A unified treatment of ODEs under Osgood and Sobolev type conditions

Huaiqian Li[1], Dejun Luo[2*]

[1]School of Mathematical Sciences, Beijing Normal University, Beijing, China

[2]Key Lab of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract

In this paper we prove the existence, uniqueness and regularity of the DiPerna–Lions flow generated by a vector field which is "almost everywhere Osgood continuous", following Crippa and de Lellis's direct method. As an application, we show the well-posedness of transport equations in the space of nonnegative integrable functions.

Keywords: DiPerna–Lions theory, Sobolev regularity, Osgood condition, regular Lagrangian flow, transport equation

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1 Introduction

In the seminal paper [7], DiPerna and Lions established the existence and uniqueness of the quasi-invariant flow of measurable maps generated by a Sobolev vector field with bounded divergence. Their method is quite indirect in the sense that they first established the well-posedness of the corresponding transport equation, from which they deduced the results on ODE. Their methodology is now called the DiPerna–Lions theory and can be seen as a generalization of the classical method of characteristics. It has subsequently been extended to the case of BV vector fields by Ambrosio [1, 2], via the well-posedness of the continuity equation. For the development of this theory in the infinite dimensional Wiener space, see [3, 9]. Recently, Crippa and de Lellis [6] obtained some a-priori estimates on the flow (called regular Lagrangian flow there) which enable them to give a direct construction of the flow (see the extension to the case of stochastic differential equations in [20, 10]).

To introduce the setting of the present work, we recall the key ingredient in Crippa–de Lellis's direct method, namely, a Sobolev vector field $b \in W^{1,p}_{loc}(\mathbb{R}^d)$ $(p \geq 1)$ is "almost everywhere Lipschitz continuous" (it holds even for BV vector fields, see [6, Lemma A.3]). More precisely, there are a negligible subset $N \subset \mathbb{R}^d$ and a constant C_d depending only on the dimension d, such that for all $x, y \notin N$ and $|x-y| \leq R$, one has

$$|b(x) - b(y)| \leq C_d |x-y| \big(M_R|\nabla b|(x) + M_R|\nabla b|(y)\big), \tag{1.1}$$

where $M_R f$ is the local maximal function of $f \in L^1_{loc}(\mathbb{R}^d)$:

$$M_R f(x) := \sup_{0<r\leq R} \frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)} |f(y)| \,\mathrm{d}y.$$

*Email: luodj@amss.ac.cn

Here $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $B(x,r)$ is the ball centered at x with radius r. If $x=0$ is the origin, we will simply write $B(r)$ instead of $B(0,r)$. For a proof of the inequality (1.1), see [10, Appendix]; one can also find a more complete discussion in [4] for higher order Sobolev spaces. Using the inequality (1.1), Crippa and de Lellis estimated the following type of quantity

$$\int_{B(R)} \log\left(\frac{|X_t(x)-\tilde X_t(x)|}{\delta}+1\right)\mathrm{d}x \tag{1.2}$$

in terms of R,δ and the L^p-norms of $\nabla b, \nabla\tilde b$ on some ball, where X_t and $\tilde X_t$ are respectively the flows associated to the Sobolev vector fields b and $\tilde b$. We refer the readers to [5] for an extension of some of the results.

On the other hand, the study of stochastic differential equations with non-Lipschitz coefficients has attracted intensive attentions in the past decade, see for instance [17, 11, 19, 16]. In particular, S. Fang and T. Zhang considered in [11] the general Osgood condition:

$$|b(x)-b(y)|\le C|x-y|r(|x-y|^2),\quad |x-y|\le c_0, \tag{1.3}$$

where $r:(0,c_0]\to[1,\infty)$ is a continuous function defined on a neighborhood of 0 and satisfies $\int_0^{c_0}\frac{\mathrm{d}s}{sr(s)}=\infty$. Under this condition and assuming that the ODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t}=b(X_t),\quad X_0=x$$

has no explosion, they proved that the solution X_t is a flow of homeomorphisms on $\mathbb{R}^d$ (see [11, Theorem 2.7]). If in addition $r(s)=\log\frac1s$ and the generalized divergence of b is bounded, then it is proved in [8, Theorem 1.8] that the Lebesgue measure $\mathcal{L}^d$ is also quasi-invariant under the flow X_t. In a recent paper [15], the second named author generalized this result to the Stratonovich SDE with smooth diffusion coefficients, using Kunita's expression for the Radon–Nikodym derivative of the stochastic flow (see [13, Lemma 4.3.1]).

Inspired by these two types of conditions (1.1) and (1.3), we consider in this work the following assumption on the time dependent measurable vector field $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$:

(H) there are $g\in L^1\big([0,T],L^1_{loc}(\mathbb{R}^d)\big)$ and negligible subsets N_t, such that for all $t\in[0,T]$ and $x,y\notin N_t$, one has
$$|b_t(x)-b_t(y)|\le\big(g_t(x)+g_t(y)\big)\rho(|x-y|), \tag{1.4}$$
where $\rho\in C(\mathbb{R}_+,\mathbb{R}_+)$ is strictly increasing, $\rho(0)=0$ and $\int_{0+}\frac{\mathrm{d}s}{\rho(s)}=\infty$.

The typical examples of the function ρ are $\rho(s)=s,\ s\log\frac1s,\ s\big(\log\frac1s\big)\big(\log\log\frac1s\big),\cdots$. Notice that the two latter functions are only well defined on some small interval $(0,c_0]$, but we can extend their domain of definition by piecing them together with radials. In this paper we fix such a function ρ. Similarly we may call a function satisfying (1.4) "almost everywhere Osgood continuous". It is clear that if we take $g_t=C_dM_R|\nabla b_t|$ and $\rho(s)=s$ for all $s\ge0$, then the inequality (1.4) is reduced to (1.1). On the other hand, if g is essentially bounded, then (1.4) becomes the general Osgood condition (1.3), except on the negligible set N_t (we can redefine b_t on this null set to get a continuous vector field). Therefore, this paper can be seen as a unified treatment of the two different types of conditions. We would like to mention that the assumptions like (H) were considered in [18], but the function ρ was always taken as $\rho(s)=s$ for all $s\ge0$.

The paper is organized as follows. In Section 2, we construct the flow of measurable maps under the condition (H) and the boundedness of the divergence of b, following Crippa and de

Lellis's direct method. We apply this result to show the well-posedness of the transport equation in the space of nonnegative integrable solutions. Then in Section 3, we prove a regularity property of the flow, which is weaker than the approximate differentiability discussed in [6]. We also prove a compactness result on the flow. To avoid technical complexities, we assume that the vector fields are bounded throughout this paper.

