\mathrm{e}^x\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{A,L}_{n,\beta_j}(x-\delta_j)\ge \mathrm{e}^{\beta_j(x-\delta_j)}\};M_n^{kill}\in I_n(x-\Delta),m^{kill,(n)}\in Z^{x-\Delta,L}\right)-c(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right|\le 5\eta.$$

Dependence in b disappears which gives exactly Lemma 4.7. □

4.4.3 Part 3, The others points

(i) results of Lemma 4.6 and Lemma 3.5 of [3]. (ii) is stated in Lemma 4.14. (iii) is simply a consequence of the change of variable

$$\begin{aligned}
&\mathrm{e}^x\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n^{kill}\in I_n(x-\Delta)\right)\\
&=\mathrm{e}^{\Delta}\mathrm{e}^{x-\Delta}\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\Delta-(\delta_j-\Delta))}\},\ M_n^{kill}\in I_n(x-\Delta-0)\right).
\end{aligned}$$

It remains (v). Let $\epsilon>0$. By Lemma 3.5 [3], there exists $p_0\ge 0$ such that for any $x\ge 0$ and $n\ge 1$

$$\sum_{p\ge p_0}\mathrm{e}^x\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n^{kill}\in I_n(x-\Delta+p)\right)\le\frac{\epsilon}{2}.$$

Moreover (iv) says that there exists $A=A(p_0+K,\frac{\epsilon}{p_0})$ such that for all $\boldsymbol{\delta}\in[-K,K]^k$ there exists N such that for any $n\ge N$ and $x\in[A,A+\frac{3}{2}\log n]$ $p\le p_0$

$$\left|\mathrm{e}^x\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n^{kill}\in I_n(x-(\Delta-p))\right)-\mathrm{e}^{\Delta-p}c(\boldsymbol{\beta},\boldsymbol{\delta}-(\Delta-p))\right|\le\frac{\epsilon}{p_0}.$$

By combining these inequalities with

$$\begin{aligned}
&\mathrm{e}^x\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n^{kill}\le a_n(x-\Delta)\right)\\
&=\mathrm{e}^x\sum_{p\ge 0}\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}^{kill}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n^{kill}\in I_n(x-\Delta+p)\right),
\end{aligned}$$

we get also the point (v). □

5 Proof of the Proposition 2.2

This section partially prove the Proposition 2.2, rigorously we will need to Proposition 2.1.

5.1 The branching random walk at the beginning

Following [3] we introduce some notations. To go from the tail distribution of $W^{kill}_{n,\beta}$ to the one of $W_{n,\beta}$, we have to control excursions inside the negative axis that can appear at the beginning of the branching random walk. This can be seen as the analogue of the "delay" mentioned by Lalley and Sellke [20]. For $z \geq A \geq 0$ and $n \geq 1$, we define the set

$$S_A := \{u \in T : \min_{k \leq |u|-1} V(u_k) > V(u) \geq A - x \text{ and } |u| \leq \sqrt{n}\}.$$

We notice that S_A depends on n and x, but we omit to write this dependency in the notation for sake of concision. For $x \geq 0$ and $u \in S_A$, we define the indicator $B_{n,z}(u)$ equal to 1 if and only if the branching random walk emanating from u and killed below $V(u)$ has its minimum below $\frac{3}{2}\log n - x$. Equivalently,

Definition 5.1 *For $u \in S_A$, we call $B_{n,x}(u)$ the indicator of the event that there exists $|v| = n, v > u$ such that $V(v_l) \geq V(u), \forall |u| \leq l \leq n$ and $V(v) \leq \frac{3}{2}\log n - x$.*

Identically for $u \in S_A$, we call $B^W_{\beta,n,x}(u)$ the indicator of the event $\{W^u_{n,\beta,kill} \geq \frac{e^{\beta(x+V(u))}}{n^{\frac{3}{2}\beta}}\}$, where

$$W^u_{n,\beta,kill} := \sum_{|z|=n, z>u} e^{-\beta(V(z)-V(u))} \mathbb{1}_{\{\min_{k\in[|u|,n]} V(z_k)-V(u)\geq 0\}}.$$

Finally, let for $|v| \geq 1$,

$$\xi(v) := \sum_{w\in\Omega(v)} (1 + (V(w) - V(\overleftarrow{v}))_+) e^{-(V(w)-V(\overleftarrow{v}))}.$$

To avoid some extra integrability conditions, we are led to consider vertices $u \in S_A$ which behave 'nicely', meaning that $\xi(u_k)$ is not too big along the path $\{u_1, ..., u_{|u|} = u\}$.

5.2 Proof of Proposition 2.2 in admitting "an italic assertion" and the Proposition 2.1

The assertion in italics is

$\forall K, \epsilon > 0\, \exists A(K,\epsilon) > 0$ such that $\forall(\delta_1, ..., \delta_j, \Delta) \in [-K, K]^{k+1}$, $\exists N(\epsilon, \boldsymbol{\delta}, \Delta)$ such that $\forall n > N$, $x \in [A, A + \log\log n]$

$$\left| \frac{e^x}{x} \mathbf{P}\left(\bigcap_{j\leq k} \{\widetilde{W}_{n,\beta_j} \geq e^{\beta_j(x-\delta_j)}\},\, M_n \leq a_n(x-\Delta) \right) - c_0\chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon.$$

Suppose that it's true.

Proof of Proposition 2.2 in admitting Proposition 2.1. We need to observe that $\forall K, \epsilon > 0$ there exists $A(K,\epsilon) > 0$ such that $\forall (i, \delta_1, ..., \delta_k) \in [-K, K]^{k+1}\, \exists N(\epsilon, i, \boldsymbol{\delta})$ such that $\forall n > N$, $x \in [A, A + \log\log n]$

$$\left| \frac{e^x}{x} \mathbf{P}\left(\bigcap_{j\leq k} \mathbb{1}_{\{e^{\beta_j(x-\delta_j)} \leq \widetilde{W}_{n,\beta_j}\}};\, M_n \in I_n(x-i) \right) - e^i c_0 c(\boldsymbol{\beta}, \boldsymbol{\delta} - i, 0) \right| \leq \epsilon. \tag{5.1}$$

Indeed it's obvious because $\mathbb{1}_{\{M_n \in I_n(x-i)\}} = \mathbb{1}_{\{M_n \leq a_n(x-i)\}} - \mathbb{1}_{\{M_n \leq a_n(x-i)-1\}}$, and $\chi(\beta, \boldsymbol{\delta}, i) - \chi(\beta, \boldsymbol{\delta}, i-1) = e^i c(\boldsymbol{\beta}, \boldsymbol{\delta} - i, 0)$. So

$$\begin{aligned} \frac{e^x}{x} \mathbf{P}\left(\bigcap_{j\leq k} \{\widetilde{W}_{n,\beta_j} \geq e^{\beta_j(x-\delta_j)}\} \right) = \frac{e^x}{x} \sum_{i\geq i_0} \mathbf{P}\left(\bigcap_{j\leq k} \{\widetilde{W}_{n,\beta_j} \geq e^{\beta_j(x-\delta_j)}\},\, M_n \in I_n(x-i) \right) + \\ \frac{e^x}{x} \mathbf{P}\left(\cap_{j\leq k} \{\widetilde{W}_{n,\beta_j} \geq e^{\beta_j(x-\delta_j)}\},\, M_n \leq a_n(x-i_0) \right). \end{aligned}$$

Now when x then n tend to infinity, the question is to know whether the sum is negligible. The answer is yes, thanks to Proposition 2.1. Recall that this Proposition says exactly there exists $N > 0$ such that for any $n \geq N$,$j \geq 1$ and $x \in [1, \log\log n]$

$$\mathbf{P}\left(\bigcap_{j\leq k} \{\widetilde{W}_{n,\beta_j} \leq e^{\beta_j(x-\delta_j)}\},\, M_n \in I_n(x-i) \right) \leq c_1 x e^{-x} e^{-\alpha i}.$$

So we get the first assertion of Proposition 2.2, with $\chi(\boldsymbol{\beta}, \boldsymbol{\delta}) = \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, -\infty)$. For the assertions (i), (ii), (iii) it's obvious thanks to Proposition 4.3. □

So it remains to prove the assertion in italics. We decompose the proof in two steps. As for the previous section, step 1 is very close to [3] and contains statements without proof. Step 2, contains some calculus which concern specifically the partition function, their aim is to ensure that step 1 is relevant for the partition function.

5.3 Step 1

Recall that R is the renewal function associated to $(S_n)_{n\in\mathbb{N}}$ and $c_0 = \lim_{n\to\infty} \frac{R(x)}{x}$. For the step 1 we want show

Proposition 5.2 *For any $K, \epsilon > 0$ there exists $A = A(K, \epsilon)$ and $X > A$ such that for any $(\delta_1, ..., \delta_k, \Delta) \in [-K, K]^{k+1}$ there exists $N(\epsilon, \boldsymbol{\delta}, \Delta)$ such that for all $n > N$, $x \in [X, X + \frac{1}{2}\log n]$*

$$\left| \frac{e^x}{x} \mathbf{P}\left(\bigcap_{j=1}^{k} \{\sum_{u\in S_A} B^W_{\beta_j, n, x-\delta_j}(u) \geq 1\};\, \sum_{u\in S_A} B_{n,x-\Delta}(u) \geq 1 \right) - c_0 \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon. \tag{5.2}$$

We see that it's identical to the assertion in italic except $\widetilde{W}_{n,\beta_j}$ which is replaced by $\sum_{u\in S_A} B^W_{\beta_j,n,x-\delta_j}$. The proof requires the following two lemmas that we suppose for the moment (the demonstration are deferring to the Appendix).

Lemma 5.3 *(i)Recall that $R(x)$ is the renewal function of $(S_n)_{n\geq 0}$ previous defined. Let $\epsilon > 0$. There exists $A \geq 0$ such that for n large enough and $z \in [A, (\log n)^{\frac{1}{5}}]$,*

$$\left| \frac{\mathrm{e}^x}{R(x-A)} \mathbf{E}\left[\sum_{u\in S_A} B_{n,x-\Delta}(u) \prod_{j\leq k} B^W_{\beta_j,n,x-\delta_j}(u) \right] - \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon. \tag{5.3}$$

(ii) For any $|u| \geq 1$, let $\Gamma(u) := \{\forall 1 \leq k \leq |u| : \xi(u_k) < \mathrm{e}^{V(u_{k-1})+x-A)/2}\}$. We have

$$\mathbf{P}\left(\sum_{u\in S_A} \widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \mathbb{1}_{\Gamma(u)^c} \geq 1 \right) \leq c_{22} \log_+(x) \mathrm{e}^{-x},$$

uniformly in $A \geq 0$, $\Delta \in [-K, K]$ and $n \geq 1$.
(iii) In particular

$$\mathbf{P}\left(\sum_{u\in S_A} B^W_{n,x} \mathbb{1}_{\Gamma(u)^c} \geq 1 \right) \leq c_{22} \log_+(x) \mathrm{e}^{-x}.$$

Next lemma serves to close the expectation in (5.3) with the probability in (5.2). Let $\theta > 1$ (θ will be better determinate later). For $u \in S_A$, we call $B^{W,\theta}_{n,x}(u)$ the indicator of the event $\{\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(x+V(u))}} \geq \mathrm{e}^{-\frac{\beta\theta}{\beta-1}\log_+\log_+ x)}\}$.

Lemma 5.4 *Set $K, \theta > 1$. There exists a constant $c_{23} > 0$ such that for any $(\delta_1, \delta_2, \Delta) \in [-K, K]^3$ $x \geq A \geq 0$, and $n \geq 1$ we get the following inequalities:*

$$\mathbf{E}\left(\sum_{u\neq v, \in S_A} B^{W,\theta}_{\beta_1,n,x-\delta_1}(u) B^{W,\theta}_{\beta_2,n,x-\delta_2}(v) \mathbb{1}_{\Gamma(u)\cap\Gamma(v)} \right) \leq c_{23} (\log x)^{\frac{\beta\theta}{\beta-1}+1} \mathrm{e}^{-x} \mathrm{e}^{-A}, \tag{5.4}$$

$$\mathbf{E}\left(\sum_{u\neq v, \in S_A} B^{W,\theta}_{\beta_1,n,x-\delta_1}(u) B_{n,x-\Delta}(v) \mathbb{1}_{\Gamma(u)\cap\Gamma(v)} \right) \leq c_{23} (\log x)^{\frac{\beta\theta}{\beta-1}+1} \mathrm{e}^{-x} \mathrm{e}^{-A}. \tag{5.5}$$

In particular as $B^{W,\theta}_{n,x} \leq B^W_{n,x}$ it is also true with $B^W_{n,x}$ at the place of $B^{W,\theta}_{n,x}$.

This Lemma implies that *for any $K>0$ there exists a constant $c_{24}>c_{23}$ such that for any $(\boldsymbol{\delta},\Delta)\in[-K,K]^{k+1}$ $x\geq A\geq 0$, and $n\geq 1$*

$$\left|\mathbf{P}\left(\bigcap_{j=1}^{k}\{\sum_{u\in S_A}B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\},\ \sum_{u\in S_A}B_{n,x-\Delta}(u)\geq 1\right)-\mathbf{P}\left(\sum_{u\in S_A}B_{n,x-\Delta}(u)\prod_{j\leq k}B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\right)\right| \leq c_{24}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-x}\mathrm{e}^{-A}$$

and we are now able to show the

Proof of Proposition 5.2. Let $\epsilon>0$. *Suppose $A>0$ large enough to apply Lemma 5.3 (i) and such that $c_{24}\mathrm{e}^{-A_1}\leq\epsilon$. Suppose also $X>0$ large enough such that for any $x\geq X\geq A$ $|R(x-A)-xc_0|\leq\epsilon x$ and $\frac{(\log x)^{\frac{\beta\theta}{\beta-1}+1}}{x}\leq\epsilon$.* Let us look at the upper bound. We have for n large enough and $x\in[X,\log n]$

$$\begin{aligned}
&\frac{\mathrm{e}^x}{x}\mathbf{P}\left(\bigcap_{j=1}^{k}\{\sum_{u\in S_A}B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\},\ \sum_{u\in S_A}B_{n,x-\Delta}(u)\geq 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\\
\leq\ &\frac{\mathrm{e}^x}{x}\mathbf{P}\left(\sum_{u\in S_A}B_{n,x-\Delta}(u)\prod_{j\leq k}B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)+\epsilon\\
\leq\ &\frac{\mathrm{e}^x}{x}\mathbf{E}\left[\sum_{u\in S_A}B_{n,x-\Delta}(u)\prod_{j\leq k}B^W_{\beta_j,n,x-\delta_j}(u)\right]-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)+\epsilon\\
\leq\ &\left(\frac{\mathrm{e}^x|R(x-A)-xc_0|}{R(x-A)x}+\frac{c_0\mathrm{e}^x}{R(x-A)}\right)\mathbf{E}\left[\sum_{u\in S_A}B_{n,x-\Delta}(u)\prod_{j\leq k}B^W_{\beta_j,n,x-\delta_j}(u)\right]-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)+\epsilon\\
\leq\ &2\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\epsilon+c_0\epsilon+\epsilon.
\end{aligned}$$

It's enough for the upper bound. It remains the lower bound. If we write $U:=\sum_{u\in S_A}B_{n,x-\Delta}(u)\prod_{j\leq k}B^W_{\beta_j,n,x-\delta_j}(u)\mathbb{1}_{\Gamma(u)}$ then by the Paley-Zygmund formula, we have $\mathbf{P}(U\geq 1)\geq\frac{\mathbf{E}[U]^2}{\mathbf{E}[U^2]}$. Under the conditions in italics, we have that $\mathbf{E}[U^2]\leq(1+\epsilon)\mathbf{E}[U]$. Hence, by Lemma 5.3 (i) and (ii) $\frac{\mathrm{e}^x}{R(x-A)}\mathbf{P}(U\geq 1)\geq\frac{\mathrm{e}^x}{R(x-A)(1+\epsilon)}\mathbf{E}[U]\geq\frac{\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)-2\epsilon}{1+\epsilon}$. It yields

$$
\begin{aligned}
&\frac{\mathrm{e}^x}{x}\mathbf{P}\left(\bigcap_{j=1}^{k}\{\sum_{u\in S_A} B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\},\ \sum_{u\in S_A} B_{n,x-\Delta}(u)\geq 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\\
\geq\ &\frac{\mathrm{e}^x}{x}\mathbf{P}\left(\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\leq k} B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\\
\geq\ &\left(-\frac{\mathrm{e}^x|\frac{R(x-A)}{c_0}-x|}{R(x-A)x}+\frac{\mathrm{e}^x}{R(x-A)}\right)\mathbf{P}\left(\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\leq k} B^W_{\beta_j,n,x-\delta_j}(u)\geq 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\\
\geq\ &-2\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\epsilon+\frac{c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)-2\epsilon}{1+\epsilon}-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\\
\geq\ &-\epsilon\left(\frac{1+c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)}{1+\epsilon}+2\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right).
\end{aligned}
$$

Thus the Proposition 5.2 follows.

