G-GRADED CASTELNUOVO MUMFORD REGULARITY

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Abstract. We develop a general graded variant of Castelnuovo-Mumford regularity for modules over a commutative ring R graded by a finitely generated abelian group G. With this aim, we establish a clear relation between supports of local cohomology modules and supports of Tor modules and Betti numbers. We give a definition of weak and very weak γ-regularity, as well as an extension of the notion of Castelnuovo-Mumford regularity which is closely related to previous ones. We provide new stability results for these regularity regions.

We extend results on Hilbert function to multigraded polynomial rings. In particular, we prove that for a finitely generated module, by Grothendieck-Serre formula, there is a numerical polynomial that coincides with its Hilbert function in the regularity region.

1. Introduction.

Castelnuovo-Mumford regularity is a fundamental invariant in commutative algebra and algebraic geometry. It is a kind of universal bound for important invariants of graded algebras such as the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules.

Intuitively, it measures the complexity of a module or sheaf: the regularity of a module approximates the largest degree of the minimal generators and the regularity of a sheaf estimates the smallest twist for which the sheaf is generated by its global sections. It has been used as a measure for the complexity of computational problems in algebraic geometry and commutative algebra (see for example [EG84] or [BM93]).

One has often tried to find upper bounds for the Castelnuovo-Mumford regularity in terms of simpler invariants. The simplest invariants which reflect the complexity of a graded algebra are the dimension and the multiplicity. However, the Castelnuovo-Mumford regularity can not be bounded in terms of the multiplicity and the dimension, what has made it computation interesting and not trivial in many cases.

The two most popular definitions of $\mathbf{Z}$-graded Castelnuovo-Mumford regularity are the one in terms of graded Betti numbers and the one using local cohomology. The equivalence of this two definitions constitutes one of the main basic results of the theory. For more discussion on the regularity, refer to the survey of Bayer and Mumford [BM93], or to [Mum66].

The aim of this paper is to develop a multigraded variant of Castelnuovo-Mumford regularity for modules over a commutative ring R graded by a finitely generated abelian group G. This notion is closely relate to previous definitions by other authors.

One motivation for studying regularity over multigraded polynomial rings comes from toric geometry. For a simplicial toric variety X, the homogeneous coordinate ring, introduced in

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[Cox95], is a polynomial ring S graded by the divisor class group G of X. The dictionary linking the geometry of X with the theory of G-graded S-modules leads to geometric interpretations and applications for multigraded regularity.

In [HW04], Hoffman and Wang define the concept of regularity for bigraded modules over a bigraded polynomial ring motivated by the geometry of $\mathbf{P}^1 \times \mathbf{P}^1$. They prove analogs of some of the classical results on m-regularity for graded modules over polynomial algebras. In [MS04], Maclagan and Smith develop a multigraded variant of Castelnuovo-Mumford regularity also motivated by toric geometry. They work with modules over a polynomial ring graded by a finitely generated abelian group, in order to establish the connection with the minimal generators of a module and its behavior in exact sequences. In this article, we extend this work restating some of the results in [MS04].

As in the standard graded case, our definition of multigraded regularity involves the vanishing of graded components of local cohomology, following [HW04] and [MS04]. In the standard graded case, it reduces to $\mathbf{Z}$-graded Castelnuovo-Mumford regularity. When S is the homogeneous coordinate ring of a product of projective spaces, multigraded regularity is the weak form of bigraded regularity defined in [HW04].

One point we are interested in remark is that Castelnuovo-Mumford regularity establish a relation between the degrees of vanishing of local cohomology modules and the degrees where Tor modules vanish. This provides a powerful tool for computing one region of $\mathbf{Z}$ in terms of the other. In this article, we handle G-graded modules over a G-graded polynomial rings, where G is a finitely generated abelian group. We exploit some of the similarities we get in multigraded regularity with standard regularity, being able to compute the regions of G where local cohomology modules vanish in terms of the supports of Tor modules, and vice-versa. We also give a definition of weak and very weak γ-regularity, and we provide new stability results for these regularity regions.

Let S be a commutative ring, G an abelian group and $R := S[X_1, \dots, X_n]$, with $\deg(X_i) = \gamma_i$ and $\deg(s) = 0$ for $s \in S$. Consider $B \subseteq (X_1, \dots, X_n)$ a finitely generated graded R-ideal and $\mathcal{C}$ the monoid generated by $\{\gamma_1, \dots, \gamma_n\}$, we propose in Definition 4.1 that:

For $\gamma \in G$, and for a graded R-module M is *very weakly γ-regular* if

$$\gamma \notin \bigcup_i \mathrm{Supp}_G(H^i_B(M)) + \mathcal{E}_i,$$

and *weakly γ-regular* if

$$\gamma \notin \bigcup_i \mathrm{Supp}_G(H^i_B(M)) + \mathcal{F}_i.$$

where $\mathcal{E}_i := \{\gamma_{j_1} + \cdots + \gamma_{j_i} \mid j_1 < \cdots < j_i\}$ and $\mathcal{F}_i := \{\gamma_{j_1} + \cdots + \gamma_{j_i} \mid j_1 \leq \cdots \leq j_i\}$.

We also set that if further, M is weakly γ'-regular for any $\gamma' \in \gamma + \mathcal{C}$, then M is *γ-regular* and

$$\mathrm{reg}(M) := \{\gamma \in G \mid M \text{ is } \gamma\text{−regular}\}.$$

We deduce from the definition that $\mathrm{reg}(M)$ is the maximal set $\mathcal{S}$ of elements in G such that $\mathcal{S} + \mathcal{C} = \mathcal{S}$ and M is γ-regular for any $\gamma \in \mathcal{S}$.

The study of Castelnuovo-Mumford regularity naturally arises questions about Hilbert functions of graded modules. Such questions comes intrinsically from the algebraic perspective, but also motivated by the geometry behind. Our main example on the second are multi-projective anisotropic spaces.

The study of Hilbert functions over standard graded algebras has taken a central role in commutative algebra and algebraic geometry since the famous paper of Hilbert [Hil90] in 1890. For a graded module M over a standard graded ring over a field, it can be proven that the Hilbert function, which measures the length of the graded components of the module M, is asymptotically polynomial.

In the last section of this article we extend the study of Hilbert functions and Hilbert polynomials to multigraded algebras over k, that, for the sake of simplicity we will assume it is a field (this hypothesis can be slightly relaxed in many cases). Precisely, in Section 6 we write for a finitely generated graded R-module M, $[M](\mu) := \dim_k(M_\mu)$ and we define

$$F_M(\mu) := [M](\mu) - \sum_i (-1)^i [H^i_B(M)](\mu).$$

We first show that the smallest abelian group of numerical functions from G to $\mathbf{Z}$ containing F_R closed by shifting $F\{\gamma\} : g \mapsto F(\gamma + g)$, coincides with the set of functions of the form $\sum_{i=0}^s (-1)^i F_{M_i}$ with $s \in \mathbf{N}$ and the M_i's in the category of finitely generated graded R-modules. This result, combined with the notion of regularity, leads to the fact that for any $\mu \in \mathbf{Z}^n$, $H^i_B(M)_\mu$ is a finite dimensional vector space and there exists a numerical polynomial P_M such that

$$[M](\mu) = P_M(\mu) + \sum_i (-1)^i [H^i_B(M)](\mu).$$

This, in turn, gives that there exists a numerical polynomial P_M such that $[M](\mu) = P_M(\mu)$, for every $\mu \in \operatorname{reg}(M)$.

In Theorem 6.5 we show that for a polynomial ring R over a field in finitely many variables, graded by an abelian group G, if the subgroup generated by the degrees is $\gamma_1\mathbf{Z} \oplus \cdots \oplus \gamma_n\mathbf{Z}$ and the degree of each variable is in $\gamma_i\mathbf{Z}_+$ for some i, with B_i for the ideal generated by the variables whose degree is a multiple of γ_i and $B := \cap_{i=1}^n B_i$. There exists positive integers $l_1, \ldots, l_n$ such that the following holds:

For any decomposition $G = \mathbf{Z}^n \oplus H$ such that the images $\epsilon_1, \ldots, \epsilon_n$ of $\gamma_1, \ldots, \gamma_n$ in $\mathbf{Z}^n = G/H$ are linearly independent, set $E := \{\sum_{i=1}^n \lambda_i \epsilon_i l_i,\ 0 \leq \lambda_i < 1\} \cap \mathbf{Z}^n$, denote by $\mathcal{E}$ the semi-group generated by the $l_i\epsilon_i$'s and by $\mathbf{Z}\mathcal{E}$ for the lattice in $\mathbf{Z}^n$ generated by $\mathcal{E}$. Then, for any finitely generated graded R-module M and any $e \in E$, there exist a polynomial P_e in $\mathbf{Q}[t_1, \ldots, t_n]$ such that $[M](\mu) = P_e(\mu)$, for all $\mu \in e + \mathbf{Z}\mathcal{E}$ such that $\mu \oplus H \subset \operatorname{reg}_B(M)$.