2 Preparations

We first give the definition of the flow associated to a vector field b (also called regular Lagrangian flow in [1, 6]).

Definition 2.1 (Regular Lagrangian flow)**.** *Let $b \in L^1_{loc}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$. A map $X : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is called the regular Lagrangian flow associated to the vector field b if*

(i) *for a.e. $x \in \mathbb{R}^d$, the function $t \to X_t(x)$ is absolutely continuous and satisfies*

$$X_t(x) = x + \int_0^t b_s(X_s(x))\,\mathrm{d}s, \quad \textit{for all } t \in [0,T];$$

(ii) *there exists a constant $L > 0$ independent of $t \in [0,T]$ such that $(X_t)_\#\mathcal{L}^d \le L\mathcal{L}^d$.*

Recall that $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $(X_t)_\#\mathcal{L}^d$ is the push-forward of $\mathcal{L}^d$ by the flow X_t. L will be called the compressibility constant of the flow X.

Next we introduce some notations and results that will be used in the subsequent sections. Denote by $\Gamma_T = C([0,T],\mathbb{R}^d)$, i.e. the space of continuous paths in $\mathbb{R}^d$. For $\gamma \in \Gamma_T$, we write $\|\gamma\|_{\infty,T}$ for its supremum norm. Let $\delta > 0$, we define an auxiliary function by (cf. [11, (2.7)])

$$\psi_\delta(\xi) = \int_0^\xi \frac{\mathrm{d}s}{\rho(s)+\delta}, \quad \xi \ge 0. \tag{2.1}$$

Note that if $\rho(s) = s$ for all $s \ge 0$, then

$$\psi_\delta(\xi) = \int_0^\xi \frac{\mathrm{d}s}{s+\delta} = \log\left(\frac{\xi}{\delta}+1\right)$$

which is the functional used in (1.2). Here are some properties of ψ_δ.

Lemma 2.2. (1) $\lim_{\delta\downarrow 0}\psi_\delta(\xi) = +\infty$ *for all $\xi > 0$;*

(2) *for any $\delta > 0$, the function ψ_δ is concave.*

Proof. Property (1) follows from the fact that $\int_{0+}\frac{\mathrm{d}s}{\rho(s)} = \infty$. To prove (2), we notice that $\psi_\delta'(s) = \frac{1}{\rho(s)+\delta}$. Since $s \mapsto \rho(s)$ is increasing, we see that the derivative $\psi_\delta'(s)$ is monotone decreasing, hence ψ_δ is concave. □

The concavity of ψ_δ will play an important role in the arguments of Sections 3 and 4. Finally we give an inequality concerning the local maximal function (see [6, Lemma A.2]).

Lemma 2.3. *Let R, λ and α be positive constants. Then there is C_d depending only on the dimension d, such that*

$$\mathcal{L}^d\{x \in B(R) : M_\lambda f(x) > \alpha\} \le \frac{C_d}{\alpha}\int_{B(R+\lambda)} |f(y)|\,\mathrm{d}y.$$

3 Existence and uniqueness of the regular Lagrangian flow

In order to prove the existence and uniqueness of the flow generated by a vector field b satisfying the assumption (H), we first establish an a-priori estimate.

Theorem 3.1. *Let b and $\tilde b$ be time dependent bounded vector fields satisfying* (H) *with g and $\tilde g$ respectively. Let X and $\tilde X$ be the regular Lagrangian flows associated to b and $\tilde b$, with the compressibility constants L and $\tilde L$ respectively. Then for any $R > 0$ and $t \le T$,*

$$\int_{B(R)} \psi_\delta\big(\|X_\cdot(x) - \tilde X_\cdot(x)\|_{\infty,T}\big)\,\mathrm{d}x \le (L+\tilde L)\|g\|_{L^1([0,T]\times B(\bar R))} + \frac{\tilde L}{\delta}\|b-\tilde b\|_{L^1([0,T]\times B(\bar R))},$$

where ψ_δ is defined in (2.1), *$\|\cdot\|_{\infty,T}$ is the supremum norm in Γ_T and $\bar R = R + T\big(\|b\|_{L^\infty} \vee \|\tilde b\|_{L^\infty}\big)$.*

Proof. By Definition 2.1(1), for a.e. $x \in \mathbb{R}^d$, the function $t \mapsto |X_t(x) - \tilde X_t(x)|$ is Lipschitz continuous, hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\psi_\delta\big(|X_t(x)-\tilde X_t(x)|\big)\big] &= \psi_\delta'\big(|X_t(x)-\tilde X_t(x)|\big)\frac{\mathrm{d}}{\mathrm{d}t}|X_t(x)-\tilde X_t(x)| \\ &\le \frac{\big|b_t(X_t(x)) - \tilde b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}.\end{aligned}$$

Integrating from 0 to t and noticing that $\psi_\delta(0)=0$, we get

$$\psi_\delta\big(|X_t(x)-\tilde X_t(x)|\big) \le \int_0^t \frac{\big|b_s(X_s(x)) - \tilde b_s(\tilde X_s(x))\big|}{\rho\big(|X_s(x)-\tilde X_s(x)|\big)+\delta}\,\mathrm{d}s.$$

As a result,

$$\sup_{0\le t\le T}\psi_\delta\big(|X_t(x)-\tilde X_t(x)|\big) \le \int_0^T \frac{\big|b_t(\tilde X_t(x)) - \tilde b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}\,\mathrm{d}t.$$

Since the function ψ_δ is continuous, we arrive at

$$\psi_\delta\big(\|X_\cdot(x)-\tilde X_\cdot(x)\|_{\infty,T}\big) \le \int_0^T \frac{\big|b_t(\tilde X_t(x)) - \tilde b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}\,\mathrm{d}t.$$

Therefore

$$\int_{B(R)}\psi_\delta\big(\|X_\cdot(x)-\tilde X_\cdot(x)\|_{\infty,T}\big)\,\mathrm{d}x \le \int_0^T\int_{B(R)} \frac{\big|b_t(\tilde X_t(x)) - \tilde b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}\,\mathrm{d}x\mathrm{d}t. \tag{3.1}$$