□

5.4 Step 2

Recall the notation $\widetilde{W}^u_{n,\beta}=n^{\frac{3}{2}\beta}W^u_{n,\beta}$ and $\widetilde{W}_{n-p,\beta}=n^{\frac{3}{2}\beta}W_{n-p,\beta}$ for $u\in S_A$, $p\in[0,\sqrt{n}]\cap\mathbb{N}$. For the step 2 our goal is:

Proposition 5.5 *For any $K>0$, $\eta>0$ there exists $A>0$ and $X>A$ such that, for any $(\boldsymbol{\delta},\Delta)\in[-K,K]^{k+1}$ $\exists N(\epsilon,\boldsymbol{\delta},\Delta)$ such that for any $n\geq N$ and $x\in[X,\frac{3}{2}log(n)-1]$*

$$
\left|\frac{\mathrm{e}^x}{x}\mathbf{E}\left(\prod_{j\leq k}\mathbb{1}_{\{\sum_{u\in S_A}\frac{\widetilde{W}^u_{n,\beta_j,kill}}{\mathrm{e}^{\beta_j(x-\delta_j)}}\geq \mathrm{e}^{\beta_j V(u)}\}}-\prod_{j\leq k}\mathbb{1}_{\{\sum_{u\in S_A}B^W_{\beta_j,n,x-\delta_j}\geq 1\}};\ \sum_{u\in S_A}B_{n,x-\Delta}(u)\geq 1\right)\right|\leq\eta. \tag{5.6}
$$

This Proposition means that only one particle u among those of S_A own a partition function $\mathrm{e}^{-\beta V(u)}\widetilde{W}^u_{n,\beta,kill}$ non negligible. Intuitively, the amounts $\mathrm{e}^{-\beta V(u)}\widetilde{W}^u_{n,\beta,kill}$ are "almost" independent and $\mathbf{P}(\mathrm{e}^{-\beta V(u)}\widetilde{W}^u_{n,\beta,kill}\geq \mathrm{e}^{\beta x})\leq \mathrm{cst}\mathrm{e}^{-(V(u)+x)}$. Thus the probability

$$
\mathbf{P}\left(\exists u,v\in S_A, u\neq v \text{ such that } \mathrm{e}^{-\beta V(u)}\widetilde{W}^u_{n,\beta,kill}\geq \mathrm{e}^{\beta x},\ \mathrm{e}^{-\beta V(v)}\widetilde{W}^v_{n,\beta,kill}\geq \mathrm{e}^{\beta x}\right)
$$

decreases fast.

This Proposition requires the subsequent Lemma. Part (. bis) will be useful only for the proof of Proposition 2.1 (see Appendix). For sake of lightness in the notation we denote $W^{u,a}_{n,\beta,kill}:=\mathrm{e}^{-\beta a}W^u_{n,\beta,kill}$.

Lemma 5.6 *(i) For any $\epsilon > 0$, $\eta > 0$ there exists $\theta > 0$ such that for all $x \geq 1$ and $n > 5$*

$$\mathbf{P}\left(\sum_{|u|=k, u\in S_A} \widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \leq \mathrm{e}^{-\frac{\beta\theta}{\beta-1}\log_+\log_+ x}\}} \geq \epsilon\right) \leq \eta x \mathrm{e}^{-x}. \tag{5.7}$$

(i bis)For the same θ there exists $c_{(1)}$ and $\alpha_{(1)}$ (The numbering is different to better remember this constant) such that for all $x \geq 1$ and $n, j > 5$,

$$\mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}} \widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \leq \mathrm{e}^{-\frac{\beta\theta}{\beta-1}\log_+\log_+ x}\}} \geq 1,\ M_n \geq a_n(x-j)\right) \leq c_{(1)}\mathrm{e}^{-\alpha_{(1)}j} x\mathrm{e}^{-x}. \tag{5.8}$$

(ii) There exists $c_{25} > 0$ such that for all $x \geq 1$, $s \leq 0$, and any integer $n > 5$, $p \leq \sqrt{n}$

$$\mathbf{E}_y\left(\mathrm{e}^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\mathbb{1}_{\{\mathrm{e}^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\leq 1\}}\right) \leq c_{25}\log_+ x\,(1+y)\mathrm{e}^{-y}\mathrm{e}^{-(x+s)}. \tag{5.9}$$

(ii bis)For all $x \geq 1$, $s \leq 0$, and any integer $n, j > 5$, $p \leq \sqrt{n}$

$$\mathbf{E}_y\left(\mathrm{e}^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\mathbb{1}_{\{\mathrm{e}^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\leq 1\}},\ M_{n-p} \geq a_n(x-j)\right) \leq c_{(1)}\mathrm{e}^{-\alpha_{(1)}j}\log_+ x\,(1+y)\mathrm{e}^{-y}\mathrm{e}^{-(x+s)}. \tag{5.10}$$

Proof of Proposition 5.5 in admitting Lemma 5.6. To obtain Proposition 5.5 we need to resume all our previous inequalities, observe that for any $\epsilon \geq 0$

$$\begin{aligned}&\frac{\mathrm{e}^x}{x}\left|\mathbf{E}\left(\prod_{j\leq k}\mathbb{1}_{\{\sum_{u\in S_A}\frac{\widetilde{W}^u_{n,\beta_j,kill}}{\mathrm{e}^{\beta_j(x-\delta_j)}}\geq \mathrm{e}^{\beta_j V(u)}\}} - \prod_{j\leq k}\mathbb{1}_{\{\sum_{u\in S_A}B^W_{\beta_j,n,x-\delta_j}\geq 1\}};\ \sum_{u\in S_A}B_{n,x-\Delta}(u)\geq 1\right)\right|\\ &\leq P_1+P_2+P_3+P_4,\end{aligned}$$

with

$$P_1 = \frac{\mathrm{e}^x}{x}\sum_{j\le k}\mathbf{P}\left(\sum_{u\in S_A}\widetilde{W}^{u,x-\delta_j+V(u)}_{n,\beta_j,kill}\mathbb{1}_{\Gamma(u)^c}\ge\frac{\epsilon}{2}\right),$$

$$P_2 = \frac{\mathrm{e}^x}{x}\sum_{j\le k}\mathbf{P}\left(\sum_{u\in S_A}\widetilde{W}^{u,x-\delta_j+V(u)}_{n,\beta_j,kill}\mathbb{1}_{\{\widetilde{W}^{u,x-\delta_j+V(u)}_{n,\beta_j,kill}\le \mathrm{e}^{-\frac{\theta\beta_j}{\beta_j-1}\log_+\log_+(x-\delta_j+V(u))}\}}\ge\frac{\epsilon}{2}\right),$$

$$P_3 = \frac{\mathrm{e}^x}{x}\sum_{j\le k}\mathbf{P}\left(\sum_{u\in S_A}\widetilde{W}^{u,x-\delta_j+V(u)}_{n,\beta_j,kill}\mathbb{1}_{\Gamma(u)}B^{W,\theta}_{\beta_j,n,x-\delta_j}(u)\in[1-\epsilon,1],\sum_{u\in S_A}B_{n,x-\Delta}(u)\ge 1\right),$$

$$P_4 = \frac{\mathrm{e}^x}{x}\sum_{j\le k}\mathbf{P}\left(\exists u,v\in S_A, u\ne v,\ B^{W,\theta}_{\beta_j,n,x}(u)\mathbb{1}_{\Gamma(u)}B^{W,\theta}_{\beta_j,n,x}(v)\mathbb{1}_{\Gamma(v)}=1\right).$$

Suppose that: *for any $K>0$, $\eta>0$ there exists $\epsilon_0\in[0,1]$ such that for any $\theta>1$ there exists $A_1>A>0$ such that for all $\delta\in[-K,K],\epsilon\in[0,\epsilon_0]$ there exists N such that* $\forall n>N$, $x\in[A_1,\frac{3}{2}\log n]$

$$P_3 = P_3^{A,n,x}(\epsilon,\theta,\delta)\le\eta, \tag{5.11}$$

is true, then the Proposition 5.5 is too. Indeed let $\eta>0$ and $K>0$. It suffices first to fix ϵ_0 for the previous affirmation. We choose θ large enough such that for any $n>0$ and $x\in[1,\frac{3}{2}\log n]$ P_2 is smaller than η. Then there exists $A_1>A>0$ such that for all $\delta\in[-K,K],\epsilon\in[0,\epsilon_0]$ there exists N such that $\forall n>N$, $x\in[A_1,\frac{3}{2}\log n]$ P_1,P_3 and $P_4\le\eta$ and we conclude.

It remains to prove this affirmation. Main difficulty holds in the multiplication of variables and quantifiers, but idea is simply. We stay rigorous if we suppose $\delta=0$. It suffices to see that

$$\left\{\sum_{u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\Gamma(u)}B^{W,\theta}_{\beta,n,x}(u)\ge 1-\alpha\right\}-\left\{\sum_{u\in S_A}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\ge 1-\alpha\}}\mathbb{1}_{\Gamma(u)}\ge 1\right\}$$
$$\subset\left\{\exists u,v\in S_A,u\ne v,\ \mathbb{1}_{\Gamma(u),\Gamma(v)}B^{W,\theta}_{\beta,n,x}(u)B^{W,\theta}_{\beta,n,x}(v)=1\right\}.$$

and that by Lemma 5.4: for any $\theta>0$ there exists $A_0>A$ such that $\forall A>A_0$, $\exists\delta\in[-K,K]$, $N>0$ such that $\forall n>N,x\in[A,\frac{3}{2}\log n]$ the probability of this event is smaller than $\eta x\mathrm{e}^{-x}$. Finally this inclusion is also true on $\left\{\sum_{u\in S_A}B_{n,x-\Delta}(u)\ge 1\right\}$, hence by Proposition 5.3 under the same quantifiers, probability of the event $\left\{\sum_{u\in S_A}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\ge 1-\alpha\}}\mathbb{1}_{\Gamma(u)}\ge 1\right\}\cap$

$\left\{\sum_{u\in S_A} B_{n,x-\Delta}(u)\geq 1\right\}$ is very close $x\mathrm{e}^{-x}\chi(\beta,\frac{1}{\beta}\log(1-\alpha),\Delta)$. Identically the probability of $\left\{\sum_{u\in S_A}\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(x+V(u))}}\mathbb{1}_{\Gamma(u)}B^{W,\theta}_{\beta,n,x}(u)\geq 1\right\}$ is very close to $x\mathrm{e}^{-x}\chi(\beta,0,\Delta)$. We conclude in keeping in mind that χ is continuous.

$\square$

Proof of Lemma 5.6 (i)

Let $\epsilon>0$ and $\eta>0$. Proof consists to seek a good decomposition of $[0,\mathrm{e}^{-\frac{\theta\beta}{\beta-1}\log_+\log_+ x}]$. Let $\max_{k\leq\sqrt{n},n\geq 5}\mathrm{e}^x\mathbf{P}(\widetilde{W}^{kill}_{n-k,\beta}\geq \mathrm{e}^{\beta x}):=c_{26}<\infty$, and c_{27} such that for any A, $x>1$ $R(x-A)\leq c_{27}x$. Let $\theta>0$ large enough such that:

$$\frac{(c_{27}+c_{26}c_{27})}{\epsilon}\sum_{k\geq 2}k^{-\theta}<\eta. \tag{5.12}$$

We define the sequence $f_0=+\infty$ and $f_l(x+V(u)):=\frac{1}{\beta^l}(x+V(u))+\theta\sum_{0\leq j\leq l}\frac{\log(l+2-j)}{\beta^j}$ for $l\in\mathbb{N}^*$. A quickly study show that

$$[0,\mathrm{e}^{-\frac{2\theta\beta}{\beta-1}\log_+\log_+ x}]\subset\bigcup_{l=0}^{\lfloor\frac{\log(x+V(u))}{\log\beta}\rfloor+1}[\mathrm{e}^{-f_l(x+V(u))},\mathrm{e}^{-f_{l+1}(x+V(u))}].$$

Observe by Lyons' change of measure that

$$\begin{aligned}\frac{1}{\epsilon}\mathbf{E}\left(\sum_{|u|=k,u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\leq \mathrm{e}^{f_{x+V(u)}(1)}\}}\right) &\leq \frac{1}{\epsilon}\mathbf{E}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A}\mathrm{e}^{-x}2^{-\theta}\mathrm{e}^{-V(u)}\right)\\ &\leq \frac{R(x-A)\mathrm{e}^{-x}2^{-\theta}}{\epsilon}\\ &\leq \frac{c_{27}2^{-\theta}}{\epsilon}\mathrm{e}^{-x}x.\end{aligned}$$

Same

$$\begin{aligned}&\frac{1}{\epsilon}\quad\mathbf{E}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\mathrm{e}^{-f_{l+1}(x+V(u))}\geq\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\geq \mathrm{e}^{-f_l(x+V(u))}\}}\right)\\ \leq&\frac{1}{\epsilon}\quad\mathbf{E}\left(\sum_{|u|=k,u\in S_A}\mathrm{e}^{-\frac{1}{\beta^{l+1}}(x+V(u))-\theta\sum_{0\leq j\leq l+1}\frac{\log(l+1+2-j)}{\beta^j}}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\geq \mathrm{e}^{-f_l(x+V(u))}\}}\right),\end{aligned}$$

which is, by Markov property, equal to

$$= \frac{1}{\epsilon}\mathbf{E}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A} \mathrm{e}^{-\frac{1}{\beta^{l+1}}(x+V(u))-\theta \sum\limits_{0\le j\le l+1}\frac{\log(l+1+2-j)}{\beta^j}} \mathbf{P}\left(\widetilde{W}^{x+s}_{n-p,\beta,kill}\ge \mathrm{e}^{-f_l(x+s)}\}\right)_{s=V(u),|u|=p,u\in S_A}\right).$$