This contributes to the understanding of Hilbert functions over toric varieties, generalizing previous results as for example those in [Hil90], and in [CN08].

2. Local Cohomology

Let R be a commutative ring and B be a finitely generated ideal. One can define the local cohomology groups of an R-module M as the homologies of the Čech complex constructed on a finite set of generators of B. These homology groups only depend on the radical of the ideal B, and correspond the sheaf cohomology with support in $V(B)$ (see [Har67, Chap. 2.3] or [CJR11]). This in particular implies that one has a Mayer-Vietoris sequence for Čech cohomology. This cohomology commutes with arbitrary direct sums. It coincide with the right derived functors of the left exact functor H^0_B in several instances (notably when R is Noetherian or B is generated by a regular sequence in R). From the Mayer-Vietoris sequence, it also follows that both coincides when B is a finitely generated monomial ideal in a polynomial ring (see below).

2.1. **Local cohomology with support on monomial ideals.** In this section we study the support of the local cohomology with support on a monomial ideal B. Thus, assume $R := S[X_1,\dots,X_n]$ is a polynomial ring over a commutative ring S, $\deg(X_i)=\gamma_i\in G$ for $1\leq i\leq n$ and $\deg(s)=0$ for $s\in S$. Take $(\gamma_1,\dots,\gamma_n)\in G^n$.

Assume that $B\subseteq (X_1,\dots,X_n)$ be a finitely generated monomial R-ideal. Since the local cohomology modules depend only on the radical of the support ideal B, assume wlog that $B=\sqrt{B}$, hence, $B=\bigcap_{i=1}^t J_i$, where $J_i=(X_{i_1},\dots,X_{i_{s(i)}})$ is an R-ideal. The motivating examples are those where S is the G-graded homogeneous coordinate ring of a toric variety with irrelevant ideal B, for a wider reference see [Cox95].

Lemma 2.1. *Let M be a graded R-module, then*

$$\operatorname{Supp}_G(H^\ell_B(M))\subset \bigcup_{1\leq i\leq t}\ \bigcup_{1\leq j_1<\cdots<j_i\leq t}\operatorname{Supp}_G(H^{\ell+i-1}_{J_{j_1}+\cdots+J_{j_i}}(M)). \tag{1}$$

Proof. Let $B=\bigcap_{i=1}^t J_i$. We induct on t. The result is obvious for $t=1$, thus, assume that $t>1$ and that (1) holds for $t-1$. Write $J_{\leq t-1}:=J_1\cap\cdots\cap J_{t-1}$. The, for $t>1$ and $\ell\geq 0$ consider the Mayer-Vietoris long exact sequence of local cohomology

$$\cdots\to H^\ell_{J_{\leq t-1}+J_t}(M)\to H^\ell_{J_{\leq t-1}}(M)\oplus H^\ell_{J_t}(M)\to H^\ell_B(M)\to H^{\ell+1}_{J_{\leq t-1}+J_t}(M)\to\cdots.$$

Hence, $\operatorname{Supp}_G(H^\ell_B(M))\subset\operatorname{Supp}_G(H^\ell_{J_{\leq t-1}}(M))\cup\operatorname{Supp}_G(H^\ell_{J_t}(M))\cup\operatorname{Supp}_G(H^{\ell+1}_{J_{\leq t-1}+J_t}(M))$. By inductive hypothesis $\operatorname{Supp}_G(H^\ell_{J_{\leq t-1}}(M))\subset\bigcup_{1\leq i\leq t-1}\bigcup_{1\leq j_1<\cdots<j_i\leq t-1}\operatorname{Supp}_G(H^{\ell+i-1}_{J_{j_1}+\cdots+J_{j_i}}(M))$. Since $J_{\leq t-1}+J_t=(J_1+J_t)\cap\cdots\cap(J_{t-1}+J_t)$, again by inductive hypothesis we obtain that $\operatorname{Supp}_G(H^{\ell+1}_{J_{\leq t-1}+J_t}(M))\subset\bigcup_{1\leq i\leq t-1}\bigcup_{1\leq j_1<\cdots<j_i\leq t-1}\operatorname{Supp}_G(H^{\ell+i-1}_{J_{j_1}+\cdots+J_{j_i}+J_{j_t}}(M))$ which complets the proof. ☐

Remark 2.2. The exact sequence

$$H^\ell_{J_1\cap\cdots\cap J_{t-1}}(M)\oplus H^\ell_{J_t}(M)\to H^\ell_B(M)\to H^{\ell+1}_{(J_1+J_t)\cap\cdots\cap(J_{t-1}+J_t)}(M)$$

applied for $\ell\geq 1$ and M injective shows, by recursion on t, that $H^\ell_B(M)=0$ in this case (the case $t=1$ is classical and follows from the fact that B is then generated by a regular sequence). This in turn shows that H^ℓ_B is the ℓ-th right derived functor of H^0_B, when B is monomial.

The approach of Mustata in [Mus00] that we now recall uses the isomorphism

$$H^i_B(M)\simeq\varinjlim_t \operatorname{Ext}^i_R(R/B^t,M)$$

which holds over any commutative ring, taking for H^i_B the i-th derived functor of H^0_B. As for a monomial ideal B this agrees with Čech cohomology we have an isomorphism in our setting. Let $B^{[t]}:=(f_1^t,\dots,f_s^t)$ where the f_i's are the minimal monomial generators of B, the Taylor resolution $T^t_\bullet$ of $R/B^{[t]}$ has a natural map to the one of $R/B^{[t']}$ for $t\geq t'$ that in turn provides a natural map : $\operatorname{Hom}_R(T^{t'}_\bullet,R)\to\operatorname{Hom}_R(T^t_\bullet,R)$. This $\mathbf{Z}^n$-graded map is an isomorphism of complexes in degree $\gamma\in(-t',\dots,-t')+\mathbf{Z}^n_{\geq 0}$ and else $\operatorname{Hom}_R(T^{t'}_\bullet,R)_\gamma=0$. For $a=(a_1,\cdots,a_n)\in\{0,1\}^n$, let $E_a:=\{i,\ a_i=0\}$ and $R^*_a=\frac{1}{X^a}S[X_i,X_j^{-1},i\in E_a,j\in\{1,\dots,n\}\setminus E_a]$.

Setting $N_{i,a}:=H^i(\operatorname{Hom}_R(T_\bullet,R)_{-a})$, were $T_\bullet:=T^1_\bullet$ is the Taylor resolution of R/B, by Mustata description one has :

$$H^i_B(R)=\oplus_{a\in\{0,1\}^n}H^i_B(R)_{-a}\otimes_S R^*_a=\oplus_{a\in\{0,1\}^n}N_{i,a}\otimes_S R^*_a.$$

and this sum is restricted by the inclusion (1) or by inspecting a little $T^1_\bullet$. For instance if $n - \#E_a = |a| < i$ then $N_{i,a} = 0$.

3. Local Cohomology and graded Betti numbers

In this chapter we aim is to establish a clear relation between supports of local cohomology modules and supports of Tor modules and Betti numbers, in order to give a general definition for Castelnuovo-Munford regularity in next chapter.

Throughout this chapter let G be a finitely generated abelian group, and let R be a commutative G-graded ring with unit. Let B be a homogeneous ideal of R.

Remark 3.1. Is of particular interest the case where R is a polynomial ring in n variables over a commutative ring whose elements have degree 0 and $G = \mathbf{Z}^n/K$, is a quotient of $\mathbf{Z}^n$ by some subgroup K. Note that, if M is a $\mathbf{Z}^n$-graded module over a $\mathbf{Z}^n$-graded ring, and $G = \mathbf{Z}^n/K$, we can give to M a G-grading coarser than its $\mathbf{Z}^n$-grading. For this, define the G-grading on M by setting, for each $\gamma \in G$, $M_\gamma := \bigoplus_{d\in\pi^{-1}(\gamma)} M_d$.