Denote by I the integral on the right hand side of (3.1). Using the triangular inequality, we obtain

$$I \le \int_0^T\int_{B(R)} \frac{\big|b_t(X_t(x)) - b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{B(R)} \frac{\big|b_t(\tilde X_t(x)) - \tilde b_t(\tilde X_t(x))\big|}{\rho\big(|X_t(x)-\tilde X_t(x)|\big)+\delta}\,\mathrm{d}x\mathrm{d}t. \tag{3.2}$$

Since the flows X_t and $\tilde X_t$ leave the Lebesgue measure absolutely continuous, we can apply the condition (H) for b_t and obtain that for a.e. $x \in B(R)$,

$$\big|b_t(X_t(x)) - b_t(\tilde X_t(x))\big| \le \big(g_t(X_t(x)) + g_t(\tilde X_t(x))\big)\rho\big(|X_t(x)-\tilde X_t(x)|\big).$$

Next it is clear that from Definition 2.1(i), one has $\|X_\cdot(x)\|_{\infty,T} \le R+T\|b\|_{L^\infty}$ and $\|\tilde{X}_\cdot(x)\|_{\infty,T} \le R+T\|\tilde{b}\|_{L^\infty}$ for a.e. $x \in B(R)$. Therefore, by the definition of the compressibility constants L and $\tilde{L}$, the first term on the right hand side of (3.2) can be estimated as follows:

$$\begin{aligned}\int_0^T\int_{B(R)} \frac{\big|b_t(X_t(x)) - b_t(\tilde{X}_t(x))\big|}{\rho\big(|X_t(x)-\tilde{X}_t(x)|\big)+\delta}\,\mathrm{d}x\mathrm{d}t &\le \int_0^T\int_{B(R)} \big(g_t(X_t(x)) + g_t(\tilde{X}_t(x))\big)\,\mathrm{d}x\mathrm{d}t \\ &\le L\int_0^T\int_{B(\bar{R})} g_t(y)\,\mathrm{d}y\mathrm{d}t + \tilde{L}\int_0^T\int_{B(\bar{R})} g_t(y)\,\mathrm{d}y\mathrm{d}t \\ &= (L+\tilde{L})\|g\|_{L^1([0,T]\times B(\bar{R}))}. \end{aligned} \tag{3.3}$$

Moreover, the second integral in (3.2) is dominated by

$$\begin{aligned}\frac{1}{\delta}\int_0^T\int_{B(R)} \big|b_t(\tilde{X}_t(x)) - \tilde{b}_t(\tilde{X}_t(x))\big|\,\mathrm{d}x\mathrm{d}t &\le \frac{\tilde{L}}{\delta}\int_0^T\int_{B(\bar{R})} |b_t(y)-\tilde{b}_t(y)|\,\mathrm{d}y\mathrm{d}t \\ &= \frac{\tilde{L}}{\delta}\|b-\tilde{b}\|_{L^1([0,T]\times B(\bar{R}))}.\end{aligned}$$

Combining this with (3.2) and (3.3), we obtain

$$I \le (L+\tilde{L})\|g\|_{L^1([0,T]\times B(\bar{R}))} + \frac{\tilde{L}}{\delta}\|b-\tilde{b}\|_{L^1([0,T]\times B(\bar{R}))}.$$

Substituting I into (3.1), we complete the proof. $\square$

Now we can prove the main result of this section.

Theorem 3.2 (Existence and uniqueness of the regular Lagrangian flow)**.** *Assume that $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is a bounded vector field satisfying* (H) *with $g \in L^1\big([0,T], L^1_{loc}(\mathbb{R}^d)\big)$. Moreover, the distributional divergence* $\mathrm{div}(b)$ *of b exists and $[\mathrm{div}(b)]^- \in L^1\big([0,T], L^\infty(\mathbb{R}^d)\big)$. Then there exists a unique regular Lagrangian flow generated by b.*

Proof. *Step 1: Uniqueness.* Suppose there are two regular Lagrangian flows X_t and $\hat{X}_t$ associated to b with compressibility constants L and $\hat{L}$ respectively. Applying Theorem 3.1, we have

$$\int_{B(R)} \psi_\delta\big(\|X_\cdot(x)-\hat{X}_\cdot(x)\|_{\infty,T}\big)\,\mathrm{d}x \le (L+\hat{L})\|g\|_{L^1([0,T]\times B(\bar{R}))}, \tag{3.4}$$

where $\bar{R} = R + T\|b\|_{L^\infty}$. If $\mathcal{L}^d\{x \in B(R) : X_\cdot(x) \ne \hat{X}_\cdot(x)\} > 0$, then there is $\varepsilon_0 > 0$ such that $K_{\varepsilon_0} := \{x \in B(R) : \|X_\cdot(x)-\hat{X}_\cdot(x)\|_{\infty,T} \ge \varepsilon_0\}$ has positive measure. Thus by (3.4),

$$(L+\hat{L})\|g\|_{L^1([0,T]\times B(\bar{R}))} \ge \int_{K_{\varepsilon_0}} \psi_\delta\big(\|X_\cdot(x)-\hat{X}_\cdot(x)\|_{\infty,T}\big)\,\mathrm{d}x \ge \psi_\delta(\varepsilon_0)\mathcal{L}^d(K_{\varepsilon_0}).$$

Letting $\delta \downarrow 0$, we deduce from Lemma 2.2(1) that

$$\infty > (L+\hat{L})\|g\|_{L^1([0,T]\times B(\bar{R}))} \ge \infty,$$

which is a contradiction. Hence $N = \{x \in B(R) : X_\cdot(x) \ne \hat{X}_\cdot(x)\}$ is $\mathcal{L}^d$-negligible. We conclude that the two flows X_t and $\hat{X}_t$ coincide with each other on the interval $[0,T]$.