As $u\in S_A$ implies $|u|\le\sqrt{n}$

$$\mathbf{P}\left(\widetilde{W}^{x+s}_{n-p,\beta,kill}\ge \mathrm{e}^{-f_l(x+s)}\right)_{s=V(u),p=|u|,u\in S_A}\le c_{26}\mathrm{e}^{-x-s}\mathrm{e}^{\frac{1}{\beta^{l+1}}(x+V(u))+\theta\sum\limits_{0\le j\le l}\frac{\log(l+1+2-j)}{\beta^j}},$$

it follows that

$$\begin{aligned}
&\frac{1}{\epsilon}\mathbf{E}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\mathrm{e}^{-f_{l+1}(x+V(u))}\ge\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\ge \mathrm{e}^{-f_l(x+V(u))}\}}\right)\\
\le\ & \frac{1}{\epsilon}\mathbf{E}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A} c_{26}\mathrm{e}^{-x-V(u)}(l+2)^{-\theta}\right)\le \frac{c_{26}R(x-A)(l+2)^{-\theta}}{\epsilon}\mathrm{e}^{-x}.\\
\le\ & \frac{c_{26}c_{27}(l+2)^{-\theta}}{\epsilon}x\mathrm{e}^{-x}.
\end{aligned}$$

So we sum all these inequalities up to ∞ to obtain

$$\mathbf{P}\left(\sum_{k=0}^{\sqrt{n}}\sum_{|u|=k,u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\le \mathrm{e}^{-\frac{2\theta\beta}{\beta-1}\log_+\log_+ x}\}}\ge\epsilon\right)\le \eta\mathrm{e}^{-x}x.$$

Hence we have proved exactly (5.7) with $\theta'=2\theta$. ☐

Proof of (i bis) Suppose that there exists c_{28},c_{29} and c_{30} such that for any $n,j\in\mathbb{N}^*$, $p\le\sqrt{n}$, $s\le 0$, $x\ge 1$

$$\mathbf{E}\left(\widetilde{W}^{s+x}_{n-p,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{s+x}_{n-p,\beta,kill}\le \mathrm{e}^{-c_{28}\log_+\log(x+j)}\}};\,M_{n-p}\ge a_n(x+s-j)\right)\le c_{30}\mathrm{e}^{-(x+s)}\mathrm{e}^{-c_{29}j}, \tag{5.13}$$

then for $x=x+\frac{c_{29}}{\beta}j$ we get trivially

$$\mathbf{E}\left(\widetilde{W}^{s+x}_{n-p,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{s+x}_{n-p,\beta,kill}\le \mathrm{e}^{-c_{28}[\log_+\log(x+j)-\frac{1}{c_{28}}j]}\}}\mathbb{1}_{\{M_{n-p}\ge a_n(x+s-(1-\frac{c_{(4.6)}}{\beta})j)\}}\right)\le c_{30}\mathrm{e}^{-(x+s)}\mathrm{e}^{-\frac{c_{29}}{\beta}j}.$$

Since $\log_+\log(x+j)-\frac{1}{c_{28}}j\le\log_+\log_+ x$ when $x,j\ge 1$, for some $c_{31}>c_{30}$ then for any $n,j\in\mathbb{N}^*$, $p\le\sqrt{n}$, $s\le 0$, $x\ge 1$

$$\mathbf{E}\left(\widetilde{W}^{s+x}_{n-p,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{s+x}_{n-|u|,\beta,kill}\le e^{-c_{28}[\log_+\log x]}\}};\, M_{n-p}\ge a_n(x+s-j)\right)\le c_{31}e^{-(x+s)}e^{-\frac{c_{29}}{\beta}j}.$$

So if the last inequality is true then

$$\begin{aligned}
&\mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\le e^{-c_{28}\log_+\log_+ x}\}}\ge 1,\ M_n\ge a_n(x-j)\right)\\
&\le \mathbf{E}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\mathbf{E}\left(\widetilde{W}^{s+x}_{n-|u|,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{s+x}_{n-|u|,\beta,kill}\le e^{-c_{28}[\log_+\log x},\, M_{n-|u|}\ge a_n(x+V(u)-j)\}}\Big| u\in S_{-\frac{\kappa j}{2}}, V(u)=s\right)\right)\\
&\le c_{31}e^{-x}e^{-\frac{c_{29}}{\beta}j}\mathbf{E}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}e^{-V(u)}\right)\le c_{31}e^{-x}e^{-\frac{c_{29}}{\beta}j}R(x+\frac{\kappa j}{2})\\
&\le c_{(4)}e^{-x}e^{-\alpha_{(4)}j}.
\end{aligned}$$

with $c_{(4)}>c_{31}$ and $c_{(4)}<c_{29}$. Thus (i bis) is true if (5.13) is too, but it's exactly the same reasoning. It's suffices to introduce $f_l(x+V(u)+j):=(\frac{1+c_{(4.6)}}{\beta})^l(x+V(u)+j)+\theta\sum_{0\le s\le l}\frac{\log(l+2-j)}{(\frac{1+c_{(4.6)}}{\beta})^s}$ and keeping in mind

$$\mathbf{E}\left(\widetilde{W}^{s+x}_{n-p,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{s+x}_{n-p,\beta,kill}\in[e^{-f_l(x+s+j)},e^{-f_{l+1}(x+V(u)+j)}],\, M_{n-p}\ge a_n(x+s-j)\}}\right)\le e^{-\theta\log(l+1+2)}e^{-x-s}e^{-c_{29}j},$$

thanks to Lemma 4.6.

Proof of (ii) By Lemma 4.6, if $n\ge 5$, $p\le\sqrt{n}$ and $s\le 0$

$$\begin{aligned}
\mathbf{E}_y\left(e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\mathbb{1}_{\{e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\le e^{-(x+s)}\}}\right) &\le e^{-(x+s)}.\\
\mathbf{E}_y\left(e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\mathbb{1}_{\{2e^{-\frac{1}{\beta^2}(x+s)}\ge e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\ge e^{-\frac{1}{\beta}(x+V(u))}\}}\right) &\le c_{26}e^{-(x+s)}(1+y)e^{-y}.\\
&\dots\\
\mathbf{E}_y\left(e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\mathbb{1}_{\{2e^{-\frac{1}{\beta^t}(x+V(u))}\ge e^{-\beta(x+s)}\widetilde{W}^{kill}_{n-p,\beta}\ge e^{-\frac{1}{\beta^{t-1}}(x+V(u))}\}}\right) &\le c_{26}e^{-(x+s)}(1+y)e^{-y}.
\end{aligned}$$

We continue until $2e^{-\frac{1}{\beta^t}(x+s)}\ge 1 \iff t\ge\frac{1}{\log\beta}(\log(x+s)-\log\log 2)$. We get at most $c\log(x+s)$ term which explain that:

$$\mathbf{E}_y\left(\widetilde{W}^{x+s}_{n-p,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{x+s}_{n,\beta,kill}\le 1\}}\right)\le c_{25}\log_+(x+s)(1+y)e^{-y}e^{-(x+s)}.$$

Remark: Proof of (ii bis) run like proof of (i bis) knowing (i). □

5.5 End of proof of assertion in italic

Now we can affirm that:

For any $K, \epsilon > 0$ there exists $A = A(K, \epsilon)$ and $X > A$ (which depend only of R) such that for any $(\delta_1, ..., \delta_k, \Delta) \in [-K, K]^{k+1}$ there exists $N(\epsilon, \Delta)$ such that for all $n > N$, $x \in [X, X + \frac{1}{2}\log n]$

$$\left| \frac{\mathrm{e}^x}{x}\mathbf{P}\left(\bigcap_{j\le k}\{\sum_{u\in S_A}\frac{\widetilde{W}^u_{n,\beta_j,kill}}{\mathrm{e}^{\beta_j(x-\delta_j)}} \geq \mathrm{e}^{\beta_j V(u)}\},\ \sum_{u\in S_A} B_{n,x-\Delta}(u) \geq 1\right) - c_0\chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta)\right| \leq \epsilon. \tag{5.14}$$

It remains to make disappear S_A to obtain our result. Let $\epsilon > 0$. We see that for any $r \geq 0$,

$$\begin{aligned}
\mathbf{P}\left(\exists |u| \geq \sqrt{n} : V(u) \in [-r, 0], \min_{j\geq|u|} V(u_j) \geq -r\right) &\leq \sum_{k\geq\sqrt{n}} \mathbf{E}\left[\sum_{|u|=k} 1_{\{V(u)\in[-r,0], \min_{j\geq k} V(u_j)\geq -r\}}\right] \\
&= \sum_{k\geq\sqrt{n}} \mathbf{E}\left[\mathrm{e}^{S_k}, S_k \in [-r, 0], \underset{j\leq k}{S}_j \geq -r\right] \\
&\leq \sum_{k\geq\sqrt{n}} \mathbf{P}\left(S_k \in [-r, 0], \min_{j\geq l} S_j \geq -r\right).
\end{aligned}$$

We notice that $\mathbf{P}\left(S_k \in [-r, 0], \underset{j\geq k}{S_j} \geq -r\right) \leq c(1 + r)^2 k^{-\frac{3}{2}}$. Therefore

$$\mathbf{P}\left(\exists |u| \geq \sqrt{n} : V(u) \in [-r, 0], \min_{j\geq|u|} V(u_j) \geq -r\right) \leq c(1 + r)^2(\sqrt{n})^{-5}.$$

We also observe that:

$$\begin{aligned}
\mathbf{P}\left(\exists u \in T : V(u) \leq -r\right) &\leq \sum_{n\geq 0} \mathbf{E}\left[\sum_{|u|=n} 1_{\{V(u)\leq -r, V(u_k) > -r\ \forall k<n\}}\right] \\
&= \sum_{n\geq 0} \mathbf{E}\left[\mathrm{e}^{S_n}, S_n \leq -r, S_k > -r\ \forall k < n\right] \\
&\leq \mathrm{e}^{-r}.
\end{aligned}$$

On the event $\{\forall |u| \geq \sqrt{n} : V(u) \geq 0\} \cap \{\forall u \in T, V(u) \geq A - x\}$, we observe that $\frac{\widetilde{W}_{n,\beta}}{\mathrm{e}^{\beta x}} \geq 1$ and$M_n \leq \frac{3}{2}(\log n) - (x - \Delta)$ if and only if $\sum_{u\in S_A} \frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(x+V(u))}} \geq 1$ and $\sum_{u\in S_A} B_{n,x-\Delta} \geq 1$. Moreover with the two previous Propositions there exists $X > A > 0$ such that for n large enough and $x \geq X$.

$$\left|\frac{\mathrm{e}^x}{x}\mathbf{P}\left(\bigcap_{j\le k}\{\widetilde{W}_{n,\beta_j}\ge \mathrm{e}^{\beta_j(x-\delta_j)}\},\ M_n\le a_n(x-\Delta)\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right|\le$$
$$\le \frac{c_{32}(1+x-A)^2}{(\sqrt{n})^5}+\mathrm{e}^{A-x}+\left|\frac{\mathrm{e}^x}{x}\mathbf{E}\left(\bigcap_{j\le k}\{\sum_{u\in S_A}B^W_{\beta_j,n,x-\delta_j}\ge 1\},\ \sum_{u\in S_A}B_{n,x-\Delta}(u)\ge 1\right)-c_0\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right|$$
$$\le \frac{c_{32}(1+x-A)^2}{(\sqrt{n})^5}+\mathrm{e}^{A-x}+\eta.$$

The assertion 2.2 follows easily, if $x\in[X,X+\log\log n]$. □

6 Appendix

6.1 Proofs for the killed branching random walk

We state and prove 6 Lemmas. Their proofs require continual references to [3]. We recall that $\xi_n=\{\forall k\le n-b,\forall v\in\Omega(w_k),\ \min_{u\ge v,|u|=n}V(u)>a_n(x)+A\}$ and $\lozenge_{A,L,b}(\beta_j,\delta_j,y):=\mathbb{1}_{\left\{\mathrm{e}^{-\beta_j(\delta_j+L)}\le\sum_{|z|=b}\mathrm{e}^{-\beta_j(V(z)+y)}\mathbb{1}_{\{V(z)+y\le\delta_j+L+A,\ \min_{k\le b}V(z_k)+y\ge\delta_j\}}\right\}}$. Remenber also that $\widetilde{W}^{A,L}_{n,\beta}(x)=n^{\frac{3}{2}\beta}W^{A,L}_{n,\beta}(x)$

Lemma 6.1 *For any $\epsilon>0$ there exists $L_0>0$ such that $\forall L>L_0$, $A,x\ge 1$, $n\in\mathbb{N}$*

$$\left|\mathbf{P}\left(\widetilde{W}^{A,L}_{n,\beta}(x)\ge \mathrm{e}^{\beta x},\ M^{kill}_n\le a_n(x)\right)-\mathbf{P}\left(M^{kill}_n\le a_n(x)\right)\right|\le\epsilon\mathrm{e}^{-x}.$$

Proof of lemma 6.1. It's a consequence of Lemma 3.3 [3], indeed keeping in mind that there exists L_0 such that $\forall L>L_0$, $x\ge 1,n\in\mathbb{N}$ we have

$$\begin{aligned}\sum_{j\ge0}\mathbf{P}\left(M^{kill}_n\in I_n(x+j),m^{kill,(n)}\notin Z^{x+j,L}\right) &\le \sum_{j\ge0}\epsilon\mathrm{e}^{-j}\mathrm{e}^{-x}\\ &\le \epsilon\mathrm{e}^{-x},\end{aligned}$$

we notice that

$$\begin{aligned}&\left|\mathbf{E}\left((\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta}\ge\mathrm{e}^{\beta x}\}}-1)\mathbb{1}_{\{M^{kill}_n\le a_n(x)\}}\right)\right|=\left|\sum_{j\ge0}\mathbf{E}\left((\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta}\ge\mathrm{e}^{\beta x}\}}-1)\mathbb{1}_{\{M^{kill}_n\in I_n(x+j)\}}\right)\right|\\
=&\left|\sum_{j\ge0}\mathbf{E}\left((\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta}\ge\mathrm{e}^{\beta x}\}}-1)(\mathbb{1}_{\{M^{kill}_n\in I_n(x+j),m^{kill,(n)}\in Z^{x+j,L}\}}+\mathbb{1}_{\{M^{kill}_n\in I_n(x+j),m^{kill,(n)}\notin Z^{x+j,L}\}})\right)\right|\\
=&\left|\sum_{j\ge0}\mathbf{E}\left((\mathbb{1}_{\{\widetilde{W}^{A,L}_{n,\beta}\ge\mathrm{e}^{\beta x}\}}-1)(\mathbb{1}_{\{M^{kill}_n\in I_n(x+j),m^{kill,(n)}\notin Z^{x+j,L}\}})\right)\right|\\
\le&\ \epsilon\mathrm{e}^{-x}.\end{aligned}$$