In order to fix the notation, we state the following definitions concerning local cohomology of graded modules, and support of a graded modules M on G. Recall that the cohomological dimension of a module M is $\mathrm{cd}_B(M) := \inf\{i \mid H^j_B(M) = 0, \forall j > i\}$.

Definition 3.2. Let M be a graded R-module, the support of the module M is $\mathrm{Supp}_G(M) := \{\gamma \in G : \; M_\gamma \neq 0\}$.

Observe that if $F_\bullet$ is a free resolution of a graded module M, much information on the module can be read from the one of the resolution. Next we present a result that permits describing the support of a graded module M in terms of some homological information of a complex which need not be a resolution of M, but M is its first non-vanishing homology.

Definition 3.3. Let $C_\bullet$ be a complex of graded R-modules. For all $i, j \in \mathbf{Z}$ we define a condition (D_{ij}) as above

(D_{ij}) $\quad H^i_B(H_j(C_\bullet)) \neq 0$ implies $H^{i+\ell+1}_B(H_{j+\ell}(C_\bullet)) = H^{i-\ell-1}_B(H_{j-\ell}(C_\bullet)) = 0$ for all $\ell \geq 1$.

We have the following result on the support of the local cohomology modules of the homologies of $C_\bullet$ assuming (D_{ij}).

Theorem 3.4. *Let $C_\bullet$ be a complex of graded R-modules and $i \in \mathbf{Z}$. If (D_{ij}) holds, then*

$$\mathrm{Supp}_G(H^i_B(H_j(C_\bullet))) \subset \bigcup_{k\in\mathbf{Z}} \mathrm{Supp}_G(H^{i+k}_B(C_{j+k})).$$

Proof. Consider the two spectral sequences that arise from the double complex $\check{\mathcal{C}}^\bullet_B C_\bullet$ of graded R-modules.

The first spectral sequence has as second screen ${}'_2E^i_j = H^i_B(H_j(C_\bullet))$. Condition (D_{ij}) implies that ${}'_\infty E^i_j = {}'_2E^i_j = H^i_B(H_j(C_\bullet))$. The second spectral sequence has as first screen ${}''_1E^i_j = H^i_B(C_j)$.

By comparing both spectral sequences, one deduces that, for all $\gamma \in G$, the vanishing of $(H^{i+k}_B(C_{j+k}))_\gamma$ for all k implies the vanishing of $({}'_\infty E^{i+\ell}_{j+\ell})_\gamma$ for all ℓ, which carries the vanishing of $(H^i_B(H_j(C_\bullet)))_\gamma$. $\square$

We next give some cohomological conditions on the complex $C_\bullet$ to imply (D_{ij}) of Definition 3.3.

Lemma 3.5. *Let $C_\bullet$ be a complex of graded R-modules. Consider the following conditions*

(1) $C_\bullet$ is a right-bounded complex, say $C_j = 0$ for $j < 0$ and, $\mathrm{cd}_B(H_j(C_\bullet)) \leq 1$ for all $j \neq 0$.
(2) For some $q \in \mathbf{Z} \cup \{-\infty\}$, $H_j(C_\bullet) = 0$ for all $j < q$ and, $\mathrm{cd}_B(H_j(C_\bullet)) \leq 1$ for all $j > q$.
(3) $H_j(C_\bullet) = 0$ for $j < 0$ and $\mathrm{cd}_B(H_k(C_\bullet)) \leq k + i$ for all $k \geq 1$.

Then,

(i) $(1) \Rightarrow (2) \Rightarrow (D_{ij})$ for all $i, j \in \mathbf{Z}$, and
(ii) $(1) \Rightarrow (3) \Rightarrow (D_{ij})$ for $j = 0$.

Proof. For proving item (i), it suffices to show that $(2) \Rightarrow (D_{ij})$ for all $i, j \in \mathbf{Z}$ since $(1) \Rightarrow (2)$ is clear.

Let $\ell \geq 1$.

Condition (2) implies that $H^i_B(H_j(C_\bullet)) = 0$ for $j > q$ and $i \neq 0, 1$ and for $j < q$. If $H^i_B(H_j(C_\bullet)) \neq 0$, either $j > q$ and $i \in \{0, 1\}$ in which case $j + \ell > q$ and $i + \ell + 1 \geq 2$ and $i - \ell - 1 < 0$, or $j = q$ in which case $j + \ell > q$ and $i + \ell + 1 \geq 2$ and $j - \ell < 0$. In both cases the asserted vanishing holds.

Condition (1) automatically implies (3). Condition (3) implies that $H^{i+\ell+1}_B(H_\ell(C_\bullet)) = 0$ and $H_{j-\ell}(C_\bullet) = 0$. □

In the following subsection we establish the relation between the support of local cohomology modules and support of Tor modules. With this purpose we will base our results on Theorem 3.4 and Lemma 3.5 with $C_\bullet$ denoting the Koszul complex $\mathcal{K}_\bullet(X_1, \ldots, X_n; M)$ as we show below.

3.1. **From Local Cohomology to Betti numbers.** In this subsection we bound the support of Tor modules in terms of the support of local cohomology modules. This generalizes the fact that for $\mathbf{Z}$-graded Castelnuovo-Mumford regularity, if $b_i(M) := \max\{\mu \mid \mathrm{Tor}^R_i(M, k)_\mu \neq 0\}$ and $a_i(M) := \max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu \neq 0\}$, then $b_i(M) - i \leq \mathrm{reg}(M) := \max_i\{a_i(M) + i\}$.

Assume $R := S[X_1, \ldots, X_n]$ is a polynomial ring over a commutative ring S, $\deg(X_i) = \gamma_i \in G$ for $1 \leq i \leq n$ and $\deg(s) = 0$ for $s \in S$. Take $(\gamma_1, \ldots, \gamma_n) \in G^n$.

Let $B \subseteq (X_1, \ldots, X_n)$ be a finitely generated graded R-ideal.

Definition 3.6. Set $\mathcal{E}_0 := \{0\}$ and $\mathcal{E}_l := \{\gamma_{i_1} + \cdots + \gamma_{i_l} \ : \ i_1 < \cdots < i_l\}$ for $l \neq 0$.

Observe that if $l < 0$ or $l > n$, then $\mathcal{E}_l = \emptyset$. If $\gamma_i = \gamma$ for all i, $\mathcal{E}_l = \{l \cdot \gamma\}$ when $\mathcal{E}_l \neq \emptyset$.

Notation 3.7. For an R-module M, we denote by $M[\gamma']$ the shifted module by $\gamma' \in G$, with $M[\gamma']_\gamma := M_{\gamma'+\gamma}$ for all $\gamma \in G$.

Let M be a graded R-module. Write $\mathcal{K}^M_\bullet := \mathcal{K}_\bullet(X_1, \ldots, X_n; M)$ for the Koszul complex of the sequence $(X_1, \ldots, X_n)$ with coefficients in M. We next establish a relationship between the support of the local cohomologies of its homologies and graded Betti numbers of M.

The Koszul complex $\mathcal{K}^M_\bullet$ is graded with $K^M_l := \bigoplus_{i_1 < \cdots < i_l} M[-\gamma_{i_1} - \cdots - \gamma_{i_l}]$. Let Z^M_i and B^M_i be the Koszul i-th cycles and boundaries modules, with the grading that makes the inclusions $Z^M_i, B^M_i \subset K^M_i$ a map of degree $0 \in G$, and set $H^M_i = Z^M_i / B^M_i$.

Theorem 3.8. *Let M be a G-graded R-module. Then*

$$\mathrm{Supp}_G(\mathrm{Tor}_j^R(M,S)) \subset \bigcup_{k\geq 0}(\mathrm{Supp}_G(H_B^k(M)) + \mathcal{E}_{j+k}),$$

for all $j \geq 0$.

Proof. Notice that $H_j^M \simeq \mathrm{Tor}_j^R(M,S)$ is annihilated by B, hence has cohomological dimension 0 relatively to B. According to Lemma 3.5 (case (1)), Theorem 3.4 applies and shows that

$$\mathrm{Supp}_G(\mathrm{Tor}_j^R(M,S)) \subset \bigcup_{\ell\geq 0}\mathrm{Supp}_G(H_B^\ell(K_{j+\ell})) = \bigcup_{k\geq 0}(\mathrm{Supp}_G(H_B^k(M)) + \mathcal{E}_{j+k}). \qquad \square$$

Notice that taking $G = \mathbf{Z}$ and $\deg(X_i) = 1$, Theorem 3.8 gives the well know bound $b_i(M) - i \leq \mathrm{reg}(M) := \max_i\{a_i(M) + i\}$.