Step 2: Existence. Let $\{\chi_n : n \ge 1\}$ be a sequence of standard convolution kernels. For $t \in [0,T]$, define $b^n_t = b_t * \chi_n$, i.e. the convolution of b_t and χ_n. Then for every $n \ge 1$, b^n is a time dependent smooth vector field, and

$$\|b^n_t\|_{L^\infty} \le \|b_t\|_{L^\infty}, \quad \big\|[\mathrm{div}(b^n_t)]^-\big\|_{L^\infty} \le \big\|[\mathrm{div}(b_t)]^-\big\|_{L^\infty}, \quad t \in [0,T].$$

Let X_t^n be the smooth flow generated by b_t^n, then it is easy to know that $(X_t^n)_\#\mathcal{L}^d \le L_n\mathcal{L}^d$, where

$$L_n = \exp\bigg(\int_0^T \big\|[\operatorname{div}(b_t^n)]^-\big\|_{L^\infty}\,\mathrm{d}t\bigg) \le \exp\bigg(\int_0^T \big\|[\operatorname{div}(b_t)]^-\big\|_{L^\infty}\,\mathrm{d}t\bigg) =: L.$$

Now we show that b_t^n satisfies (H) with $g_t^n = g_t * \chi_n$ for all $n \ge 1$. To this end, we fix any two points $x, y \in \mathbb{R}^d$. We have by the definition of convolution,

$$|b_t^n(x) - b_t^n(y)| \le \int_{\mathbb{R}^d} |b_t(x-z) - b_t(y-z)|\chi_n(z)\,\mathrm{d}z.$$

Now we shall make use of the condition (H). Note that $(x - N_t) \cup (y - N_t)$ is a negligible subset. When $z \notin (x - N_t) \cup (y - N_t)$, one has $x - z \notin N_t$ and $y - z \notin N_t$, hence by (H),

$$|b_t(x-z) - b_t(y-z)| \le \big(g_t(x-z) + g_t(y-z)\big)\rho(|x-y|).$$

As a result,

$$\begin{aligned} |b_t^n(x) - b_t^n(y)| &\le \int_{\mathbb{R}^d} \big(g_t(x-z) + g_t(y-z)\big)\rho(|x-y|)\chi_n(z)\,\mathrm{d}z \\ &= \big(g_t^n(x) + g_t^n(y)\big)\rho(|x-y|). \end{aligned} \tag{3.5}$$

Thus b_t^n satisfies (H) with the function g_t^n. Notice that (3.5) holds for all $x, y \in \mathbb{R}^d$.

From the above arguments, we can apply Theorem 3.1 to the sequence of smooth flows $\{X_t^n : n \ge 1\}$ and get

$$\begin{aligned} &\int_{B(R)} \psi_\delta\big(\|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\big)\,\mathrm{d}x \\ &\quad\le (L_n + L_m)\|g_n\|_{L^1([0,T]\times B(\bar R))} + \frac{L_m}{\delta}\|b^n - b^m\|_{L^1([0,T]\times B(\bar R))} \\ &\quad\le 2L\|g\|_{L^1([0,T]\times B(\bar R+1))} + \frac{L}{\delta}\|b^n - b^m\|_{L^1([0,T]\times B(\bar R))}. \end{aligned} \tag{3.6}$$

Set

$$\delta = \delta_{n,m} := \|b^n - b^m\|_{L^1([0,T]\times B(\bar R))}$$

which tends to 0 as $n, m \to \infty$, since b^n converges to b in $L^1\big([0,T], L^1_{loc}(\mathbb{R}^d)\big)$. Then we obtain

$$\int_{B(R)} \psi_{\delta_{n,m}}\big(\|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\big)\,\mathrm{d}x \le 2L\|g\|_{L^1([0,T]\times B(\bar R+1))} + L =: C < \infty. \tag{3.7}$$

We will show that $\{X_\cdot^n : n \ge 1\}$ is a Cauchy sequence in $L^1(B(R), \Gamma_T)$. For any $\eta > 0$, let

$$\begin{aligned} K_{n,m} &= \{x \in B(R) : \|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T} \le \eta\} \\ &= \big\{x \in B(R) : \psi_{\delta_{n,m}}\big(\|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\big) \le \psi_{\delta_{n,m}}(\eta)\big\}. \end{aligned}$$

By Chebyshev's inequality and (3.7),

$$\mathcal{L}^d(B(R) \setminus K_{n,m}) \le \frac{1}{\psi_{\delta_{n,m}}(\eta)} \int_{B(R)} \psi_{\delta_{n,m}}\big(\|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\big)\,\mathrm{d}x \le \frac{C}{\psi_{\delta_{n,m}}(\eta)}.$$

Therefore

$$\begin{aligned} \int_{B(R)} \|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\,\mathrm{d}x &= \bigg(\int_{K_{n,m}} + \int_{B(R)\setminus K_{n,m}}\bigg)\|X_\cdot^n(x) - X_\cdot^m(x)\|_{\infty,T}\,\mathrm{d}x \\ &\le \eta\mathcal{L}^d(K_{n,m}) + 2(R + T\|b\|_{L^\infty})\mathcal{L}^d(B(R) \setminus K_{n,m}) \\ &\le \eta\mathcal{L}^d(B(R)) + 2\bar R\frac{C}{\psi_{\delta_{n,m}}(\eta)}. \end{aligned}$$

Note that as $n,m\to\infty$, $\delta_{n,m}\to 0$, thus by Lemma 2.2(1), $\psi_{\delta_{n,m}}(\eta)\to\infty$ for any $\eta>0$. Consequently,

$$\varlimsup_{n,m\to\infty}\int_{B(R)}\|X^n_\cdot(x)-X^m_\cdot(x)\|_{\infty,T}\,\mathrm{d}x\le \eta\mathcal{L}^d(B(R)).$$

By the arbitrariness of $\eta>0$, we conclude that $\{X^n_\cdot : n\ge 1\}$ is a Cauchy sequence in $L^1(B(R),\Gamma_T)$ for any $R>0$. Therefore, there exists a measurable map $X:\mathbb{R}^d\to\Gamma_T$ which is the limit in $L^1_{loc}(\mathbb{R}^d,\Gamma_T)$ of X^n. We can find a subsequence $\{n_k : k\ge 1\}$, such that for a.e. $x\in\mathbb{R}^d$, $X^{n_k}_t(x)$ converges to $X_t(x)$ uniformly in $t\in[0,T]$. Hence we still have

$$\|X_\cdot(x)\|_{\infty,T}\le R+T\|b\|_{L^\infty}=\bar R,\quad \text{for a.e. } x\in B(R). \tag{3.8}$$

Now we prove that X_t is a regular Lagrangian flow generated by b. Firstly, for any $\phi\in C_c(\mathbb{R}^d,\mathbb{R}_+)$, we have by the Fatou lemma,

$$\int_{\mathbb{R}^d}\phi(X_t(x))\,\mathrm{d}x\le \varliminf_{k\to\infty}\int_{\mathbb{R}^d}\phi(X^{n_k}_t(x))\,\mathrm{d}x\le L\int_{\mathbb{R}^d}\phi(y)\,\mathrm{d}y.$$