Last equality follows from $M_n^{kill} \in I_n(x+j), m^{kill,(n)} \in Z^{x+j,L} \Rightarrow \widetilde{W}_{n,\beta}^{A,L} \geq \mathrm{e}^{\beta x}$. $\square$

Lemma 4.11 *$\forall K, \eta, L, A > 0$ $\exists D(A,\eta) > 0$ and $B(A,L,K,\eta) \geq 1$ such that $\forall b \geq B$, $n \geq b$, $x \geq D$ and $\Delta \in [-K, K]$*

$$\mathbf{Q}((\xi_n)^c, \omega_n \in Z_n^{x-\Delta,L}) \leq \eta n^{-\frac{3}{2}}. \tag{6.1}$$

Proof of lemma 4.11. Let $K, L, A, \eta > 0$, and $\Delta \in [-K, K]$. We take numbers $(e_k, 0 \leq k \leq n)$ such that

$$e_k = e_k^{(n)} = \begin{cases} k^{1/2} & \text{if} \quad \frac{n}{2} < k \leq \frac{3}{4}n \\ (n-k)^{1/2} & \text{if} \quad \frac{3}{4}n < k \leq n-a \end{cases}, \tag{6.2}$$

and denote

$$d_k = d_k(n, x-\Delta, L) := \begin{cases} 0 & \text{if} \quad 0 \leq k \geq \frac{n}{2} \\ \max(\frac{3}{2}\log n - x + \Delta - L - 1, 0) & \text{if} \quad \frac{n}{2} < k \leq n \end{cases}. \tag{6.3}$$

We say that $|u| = n$ is a good vertex if $u \in Z^{x-\Delta,L}$ and

$$\sum_{w \in \Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)}\{1 + (V(v) - d_k)_+\} \leq B\mathrm{e}^{-e_k} \qquad \forall 1 \leq k \leq n.$$

According to Lemma C.1 [3], there exists $B(= B(L))$ such that, for $n \geq 1$ and $x - \Delta \geq 0$

$$\mathbf{Q}(w_n \in Z^{x-\Delta,L}, w_n \text{ is not a good vertex }) \leq \frac{\eta}{n^{\frac{3}{2}}}. \tag{6.4}$$

For ξ_n to happen, every brother of the spine at generation less than $n - b$ must have its descendants at time n greater than $a_n(x) + A$. In others words,

$$\mathbf{Q}((\xi_n)^c, \omega_n \text{ is a good vertex}) = \mathbf{Q}\left[1 - \prod_{k=1}^{n-b}\prod_{u \in \Omega(\omega_k)} p(u, x-A), \omega_n \text{ is a good vertex}\right], \tag{6.5}$$

where $p(u, x-A) = \mathbf{P}_{V(u)}(M_{n-|u|}^{kill} \geq a_n(x-A))$ is the probability that the killed branching random walk rooted at u has its minimum greater $a_n(x) + A$ at time $n - |u|$. From Lemma 3.5 [3], we see that

$$-\log p(u, x-A) \leq 1 - p(u, x-A) \leq c_{33}(1 + V(u)_+)\mathrm{e}^{-(x-A)-V(u)}.$$

Since w_n is a good vertex, we have for $k \leq n/2$ (hence $d_k = 0$), $\sum_{u \in \Omega(\omega_k)} (1 + V(u)_+)\mathrm{e}^{-V(u)} \leq B\mathrm{e}^{-e_k} = B\mathrm{e}^{-k^{\frac{1}{12}}}$. It implies that for x large enough and $1 \leq k \leq n/2$,

$$\prod_{u \in \Omega(\omega_k)} p(u, x - A) \geq \exp(-c_{34} B\mathrm{e}^{-(x-A)}\mathrm{e}^{-k^{\frac{1}{12}}}).$$

It yields that

$$\prod_{k=1}^{\lfloor n/2 \rfloor} \prod_{u \in \Omega(\omega_k)} p(u, x - A) \geq \exp(-c_{34} B\mathrm{e}^{-(x-A)} \sum_{k=1}^{n/2} \mathrm{e}^{-k^{\frac{1}{12}}}) \geq \exp(-c_{35} B\mathrm{e}^{-(x-A)}).$$

Therefore, there exists $D_1(A) > 0$ such that for any $x \geq D_1$

$$\prod_{k=1}^{\lfloor n/2 \rfloor} \prod_{u \in \Omega(\omega_k)} p(u, x - A) \leq (1 - \eta)^{1/2}. \tag{6.6}$$

If $k > n/2$, we simply observe that if $M_k^{kill} \leq x$, a fortiori $M_l \leq x$. Since W_n (defined in (4.2) is a martingale, we have $1 = \mathbf{E}[W_l] \geq \mathbf{E}[\mathrm{e}^{-M_l}] \geq \mathrm{e}^{-x}\mathbf{P}(M_l \leq x)$ for any $l \geq 1$ and $x \in \mathbb{R}$. We get that

$$1 - p(u, x - A) \leq \mathbf{P}\left(M_{n-|u|} \leq a_n(x) + A - V(u)\right) \leq \mathrm{e}^{a_n(x-A)}\mathrm{e}^{-V(u)}.$$

We rewrite it (we have $x - A \geq 0$), $1 - p(u, x - A) \leq n^{\frac{3}{2}} e^{-V(u)}\mathrm{e}^{-x+A} = \mathrm{e}^{-(V(u)-d_k)}\mathrm{e}^{A-\Delta+L}$ for $n/2 < k \leq n$. Since w_n is a good vertex, we get that $\prod_{u \in \Omega(w_k)} p(u, x - A) \geq \mathrm{e}^{-c_{36} e_k \mathrm{e}^{A-\Delta+L}} = \mathrm{e}^{-c_{36}(n-k)^{1/12}\mathrm{e}^{A-\Delta+L}}$. Consequently,

$$\prod_{k=\lfloor n/2 \rfloor + 1}^{n-b} \prod_{u \in \Omega(\omega_k)} p(u, x - A) \geq \mathrm{e}^{-c_{36}\mathrm{e}^{+A+K+L} \sum_{\lfloor n/2 \rfloor + 1}^{n-b} \mathrm{e}^{-(n-k)^{\frac{1}{12}}}}.$$

It yields that there exists $B(A, \eta, K, L) \geq 1$ large enough such that $\forall b \geq B$, $n > b$, we have,

$$\prod_{k=\lfloor n/2 \rfloor + 1}^{n-b} \prod_{u \in \Omega(\omega_k)} p(u, x - A) \geq (1 - \eta)^{\frac{1}{2}}. \tag{6.7}$$

In view of (6.6) and (6.7), we have for $b \geq B$, $x \geq D_1$ and $n \geq b$, $\prod_{k=1}^{n-b} \prod_{u \in \Omega(w_k)} p(u, x - A) \geq (1 - \eta)$. Plugging into (6.5) yields that

$$\mathbf{Q}((\xi_n)^c, w_n \text{ is a good vertex}) \leq \eta \mathbf{Q}(w_n \text{ is a a good vertex}) \leq \eta \mathbf{Q}(w_n \in Z^{x-\Delta,L}).$$

It follows from (6.4) that

$$Q((\xi_n)^c, w_n \in Z^{x-A,L}) \leq \eta(\mathbf{Q}(w_n \in Z^{x-\Delta,L}) + n^{-\frac{3}{2}}).$$

Remember that the spine behaves as a centred random walk. Then apply (4.7) to see that $\mathbf{Q}(w_n \in Z^{x-\Delta,L}) \leq c_{37} n^{-\frac{3}{2}}$with c_{37} which run for any $\Delta \in [-K, K]$), which completes the proof of the Lemma. □

Lemma 4.12 $y \mapsto F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y)$ *is Riemann integrable and there exists a non-increasing function* $\bar{F} : \mathbb{R}_+ \to \mathbb{R}$ *such that* $|F(x)| \leq \bar{F}(x)$ *for any* $x \geq 0$ *and* $\int_{x \geq 0} x \bar{F}(x) < \infty$.

Proof of Lemma 4.12. We recall that by Proposition 4.1 the spine has the law of $(S_n)_{n \geq 0}$. We see that $\frac{\mathbb{1}_{\{V(\omega_b) = M_b\}}}{\sum_{|u|=b} \mathbb{1}_{\{V(u) = M_b\}}}$ is smaller than 1, and $\mathrm{e}^{V(\omega_b) - L} \leq \mathrm{e}^{\Delta}$. Hence, $|F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y)| \leq$

$\mathbf{P}(S_b \leq L - x) =: \bar{F}(x)$ which is non-increasing in x, and $\int_{x \geq 0} \bar{F}(x) x dx = \frac{1}{2} \mathbf{E}[(L - S_b)^2 1_{\{S_b \leq L\}}] < \infty$. Moreover, observe that

$$F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y) := \mathbf{E}_{\mathbf{Q}} \left[\frac{\mathrm{e}^{V(\omega_b) + y - L} \mathbb{1}_{\{V(\omega_b) = M_b\}}}{\sum_{|u|=b} \mathbb{1}_{\{V(u) = M_b\}}} \mathbb{1}_{\{V(\omega_b) + y \in [L + \Delta - 1, L + \Delta], \min_{k \leq b} V(\omega_k) + y \geq \Delta\}} \prod_{j \leq k} \Diamond_{A,L,b}(\beta_j, \delta_j, y) \right].$$

The fraction in the expectation is smaller than e^{Δ}. Using the identity $|\mathbb{1}_E - a \mathbb{1}_F| \leq 1 - a + |\mathbb{1}_E - \mathbb{1}_F|$ for $a \in (0, 1)$, it yields that for $y_2 \geq 0$,$\epsilon > 0$ and any $y_1 \in [y_2, y_2 + \epsilon]$,

$$\frac{1}{\mathrm{e}^{\Delta}} |F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y_1) - F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y_2)| \leq 1 - \mathrm{e}^{-\epsilon} + \sum_{j \in [0,k]} \mathbf{E}_{\mathbf{Q}} [|\Diamond_{A,L,b}(\beta_j, \delta_j, y_1) - \Diamond_{A,L,b}(\beta_j, \delta_j, y_2)|] +$$

$$\mathbf{E}_{\mathbf{Q}} \left[|\mathbb{1}_{\{V(\omega_b) + y_1 \in [L, L-1], \min_{k \leq b} V(\omega_b) + y_1 \geq 0\}} - \mathbb{1}_{\{V(\omega_b) + y_2 \in [L, L-1], \min_{k \leq b} V(\omega_b) + y_2 \geq 0\}}| \right].$$

We easily deduce that $y \mapsto F_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta, y)$ is Riemann integrable. □

Remark: The interest of this Lemma is to allow the application of the Lemma 2.2 [3] to the function $F_{A,L,b}$.

Lemma 4.13 *Let* $L, A > 0$ *and* $K, \eta > 0$. *Let* D *and* B *be as in Lemma 4.11 then* $\forall b \geq B, (\boldsymbol{\delta}, \Delta) \in [-K, K]^{k+1}$ $\exists N(b, L, \boldsymbol{\delta}, \Delta, \eta) > 0$ *such that* $\forall n > N$ *and* $\forall x \in [D, \log n]$

(6.8)

$$\left| \mathrm{e}^x \mathbf{E}_{\mathbf{Q}} \left[\frac{\mathrm{e}^{V(\omega_n)} 1_{\{V(\omega_n) = M_n^{kill}, \omega_n \in Z^{x-\Delta,L}\}}}{\sum_{|u|=n} 1_{\{V(u) = M_n^{kill}\}}} \prod_{j \leq k} \mathbb{1}_{\{\widetilde{W}_{n,\beta_j}^{A,L}(x - \delta_j) \geq \mathrm{e}^{\beta_j (x - \delta_j)}\}}, \xi_n \right] - c_{A,L,b}(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq (2 + \mathrm{e}^{K+1}) \eta.$$

Proof of lemma 4.13. Let Δ, L, A, η, D, B be as in the Lemma 4.11. Let $n > b > B$ and $x \geq D$. We denote by $\mathbf{Q}_{(6.8)}$ the expectation in (6.8). By the Markov property at time $n-b$ (for $n \geq 2b$), we have

$$\mathbf{Q}_{(6.8)} = \mathbf{E}_{\mathbf{Q}}\left[F^{kill}(V(\omega_{n-b})), V(\omega_k) \geq d_k \ \forall k \geq n-b, \xi_n\right],$$

where we recall that

$$d_k = d_k(n, x-\Delta, L) := \begin{cases} 0 & \text{if} \quad 0 \leq k \geq \frac{n}{2} \\ \max(\frac{3}{2}\log n - x + \Delta - L - 1, 0) & \text{if} \quad \frac{n}{2} < k \leq n \end{cases}$$

and $F^{kill}(y)$ is defined by

(6.9)
$$\mathbf{E}_{\mathbf{Q}_y}\left[\frac{\mathrm{e}^{V(\omega_b)}\mathbb{1}_{\{V(\omega_b)=M_b^{kill}\}}}{\sum_{|u|=b}\mathbb{1}_{\{V(u)=M_b^{kill}\}}}\prod_{j\leq k}\Diamond_{A,L,b}(\beta_j, \delta_j, 0 - a_n(x+L))\mathbb{1}_{\{V(\omega_b)\in I_n(x-\Delta), \min_{k\leq b}V(\omega_b)\geq a_n(x-\Delta+L)\}}\right],$$

$$= \mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{y+V(\omega_b)}\mathbb{1}_{\{V(\omega_b)=M_b^{kill}\}}}{\sum_{|u|=b}\mathbb{1}_{\{V(u)=M_b^{kill}\}}}\prod_{j\leq k}\Diamond_{A,L,b}(\beta_j, \delta_j, y - a_n(x+L))\mathbb{1}_{\{y+V(\omega_b)\in I_n(x-\Delta), y+\min_{k\leq b}V(\omega_b)\geq a_n(x-\Delta+L)\}}\right].$$

Notice that $F^{kill}(y) \leq n^{\frac{3}{2}}\mathrm{e}^{-x}\mathrm{e}^{\Delta}\mathbf{Q}_y\left(\min_{k\in[0,b]}V(\omega_k) \geq a_n(x+L), V(\omega_b) \in I_n(x-\Delta)\right)$. Hence

$$\begin{aligned}
&\left|\mathbf{Q}_{(6.8)} - \mathbf{E}_{\mathbf{Q}}\left[F^{kill}(V(\omega_{n-b})),\, V(\omega_k) \geq d_k,\, \forall k \geq n-b\right]\right| \\
=\ & \mathbf{E}_{\mathbf{Q}}\left[F^{kill}(V(\omega_{n-b})), V(\omega_k) \geq d_k,\, \forall k \geq n-b, (\xi_n)^c\right] \\
\leq\ & \frac{n^{\frac{3}{2}}}{\mathrm{e}^x}\mathrm{e}^{\Delta}\mathbf{E}_{\mathbf{Q}}\left[\mathbf{Q}_{V(\omega_{n-b})}\left(\min_{k\in[0,b]}V(\omega_k) \geq a_n(x-\Delta+L), V(\omega_b) \in I_n(x-\Delta)\right)\mathbb{1}_{\{V(\omega_k)\geq d_k \forall k\leq n-b; (\xi_n)^c\}}\right].
\end{aligned}$$