3.2. **From Betti numbers to Local Cohomology.** In this subsection we bound the support of local cohomology modules in terms of the support of Tor modules. This generalizes the fact that for **Z**-graded Castelnuovo-Mumford regularity, if $a_i(M) + i \leq \mathrm{reg}(M) := \max_i\{b_i(M) - i\}$.

We keep same hypotheses and notation as in Section 3.1

Next result gives an estimate of the support of local cohomology modules of a graded R-module M in terms of the supports of those of base ring and the twists in a free resolution. This permits (combined with Lemma 3.10) to give a bound for the support of local cohomology modules in terms of Betti numbers.

The key technical point is that Lemma 3.10 part (1) and (2) give a general version of Nakayama Lemma in order to relate shifts in a resolution with support of Tor modules; while part (3) is devoted to give a 'base change lemma' in order to pass easily to localization.

Theorem 3.9. *Let M be a graded R-module and $F_\bullet$ be a graded complex of free R-modules, with $H_0(F_\bullet) = M$. Write $F_i = \bigoplus_{j\in E_i} R[-\gamma_{ij}]$ and $T_i := \{\gamma_{ij} \mid j \in E_i\}$. Let $\ell \geq 0$ and assume $\mathrm{cd}_B(H_j(F_\bullet)) \leq \ell + j$ for all $j \geq 1$. Then,*

$$\mathrm{Supp}_G(H_B^\ell(M)) \subset \bigcup_{i\geq 0}(\mathrm{Supp}_G(H_B^{\ell+i}(R)) + T_i).$$

Proof. Lemma 3.5 (case (3)) shows that Theorem 3.4 applies for estimating the support of local cohomologies of $H_0(F_\bullet)$, and provides the quoted result as local cohomology commutes with arbitrary direct sums

$$\mathrm{Supp}_G(H_B^p(R[-\gamma])) = \mathrm{Supp}_G(H_B^p(R)) + \gamma, \text{ and } \mathrm{Supp}_G(\oplus_{i\in E}N_i) = \cup_{i\in E}\mathrm{Supp}_G(N_i)$$

for any set of graded modules N_i, $i \in E$. $\square$

Lemma 3.10. *Let M be a graded R-module.*

(1) Let S be a field and let $F_\bullet$ be a G-graded free resolution of a finitely generated module M. Then

$$F_i = \bigoplus_{\gamma\in T_i} R[-\gamma]^{\beta_{i,\gamma}}, \quad \textit{and} \quad T_i = \mathrm{Supp}_G(\mathrm{Tor}_i^R(M,S)).$$

(2) *Assume that there exists* $\phi \in \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{R})$ *such that* $\phi(\deg(x_i)) > 0$ *for all* i. *If* $\phi(\deg(a)) > m$ *for some* $m \in \mathbf{R}$ *and any* $a \in M$, *then there exists a* G*-graded free resolution* $F_\bullet$ *of* M *such that*

$$F_i = \bigoplus_{j \in E_i} R[-\gamma_{ij}] \quad \textit{with} \quad \gamma_{ij} \in \bigcup_{0 \leq \ell \leq i} \mathrm{Supp}_G(\mathrm{Tor}_\ell^R(M, S)) \ \forall j.$$

If, furthermore, there exists p *such that* F_i *is finitely generated for* $i \leq p$, *then* E_i *is finite for* $i \leq p$.

(3) *Assume that* $(S, \mathfrak{m}, k)$ *is local. Then*

$$\mathrm{Supp}_G(\mathrm{Tor}_i^R(M, k)) \subseteq \bigcup_{j \leq i} \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S)).$$

Proof. For Part (1) and (2) see [CJR11]. Part (3) follows from the fact that if $(S, \mathfrak{m}, k)$ is local there is an spectral sequence $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(M, S), k) \Rightarrow \mathrm{Tor}_{p+q}^R(M, k)$ and the fact that $S \subset R_0$. □

Combining Theorem 3.9 with Lemma 3.10 (case (1)) one obtains:

Corollary 3.11. *Assume that* S *is a field and let* M *be a finitely generated graded* R*-module. Then, for any* ℓ,

$$\mathrm{Supp}_G(H_B^\ell(M)) \subset \bigcup_{i \geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_i^R(M, S))).$$

If S is Noetherian, Lemma 3.10 (case (3)) implies the following:

Corollary 3.12. *Assume that* $(S, \mathfrak{m}, k)$ *is local Noetherian and let* M *be a finitely generated graded* R*-module. Then, for any* ℓ,

$$\begin{aligned} \mathrm{Supp}_G(H_B^\ell(M)) \quad & \subset \textstyle\bigcup_{i \geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_i^R(M, k))) \\ & \subset \textstyle\bigcup_{i \geq j \geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S))). \end{aligned}$$

After passing to localization, Corollary 3.12 shows that:

Corollary 3.13. *Let* M *be a finitely generated graded* R*-module, with* S *Noetherian. Then, for any* ℓ,

$$\mathrm{Supp}_G(H_B^\ell(M)) \subset \bigcup_{i \geq j \geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S))).$$

Proof. Let $\gamma \in \mathrm{Supp}_G(H_B^\ell(M))$. Then $H_B^\ell(M)_\gamma \neq 0$, hence there exists $\mathfrak{p} \in \mathrm{Spec}(S)$ such that $(H_B^\ell(M)_\gamma) \otimes_S S_\mathfrak{p} = H^\ell_{B \otimes_S S_\mathfrak{p}}(M \otimes_S S_\mathfrak{p}) \neq 0$. Applying Corollary 3.12 the result follows since both the local cohomology functor and the Tor functor commute with localization in S, and preserves grading as $S \subset R_0$. □

Finally, Lemma 3.10 (case (2)) gives:

Corollary 3.14. *Let* M *be a graded* R*-module, and assume that there exists* $\phi \in \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{R})$ *such that* $\phi(\deg(x_i)) > 0$ *for all* i. *If* $\phi(\deg(a)) > m$ *for some* $m \in \mathbf{R}$ *and any* $a \in M$, *then, for any* ℓ,

$$\mathrm{Supp}_G(H_B^\ell(M)) \subset \bigcup_{i \geq j \geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S))).$$

Notice that taking $G = \mathbf{Z}$ and $\deg(X_i) = 1$, Corollaries 3.11, 3.12, 3.13 and 3.14 give the well know bound $a_i(M) + i \leq \max_i\{b_i(M) - i\}$.

4. Castelnuovo-Mumford regularity

One point we are interested in remark is that Castelnuovo-Mumford regularity establishes a relation between the degrees of vanishing of local cohomology modules and the degrees where Tor modules vanish. It is clear that this provides a powerful tool for computing one region of $\mathbf{Z}$ in terms of the other.

In this section we give a definition for a G-graded R-module M and $\gamma \in G$ to be *weakly γ-regular* or just *γ-regular*, depending if γ is or is not on the shifted support of some local cohomology modules of M (cf. 4.1). This definition allows us to generalize the classical fact that weak regularity implies regularity. Lemma 4.2 and Theorem 4.3 provide new stability results for these regularity regions.

In the later part of this section, in Theorem 4.5, we prove that for $j \geq 0$, the supports of $\mathrm{Tor}_j^R(M, S)$ does not meet the support of any shifted regularity region $\mathrm{reg}(M) + \gamma$ for γ moving on $\mathcal{E}_j$. As we have mentioned in the introduction of this chapter, this result generalizes the fact that when $G = \mathbf{Z}$ and the grading is standard, $\mathrm{reg}(M) + j \geq \mathrm{end}(\mathrm{Tor}_j^R(M, S))$.

4.1. **Regularity for Local Cohomology modules.** Let S be a commutative ring, G an abelian group and $R := S[X_1, \ldots, X_n]$, with $\deg(X_i) = \gamma_i$ and $\deg(s) = 0$ for $s \in S$. Let $B \subseteq R_+ := (X_1, \ldots, X_n)$ be a graded R-ideal and $\mathcal{C}$ be the monoid generated by $\{\gamma_1, \ldots, \gamma_n\}$.

In addition to the definition of $\mathcal{E}_i$, we introduce the following sets already used by Hoffman and Wang, Maclagan and Smith and other authors:

$$\mathcal{F}_i := \{\gamma_{j_1} + \cdots + \gamma_{j_i} \mid j_1 \leq \cdots \leq j_i\}.$$

It is clear that $\mathcal{E}_i \subset \mathcal{F}_i$.