This implies

$$(X_t)_\#\mathcal{L}^d\le L\mathcal{L}^d,\quad \text{for all } t\in[0,T]; \tag{3.9}$$

thus Definition 2.1(ii) is satisfied. Secondly, we show that for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$, $t\to X_t(x)$ is an integral curve of the vector field b_t. To this end, we estimate the quantity

$$J^n:=\int_{B(R)}\sup_{0\le t\le T}\left|\int_0^t b^n_s(X^n_s(x))\,\mathrm{d}s-\int_0^t b_s(X_s(x))\,\mathrm{d}s\right|\mathrm{d}x.$$

By the triangular inequality, J^n is dominated by the sum of

$$J^n_1:=\int_{B(R)}\int_0^T\left|b^n_s(X^n_s(x))-b_s(X^n_s(x))\right|\mathrm{d}s\mathrm{d}x$$

and

$$J^n_2:=\int_{B(R)}\int_0^T\left|b_s(X^n_s(x))-b_s(X_s(x))\right|\mathrm{d}s\mathrm{d}x.$$

For the first term, we have

$$J^n_1=\int_0^T\int_{B(R)}\left|b^n_s(X^n_s(x))-b_s(X^n_s(x))\right|\mathrm{d}x\mathrm{d}s\le L\int_0^T\int_{B(\bar R)}\left|b^n_s(y)-b_s(y)\right|\mathrm{d}y\mathrm{d}s.$$

Hence

$$\lim_{n\to\infty}J^n_1=0. \tag{3.10}$$

For any $\varepsilon>0$, we take a vector field $\hat b\in C^1([0,T]\times B(\bar R),\mathbb{R}^d)$ such that

$$\int_0^T\int_{B(\bar R)}\left|\hat b_s(x)-b_s(x)\right|\mathrm{d}x\mathrm{d}s<\varepsilon.$$

Again by the triangular inequality,

$$\begin{aligned}J^n_2&\le\int_0^T\int_{B(R)}\left|b_s(X^n_s(x))-\hat b_s(X^n_s(x))\right|\mathrm{d}x\mathrm{d}s+\int_0^T\int_{B(R)}\left|\hat b_s(X^n_s(x))-\hat b_s(X_s(x))\right|\mathrm{d}x\mathrm{d}s\\&\quad+\int_0^T\int_{B(R)}\left|\hat b_s(X_s(x))-b_s(X_s(x))\right|\mathrm{d}x\mathrm{d}s\\&=:J^n_{2,1}+J^n_{2,2}+J^n_{2,3}.\end{aligned}$$

Since $(X_s^n)_\# \mathcal{L}^d \le L\mathcal{L}^d$ for all $s \in [0,T]$, we have

$$J_{2,1}^n \le L \int_0^T \int_{B(\bar R)} \big| b_s(y) - \hat b_s(y) \big| \,\mathrm{d}y\mathrm{d}s < L\varepsilon.$$

By (3.8) and (3.9), the same argument leads to

$$J_{2,3}^n < L\varepsilon.$$

Moreover, by the choice of $\hat b$, there is $C_1 > 0$ such that $\sup_{0\le s\le T} \big\| \nabla \hat b_s \big\|_{L^\infty(B(\bar R))} \le C_1$. Therefore

$$J_{2,2}^n \le C_1 \int_0^T \int_{B(R)} |X_s^n(x) - X_s(x)| \,\mathrm{d}x\mathrm{d}s \le C_1 T \int_{B(R)} \|X_\cdot^n(x) - X_\cdot(x)\|_{\infty,T} \,\mathrm{d}x \to 0$$

as n goes to ∞. Summing up the above arguments, we get

$$\overline{\lim_{n\to\infty}}\, J_2^n \le 2L\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\lim_{n\to\infty} J_2^n = 0$. Combining this with (3.10), we finally obtain $\lim_{n\to\infty} J^n = 0$. Therefore, letting $n \to \infty$ in the equality

$$X_t^n(x) = x + \int_0^t b_s^n(X_s^n(x))\,\mathrm{d}s \quad \text{for all } t \le T,$$

we see that both sides converge in $L^1_{loc}(\mathbb{R}^d, \Gamma_T)$ to $X_\cdot(x)$ and $x + \int_0^\cdot b_s(X_s(x))\,\mathrm{d}s$ respectively. Hence for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, it holds

$$X_t(x) = x + \int_0^t b_s(X_s(x))\,\mathrm{d}s \quad \text{for all } t \in [0,T];$$

that is, $t \to X_t(x)$ is an integral curve of the vector field b_t. To sum up, X_t is a regular Lagrangian flow generated by b. □

Remark 3.3. *We can relax the condition* $[\mathrm{div}(b)]^- \in L^1\big([0,T], L^\infty(\mathbb{R}^d)\big)$ *to be* $[\mathrm{div}(b)]^- \in L^1\big([0,T], L^\infty_{loc}(\mathbb{R}^d)\big)$*, since we have good estimate on the growth of the flow* X_t*.*

As a byproduct of the above result, we have the well-posedness of the corresponding transport equation

$$\frac{\partial}{\partial t} u_t + b_t \cdot \nabla u_t + \mathrm{div}(b_t)\, u_t = 0, \quad u|_{t=0} = u_0. \tag{3.11}$$

Corollary 3.4 (Well-posedness of the transport equation)**.** *Assume the conditions of Theorem 3.2. Then for any integrable function* $u_0 \ge 0$*, the transport equation* (3.11) *has a unique nonnegative solution in* $L^\infty([0,T], L^1(\mathbb{R}^d))$*.*

Proof. The proof of the existence part is quite standard, see for instance the case $p = 1$ in [7, Proposition II.1] (it is easy to see that the nonnegative property of solutions is preserved in the limit process).