By Markov property, the last term is equal to

$$\frac{n^{\frac{3}{2}}}{\mathrm{e}^x}\mathrm{e}^{\Delta}\mathbf{E}_{\mathbf{Q}}\left(\omega_n \in Z^{x-\Delta,L}; (\xi_n)^c\right) \leq \eta\mathrm{e}^{-x}\mathrm{e}^{K+1},$$

by our choice of D and B. Therefore

$$\left|\mathbf{Q}_{(6.8)} - \mathbf{E}_{\mathbf{Q}}\left[F^{kill}(V(w_{n-b}), V(w_l) \geq d_l, \quad \forall l \leq n-b\right]\right| \leq \eta\mathrm{e}^{-x}. \tag{6.10}$$

We would like to replace $F^{kill}(y)$ by $n^{\frac{3}{2}}\mathrm{e}^{-x}F_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,y-a_n(x+L))$. We notice that

$$\begin{aligned}
&\frac{n^{\frac{3}{2}}}{\mathrm{e}^{x}}F_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,y-a_n(x+L))\\
&=\mathbf{E}_{\mathbf{Q}_y}\left[\frac{\mathrm{e}^{V(\omega_b)}\mathbb{1}_{\{V(\omega_b)=M_b\}}}{\sum\limits_{|u|=b}\mathbb{1}_{\{V(u)=M_b\}}}\prod_{j\le k}\Diamond_{A,L,b}(\beta_j,\delta_j,-a_n(x+L))\mathbb{1}_{\{V(\omega_b)\in I_n(x-\Delta),\min\limits_{k\le b}V(\omega_b)\ge a_n(x-\Delta+L)\}}\right].
\end{aligned}$$

We observe that the only difference with (6.9) is that the branching random walk is not killed any more. Since $\left|\frac{\mathbb{1}_{\{V(\omega_b)=M_b\}}}{\sum\limits_{|u|=b}\mathbb{1}_{\{V(u)=M_b\}}}-\frac{\mathbb{1}_{\{V(\omega_b)=M_b^{kill}\}}}{\sum\limits_{|u|=b}\mathbb{1}_{\{V(u)=M_b^{kill}\}}}\right|$ is always smaller than 1 and is equal to zero if no particle touched the barrier 0, we have that, for any $H\ge 0$ such that $H\le a_n(x+L)$

$$\left|\frac{\mathbb{1}_{\{V(\omega_b)=M_b\}}}{\sum\limits_{|u|=b}\mathbb{1}_{\{V(u)=M_b\}}}-\frac{\mathbb{1}_{\{V(\omega_b)=M_b^{kill}\}}}{\sum\limits_{|u|=b}\mathbb{1}_{\{V(u)=M_b^{kill}\}}}\right|\le \mathbb{1}_{\{\exists|u|\le b:V(u)\le a_n(x+L+H)\}}.$$

Consequently,

$$\begin{aligned}
&\left|F^{kill}(x)-n^{\frac{3}{2}}\mathrm{e}^{-x}F_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,x-a_n(x+L))\right|\\
\le\;&\mathbf{E}_{\mathbf{Q}_x}\left[\mathrm{e}^{V(w_b)}\mathbb{1}_{\{\exists|u|\le b:V(u)\le a_n(x+L+H),\min\limits_{k\in[0,b]}V(w_k)\ge a_n(x-\Delta+L),V(w_b)\in I_n(x-\Delta)\}}\right]\\
\le\;&\frac{n^{\frac{3}{2}}\mathrm{e}^{\Delta}}{\mathrm{e}^{x}}\mathbf{E}_{\mathbf{Q}_x}\left[\mathbb{1}_{\{\exists|u|\le b:V(u)\le a_n(x+L+H),\min\limits_{k\in[0,b]}V(w_k)\ge a_n(x-\Delta+L),V(w_b)\in I_n(x-\Delta)\}}\right]\\
=\;&n^{\frac{3}{2}}\mathrm{e}^{\Delta-x}G_H(x-a_n(x+L)),
\end{aligned}$$

with

$$G_H(y):=\mathbf{Q}_y\left(\{\exists|u|\le b:V(u)\le -H\}\cap\{\min_{k\in[0,b]}V(w_k)\ge\Delta,V(w_b)\in[\Delta+L-1,\Delta+L]\}\right).$$

It shows that, for any $H\in[0,a_n(x+L)]$

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[\left|F^{kill}(V(w_{n-b}))-n^{\frac{3}{2}}\mathrm{e}^{-x}F_{A,L,b}\left(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,V(w_{n-b})-a_n(x+L)\right)\right|\mathbb{1}_{\{V(w_l)\ge d_l,\forall l\le n-b\}}\right]\\
\le\;&n^{\frac{3}{2}}\mathrm{e}^{\Delta-x}\mathbf{E}_{\mathbf{Q}}\left[G_H\left(V(w_{n-b})-a_n(x+L)\right)\mathbb{1}_{\{V(w_l)\ge d_l,\forall l\le n-b\}}\right],
\end{aligned}$$

we choose H such that $\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\int_{y\ge 0}G_H(y)R_-(y)dy\le\frac{\eta}{2\mathrm{e}^{K+1}}$. The function G_H satisfies the conditions of Lemma 2.2 [3] for the same reasons than $F_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,.)$. By Lemma 2.2 [3], it yields that

$$\mathbf{E_Q}\left[\left|F^{kill}(V(\omega_{n-b})) - n^{\frac{3}{2}}\mathrm{e}^{-x}F_{A,L,b}\left(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,V(\omega_{n-b})-a_n(x+L)\right)\right|\mathbb{1}_{\{V(\omega_l)\geq d_l,\forall l\leq n-b\}}\right] \leq \eta\mathrm{e}^{-x},$$

for n large enough and $x\in[0,\log n]$. Combined with (6.10), we get

(6.11)
$$|\mathbf{Q}_{(6.8)} - n^{\frac{3}{2}}\mathrm{e}^{-x}\mathbf{E_Q}\left[F_{A,L,b}\left(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,V(w_{n-b})-a_n(x+L)\right)\mathbb{1}_{\{V(\omega_l)\geq d_l,\forall l\leq n-b\}}\right]| \leq 2\eta\mathrm{e}^{-x}.$$

Remenber the definition of $c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)$, we apply again Lemma 2.2 [3] to see that:

$$\mathbf{E_Q}\left[F_{A,L,b}\left(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,V(w_{n-b})-a_n(x+L)\right)\mathbb{1}_{\{V(\omega_l)\geq d_l,\forall l\leq n-b\}}\right] \sim \frac{c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)}{n^{\frac{3}{2}}},$$

as $n\to\infty$ uniformly in $x\in[0,\log n]$. Consequently, we have for n large enough and $x\in[0,\log n]$,

$$\left|n^{\frac{3}{2}}\mathrm{e}^{-x}\mathbf{E_Q}\left[F_{A,L,b}\left(\boldsymbol{\beta},\boldsymbol{\delta},\Delta,V(w_{n-b})-a_n(x+L)\right)\mathbb{1}_{\{V(\omega_l)\geq d_l,\forall l\leq n-b\}}\right] - \mathrm{e}^{-x}c_{A,L,b}(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\right| \leq \eta\mathrm{e}^{-x}.$$

The Lemma follows from (6.11). □

Yet two very close Lemma.

Lemma 6.2 *For all $L>0$*

$$\lim_{x\to\infty}\lim_{n\to\infty}\mathrm{e}^x\mathbf{P}(M_n^{kill}\in I_n(x), m^{kill,(n)}\in Z^{x,L}) = C_L. \tag{6.12}$$

C_L is defined in [3] p 22.

Lemma 6.3 *For all $A,L>0$, $(\delta,\Delta)\in\mathbb{R}^2$*

(6.13)
$$\lim_{x\to\infty}\lim_{n\to\infty}\mathrm{e}^x\mathbf{P}\left(\widetilde{W}_{n,\beta}^{A,L}(x-\delta)\geq \mathrm{e}^{\beta(x-\delta)}, M_n^{kill}\in I_n(x), m^{kill,(n)}\in Z^{x-\Delta,L}\right) = c_{A,L}(\beta,\delta,\Delta).$$

We give only proof of 6.2, 6.3 is identical.

Proof of Lemma 6.2. Let $\eta>0$. With Lemma 3.7 and 3.8 of [3] there exists $A>0$ such that there exists B_0 such that for any $b\geq B_0$, $n\geq 1$ and $x\geq A$

$$\begin{aligned}
\mathbf{Q}\left((\xi_n)^c, w_n\in Z^{x,L}\right) &\leq \frac{\eta}{n^{\frac{3}{2}}},\\
\left|\mathrm{e}^x\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_n)}\mathbb{1}_{\{V(w_n)=M_n\}}}{\sum_{|u|=n}\mathbb{1}_{\{V(u)=M_n^{kill}\}}}, w_n\in Z^{x,L}, \xi_n\right] - C_{L,b}\right| &\leq \eta,\\
|C_{L,b}-C_L| &\leq \eta.
\end{aligned}$$

By combining this three inequalities we get: $\forall \eta > 0$, $\exists N, A > 0$ such that for any $n > N$ and $x \in [A, \log n]$

$$\mathrm{e}^x \left| \mathbf{P}(M_n^{kill}, m^{kill,(n)} \in Z^{z,L}) - C_L \right| \leq 3\eta.$$

$\square$

6.2 Proofs for the section 5 and proof of Proposition 2.1

Careful reader knows that it remains two lemma without proof. We want also prove in Proposition 2.1. For this purpose it will be convenient to extent the statements of this two lemma with additional results. Extensions will be recognized as the assertions starting with (. bis). Recall that the event of particles S_A has been introduced to give a precise estimation of

$$\mathbf{P}\left(\bigcap_{j=1}^{k} \{ \sum_{u \in S_A} B^W_{\beta_j,n,x-\delta_j}(u) \geq 1 \}, \sum_{u \in S_A} B_{n,x-\Delta}(u) \geq 1 \right),$$

According to proposition 4.3, which treat the tail of distribution of $W^{kill}_{n,\beta}$, it was necessary to suppose A large. For Proposition 2.1 we need simply a bound, but it must be uniform in $n \in \mathbb{N}$. This requirement force us to study also

$$\mathbf{E}\left(\bigcap_{j=1}^{k} \{ \sum_{u \in S_{-\kappa j/2}} B^W_{\beta_j,n,x-\delta_j}(u) \geq 1 \}, \sum_{u \in S_{-j/2}} B_{n,x-\Delta}(u) \geq 1 \right),$$

with j large and $\kappa < c_{19}$ (c_{19} the constant from Lemma 4.6). Thus in the following our statements will include two part, first for the precise estimation second for our uniform bound. Proof of the second part which are very similar, are not always given with all details.

We recall some notations for $u \in S_A$:

$$W^u_{n,\beta,kill} = \sum_{|z|=n, z>u} \mathrm{e}^{-\beta(V(z)-V(u))} \mathbb{1}_{\{\min_{k \in [|u|,n]} V(z_k) - V(u) \geq 0\}},$$

$$B^W_{\beta_j,n,x}(u) = \mathbb{1}_{\{\widetilde{W}^u_{n,\beta,kill} \geq \mathrm{e}^{\beta(x+V(u))}\}}, \qquad B^{W,\theta}_{\beta_j,n,x}(u) = \mathbb{1}_{\{\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(x+V(u))}} \geq \mathrm{e}^{-\frac{\beta\theta}{\beta-1} \log_+ \log_+ x}\}},$$

$$\widetilde{W}^{u,a}_{n,\beta,kill} = \mathrm{e}^{-\beta a} W^u_{n,\beta,kill} \qquad \text{for any } a \in \mathbb{R}.$$

and “tilde“ means always $\times n^{\frac{3}{2}\beta}$.

Lemma 5.3 *(i)Recall that $R(x)$ is the renewal function of $(S_n)_{n\geq 0}$ previous defined. Let $\epsilon > 0$. There exists $A \geq 0$ such that for n large enough and $z \in [A, (\log n)^{\frac{1}{5}}]$,*

$$\left| \frac{\mathrm{e}^x}{R(x-A)} \mathbf{E}\left[\sum_{u \in S_A} B_{n,x-\Delta}(u) \prod_{j \leq k} B^W_{\beta_j,n,x-\delta_j}(u) \right] - \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta) \right| \leq \epsilon. \tag{6.14}$$

(i bis) There exists $c_{(3)}, \alpha_{(3)} > 0$ such that $\forall j, n \geq 0$

$$(3) := \mathbf{P}\left(\sum_{u \in S_{-\frac{\kappa j}{2}}} B^W_{\beta,n,x} \geq 1, M_n \geq a_n(x-j)\right) \leq c_{(3)} \mathrm{e}^{-\alpha_{(3)} j} x \mathrm{e}^{x}.$$

(ii) For any $|u| \geq 1$, let $\Gamma(u) := \{\forall 1 \leq k \leq |u| : \xi(u_k) < \mathrm{e}^{V(u_{k-1})+x-A)/2}\}$. We have

$$\mathbf{P}(\sum_{u \in S_A} \widetilde{W}^{u,x+V(u)}_{n,\beta,kill} \mathbb{1}_{\Gamma(u)^c} \geq 1) \leq c_{22} \log_+(x) \mathrm{e}^{-x},$$

uniformly in $A \geq 0$, $\Delta \in [-K, K]$ and $n \geq 1$. In particular $\mathbf{P}(\sum_{u \in S_A} B^W_{n,x} \mathbb{1}_{\Gamma(u)^c} \geq 1) \leq$ $c_{22} \log_+(x) \mathrm{e}^{-x}$.
(ii bis) There exists $c_{(2)}, \alpha_{(2)} > 0$ such that $\forall j, n \geq 0$

$$(2) := \mathbf{P}\left(\sum_{u \in S_{-\frac{\kappa j}{2}}} \frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(V(u)+x)}} \mathbb{1}_{\Gamma(u)^c} \geq 1,\ M_n \geq a_n(x-j)\right) \leq c_{(2)} \mathrm{e}^{-\alpha_{(2)} j} \log_+(x) \mathrm{e}^{-x}.$$

Be careful, here $\Gamma(u) := \{\forall 1 \leq k \leq |u| : \xi(u_k) < \mathrm{e}^{(V(u_{k-1})+x+\frac{\kappa j}{2})/2}\}$.