Definition 4.1. For $\gamma \in G$ and $\ell \in \mathbf{N}$, a graded R-module M is *very weakly γ-regular at level ℓ* if

$$\gamma \notin \bigcup_{i \geq \ell} \mathrm{Supp}_G(H_B^i(M)) + \mathcal{E}_i.$$

M is very weakly *γ-regular* if it is very weakly γ-regular at level 0.

M is *weakly γ-regular at level ℓ* if

$$\gamma \notin \bigcup_{i \geq \ell} \mathrm{Supp}_G(H_B^i(M)) + \mathcal{F}_i.$$

M is weakly *γ-regular* if it is weakly γ-regular at level 0.

If further M is weakly γ'-regular (resp. weakly γ'-regular at level ℓ) for any $\gamma' \in \gamma + \mathcal{C}$, then M is γ-regular (resp. γ-regular at level ℓ). One writes $\mathrm{reg}(M) := \mathrm{reg}^0(M)$ with

$$\mathrm{reg}^\ell(M) := \{\gamma \in G \mid M \text{ is } \gamma\text{−regular at level } \ell\}.$$

It immediately follows from the definition that $\mathrm{reg}^\ell(M)$ is the maximal set $\mathcal{S}$ of elements in G such that $\mathcal{S} + \mathcal{C} = \mathcal{S}$ and M is weakly γ-regular at level ℓ for any $\gamma \in \mathcal{S}$. Also notice that, as $\mathcal{F}_i + \mathcal{C} = \mathcal{E}_i + \mathcal{C}$, one may replace "weakly γ-regular" by "very weakly γ-regular" in the previous sentence.

The following lemma will give cases where weak regularity implies regularity under some extra requirement.

Lemma 4.2. *Let M be a graded R-module, I be a graded ideal generated by elements of degrees $\delta_1,\dots,\delta_s$ and set $\mathscr{E}_0^I := \{0\}$ and $\mathscr{E}_i^I := \{\delta_{j_1}+\cdots+\delta_{j_i}, j_1<\cdots<j_i\}$ and ℓ be an integer. If $\ell > \mathrm{cd}_B(R/(I+\mathrm{ann}_R(M)))$,*

$$\gamma \notin \bigcup_{i\geq 0} \mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathscr{E}_{i+1}^I \;\Rightarrow \gamma \notin \mathrm{Supp}_G(H_B^\ell(M)).$$

If $\ell = \mathrm{cd}_B(R/(I+\mathrm{ann}_R(M)))$ and $\gamma \notin \bigcup_{i>0}\mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathscr{E}_{i+1}^I$, then

$$(H_B^\ell(M)/IH_B^\ell(M))_\gamma \subseteq H_B^\ell(M/IM)_\gamma$$

and equality holds if $\gamma \notin \bigcup_{i>0}\mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathscr{E}_i^I$.

Proof. Write $I=(f_1,\dots,f_s)$, $\mathcal{K}_\bullet := \mathcal{K}_\bullet(f_1,\dots,f_s;M)$ and H_j for the j-th homology module of $\mathcal{K}_\bullet$. Consider the two spectral sequences that arise from the double Čech-Koszul complex $\check{\mathcal{C}}_B^\bullet\mathcal{K}_\bullet$ of graded R-modules.

The first spectral sequence has as second screen ${}'_2E_j^i = H_B^i(H_j)$. As I and $\mathrm{ann}_R(M)$ annihilate H_j, $\mathrm{cd}_B(H_j) \leq \mathrm{cd}_B(R/(I+\mathrm{ann}_R(M))) < \ell$, which shows that ${}'_2E_j^i = 0$ for $i-j=\ell$ unless $\ell = \mathrm{cd}_B(M/IM) = \mathrm{cd}_B(R/(I+\mathrm{ann}_R(M)))$, in which case ${}'_2E_j^i = 0$ for $j\neq 0$ and ${}'_2E_0^\ell = {}'_\infty E_0^\ell = H_B^\ell(M/IM)$. The second spectral sequence has as first screen $({}''_1E_j^i)_\mu = \oplus_{\gamma\in\mathscr{E}_j^I} H_B^i(M)_{\mu-\gamma}^{b_{j,\gamma}}$ for some positive $b_{j,\gamma}\in\mathbf{Z}$ ($b_{00}=1$).

By hypothesis $({}''_1E_{i+1}^{\ell+i})_\mu = 0$ for all $i\geq 0$. As $({}''_1E_{-i-1}^{\ell-i}) = 0$ for $i\geq 0$, we deduce that $({}''_1E_0^\ell)_\mu = ({}''_\infty E_0^\ell)_\mu$. As $({}''_1E_0^\ell)_\mu = H_B^\ell(M)_\mu$ and ${}''_\infty E_j^i = {}'_\infty E_j^i = 0$ for $i-j=\ell$, the conclusion follows. □

Recall that $\mathrm{cd}_B(R/J) \leq \mathrm{cd}_B(R/I)$ if $I\subseteq J$. Also, by [CJR11, Thm. 4.3] $\mathrm{cd}_B(M) \leq \mathrm{cd}_B(R/\mathrm{ann}_R(M))$ for any R-module M and equality holds if M is finitely generated. Furthermore, by [CJR11, Lem. 4.6], $\mathrm{cd}_B(N) \leq \mathrm{cd}_B(M)$ if M is finitely presented and $\mathrm{Supp}_R(N)\subseteq \mathrm{Supp}_R(M)$; which implies that Lemma 4.2 holds with $\mathrm{cd}_B(R/(I+\mathrm{ann}_R(M)))$ replaced by $\mathrm{cd}_B(M/IM)$ in the case where M is finitely presented.

If $G=\mathbf{Z}$ and R has the standard grading, we have that if $\ell > \mathrm{cd}_B(R/(I+\mathrm{ann}_R(M)))$ or $\ell > \mathrm{cd}_B(M/IM)$ if M is finitely presented, then, $\gamma > \mathrm{reg}(M)-\ell+1$ implies $\gamma > \mathrm{end}(H_{R_+}^\ell(M))$. Equivalently, $\gamma > \mathrm{reg}(M)$ implies $\gamma > a_\ell - 1$.

Before establishing the relation between *weak γ-regularity at level ℓ* and *γ-regularity at level ℓ*, we introduce some notation. Let $\{\gamma_1,\dots,\gamma_n\} = \{\mu_1,\dots,\mu_m\}$, with $\mu_i\neq\mu_j$ for $i\neq j$. Denote by B_i, for i from 1 to m, the ideal generated by the variables of degree μ_i.

Theorem 4.3. *Assume that $B\subset B_i$ for every i. Let M be a graded R-module, $\ell\in\mathbf{N}$ and assume that M is weakly γ-regular at level ℓ. Then,*

(1) If $\ell\geq 1$, then M is γ-regular at level ℓ.

(2) If $\ell=0$, then M is γ-regular at level 1, $(M/R_+M)_\mu = (H_B^0(M)/R_+H_B^0(M))_\mu$ for any $\mu\in\gamma+\mathcal{C}$ and the following conditions are equivalent :

(i) $H_B^0(M)_{\gamma+\mathcal{C}} = 0$,

(ii) M is γ-regular.

As a consequence, if condition (i) or (ii) holds, then $(M/R_+M)_{\gamma+\mathcal{C}} = 0$.

Proof. For $1\leq p\leq m$ let $\mathcal{F}_0^p = \mathcal{F}_i^{(p)} = \{0\}$, $\mathcal{F}_i^p := \{i\mu_p\}$,

$$\mathcal{F}_i^{(p)} := \{\gamma_{j_1}+\cdots+\gamma_{j_i} \mid j_1\leq\cdots\leq j_i \text{ and } \gamma_{j_l}\neq\mu_p,\ \forall l\}.$$

Applying Lemma 4.2 with $I := B_p$, one gets

$$\begin{aligned}\gamma \notin \bigcup_{i\geq 0} \mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathcal{F}_i^p \quad &\Leftrightarrow \gamma + \mu_p \notin \bigcup_{i\geq 0}\bigcup_{j\geq 0} (\mathrm{Supp}_G(H_B^{(\ell+i)+j}(M)) + \mathcal{F}_{j+1}^p) + \mathcal{F}_i^p \\ &\Rightarrow \gamma + \mu_p \notin \bigcup_{i\geq 0} \mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathcal{F}_i^p.\end{aligned}$$

For any p one can write

$$\bigcup_{i\geq \ell} \mathrm{Supp}_G(H_B^i(M)) + \mathcal{F}_i \quad = \bigcup_{j\geq \ell}\bigcup_{i\geq 0} (\mathrm{Supp}_G(H_B^{j+i}(M)) + \mathcal{F}_i^p) + \mathcal{F}_j^{(p)}$$

which shows that $\gamma \notin \bigcup_{i\geq 0} \mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathcal{F}_i \Rightarrow \gamma + \mu_p \notin \bigcup_{i\geq 0} \mathrm{Supp}_G(H_B^{\ell+i}(M)) + \mathcal{F}_i$ for any p and concludes the proof. □

Next example illustrates Theorem 4.3 in the standard multigraded case.