For the uniqueness of solutions, noticing that the equation (3.11) is equivalent to the continuity equation

$$\frac{\partial}{\partial t} u_t + D_x \cdot (b_t\, u_t) = 0, \quad u|_{t=0} = u_0,$$

where $D_x\cdot$ is the generalized divergence operator. Next since $u \in L^\infty([0,T], L^1(\mathbb{R}^d, \mathbb{R}_+))$, one has

$$\int_0^T \int_{\mathbb{R}^d} \frac{|b_t(x)|}{1+|x|} u_t(x)\,\mathrm{d}x\mathrm{d}t \le \|b\|_{L^\infty([0,T]\times\mathbb{R}^d)} \|u\|_{L^\infty([0,T],L^1(\mathbb{R}^d))} < \infty.$$

By [2, Theorem 3.2], u is a superposition solution, that is, there exists a measure $\boldsymbol{\eta} \in \mathcal{M}_+(\mathbb{R}^d \times \Gamma_T)$ concentrated on the set of pairs (x,γ) such that γ is an absolutely continuous integral curve of b_t with $\gamma(0) = x$, and

$$\int_{\mathbb{R}^d} \varphi\, u_t \,\mathrm{d}x = \int_{\mathbb{R}^d\times\Gamma_T} \varphi(\gamma(t))\,\mathrm{d}\boldsymbol{\eta}(x,\gamma), \quad \text{for all } \varphi \in C_b(\mathbb{R}^d).$$

Recall that $\Gamma_T = C([0,T], \mathbb{R}^d)$. Let $\{\boldsymbol{\eta}_x : x \in \mathbb{R}^d\}$ be the disintegration of $\boldsymbol{\eta}$ with respect to the measure $\mu_0(\mathrm{d}x) := u_0(x)\,\mathrm{d}x$. Then for μ_0-a.e. $x \in \mathbb{R}^d$, $\boldsymbol{\eta}_x$ concentrates on the integral curve of b_t starting from x. By the uniqueness of the regular Lagrangian flow X_t proved in Theorem 3.2, we have $\boldsymbol{\eta}_x\{X_\cdot(x)\} = 1$ for μ_0-a.e. $x \in \mathbb{R}^d$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi\, u_t\,\mathrm{d}x &= \int_{\mathbb{R}^d} \bigg(\int_{\Gamma_T} \varphi(\gamma(t))\,\mathrm{d}\boldsymbol{\eta}_x(\gamma)\bigg) u_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \varphi(X_t(x))\, u_0(x)\,\mathrm{d}x. \end{aligned}$$

This gives the uniqueness of nonnegative integrable solutions to (3.11). □

Remark 3.5. *If one wants to prove the uniqueness of solutions for any initial condition $u_0 \in L^1(\mathbb{R}^d)$, then one has to extend the superposition principle proved in [2, Theorem 3.2] to the case of solutions of the continuity equation consisting of signed measures with finite total variation.*

Under the condition (H), it seems to the authors that one is unable to prove the well posedness of the transport equation (3.11) by following DiPerna-Lions's original approach, that is, by proving an estimate on the commutator

$$r_n(b_t, u_t) = (b_t \cdot \nabla u_t) * \chi_n - b_t \cdot \nabla(u_t * \chi_n),$$

where χ_n is the standard convolution kernel. This can be seen from the proof of [7, Lemma II.1] (or [2, Proposition 4.1]), which essentially relies on the "almost everywhere Lipschitz continuity" of Sobolev vector fields.

4 Regularity of the flow

In this section, we first prove a regularity result on the regular Lagrangian flow, a property much weaker than the approximate differentiability discussed in [6]. We need the following notation: for a bounded measurable subset U with positive measure, define the average of $f \in L^1_{loc}(\mathbb{R}^d)$ on U by

$$⨍_U f\,\mathrm{d}x = \frac{1}{\mathcal{L}^d(U)} \int_U f\,\mathrm{d}x.$$

Then the local maximal function

$$M_R f(x) = \sup_{0<r\le R} ⨍_{B(x,r)} |f(y)|\,\mathrm{d}y.$$

Now we can prove

Theorem 4.1. *Let b be a bounded vector field satisfying* (H), *and* $[\mathrm{div}(b)]^- \in L^1\big([0,T], L^\infty(\mathbb{R}^d)\big)$. *Let X be the unique regular Lagrangian flow associated to b. Then for any $R>0$ and sufficiently small ε, there are a measurable subset $E\subset B(R)$ and some constant C depending on R, d and g, such that $\mathcal{L}^d(B(R)\setminus E)\le \varepsilon$ and for all $x,y\in E$, one has*

$$|X_t(x)-X_t(y)|\le \psi^{-1}_{|x-y|}(C/\varepsilon).$$

Here ψ_r^{-1} is the inverse function of ψ_r. Note that by Lemma 2.2(1), we have $\lim_{r\downarrow 0}\psi_r^{-1}(\xi)=0$ for all $\xi>0$. Therefore this theorem implies that X_t is uniformly continuous in E, since when $y\to x$ in the subset E, the quantity $\psi^{-1}_{|x-y|}(C/\varepsilon)$ decreases to 0. Unfortunately, the function ψ_r^{-1} does not have an explicit expression, unless $\rho(s)=s$ for all $s\ge 0$ (see Remark 4.2).

Proof of Theorem 4.1. We follow the ideas of [6, Remark 2.4] (see also [14, Proposition 5.2]). For $0\le t\le T$, $0<r\le 2R$ and $x\in B(R)$, define

$$Q(t,x,r)=\fint_{B(x,r)}\psi_r(|X_t(x)-X_t(y)|)\,\mathrm{d}y.$$

Then

$$Q(0,x,r)=\fint_{B(x,r)}\psi_r(|x-y|)\,\mathrm{d}y\le \fint_{B(x,r)}\psi_r(r)\,\mathrm{d}y\le 1.$$

By Definition 2.1(i), we see that $t\to Q(t,x,r)$ is Lipschitz and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Q(t,x,r)&=\fint_{B(x,r)}\psi_r'(|X_t(x)-X_t(y)|)\,\frac{\mathrm{d}}{\mathrm{d}t}|X_t(x)-X_t(y)|\,\mathrm{d}y\\ &\le \fint_{B(x,r)}\frac{\big|b_t(X_t(x))-b_t(X_t(y))\big|}{\rho(|X_t(x)-X_t(y)|)+r}\,\mathrm{d}y.\end{aligned}$$

Using the assumption (H) on b, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t,x,r)\le \fint_{B(x,r)}\big(g_t(X_t(x))+g_t(X_t(y))\big)\,\mathrm{d}y=g_t(X_t(x))+\fint_{B(x,r)}g_t(X_t(y))\,\mathrm{d}y.$$

Integrating both sides with respect to time from 0 to t, we arrive at

$$\begin{aligned}Q(t,x,r)&\le Q(0,x,r)+\int_0^t g_s(X_s(x))\,\mathrm{d}s+\int_0^t\fint_{B(x,r)}g_s(X_s(y))\,\mathrm{d}y\mathrm{d}s\\ &\le 1+\int_0^T g_s(X_s(x))\,\mathrm{d}s+\int_0^T\fint_{B(x,r)}g_s(X_s(y))\,\mathrm{d}y\mathrm{d}s.\end{aligned}\tag{4.1}$$