Proof of lemma 5.3. Start by (i), let $k \leq \sqrt{n}$. By the Markov property at time k, we have

$$\mathbf{E}\left[\sum_{u \in S_A} B_{n,x-\Delta}(u) \prod_{j \leq k} B^W_{\beta_j,n,x-\delta_j}(u) \mathbb{1}_{\{|u|=k\}}\right] =$$

$$\mathbf{E}\left[\sum_{u \in S_A} \mathbb{1}_{\{|u|=k\}} \mathbf{E}\left(\prod_{j \leq k} \mathbb{1}_{\{\mathrm{e}^{\beta_j(x-\delta_j+r)} \leq \widetilde{W}^{kill}_{n-k,\beta_j}\}};\ M^{kill}_{n-k} \leq a_n(x+r-\Delta)\right)_{r=V(u)}\right]. \tag{6.15}$$

We observe that $V(u) \in [A - x, 0]$ when $u \in S_A$. By Proposition 4.3 there exists $A > 0$ and $N \geq 1$ such that for any $n \geq N, k \leq \sqrt{n}$ and $x + r \in [A, \log n]$,

$$|\mathrm{e}^{x+r} \mathbf{E}\left(\prod_{j \leq k} \mathbb{1}_{\{\mathrm{e}^{\beta_j(x-\delta_j+r)} \leq \widetilde{W}^{kill}_{n-k,\beta_j}\}};\ M^{kill}_{n-k} \leq a_n(x+r-\Delta)\right) - \chi(\boldsymbol{\beta}, \boldsymbol{\delta}, \Delta)| \leq \epsilon.$$

Plugging it into (6.15), it implies that, for $n \geq N, k \leq \sqrt{n}$ and $z \in [A, \log n]$

$$\left|\mathrm{e}^x\mathbf{E}\left[\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\le k} B^W_{\beta_j,n,x-\delta_j}(u)\mathbb{1}_{\{|u|=k\}}\right] - \chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\mathbf{E}\left[\sum_{u\in S_A}\mathrm{e}^{-V(u)}\mathbb{1}_{\{|u|=k\}}\right]\right|$$
$$\le \epsilon\mathbf{E}\left[\sum_{u\in S_A}\mathrm{e}^{-V(u)}\mathbb{1}_{\{|u|=k\}}\right].$$

From the definition of S_A, we observe that by Lyons' change of measure $\mathbf{E}\left[\sum_{u\in S_A}\mathrm{e}^{-V(u)}\mathbb{1}_{\{|u|=k\}}\right] = \mathbf{P}(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1)$. Hence, we can rewrite the inequality above as

$$\left|\mathrm{e}^x\mathbf{E}\left[\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\le k} B^W_{\beta_j,n,x-\delta_j}(u)1_{\{|u|=k\}}\right] - \chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\mathbf{P}(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1)\right|$$
$$\le \epsilon\mathbf{P}\left(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1\right)$$

By definition of the renewal function $R(x)$, we have $R(x-A) = \sum_{k\ge 0} P\left(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1\right)$. Therefore, summing over $k\le\sqrt{n}$ (and since $|u|\le\sqrt{n}$ if $u\in S_A$), we get

$$\left|\mathrm{e}^x\mathbf{E}\left[\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\le k} B^W_{\beta_j,n,x-\delta_j}(u)\right] - \chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)R(x-A)\right|$$
$$\le \epsilon R(x-A) + \chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)\sum_{k>\sqrt{n}}\mathbf{P}\left(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1\right).$$

Observe that

$$\begin{aligned}
\mathbf{P}\left(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1\right) &\le \mathbf{P}\left(S_k \in]A-x,0], \min_{l<k} S_l \ge A-x\right)\\
&\le c_{38}(1+x-A)^3(1+k)^{-\frac{3}{2}}\\
&\le c_{38}(1+\log n)^3(1+k)^{-\frac{3}{2}},
\end{aligned}$$

for $n\ge 1$ and $x\in[A,\log n]$. Therefore, $P(S_k \ge A-x, S_k < S_l, \forall 0\le l<k-1) \le c_{39}\frac{1}{\sqrt{n}} \le \epsilon$ for n large enough. Since $R(x-A)$ is always bigger than 1, we obtain for $n\ge N$, and $x\in[A,\log n]$,

$$\left|\mathrm{e}^x\mathbf{E}\left[\sum_{u\in S_A} B_{n,x-\Delta}(u)\prod_{j\le k} B^W_{\beta_j,n,x-\delta_j}(u)\right] - \chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta)R(x-A)\right| \le \epsilon R(x-A)(1+\chi(\boldsymbol{\beta},\boldsymbol{\delta},\Delta))$$

This ends the proof of (i). For (i bis),

$$
\begin{aligned}
(3) \quad &\leq \mathbf{E}\left[\sum_{u\in S_{-\frac{\kappa j}{2}}} \mathbf{E}(\mathbb{1}_{\{\mathrm{e}^{\beta(x+r)}\leq \widetilde{W}^{kill}_{n-k,\beta}\}};\, M^{kill}_{n-k}\geq a_n(x+r-j))_{r=V(u)}\right] \\
&\leq c_{18}\mathrm{e}^{-c_{19}j}\mathbf{E}\left[\sum_{u\in S_{-\frac{\kappa j}{2}}} \mathrm{e}^{-V(u)-x}\right] \\
&\leq c_{18}\mathrm{e}^{-c_{19}j}\mathrm{e}^{-x}R(x+\frac{\kappa j}{2}) \\
&\leq c_{(3)}\mathrm{e}^{-\alpha_{(3)}j}x\mathrm{e}^{-x}.
\end{aligned}
$$

Now we treat (ii) and (ii bis). Similarly, we have by the Markov property, Lemma 4.6 and Lemma 5.6

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\Gamma(u)^c}\geq 1\right) \\
\leq\ &\mathbf{P}\left(\sum_{u\in S_A}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\leq 1\}}\mathbb{1}_{\Gamma(u)^c}\geq 1/2\right)+\mathbf{P}\left(\sum_{u\in S_A}B^W_{\beta,n,x}\mathbb{1}_{\Gamma(u)^c}\geq 1\right) \\
\leq\ &\mathbf{E}\left(\sum_{u\in S_A}\mathbf{E}\left(\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\{\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\leq 1\}}+B^W_{\beta,n,x}\right)\mathbb{1}_{\Gamma(u)^c}\right) \\
\leq\ &c_{40}\log_+(x)\mathrm{e}^{-x}\mathbf{E}(\sum_{u\in S_A}\mathrm{e}^{-V(u)}\mathbb{1}_{\Gamma(u)^c}).
\end{aligned}
$$

The application of Lemma is justified because $u\in S_A$ implies $|u|\leq\sqrt{n}$. Same, for (ii bis) note that

$$
\mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\widetilde{W}^{u,x+V(u)}_{n,\beta,kill}\mathbb{1}_{\Gamma(u)^c}\geq 1,\, M_n\geq a_n(x-j)\right)\leq \mathrm{e}^{-\alpha_{(2)}j}c_{41}\log_+(x)\mathrm{e}^{-x}\mathbf{E}(\sum_{u\in S_{-\frac{\kappa j}{2}}}\mathrm{e}^{-V(u)}\mathbb{1}_{\Gamma(u)^c}).
$$

Conclusion follows from this affirmation: *there exists c_{42} such that for any $X\in\mathbb{R}$*

$$
\sum_{k=0}^{\sqrt{n}}\mathbf{E}(\sum_{|u|=k,u\in S_X}\mathbb{1}_{\Gamma(u)^c}\mathrm{e}^{-V(u)})\leq c_{42}.
$$

With our integrability condition (1.4), this assertion is included in a proof of [3] (see page 28,29 and 30). $\square$

Lemma 5.4 *(i)Set $K,\theta>0$. There exists a constant $c_{23}>0$ such that for any $(\delta_1,\delta_2,\Delta)\in[-K,K]^3$ $x\geq A\geq 0$, and $n\geq 1$ we get the following inequalities:*

$$\mathbf{E}\left(\sum_{u\neq v,\in S_A} B^{W,\theta}_{\beta_1,n,x-\delta_1}(u)B^{W,\theta}_{\beta_2,n,x-\delta_2}(v)\mathbb{1}_{\Gamma(u)\cap\Gamma(v)}\right)\leq c_{23}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-x}\mathrm{e}^{-A}, \tag{6.16}$$

$$\mathbf{E}\left(\sum_{u\neq v,\in S_A} B^{W,\theta}_{\beta_1,n,x-\delta_1}(u)B_{n,x-\Delta}(v)\mathbb{1}_{\Gamma(u)\cap\Gamma(v)}\right)\leq c_{23}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-x}\mathrm{e}^{-A}. \tag{6.17}$$

(i bis) There exists constant $\alpha_{(4)},c_{(4)}>0$ such that for any $j,x\geq 0$, and $n\geq 1$ we get the following inequalities:
(6.18)

$$\mathbf{E}\left(\sum_{u\neq v,\in S_A} B^{W,\theta}_{\beta_1,n,x}(u)B_{n,x}(v)\mathbb{1}_{\Gamma(u)\cap\Gamma(v)};\, M_n\geq a_n(x-j)\right)\leq c_{(4)}\mathrm{e}^{-\alpha_{(4)}j}(\log x)^{\frac{(\beta+1)\theta}{\beta-1}+1}\mathrm{e}^{-x}.$$

Proof of Lemma 5.4. (6.16) and (6.17) have quasi-identical proofs. We will thus treat only the first, in the particular case $\delta_1=\delta_2=0$ (case different to 0 is identical). For (i bis) we will make some checkpoint (signalled by a **For (i bis)**) at the important moments for explain the proof, but in a sake of concision we don't reproduce entirely the proof. First observe that

$$\mathbf{E}\left(\sum_{u\neq v\in S_A} B^{W,\theta}_{n,x}(u)B^{W,\theta}_{n,x}(v)\mathbb{1}_{\Gamma(u)}\mathbb{1}_{\Gamma(v)}\right)\leq 2\mathbf{E}\left(\sum_{u\neq v\in S_A,|u|\leq|v|} B^{W,\theta}_{n,x}(u)B^{W,\theta}_{n,x}(v)\mathbb{1}_{\Gamma(u)}\right). \tag{6.19}$$

Then for $|u|\geq|v|$, and $u\neq v$, notice that $B^{W,\theta}_{n,x}(u)$ depends on the branching random walk rooted at u, whereas $B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{u\in S_A\}}$ is independent of it (even if v is a (strict) ancestor of u). Therefore, by the branching property,

$$\mathbf{E}\left(\sum_{u\neq v\in S_A} B^{W,\theta}_{n,x}(u)B^{W,\theta}_{n,x}(v)\mathbb{1}_{\Gamma(u)}\mathbb{1}_{\Gamma(v)}\right)\leq 2\mathbf{E}\left(\sum_{u\neq v,|u|\leq|v|}\Phi^{\theta}(V(u)+x,n-|u|)B^{W,\theta}_{n,z}(v)\mathbb{1}_{\{u,v\in S_A\}}\mathbb{1}_{\Gamma(u)}\right)$$

with $\forall x\geq 0$ and $l\leq n$

$$\Phi^{\theta}(r,l) := \mathbf{P}\left(\frac{\widetilde{W}^{kill}_{l,\beta}}{\mathrm{e}^{\beta r}} \geq \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log_+\log_+ r]}\right). \tag{6.20}$$

By Lemma 4.6, we have $\Phi(V(u)+x, n-|u|) \leq c_{20}\mathrm{e}^{-x-V(u)}(\log(x+V(u)))^{\frac{\theta}{\beta-1}} \leq c_{20}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x-V(u)}$ for $|u| \leq \sqrt{n}$, which is the case when $u \in S_A$. It gives that

$$\begin{aligned}
&\mathbf{E}\left[\sum_{u\neq v, |u|\geq|v|} B^{W,\theta}_{n,x}(u)B^{W,\theta}_{n,x}(v)\mathbb{1}_{\Gamma(u)}\mathbb{1}_{\Gamma(v)}\mathbb{1}_{\{u,v\in S_A\}}\right] \\
&\leq c_{20}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x}\sum_{k\geq 0}\mathbf{E}\left[\sum_{|u|=k}\mathrm{e}^{-V(u)}\mathbb{1}_{\{u\in S_A\}}\mathbb{1}_{\{\Gamma(u)\}}\sum_{v\neq u, |v|\leq k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_A\}}\right].
\end{aligned}$$

The weight $\mathrm{e}^{-V(u)}$ hints for a change of measure from $\mathbf{P}$ to $\mathbf{Q}$. For any $k \geq 0$, we have by proposition 4.1 (ii)

$$\mathbf{E}\left[\sum_{|u|=k}\mathrm{e}^{-V(u)}\mathbb{1}_{\{u\in S_A\}}\mathbb{1}_{\{\Gamma(u)\}}\sum_{v\neq u,|v|\leq k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_A\}}\right] = \mathbf{E}_{\mathbf{Q}}\left[1_{\{\omega_k\in S_A\}}\mathbb{1}_{\Gamma(\omega_k)}\sum_{v\neq\omega_k,|v|\leq k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_A\}}\right]$$

We have to discuss on the location of the vertex v with respect to ω_k. We say that 'u non eq v' if v is not an ancestor of u, nor u is an ancestor of v. If $v \neq u$ and $|v| \leq k = |u|$, then either 'v non eq u', or $v = \omega_l$ for some $l < k$. The Lemma will be proved once the following two estimates are shown:

$$\sum_{k\geq 0}\mathbf{E}_{\mathbf{Q}}\left[\sum_{v \text{ non eq } \omega_k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_A\}}, \omega_k\in S_A, \Gamma(\omega_k)\right] \leq c_{43}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-A}, \tag{6.21}$$

$$\sum_{k\geq 0}\sum_{l=0}^{k-1}\mathbf{E}_{\mathbf{Q}}\left[B^{W,\theta}_{n,x}(\omega_l), \omega_k\in S_A, \Gamma(\omega_k)\right] \leq c_{44}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}. \tag{6.22}$$

For (i bis) $\Phi^{\theta}(r,l) := \mathbf{P}\left(\frac{\widetilde{W}^{kill}_{l,\beta}}{\mathrm{e}^{\beta x}} \geq \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log_+\log_+ r]},\ M^{kill}_l \geq a_n(x+r-j)\right)$ and $\Phi^{\theta}(V(u), n-|u|) \leq c_{18}\mathrm{e}^{-c_{19}j}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x-V(u)}$. We obtain also

$$\mathbf{E}\left(\sum_{u\neq v,\in S_{-\frac{\kappa j}{2}}} B^{W,\theta}_{\beta_1,n,x}(u)B_{n,x}(v)\mathbb{1}_{\Gamma(u)\cap\Gamma(v)};\, M_n\geq a_n(x-j)\right)$$
$$\leq c_{18}\mathrm{e}^{-c_{19}j}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x}\sum_{k\geq 0}\mathbf{E}_{\mathbf{Q}}\left[\mathbb{1}_{\{\omega_k\in S_A\}}\mathbb{1}_{\Gamma(\omega_k)}\sum_{v\neq\omega_k,|v|\leq k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_{-\frac{\kappa j}{2}}\}};\, M_n\geq a_n(x-j)\right].$$

So it suffices to prove

$$\sum_{k\geq 0}\mathbf{E}_{\mathbf{Q}}\left[\sum_{v \text{ non eq } \omega_k} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_{-\frac{\kappa j}{2}}\}},\omega_k\in S_{-\frac{\kappa j}{2}},\Gamma(\omega_k)\right]\leq c_{45}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{\frac{\kappa j}{2}}, \tag{6.23}$$

$$\sum_{k\geq 0}\sum_{l=0}^{k-1}\mathbf{E}_{\mathbf{Q}}\left[B^{W,\theta}_{n,x}(\omega_l),\omega_k\in S_{-\frac{\kappa j}{2}},\Gamma(\omega_k)\right]\leq c_{46}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{\frac{\kappa j}{2}}. \tag{6.24}$$

Since $\kappa < c_{19}$ we will get well (6.18) with $\alpha_{(4)}=\frac{c_{19}}{2}$.