Example 4.4. Assume that $R = S[X_{ij}, 1 \leq i \leq m, 0 \leq j \leq r_i]$ is a finitely generated standard multigraded ring, $B_i := (X_{ij}, 0 \leq j \leq r_i)$, $B := B_1 \cap \cdots \cap B_m$ and $R_+ := B_1 + \cdots + B_m$. Let M be a graded R-module.

If M is weakly γ-regular, then
(a) $M/H_B^0(M)$ is γ-regular,
(b) $(H_B^0(M)/R_+H_B^0(M))_{\gamma'} = (M/R_+M)_{\gamma'}$, for any $\gamma' \in \gamma + \mathcal{C}$.

Next Theorem substantiate our results in Section 3 on regularity. Organized with the subsequent ones, they exhibit the importance of (weak) γ-regularity (at level ℓ).

Theorem 4.5. *Let M be a G-graded R-module. Then*

$$\bigcap_{\gamma\in\mathcal{E}_j} (\mathrm{reg}(M) + \gamma) \bigcap \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S)) = \emptyset$$

for all $j \geq 0$.

When $G = \mathbf{Z}$ and the grading is standard, this reads with the usual definition of $\mathrm{reg}(M) \in \mathbf{Z}$:

$$\mathrm{reg}(M) + j \geq \mathrm{end}(\mathrm{Tor}_j^R(M, S))$$

for all $j \geq 0$. Thus, $\mathrm{reg}(M) \geq \max_i\{b_i\}$ that gives by local duality the equivalence of both definitions of regularity.

Proof. If $\gamma \in \mathrm{Supp}_G(\mathrm{Tor}_j^R(M, S))$, then it follows from Theorem 3.8 that $\gamma \in \mathrm{Supp}_G(H_B^\ell(M)) + \mathcal{E}_{j+\ell}$ for some ℓ. Hence

$$\gamma - \gamma_{i_1} - \cdots - \gamma_{i_{j+\ell}} \in \mathrm{Supp}_G(H_B^\ell(M))$$

for some $i_1 < \cdots < i_{j+\ell}$. By definition it follows that if $\mu \in \mathrm{reg}(M)$ and $t_1 < \cdots < t_\ell$, then

$$\gamma - \gamma_{i_1} - \cdots - \gamma_{i_{j+\ell}} \neq \mu - \gamma_{t_1} - \cdots - \gamma_{t_\ell}$$

in particular choosing $t_k := i_{j+k}$ one has

$$\gamma - \gamma_{i_1} - \cdots - \gamma_{i_j} \notin \mathrm{reg}(M). \quad \square$$

The following result evidence the relation between regularity and vanishing of Betti numbers.

Corollary 4.6. *Assume $(S, \mathfrak{m}, k)$ is locally Noetherian, and let $F_\bullet$ be a minimal free R-resolution of a finitely generated R-module M. Then,*

$$\mathrm{Supp}_G(F_i) \subset \bigcup_{\gamma\in\mathcal{E}_i} \complement(\mathrm{reg}(M)) + \gamma.$$

On the other hand, Corollary 3.13 shows that:

Proposition 4.7. *Assume S is Noetherian, let M be a finitely generated G-graded R-module and set $T_i := \mathrm{Supp}_G(\mathrm{Tor}_i^R(M,S))$. Then, for any ℓ,*

$$\mathrm{Supp}_G(H_B^\ell(M)+\mathcal{E}_\ell) \subset \bigcup_{i\geq j}(\mathrm{Supp}_G(H_B^{\ell+i}(R))+\mathcal{E}_\ell+T_j).$$

If further S is a field,

$$\mathrm{Supp}_G(H_B^\ell(M)+\mathcal{E}_\ell) \subset \bigcup_{i}(\mathrm{Supp}_G(H_B^{\ell+i}(R))+\mathcal{E}_\ell+T_i).$$

Proposition 4.7 was stated requesting S to be Noetherian and M be a finitely generated, for the sake of simplicity. It can be proven that these hypotheses can be relinquished, by asking that there exists a function $f : G \to \mathbf{R}$ such that $f(\gamma_i) > 0$ for all i as in the case (2) of Lemma 3.10 (cf. [CJR11]). This generalization include many rings from algebraic geometry as toric rings, in particular, product of anisotropic projective spaces.

In some applications it is useful to consider local cohomologies of indices at least equal to some number, for instance positive values or values at least two. In view of Lemma 4.3, most of the time weak regularity and regularity agrees in this case. We set :

$$\mathrm{reg}^\ell(M) := \{\gamma \mid \forall \gamma' \in \mathcal{C},\ \gamma+\gamma' \notin \bigcup_{i\geq \ell}\mathrm{Supp}_G(H_B^i(M))+\mathcal{E}_i\}.$$

With this notation, Proposition 4.7 implies the following

Theorem 4.8. *Assume S is Noetherian, let M be a finitely generated G-graded R-module and set $T_i := \mathrm{Supp}_G(\mathrm{Tor}_i^R(M,S))$. Then, for any ℓ,*

$$\mathrm{reg}^\ell(M) \supseteq \bigcap_{j\leq i, \gamma\in T_j, \gamma'\in\mathcal{E}_i} \mathrm{reg}^{\ell+i}(R)+\gamma-\gamma' \supseteq \mathrm{reg}^\ell(R) + \bigcap_{j\leq i, \gamma\in T_j, \gamma'\in\mathcal{E}_i} \gamma-\gamma'+\mathcal{C}.$$

The above intersection can be restricted to $i \leq \mathrm{cd}_B(R)-\ell$. If further S is a field,

$$\mathrm{reg}^\ell(M) \supseteq \bigcap_{i, \gamma\in T_i, \gamma'\in\mathcal{E}_i} \mathrm{reg}^{\ell+i}(R)+\gamma-\gamma' \supseteq \mathrm{reg}^\ell(R) + \bigcap_{i, \gamma\in T_i, \gamma'\in\mathcal{E}_i} \gamma-\gamma'+\mathcal{C}.$$

Proof. If $\mu \notin \mathrm{reg}^\ell(M)$, by Proposition 4.7, there exists $i \geq j$ such that

$$\mu \in \mathrm{Supp}_G(H_B^{\ell+i}(R))+\mathcal{E}_\ell+T_j$$

hence there exists $\gamma' \in \mathcal{E}_i$ and $\gamma \in T_j$ such that

$$\mu+\gamma'-\gamma \in \mathrm{Supp}_G(H_B^{\ell+i}(R))+\mathcal{E}_{i+\ell}.$$

Therefore $\mu \notin \mathrm{reg}^{\ell+i}(R)+\gamma-\gamma'$. ☐

When $G = \mathbf{Z}$ and the grading is standard, this reads with the usual definition of $\mathrm{reg}^\ell(M) \in \mathbf{Z}$:

$$\mathrm{reg}^\ell(M) \leq \mathrm{reg}^\ell(R) + \max_i\{\mathrm{end}(\mathrm{Tor}_i^R(M,S))-i\} = \mathrm{reg}^\ell(R)+\max_i\{b_i\}.$$

5. Local cohomology of multigraded polynomial rings

This chapter aims to clarify Castelnuovo-Mumford regularity in the multigraded context. It is mainly focused on the advantages of having a well understood decomposition of local cohomology modules, given by Künneth formula. These isomorphisms will provide a clear description of regularity region.

Let S be a commutative ring, s and m be fixed positive integers, $r_1 \leq \cdots \leq r_s$ non-negative integers, and write $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,r_i})$ for $1 \leq i \leq s$. Define $R_i := S[\mathbf{x}_i]$, the standard $\mathbf{Z}$-graded polynomial ring in the variables $\mathbf{x}_i$ for $1 \leq i \leq s$, $R = \bigotimes_S R_i$, and $R_{(a_1,\ldots,a_s)} := \bigotimes_S (R_i)_{a_i}$ stands for its multigraded part of multidegree $(a_1, \ldots, a_s)$.