Denote by $\Phi(x)=\int_0^T g_s(X_s(x))\,\mathrm{d}s$ for a.e. $x\in\mathbb{R}^d$. Then for all $t\le T$,

$$Q(t,x,r)\le 1+\Phi(x)+\fint_{B(x,r)}\Phi(y)\,\mathrm{d}y.$$

Therefore

$$\sup_{0\le t\le T}\ \sup_{0<r\le 2R}Q(t,x,r)\le 1+\Phi(x)+M_{2R}\Phi(x).\tag{4.2}$$

For $\eta>0$ sufficiently small, we have

$$\begin{aligned}I&:=\mathcal{L}^d\{x\in B(R):1+\Phi(x)+M_{2R}\Phi(x)>1/\eta\}\\ &\le \mathcal{L}^d\{x\in B(R):\Phi(x)>1/(3\eta)\}+\mathcal{L}^d\{x\in B(R):M_{2R}\Phi(x)>1/(3\eta)\}.\end{aligned}$$

By Chebyshev's inequality and Lemma 2.3, we have

$$\begin{aligned} I &\le 3\eta \int_{B(R)} \Phi(x)\,\mathrm{d}x + 3\eta C_d \int_{B(3R)} \Phi(y)\,\mathrm{d}y \\ &\le 3\eta(1 + C_d) \int_{B(3R)} \Phi(y)\,\mathrm{d}y. \end{aligned}$$

By the definition of Φ, one has

$$\begin{aligned} I &\le 3\eta(1 + C_d) \int_0^T \int_{B(3R)} g_t(X_t(y))\,\mathrm{d}y\mathrm{d}t \\ &\le 3\eta(1 + C_d)L \int_0^T \int_{B(3R+T\|b\|_{L^\infty})} g_t(x)\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Let $\bar{C} := 3(1 + C_d)L\|g\|_{L^1([0,T]\times B(3R+T\|b\|_{L^\infty}))}$; then $I \le \eta\bar{C}$.

Now for any $\varepsilon > 0$, set $\eta = \varepsilon/\bar{C}$. Then by (4.2) and the definition of I,

$$\mathcal{L}^d\left\{x \in B(R) : \sup_{0\le t\le T} \sup_{0<r\le 2R} Q(t, x, r) > \frac{\bar{C}}{\varepsilon}\right\} \le I \le \eta\bar{C} = \varepsilon.$$

Let

$$E = \left\{x \in B(R) : \sup_{0\le t\le T} \sup_{0<r\le 2R} Q(t, x, r) \le \frac{\bar{C}}{\varepsilon}\right\}.$$

Then $\mathcal{L}^d(B(R) \setminus E) \le \varepsilon$ and for any $x \in E$, $0 \le t \le T$ and $0 < r \le 2R$, one has

$$\fint_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|)\,\mathrm{d}y \le \frac{\bar{C}}{\varepsilon}. \tag{4.3}$$

Now fix any $x, y \in E$ and let $r = |x - y|$ which is less than $2R$. Lemma 2.2(2) tells us that ψ_r is concave, hence $\psi_r(a + b) \le \psi_r(a) + \psi_r(b)$ for any $a, b \ge 0$. As a result,

$$\psi_r(|X_t(x) - X_t(y)|) \le \psi_r(|X_t(x) - X_t(z)|) + \psi_r(|X_t(z) - X_t(y)|).$$

Therefore,

$$\begin{aligned} \psi_r(|X_t(x) - X_t(y)|) &= \fint_{B(x,r)\cap B(y,r)} \psi_r(|X_t(x) - X_t(y)|)\,\mathrm{d}z \\ &\le \fint_{B(x,r)\cap B(y,r)} \psi_r(|X_t(x) - X_t(z)|)\,\mathrm{d}z \\ &\quad + \fint_{B(x,r)\cap B(y,r)} \psi_r(|X_t(z) - X_t(y)|)\,\mathrm{d}z. \end{aligned}$$

Let $\tilde{C}_d = \mathcal{L}^d(B(x, r))/\mathcal{L}^d(B(x, r) \cap B(y, r))$ which only depends on the dimension d; then

$$\begin{aligned} \psi_r(|X_t(x) - X_t(y)|) &\le \tilde{C}_d \fint_{B(x,r)} \psi_r(|X_t(x) - X_t(z)|)\,\mathrm{d}z + \tilde{C}_d \fint_{B(y,r)} \psi_r(|X_t(z) - X_t(y)|)\,\mathrm{d}z \\ &\le 2\tilde{C}_d\bar{C}/\varepsilon, \end{aligned}$$

where the last inequality follows from (4.3). Consequently, for all $x, y \in E$,

$$|X_t(x) - X_t(y)| \le \psi_r^{-1}(2\tilde{C}_d\bar{C}/\varepsilon) = \psi_{|x-y|}^{-1}\big(2\tilde{C}_d\bar{C}/\varepsilon\big).$$

Remark 4.2. *If $\rho(s) = s$ for all $s \geq 0$, then $\psi_r(s) = \log\left(\frac{s}{r} + 1\right)$ and $\psi_r^{-1}(t) = r(e^t - 1)$. Thus the last inequality in the proof of Theorem 4.1 becomes*

$$|X_t(x) - X_t(y)| \leq |x-y|\left(e^{2C_d\bar{C}/\varepsilon} - 1\right) \leq |x-y|e^{2C_d\bar{C}/\varepsilon},$$

which is the estimate given in [6, Proposition 2.3] in the case $p = 1$.

We complete this section by discussing the compactness of the regular Lagrangian flow, following the ideas in [6, Section 4]. For fixed $R > 0$ and $0 < r < R/2$, set

$$a(r, R, X) = \int_{B(R)} \sup_{0\leq t\leq T} ⨍_{B(x,r)} \psi_r(|X_t(x) - X_t(y)|)\,\mathrm{d}y\mathrm{d}x.$$

Proposition 4.3. *Let b be a bounded vector field satisfying the condition* (H) *with some function $g \in L^1([0,T], L^1_{loc}(\mathbb{R}^d))$. Let X be a regular Lagrangian flow associated to b with the compressibility constant L. Then*

$$a(r, R, X) \leq \mathcal{L}^d(B(R)) + 2L\|g\|_{L^1([0,T]\times B(\bar{R}))},$$

where $\bar{R} = 3R/2 + 2T\|b\|_{L^\infty}$.