Back to (i) Let us prove (6.22). We have

$$\begin{aligned}
&\sum_{k\geq 0}\sum_{l=0}^{k-1}\mathbf{E}_{\mathbf{Q}}\left[B^{W,\theta}_{n,x}(\omega_l),\omega_k\in S_A,\Gamma(\omega_k)\right]\\
=\ &\sum_{l\geq 0}\sum_{k>l}\mathbf{E}_{\mathbf{Q}}\left[B^{W,\theta}_{n,x}(\omega_l),\omega_k\in S_A,\Gamma(\omega_k)\right]\\
=\ &\sum_{l\geq 0}\mathbf{E}_{\mathbf{Q}}\left[B^{W,\theta}_{n,x}(w_l)\mathbb{1}_{\{w_l\in S_A\}}\sum_{k>l}1_{\{w_k\in S_A\}\cap\Gamma(w_k)}\right].
\end{aligned}$$

Let t_l be the first time t after l such that $V(w_t)<V(w_l)$. If $k>l$ and $w_k\in S_A$, then $V(w_k)<V(w_l)$, which means that necessarily $k\geq t_l$ (and $t_l<\sqrt{n}$). Moreover, we have $\Gamma(w_i)\subset\Gamma(w_j)$ if $i\leq j$. Thus,

$$\begin{aligned}
\sum_{k>l}\mathbb{1}_{\{w_k\in S_A\}\cap\Gamma(w_k)} &= \mathbb{1}_{\{w_{t_l}\in S_A,t_l<\sqrt{n}\}}\sum_{k\geq t_l}\mathbb{1}_{\{w_k\in S_A\}\cap\Gamma(w_k)}\\
&\leq \mathbb{1}_{\{w_{t_l}\in S_A,t_l<\sqrt{n}\}\cap\Gamma(w_{t_l})}\sum_{k\geq t_l}\mathbb{1}_{\{\min_{t_l\leq j<k}V(w_j)>V(w_k)\geq A-x\}}
\end{aligned}$$

We observe that $B^{W,\theta}_{n,x}$ is a function of the branching random walk killed below $V(w_l)$ and therefore is independant of the subtree rooted at w_{t_l}. Therefore, applying the branching property, we get

$$
\begin{aligned}
& \mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\sum_{k>l}\mathbb{1}_{\{\omega_k\in S_A\}\cap\Gamma(\omega_k)}\right] \\
\leq\ & \mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{w_{t_l}\in S_A, t_l<\sqrt{n}\}\cap\Gamma(w_{t_l})}\sum_{k\geq t_l}\mathbb{1}_{\{\min_{t_l\leq j<k}V(w_j)>V(w_k)\geq A-x\}}\right] \\
=\ & \mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{w_{t_l}\in S_A, t_l<\sqrt{n}\}\cap\Gamma(w_{t_l})}R(x-A+V(w_{t_l}))\right].
\end{aligned}
$$

We have $V(w_{t_l}) < V(w_l)$. Since R is a non-decreasing function, we obtain

$$
\begin{aligned}
& \mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\sum_{k>l}\mathbb{1}_{\{\omega_k\in S_A\}\cap\Gamma(\omega_k)}\right] \\
\leq\ & \mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\mathbb{1}_{\{\omega_{t_l}\in S_A, t_l<\sqrt{n})\}\cap\Gamma(\omega_{t_l})}R(x-A+V(\omega_l))\right] \\
\leq\ & \mathbf{E_Q}\left[\mathbb{1}_{\{\omega_l\in S_A\}}R(x-A+V(\omega_l))\widetilde{\Phi}(V(\omega_l),n-l)\right] \\
\leq\ & \mathbb{1}_{\{l\leq\sqrt{n}\}}\mathbf{E_Q}\left[\mathbb{1}_{\{\min_{j<l}V(\omega_j)>V(\omega_l)\geq A-x\}}R(x-A+V(\omega_l))\widetilde{\Phi}(V(\omega_l),n-l)\right],
\end{aligned}
$$

where, $\tau_0^- := \min\{j\geq 0 : V(\omega_j)<0\}$, then (when $i>n-\sqrt{n}$) and $\widetilde{\Phi}(r,i)$ is:

$$
\mathbf{Q}\left(\tau_0^-<\sqrt{n}, \frac{\widetilde{W}_{i,\beta}^{kill}}{\mathrm{e}^{\beta(x+r)}}\geq \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log_+\log_+ r]}, \forall 1\leq j\leq \tau_0^-, \xi(\omega_j)\leq \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}\right)
$$

By Proposition 4.1 (iii), it implies that

$$
\mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\sum_{k>l}\mathbb{1}_{\{\omega_k\in S_A\}\cap\Gamma(\omega_k)}\right]
$$

$$
\leq \mathbb{1}_{\{l\leq\sqrt{n}\}}\mathbf{E}\left[\mathbb{1}_{\{\min_{j<l}V(S_j)>V(S_l)\geq A-x\}}R(x-A+V(S_l))\widetilde{\Phi}(V(S_l),n-l)\right]. \tag{6.25}
$$

Let us estimate $\widetilde{\Phi}(r,i)$. We have to decompose along the spine. Notice that

$$
\widetilde{\Phi}(r,i) = \mathbf{Q}\left[\tau_0^-<\sqrt{n}, \sum_{j=1}^{\tau_0^-}\sum_{z\in\Omega(\omega_j)}\frac{\widetilde{W}_{i-j,\beta}^{V(z),kill}}{\mathrm{e}^{\beta(x+r)}}\geq \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log\log(x+r)]}, \forall 1\leq j\leq\tau_0^-, \xi(\omega_j)\leq \mathrm{e}^{\frac{r+V(\omega_{j-1})+x-A}{2}}\right],
$$

with by definition: $W^{y,kill}_{i-j,\beta} := \sum_{|y|=i-j} \mathrm{e}^{-\beta(y+V(z))} \mathbb{1}_{\{\min_{k\le i-j} V(z_k)+y\ge 0\}}$. It's smaller than than

$$\mathbf{Q}\left[\sum_{j=1}^{\tau_0^-\wedge\sqrt{n}} \sum_{z\in\Omega(\omega_j)} \frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}} \mathbb{1}_{\{\frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\le 1\}} \ge \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log\log(x+r)]}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}\right]$$
$$+\mathbf{Q}\left[\sum_{j=1}^{\tau_0^-\wedge\sqrt{n}} \sum_{z\in\Omega(\omega_j)} \frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}} \mathbb{1}_{\{\frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\ge 1\}} \ge \mathrm{e}^{-[\frac{\beta\theta}{\beta-1}\log\log(x+r)]}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}\right].$$

We treat each term separately. First is smaller than

$$\sum_{j=1}^{\sqrt{n}} \mathbf{Q}\left[\sum_{z\in\Omega(\omega_j)} \frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}} \mathbb{1}_{\{\frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\le 1\}}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right] (\log(x+r))^{\frac{\beta\theta}{\beta-1}}$$
$$= \sum_{j=1}^{\sqrt{n}} \mathbf{Q}\left[\sum_{z\in\Omega(\omega_j)} \mathbf{E}_{V(z)}\Big(\frac{\widetilde{W}^{kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}} 1_{\{\frac{\widetilde{W}^{kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\le 1\}}\Big), \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right] (\log(x+r))^{\frac{\beta\theta}{\beta-1}},$$

which, according to Lemma 5.6 line (5.9), is smaller than

$$c_{25}(\log(x+r))^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-(x+r)} \sum_{j=1}^{\sqrt{n}} \mathbf{Q}\left[\sum_{z\in\Omega(\omega_j)} (1+V(z)_+)\mathrm{e}^{-V(z)}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right].$$

The second is equal to

$$\mathbf{Q}\left[\tau_0^- < \sqrt{n}, \sum_{j=1}^{\tau_0^-} \sum_{z\in\Omega(\omega_j)} \mathbb{1}_{\{\frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\ge 1\}} \ge 1, \forall 1\le j\le \tau_0^-, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}\right]$$
$$\le \sum_{j=1}^{\sqrt{n}} \mathbf{E}_{\mathbf{Q}}\left[\sum_{z\in\Omega(\omega_j)} \mathbb{1}_{\{\frac{\widetilde{W}^{V(z),kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}}\ge 1\}}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right]$$
$$= \sum_{j=1}^{\sqrt{n}} \mathbf{E}_{\mathbf{Q}}\left[\sum_{z\in\Omega(\omega_j)} \mathbf{P}_{V(z)}\left(\frac{\widetilde{W}^{kill}_{i-j,\beta}}{\mathrm{e}^{\beta(x+r)}} \ge 1\right), \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right]$$
$$\le c_{20}\mathrm{e}^{-(x+r)} \sum_{j=1}^{\sqrt{n}} \mathbf{Q}\left[\sum_{z\in\Omega(\omega_j)} (1+V(z)_+)\mathrm{e}^{-V(z)}, \xi(\omega_j) \le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2}, j<\tau_0^-\right].$$

The sum of the two terms is less than:

$$\mathrm{e}^{-(x+r)}\left((\log(x+r))^{\frac{\beta\theta}{\beta-1}+1}+Mc_{13}\right)\sum_{j=1}^{\sqrt{n}}\mathbf{Q}\left[\sum_{z\in\Omega(\omega_j)}(1+V(z)_+)\mathrm{e}^{-V(z)},\xi(\omega_j)\le \mathrm{e}^{(r+V(\omega_{j-1})+x-A)/2},j<\tau_0^-\right]$$

$$\le c_{30}(\log(x+r))^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-(x+r)}\sum_{j=1}^{\sqrt{n}}\mathbf{E}_{\mathbf{Q}}\left[\mathrm{e}^{-V(w_{j-1})}(1+V(w_{j-1}))\mathrm{e}^{(r+V(w_{j-1})+x-A)/2},j<\tau_0^-\right].$$

It follows by Lemma B.2 (ii) of [3] that

$$\begin{aligned}\widetilde{\Phi}(r,i) &\le c_{47}(\log(x+r))^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\mathrm{e}^{-(r+x-A)/2}\sum_{j\ge1}\mathbf{E}[\mathrm{e}^{-S_{j-1}/2}(1+S_{j-1}),j<\tau_0^-]\\ &\le c_{48}(\log(x+r))^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\mathrm{e}^{-(r+x-A)/2}\end{aligned}$$

Going back to (6.25), we obtain

$$\begin{aligned}&\mathbf{E}_{\mathbf{Q}}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\sum_{k>l}\mathbb{1}_{\{\omega_k\in S_A\}\cap\Gamma(\omega_k)}\right]\\ &\le c_{48}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\mathbf{E}\left[\mathbb{1}_{\{\min_{j<l}S_j>S_l\ge A-x\}}R(x-A+S_l)\mathrm{e}^{-(S_l+x-A)/2}\right].\end{aligned}$$

We conclude with

$$\begin{aligned}&\sum_{l\ge0}\mathbf{E}_{\mathbf{Q}}\left[B_{n,x}^{W,\theta}(\omega_l)\mathbb{1}_{\{\omega_l\in S_A\}}\sum_{k>l}\mathbb{1}_{\{\omega_k\in S_A\}\cap\Gamma(\omega_k)}\right]\\ \le\;& c_{48}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\sum_{l\ge0}\mathbf{E}\left[\mathbb{1}_{\{\min_{j<l}S_j>S_l\ge A-x\}}R(x-A+S_l)\mathrm{e}^{-(S_l+x-A)/2}\right]\\ =\;& c_{48}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\sum_{l\ge0}\mathbf{E}_{x-A}\left[\mathbb{1}_{\{\min_{j<l}S_j>S_l\ge0\}}R(S_l)\mathrm{e}^{-(S_l)/2}\right]\\ =\;& c_{48}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A}\int_{-(x-A)}^{0}\mathrm{e}^{-(x-A+y)/2}R(x-A+y)U(dy)\\ \le\;& c_{44}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{-A},\end{aligned}$$

where U denote the renewal measure, and the last inequality comes from Section XI.1 of [13].

For (i bis) It suffices to replace A by $\kappa\frac{j}{2}$ to obtain

$$\sum_{k\geq 0}\sum_{l=0}^{k-1}\mathbf{E_Q}\left[B_{n,x}^{W,\theta}(\omega_l),\omega_k\in S_{-\frac{\kappa j}{2}},\Gamma(\omega_k)\right]\leq c_{46}(\log x)^{\frac{\beta\theta}{\beta-1}+1}\mathrm{e}^{\frac{\kappa j}{2}}.$$

Back to (i) It remains to treat (6.22). Decomposing the sum $\sum\limits_{v \text{ non eq } w_k}$ along the spine, we see that

$$\sum_{v \text{ non eq } w_k} B_{n,x}^{W,\theta}(v)\mathbb{1}_{\{v\in S_A\}}=\sum_{l=1}^{k}\sum_{x\in\Omega(\omega_l)}\sum_{v\geq x}B_{n,x}^{W,\theta}(v)\mathbb{1}_{\{v\in S_A\}},$$

where $\Omega(\omega_l)$ is as usual the set of brothers of ω_l. The branching random walk rooted at $x\in\Omega(\omega_l)$ has the same law under $\mathbf{P}$ and $\mathbf{Q}$. Let as before $G_\infty:=\sigma\{\omega_j,\Omega(\omega_j),V(\omega_j),V(x),x\in\Omega(\omega_j),j\geq 0\}$ be the sigma-algebra associated to the spine and its brothers. We have, for $z\in\Omega(\omega_l)$

$$\mathbf{E_Q}\left[\sum_{v\geq z}B_{n,x}^{W,\theta}(v)\mathbb{1}_{\{v\in S_A\}}|G_\infty\right]=\mathbf{E_Q}\left[\sum_{v\geq z}\Phi^\theta(V(v),n-|v|)\mathbb{1}_{\{v\in S_A\}}|G_\infty\right]\tag{6.26}$$

with the notation of (6.20), which is

$$\leq c_{32}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x}\mathbf{E_Q}\left[\sum_{v\geq z}\mathrm{e}^{-V(v)}\mathbb{1}_{\{v\in S_A\}}|G_\infty\right]$$

We observe now that if $v\geq x$ and $v\in S_A$, then $\min\limits_{|z|\leq j\leq |v|-1}V(v_j)>V(v)>A-x$. Therefore

$$\mathbf{E_Q}\left[\sum_{v\geq z}\mathrm{e}^{-V(v)}\mathbb{1}_{\{v\in S_A\}}\big|G_\infty\right]\leq\mathbf{E}_{V(z)}\left[\sum_{v\in T}\mathrm{e}^{-V(v)}\mathbb{1}_{\{\min\limits_{|x|\leq j\leq |v|-1}V(v_j)>V(v)>A-x\}}\right].$$