We define $\check{R}_i := \frac{1}{x_{i,1}\cdots x_{i,r_i}} S[x_{i,1}^{-1}, \ldots, x_{i,r_i}^{-1}]$. Given integers $1 \leq i_1 < \cdots < i_t \leq s$, take $\alpha = \{i_1, \ldots, i_t\}$, and set $\check{R}_\alpha := \left(\bigotimes_{j\in\alpha} \check{R}_j\right) \otimes_S \left(\bigotimes_{j\notin\alpha} R_j\right)$. Observe that $\check{R}_{\{i\}} \cong \check{R}_i \otimes_S \bigotimes_{j\neq i} R_j$. Given integers $1 \leq i_1 < \cdots < i_t \leq s$, take $\alpha = \{i_1, \ldots, i_t\}$. For any integer j write $\mathrm{sg}(j) := 1$ if $j \in \alpha$ and $\mathrm{sg}(j) := 0$ if $j \notin \alpha$. We define

$$Q_\alpha := \prod_{1\leq j\leq s} (-1)^{\mathrm{sg}(j)}\mathbf{N} - \mathrm{sg}(j) r_j \mathbf{e}_j \subset \mathbf{Z}^s,$$

the shift of the orthant whose coordinates $\{i_1, \ldots, i_t\}$ are negative and the rest are all positive. Following the notation in section 4, before Theorem 4.3, we set B_i for the R-ideal generated by the elements in $\mathbf{x}_i$, $B := B_1 \cdots B_s$, $B_\alpha := B_{i_1} + \cdots + B_{i_t}$ and $|\alpha| = r_{i_1} + \cdots + r_{i_t}$.

For every $\alpha \subset \{1, \ldots, s\}$, we have $\mathrm{Supp}_{\mathbf{Z}^s}(\check{R}_\alpha) = Q_\alpha$. Notice that for $\alpha, \beta \subset \{1, \ldots, s\}$, if $\alpha \neq \beta$, then $Q_\alpha \cap Q_\beta = \emptyset$.

Lemma 5.1. *Given integers $1 \leq i_1 < \cdots < i_t \leq s$, let $\alpha = \{i_1, \ldots, i_t\}$. There are graded isomorphisms of R-modules*

$$H^{|\alpha|}_{B_\alpha}(R) \cong \check{R}_\alpha. \tag{2}$$

Proof. Recall that for any ring S and any S-module M, if $x_1, \ldots, x_n$ are variables, then

$$H^i_{(x_1,\ldots,x_n)}(M[x_1, \ldots, x_n]) = \begin{cases} 0 & \text{if } i \neq n \\ \frac{1}{x_1\cdots x_n} M[x_1^{-1}, \ldots, x_n^{-1}] & \text{for } i = n. \end{cases} \tag{3}$$

We induct on $|\alpha|$. The result is obvious for $|\alpha| = 1$. Assume that $|\alpha| \geq 2$ and (2) holds for $|\alpha|-1$. Take $I = B_{i_1} \cdots B_{i_{t-1}}$ and $J = B_{i_t}$. There is a spectral sequence $H^p_J(H^q_I(R)) \Rightarrow H^{p+q}_{I+J}(R)$. By (3), $H^p_J(R) = 0$ for $p \neq r_{i_t}$. Hence, the spectral sequence stabilizes in degree 2, and gives $H^{r_{i_t}}_J(H^{|\alpha|-r_{i_t}}_I(R)) \cong H^{|\alpha|}_{I+J}(R)$. The result follows by applying (3) with $M = H^{|\alpha|-r_{i_t}}_I(R)$, and inductive hypothesis. □

Lemma 5.2. *With the above notations,*

$$H^\ell_B(R) \cong \bigoplus_{\substack{1\leq i_1<\cdots<i_t\leq s \\ r_{i_1}+\cdots+r_{i_t}-(t-1)=\ell}} H^{r_{i_1}+\cdots+r_{i_t}}_{B_{i_1}+\cdots+B_{i_t}}(R) \cong \bigoplus_{\substack{\alpha\subset\{1,\ldots,s\} \\ |\alpha|-(\#\alpha-1)=\ell}} \check{R}_\alpha. \tag{4}$$

Lemma 5.2 can be proven sheaf-theoretically, by means of Kunneth formula (cf. [Gro63, Thm. 6.7.3] or [SW59, Thm. 1]) or by induction on the Mayer-Vietoris sequence of local cohomology (cf. [Bot10, Lem. 6.4.7.]). It follows from Corollary 3.12 and Lemma 5.2 that:

Corollary 5.3. *Assume that* $(S, \mathfrak{m}, k)$ *is local Noetherian and let* M *be a finitely generated* G*-graded* R*-module. Then, for any* ℓ,

$$\begin{aligned} \mathrm{Supp}_G(H_B^\ell(M)) &\subset \bigcup_{i\geq 0} (\mathrm{Supp}_G(H_B^{\ell+i}(R)) + \mathrm{Supp}_G(\mathrm{Tor}_i^R(M,k))) \\ &= \bigcup_{i\geq 0} \bigcup_{\substack{1\leq i_1<\cdots<i_t\leq s \\ r_{i_1}+\cdots+r_{i_t}-(t-1)=\ell+i}} \left(Q_{\{i_1,\ldots,i_t\}} + \mathrm{Supp}_G(\mathrm{Tor}_i^R(M,k))\right). \end{aligned}$$

Whenever S is Noetherian, Corollary 3.13 provides an estimate of $\mathrm{Supp}_G(H_B^\ell(M))$ in terms of the sets $\mathrm{Supp}_G(\mathrm{Tor}_i^R(M,S))$. Theorem 3.4 asserts that if $C_\bullet$ is a complex of graded R-modules, assuming (D_{ij}) we have that for all $i \in \mathbf{Z}$

$$\mathrm{Supp}_G(H_B^i(H_j(C_\bullet))) \subset \bigcup_{k\in\mathbf{Z}} \mathrm{Supp}_G(H_B^{i+k}(C_{j+k})).$$

For $i = 1, \ldots, m$, take $f_i \in R$ homogeneous of the same degree γ for all i. Let M be a graded R-module. Denote by $\mathcal{K}_\bullet^M$ the Koszul complex $\mathcal{K}_\bullet(f_1, \ldots, f_m; R) \otimes_R M$. The Koszul complex $\mathcal{K}_\bullet^M$ is graded with $K_i := \bigoplus_{l_0<\cdots<l_i} R(-i\cdot\gamma)$. Set $H_i^M := H_i(\mathcal{K}_\bullet^M)$ the i-th homology module of $\mathcal{K}_\bullet^M$.

Corollary 5.4. *If* $\mathrm{cd}_B(H_i^M) \leq 1$ *for all* $i > 0$*. Then, for all* $j \geq 0$

$$\mathrm{Supp}_G(H_B^i(H_j^M)) \subset \bigcup_{k\in\mathbf{Z}} (\mathrm{Supp}_G(H_B^k(M)) + k\cdot\gamma) + (j-i)\cdot\gamma.$$

Proof. This follows by a change of variables in the index k in Lemma 3.4. Since $C_\bullet$ is $\mathcal{K}_\bullet^M$ and $K_i^M := \bigoplus_{l_0<\cdots<l_i} M(-i\cdot\gamma)$, we get that

$$\mathrm{Supp}_G(H_B^i(H_j^M)) \subset \bigcup_{k\in\mathbf{Z}} \mathrm{Supp}_G(H_B^k(K_{k+j-i}^M)) = \bigcup_{k\in\mathbf{Z}} (\mathrm{Supp}_G(H_B^k(M)[-(k+j-i)\cdot\gamma]).$$

The conclusion follows from 3.7. □

Remark 5.5. In the special case where $M = R$, we deduce that if $\mathrm{cd}_B(H_i) \leq 1$ for all $i > 0$,

$$\mathrm{Supp}_G(H_B^i(H_j)) \subset \bigcup_{k\in\mathbf{Z}} (\mathrm{Supp}_G(H_B^k(R)) + k\cdot\gamma) + (j-i)\cdot\gamma, \quad \text{for all } i,j.$$

Take $j = 0$ and write $I := (f_1, \ldots, f_m)$, we get

$$\mathrm{Supp}_G(H_B^i(R/I)) \subset \bigcup_{k\in\mathbf{Z}} (\mathrm{Supp}_G(H_B^k(R)) + (k-i)\cdot\gamma), \quad \text{for all } i.$$

Example 5.6. Let k be a field. Take $R_1 := k[x_1, x_2]$, $R_2 := k[y_1, y_2, y_3, y_4]$, and $G := \mathbf{Z}^2$. Write $R := R_1 \otimes_k R_2$ and set $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$ for all i. Set $B_1 := (x_1, x_2)$, $B_2 := (y_1, y_2, y_3, y_4)$ and define $B := B_1 \cdot B_2 \subset R$ the irrelevant ideal of R, and $\mathfrak{m} := B_1 + B_2 \subset R$, the ideal corresponding to the origin in $\mathrm{Spec}(R)$. From Lemma 5.2, it follows that $H_B^2(R) \cong \check{R}_{\{1\}} \cong H_{B_1}^2(R) = \omega_{R_1}^\vee \otimes_k R_2$, $H_B^4(R) \cong \check{R}_{\{2\}} \cong H_{B_2}^4(R) = R_1 \otimes_k \omega_{R_2}^\vee$, $H_B^5(R) \cong \check{R}_{\{1,2\}} \cong H_{\mathfrak{m}}^6(R) = \omega_R^\vee$, and $H_B^\ell(R) = 0$ for all $\ell \neq 2, 4$ and 5.