Proof. Recall the definition of $Q(t, x, r)$ at the beginning of the proof of Theorem 4.1. We still have the estimate (4.1): for any $t \leq T$,

$$Q(t, x, r) \leq 1 + \int_0^T g_s(X_s(x))\,\mathrm{d}s + \int_0^T ⨍_{B(x,r)} g_s(X_s(y))\,\mathrm{d}y\mathrm{d}s.$$

Therefore

$$\begin{aligned} a(r, R, X) &= \int_{B(R)} \sup_{0\leq t\leq T} Q(t, x, r)\,\mathrm{d}x \\ &\leq \mathcal{L}^d(B(R)) + \int_{B(R)}\int_0^T g_s(X_s(x))\,\mathrm{d}s\mathrm{d}x + \int_{B(R)}\int_0^T ⨍_{B(x,r)} g_s(X_s(y))\,\mathrm{d}y\mathrm{d}s\mathrm{d}x. \end{aligned}$$

We denote by I_1 and I_2 the two integrals on the right hand side respectively. First

$$I_1 = \int_0^T \int_{B(R)} g_s(X_s(x))\,\mathrm{d}x\mathrm{d}s \leq \int_0^T L\int_{B(\bar{R})} g_s(y)\,\mathrm{d}y\mathrm{d}s = L\|g\|_{L^1([0,T]\times B(\bar{R}))}.$$

For the second integral, by changing the order of integration, we have

$$I_2 = \int_{B(R)}\int_0^T ⨍_{B(r)} g_s(X_s(x+z))\,\mathrm{d}z\mathrm{d}s\mathrm{d}x = ⨍_{B(r)}\int_0^T \int_{B(R)} g_s(X_s(x+z))\,\mathrm{d}x\mathrm{d}s\mathrm{d}z.$$

Therefore,

$$I_2 \leq ⨍_{B(r)}\int_0^T L\int_{B(\bar{R})} g_s(y)\,\mathrm{d}y\mathrm{d}s\mathrm{d}z = L\|g\|_{L^1([0,T]\times B(\bar{R}))}.$$

Combining the above two estimates, we arrive at the conclusion. □

Now we can prove

Theorem 4.4 (Compactness of the flow)**.** *Let $\{b^n : n \geq 1\}$ be a sequence of vector fields equi-bounded in $L^\infty([0,T]\times\mathbb{R}^d)$. For every $n \geq 1$, assume that b^n satisfies* (H) *with the function g^n, and the family $\{g^n : n \geq 1\}$ is equi-bounded in $L^1([0,T], L^1_{loc}(\mathbb{R}^d))$. Let X^n be a regular Lagrangian flow associated to b^n with the compressibility constant L_n. Suppose $\sup_{n\geq 1} L_n \leq L < \infty$. Then the sequence $\{X^n : n \geq 1\}$ is strongly precompact in $L^1_{loc}([0,T]\times\mathbb{R}^d)$.*

Proof. Applying the estimate in Proposition 4.3 to the flow X^n, we get

$$a(r,R,X^n)\le \mathcal{L}^d(B(R))+2L_n\|g^n\|_{L^1([0,T]\times B(\bar R_n))},$$

where $\bar R_n=3R/2+2T\|b^n\|_{L^\infty}$. Since $\{b^n\}$ is equi-bounded, we see that $\tilde R:=\sup_{n\ge1}\bar R_n<\infty$. Moreover, by the boundedness of the sequence $\{g^n\}$ in $L^1([0,T],L^1_{loc}(\mathbb{R}^d))$, we obtain

$$\sup_{n\ge1}a(r,R,X^n)\le \mathcal{L}^d(B(R))+2L\sup_{n\ge1}\|g^n\|_{L^1([0,T]\times B(\tilde R))}=:C_{d,R,T}<\infty. \tag{4.4}$$

For $0<z\le\tilde R$, by Lemma 2.2(2), one has

$$\frac{\psi_r(z)}{z}\ge\frac{\psi_r(\tilde R)}{\tilde R},\quad\text{or equivalently,}\quad z\le\frac{\tilde R}{\psi_r(\tilde R)}\,\psi_r(z).$$

Since $y\in B(x,r)$ and $r\le R/2$, it holds $|X_t^n(x)-X_t^n(y)|\le\bar R_n\le\tilde R$. Hence by (4.4),

$$\begin{aligned}&\int_{B(R)}\sup_{0\le t\le T}\fint_{B(x,r)}|X_t^n(x)-X_t^n(y)|\,\mathrm{d}y\mathrm{d}x\\ &\quad\le\frac{\tilde R}{\psi_r(\tilde R)}\int_{B(R)}\sup_{0\le t\le T}\fint_{B(x,r)}\psi_r(|X_t^n(x)-X_t^n(y)|)\,\mathrm{d}y\mathrm{d}x\\ &\quad=\frac{\tilde R}{\psi_r(\tilde R)}\,a(r,R,X^n)\le\frac{\tilde R}{\psi_r(\tilde R)}\,C_{d,R,T}=:g(r),\end{aligned}$$

where the function $g(r)$ does not depend on n and satisfies $g(r)\downarrow 0$ as r decreases to 0, by Lemma 2.2(1). Similar to the estimate of I_2 in the proof of Proposition 4.3, we change the order of integration and obtain

$$\sup_{n\ge1}\sup_{0\le t\le T}\int_{B(r)}\int_{B(R)}|X_t^n(x)-X_t^n(x+z)|\,\mathrm{d}x\mathrm{d}z\le g(r)\mathcal{L}^d(B(r)). \tag{4.5}$$

The rest of the proof is similar to that of [6, Corollary 4.2], hence we omit it. □

With the above compactness result in mind, we can give another proof of the existence of the regular Lagrangian flow.

Corollary 4.5 (Existence of the flow)**.** *Let b be a bounded vector field satisfying* (H) *with the function g. Assume that* $[\mathrm{div}(b)]^-\in L^1([0,T],L^\infty(\mathbb{R}^d))$*. Then there exists a regular Lagrangian flow associated to b.*

Proof. We regularize the vector field b as in *Step 2* of the proof of Theorem 3.2. It is clear that the conditions of Theorem 4.4 are satisfied by the smooth vector fields $\{b^n\}$ and the corresponding flows $\{X^n\}$. As a result, the sequence $\{X^n\}$ is strongly precompact in $L^1_{loc}([0,T]\times\mathbb{R}^d)$, and every limit point of $\{X^n\}$ is a regular Lagrangian flow associated to b. □

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