Thanks to (4.1), we have

$$\begin{aligned}\mathbf{E}_{V(z)}\left[\sum_{v\in T}\mathrm{e}^{-V(v)}\mathbb{1}_{\{\min\limits_{|z|\leq j\leq |v|-1}V(v_j)>V(v)>A-x\}}\right] &= \mathrm{e}^{-V(z)}\mathbf{E}\left[\sum_{i\geq 0}\mathbb{1}_{\{\min\limits_{|z|\leq j\leq |v|-1}S_j>S_i>A-x-r\}}\right]_{r=V(z)}\\ &= \mathrm{e}^{-V(z)}R(x-A+V(z))\end{aligned}$$

by definition of the renewal function R. Going back to (6.26), we get that for any $z\in\Omega(w_l)$

$$\mathbf{E_Q}\left[\sum_{v\geq z}B_{n,x}^{W}(v)\mathbb{1}_{\{v\in S_A\}}|G_\infty\right]\leq c_{49}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x}\mathrm{e}^{-V(z)}R(x-A+V(z)).\tag{6.27}$$

and **For (i bis)** it suffices to replace A by $\kappa\frac{j}{2}$ to obtain

$$\mathbf{E_Q}\left[\sum_{v\geq z} B^{W,\theta}_{n,x}(v)\mathbb{1}_{\{v\in S_{-\frac{\kappa j}{2}}\}}|G_\infty\right] \leq c_{50}(\log x)^{\frac{\theta}{\beta-1}}\mathrm{e}^{-x}\mathrm{e}^{-V(z)}R(x+\frac{\kappa j}{2}+V(z)). \tag{6.28}$$

Conclusion follows from this affirmation: *there exists $c_{51}\geq 0$ such that for any $X\in\mathbb{R}$*

$$\sum_{k\geq 0}\sum_{l=1}^{k}\mathbf{E_Q}\left[\sum_{z\in\Omega(w_l)}\mathrm{e}^{-V(z)}R(x+X+V(z)), w_k\in S_{-X}, \Gamma(w_k)\right] \leq c_{51}\mathrm{e}^{X}. \tag{6.29}$$

This assertion is included in a proof of [3] (see p32, 33). For $X=-A$ and $X=\frac{\kappa j}{2}$ and by combining (6.29) with(6.27) we get (6.21) and (6.23) both. □

We can now prove Proposition 2.1. Recall the statement,
There exists $c_1>0$, $\alpha>0$ and $N>0$ such that for any $n>N$, $j\geq 0$ and $x\in[1,\log\log n]$

$$\mathbf{P}(\widetilde{W}_{n,\beta}\geq \mathrm{e}^{\beta x},\ M_n\in I_n(x-j)) \leq c_1 x\mathrm{e}^{-x}\mathrm{e}^{-\alpha j} \tag{6.30}$$

Proof of Proposition 2.1. Let $c_{52}>\frac{6}{2(\beta-1)}$. We divide the proof in two case
First case, $j>c_{52}\log n$.

$$\begin{aligned}\mathbf{P}\left(\mathrm{e}^{\beta x}\leq \widetilde{W}_{n,\beta_1}, M_n\in I_n(x-j)\right) &\leq \frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathbf{E}\left(\sum_{|z|=n}\mathrm{e}^{-\beta V(z)}1_{\{M_n\in I_n(x-j)\}}\right)\\ &\leq \frac{n^{\frac{3}{2}\beta}}{\mathrm{e}^{\beta x}}\mathbf{E}\left(\mathrm{e}^{(1-\beta)S_n}1_{\{S_n>a_n(x-j)-1\}}\right)\\ &\leq \mathrm{e}^{-x}\mathrm{e}^{\beta-1}\mathrm{e}^{(\beta-1)(\frac{3}{2(\beta-1)}\log n-j)},\end{aligned}$$

but $j-\frac{3}{2(\beta-1)}\log n\geq j-j/2=j/2$. Thus $\mathbf{P}\left(\mathrm{e}^{\beta x}\leq \widetilde{W}_{n,\beta_1}, M_n\in I_n(x-j)\right)\leq c_{53}\mathrm{e}^{-x}\mathrm{e}^{-\alpha j}$.
Second case, $j\leq c_{52}\log n$.

$$\mathbf{P}\left(W_{n,\beta_1}n^{\frac{3}{2}\beta_1}\geq \mathrm{e}^{\beta_1 x}, M_n\in I_n(x-j)\right)\leq \mathbf{P}\left(\exists u\in T: V(u)\leq -(x+\frac{\kappa j}{2})\right)+$$
$$\mathbf{P}\left(\exists |u|\geq\sqrt{n}, V(u)\leq 0, \min_{j\leq|u|}V(u_j)\geq -(x+\frac{\kappa j}{2})\right)+\mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\frac{\widetilde{W}^u_{n,\beta,kill}\mathbb{1}_{\{M_{n-u}+V(u)\geq a_n(x-j)\}}}{\mathrm{e}^{\beta(V(u)+x)}}\geq 1\right).$$

Two first term are similar to those encountered p37 in section "End of proof of an assertion in italic". The same approach lead to

$$\mathbf{P}\left(\exists u\in T: V(u)\leq -(x+\frac{\kappa j}{2})\right)+\mathbf{P}\left(\exists |u|\geq \sqrt{n}, V(u)\leq 0, \min_{j\leq |u|}V(u_j)\geq -(x+\frac{\kappa j}{2})\right)\leq c_{54}\mathrm{e}^{-x}\mathrm{e}^{-\kappa j}$$

once $\kappa c_{52}\leq \frac{1}{4}$. It remains to bound the third probability. $\mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\frac{n^{\frac{3}{2}\beta}W^{u,kill}_{n,\beta}}{\mathrm{e}^{\beta(V(u)+x)}}\geq 1,\ M_n\geq a_n(x-j)\right)$ is smaller than (1)+(2)+(3)+(4) where:

$$\begin{aligned}
(1) &:= \ \mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(V(u)+x)}}\mathbb{1}_{\Gamma(u)}\mathbb{1}_{\left\{\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(V(u)+x)}}\leq \mathrm{e}^{-\frac{\theta\beta}{\beta-1}\log_+\log_+ x}\right\}}\geq \frac{1}{2}, M_n\geq a_n(x-j)\right),\\
(2) &:= \ \mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}\frac{\widetilde{W}^u_{n,\beta,kill}}{\mathrm{e}^{\beta(V(u)+x)}}\mathbb{1}_{\Gamma(u)^c}\geq \frac{1}{2}, M_n\geq a_n(x-j)\right),\\
(3) &:= \ \mathbf{P}\left(\sum_{u\in S_{-\frac{\kappa j}{2}}}B^W_{\beta,n,x}\geq 1, M_n\geq a_n(x-j)\right),\\
(4) &:= \ \mathbf{P}\left(\exists u,v,u\neq v\in S_{-\frac{\kappa j}{2}} \text{ such that } B^{W,\theta}_{n,x}(u)B^{W,\theta}_{n,x}(v)\mathbb{1}_{\Gamma(u)}\mathbb{1}_{\Gamma(v)}=1, M_n\geq a_n(x-j)\right).
\end{aligned}$$

Back to Lemma 5.6, 5.3 and 5.4 we have proved that there exists $\alpha_{(1)}$, $\alpha_{(2)}$, $\alpha_{(3)}$, $\alpha_{(4)}$ and $c_{(1)}$, $c_{(2)}$, $c_{(3)}$, $c_{(4)}$ such that

$$\begin{aligned}
&(1)\leq c_{(1)}\mathrm{e}^{-\alpha_{(1)}j}x\mathrm{e}^{-x}, \quad (2)\leq c_{(2)}\mathrm{e}^{-\alpha_{(2)}j}x\mathrm{e}^{-x}\\
&(3)\leq c_{(3)}\mathrm{e}^{-\alpha_{(3)}j}x\mathrm{e}^{-x}, \quad (4)\leq c_{(4)}\mathrm{e}^{-\alpha_{(4)}j}(\log x)^{\frac{(\beta+1)\theta}{\beta-1}+1}\mathrm{e}^{-x}.
\end{aligned}$$

We have thus all the elements and the Proposition follows. $\square$

6.3 Proof for section 2

Proof of Lemma 2.5 According to Kallenberg [17] Lemma 5.1, it suffices to show that for any $f\in C_c(\mathbb{R})$, $\left(\int_{\mathbb{R}}f(x)d\mu_n(x)\right)_{n\in\mathbb{N}}$ converge in law to some random variable $\mu(f)$. Or equivalently, for any $f\in C_c(\mathbb{R})$ there exists $\Psi_f:\mathbb{R}\to\mathbb{C}$ continuous at 0 such that

$$\lim_{n\to\infty}\mathbf{E}\left(\mathrm{e}^{i\theta\int_{\mathbb{R}}f(x)\mathrm{e}^{-2x}d\mu_n(x)}\right)=\Psi_f(\theta)\qquad \forall\theta\in\mathbb{R}.$$

By property of the Fourrier transform and the fact that $\{f\in C_c(\mathbb{R})\}=\{x\mapsto f(x)\mathrm{e}^{-2x}\in C_c(\mathbb{R})\}$. If $f\in\mathbb{R}[X]$ and f(0)=0, it's true. Let $f\in C_c(\mathbb{R})$ and $b>0$ such that $supp(f)\in$

$[-b_0, b_0]$. First we will prove that the sequence $\left(\mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)d\mu_n(x)}\right)\right)_{n\in\mathbb{N}}$ admit a limit $\Psi_f(\theta)$ for any θ. Two we show that Ψ_f is continuous at 0.

Step1 Let $M, \epsilon > 0$, A associated to (ii) and $b > b_0$ associated to (i). According to Stone-Weierstrass theorem there exists a polynomial function $Q \in \mathbb{R}[X]$ such that

$$M \sup_{x\in[-b,+\infty]} \left|Q(e^{-x}) - f(x)\right| = M \sup_{y\in[0,e^b]} \left|Q(y) - f(\log\frac{1}{y})\right| \leq \frac{\epsilon}{A}$$

Let $|\theta| \leq M$, $\forall n, p \in \mathbb{N}^*$

$$\left|\mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)e^{-2x}d\mu_n(x)}\right) - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)e^{-2x}d\mu_p(x)}\right)\right| \leq \left|1 - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)-Q(e^{-x})e^{-2x}d\mu_n(x)}\right)\right|$$
$$+\left|1 - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)-Q(e^{-x})e^{-2x}d\mu_p(x)}\right)\right| + \left|\mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} Q(x)e^{-x}d\mu_n(x)}\right) - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} Q(x)e^{-2x}d\mu_p(x)}\right)\right|$$

So there exists $N > 0$ such that for any $n, p \geq N$ $\left|\mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} Q(x)e^{-2x}d\mu_n(x)}\right) - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} Q(x)e^{-2x}d\mu_p(x)}\right)\right| \leq \epsilon$. By a trivial inequality, for any $n \in \mathbb{N}^*$

$$\begin{aligned} & \left|1 - \mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)-Q(e^{-x})e^{-2x}d\mu_n(x)}\right)\right| \leq 2\mathbf{E}\left(\mathbb{1}_{\{\int_{\mathbb{R}} e^{-2x}d\mu_n(x)>A\}} + \mathbb{1}_{\{\mu_n([-\infty,-b]>0\}}\right) \\ & +\mathbf{E}\left(\left(e^{i\theta\int_{\mathbb{R}}[f(x)-Q(e^{-x})]e^{-2x}d\mu_n(x)} - 1\right)\mathbb{1}_{\{\int_{\mathbb{R}} e^{-2x}d\mu_n(x)\leq A,\, \mu_n([-\infty,-b]=0\}}\right) \\ \leq\ & 4\epsilon + A\frac{\epsilon}{A} \\ \leq\ & 5\epsilon. \end{aligned}$$

The sequence $\mathbf{E}\left(e^{i\theta\int_{\mathbb{R}} f(x)e^{-2x}d\mu_n(x)}\right)$ is Cauchy, hence admits a limit that we denote $\Psi_f(\theta)$.

Step 2. Let $\epsilon > 0$. Let Q such that $M \sup_{x\in[-b,+\infty]} |Q(e^{-x}) - f(x)| \leq \frac{\epsilon}{A}$. It's clear by the previous inequality that $\forall\theta \in [-M, M], |\Psi_f(\theta) - \Psi_Q(\theta)| \leq 5\epsilon$. We can resume by $\forall\epsilon > 0$ $\exists Q \in \mathbb{R}[X]$ such that $\forall\theta \in [-M, M]$

$$|\Psi_f(\theta) - \Psi_Q(\theta)| \leq \epsilon.$$

Hence Ψ_Q is continuous at 0, we deduce that Ψ_f is too. □

Acknowledgements. I wish to thank my supervisor Yueyun Hu for introducing me to this subject and constantly finding time for useful discussions and advice.

References

[1] Louigi Addario-Berry and Bruce Reed. Minima in branching random walks. *Ann. Probab.*, 37(3):1044–1079, 2009.

[2] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. The branching Brownian motion seen from its tip. *ArXiv e-prints*, April 2011.

[3] Elie Aidekon. Convergence in law of the minimum of a branching random walk. *Arxiv*, 2011.

[4] Elie Aïdékon and Zhan Shi. Weak convergence for the minimal position in a branching random walk: a simple proof. *Period. Math. Hungar.*, 61(1-2):43–54, 2010.

[5] L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian Statistics in the Extremal Process of Branching Brownian Motion. *ArXiv e-prints*, October 2010.

[6] L.-P. Arguin, A. Bovier, and N. Kistler. The Extremal Process of Branching Brownian Motion. *ArXiv e-prints*, March 2011.

[7] J. D. Biggins. Chernoff's theorem in the branching random walk. *J. Appl. Probability*, 14(3):630–636, 1977.

[8] J. D. Biggins and A. E. Kyprianou. Measure change in multitype branching. *Adv. in Appl. Probab.*, 36(2):544–581, 2004.

[9] Maury Bramson and Ofer Zeitouni. Tightness for a family of recursion equations. *Ann. Probab.*, 37(2):615–653, 2009.

[10] É. Brunet and B. Derrida. A Branching Random Walk Seen from the Tip. *Journal of Statistical Physics*, 143:420–446, May 2011.

[11] É. Brunet and B. Derrida. A Branching Random Walk Seen from the Tip. *Journal of Statistical Physics*, 143:420–446, May 2011.

[12] Brigitte Chauvin and Alain Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields*, 80(2):299–314, 1988.

[13] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[14] J. M. Hammersley. Postulates for subadditive processes. *Ann. Probability*, 2:652–680, 1974.

[15] Yueyun Hu and Zhan Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.*, 37(2):742–789, 2009.

[16] B. Jaffuel. The critical random barrier for the survival of branching random walk with absorption. *ArXiv e-prints*, November 2009.

[17] Olav Kallenberg. *Random measures.* Akademie-Verlag, Berlin, fourth edition, 1986.

[18] J. F. C. Kingman. The first birth problem for an age-dependent branching process. *Ann. Probability*, 3(5):790–801, 1975.

[19] M. V. Kozlov. The asymptotic behavior of the probability of non-extinction of critical branching processes in a random environment. *Teor. Verojatnost. i Primenen.*, 21(4):813–825, 1976.

[20] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[21] Russell Lyons. A simple path to Biggins' martingale convergence for branching random walk. 84:217–221, 1997.

[22] Russell Lyons, Robin Pemantle, and Yuval Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.*, 23(3):1125–1138, 1995.

[23] P. Maillard. A characterisation of superposable random measures. *ArXiv e-prints*, February 2011.