Hence, $\mathrm{Supp}_G(H_B^2(R)) = -\mathbf{N} \times \mathbf{N} + (-2, 0)$, $\mathrm{Supp}_G(H_B^4(R)) = \mathbf{N} \times -\mathbf{N} + (0, -4)$, and $\mathrm{Supp}_G(H_B^5(R)) = -\mathbf{N} \times -\mathbf{N} + (-2, -4)$, .

Take $f_1, \dots, f_m$ homogeneous elements of bidegree γ, and write $I := (f_1, \dots, f_m)$. Assume $\mathrm{cd}_B(R/I) \leq 1$, hence $\mathrm{cd}_B(H_i) \leq 1$ for all i. We will compute $\mathrm{reg}(R/I)$. Set for every $\gamma \in G$,

$$\mathfrak{S}_B(\gamma) := \bigcup_{k \geq 0} (\mathrm{Supp}_G(H^k_B(R)) + k \cdot \gamma). \tag{5}$$

Since $H^\ell_B(R) = 0$ for all $\ell \neq 2, 4$ and 5, from Remark 5.5 we get that for all i, $\mathrm{Supp}_G(H^i_B(R/I)) \subset \mathfrak{S}_B(\gamma) - i \cdot \gamma$, and, $\mathrm{reg}(R/I) \supset \complement \mathfrak{S}_B(\gamma)$.

6. Hilbert functions over graded rings

Let R be a polynomial ring over a field k, graded by an abelian group G and B be a non trivial graded ideal. Assume that $H^i_B(R)_\mu$ is a finite dimensional k-vector space for any $\mu \in G$.

For a finitely generated graded R-module M set $[M](\mu) := \dim_k(M_\mu)$ and

$$F_M(\mu) := [M](\mu) - \sum_i (-1)^i [H^i_B(M)](\mu).$$

It follows from the proof of Lemma 6.1 below that $[H^i_B(M)](\mu)$ is finite for any i and μ. Recall that in the standard graded situation, F_M is a polynomial function, called the Hilbert polynomial of M.

Lemma 6.1. *Let $\mathbf{C}$ be the smallest set of numerical functions from G to $\mathbf{Z}$ containing F_R such that for any $F, G \in \mathbf{C}$ and $\gamma \in G$, the function $F + G$, $-F$ and $F\{\gamma\} : g \mapsto F(\gamma + g)$ are in $\mathbf{C}$.*

Then $\mathbf{C}$ coincides with the set of functions of the form $\sum_{i=0}^s (-1)^i F_{M_i}$ with $s \in \mathbf{N}$ and the M_i's in the category of finitely generated graded R-modules.

Proof. First notice that any function in $\mathbf{C}$ can be written in the form $\sum_{i=0}^s (-1)^i F_{M_i}$, with $M_i = R[\gamma_i]$ for some i. On the other hand if $F_\bullet$ is a graded finite free R-resolution of M, $[M] = \sum_j (-1)^j [F_j]$ and the spectral sequence $H^i_B(F_j) \Rightarrow H^{i-j}_B(M)$ shows that $H^i_B(M)_\mu$ is a finite dimensional vector space for any μ and that

$$\sum_{i,j} (-1)^{i-j} [H^i_B(F_j)] = \sum_\ell (-1)^\ell [H^\ell_B(M)].$$

Since $F_j = \oplus_{q \in E_j} R[\gamma_{j,q}]$, it follows that

$$F_M = \sum_j (-1)^j [F_j] + \sum_{i,j} (-1)^{i-j} [H^i_B(F_j)] = \sum_j (-1)^j \sum_{q \in E_j} F_R\{\gamma_{j,q}\} \in \mathbf{C}.$$

□

In the case $R := k[X_{i,j} \mid 1 \leq i \leq n, 0 \leq j \leq r_i]$ is a standard $\mathbf{Z}^n$-graded polynomial ring over a field k, $\deg(X_{i,j}) = e_i$, let $B := \cap_{i=1}^n (X_{i,j},\ 0 \leq j \leq r_i)$. The following result generalizes [JV02, Thm. 2.4], which considers the case $n = 2$. It is proved in [CN08, Prop. 2.4.3].

Proposition 6.2. *For any $\mu \in \mathbf{Z}^n$, $H^i_B(M)_\mu$ is a finite dimensional vector space and there exists a numerical polynomial P_M such that*

$$[M](\mu) = P_M(\mu) + \sum_i (-1)^i [H^i_B(M)](\mu).$$

Following [CN08, Thm. 2.4.8], from Proposition 6.2 we conclude that

Corollary 6.3. *There exists a numerical polynomial P_M such that*

$$[M](\mu) = P_M(\mu).$$

For every $\mu \in \mathrm{reg}(M)$.

The proof of Proposition 6.2 follows directly from Lemma 6.1 and the following result

Lemma 6.4. *With the above notations,*

$$F_R(a_1, \dots, a_s) = \prod_{1 \le i \le s} \binom{r_i + a_i}{r_i}.$$

and $\mathbf{C}$ is the set of numerical polynomials of multidegree $\le (r_1, \dots, r_n)$.

Proof. The second follows from the first and [Rob98, Thm. 2.1.7].

From Lemma 1 we have that

$$H^\ell_B(R) \cong \bigoplus_{\substack{1 \le i_1 < \cdots < i_t \le s \\ r_{i_1} + \cdots + r_{i_t} + 1 = \ell}} H^{r_{i_1} + \cdots + r_{i_t} + t}_{B_{i_1} + \cdots + B_{i_t}}(R)$$

Take $(a_1, \dots, a_s) \in \mathbf{Z}^s$. If there exists i such that $-r_i \le a_i \le -1$, then both sides are zero, hence equal.

If $a_i \ge 0$ for all i, then $F_R(a_1, \dots, a_s) = [R](a_1, \dots, a_s) = \prod_{1 \le i \le s} \binom{r_i + a_i}{r_i}$.

Denote by $E(i_1, \dots, i_t)$ the set of (i, j) with $i \in \{i_1, \cdots, i_t\}$, $0 \le j \le r_{i_t}$.

$$H^{r_{i_1} + \cdots + r_{i_t} + t}_{B_{i_1} + \cdots + B_{i_t}}(R) \cong \frac{1}{\prod_{(i,j) \in E(i_1, \dots, i_t)} x_{i,j}} k[x^{-1}_{i,j}, x_{i',j'}]_{(i,j) \in E(i_1, \dots, i_t), (i',j') \notin E(i_1, \dots, i_t)}$$

If $a_{i_j} \ge -r_{i_j}$ for some $j \in \{1, \dots, t\}$ or $a_{i_j} < 0$ for some $j \in \{t+1, \dots, s\}$, then one obtains $H^\ell_B(R)_{(a_1, \dots, a_s)} = 0$.

Next assume that $a_{i_j} < -r_{i_j}$ for $j = 1, \dots, t$ and $a_{i_j} \ge 0$ for $j = t+1, \dots, s$. Then $[H^\ell_B(R)](a_1, \dots, a_s) = 0$ unless $\ell = r_{i_1} + \cdots + r_{i_t} + 1$. In the latter case, the identity of polynomials in a

$$\binom{r + (-a - r - 1)}{r} = (-1)^r \binom{r + a}{r}$$

shows that

$$\begin{aligned} F_R(a_1, \dots, a_s) &= (-1)^{\ell+1} [H^\ell_B(R)](a_1, \dots, a_s) \\ &= (-1)^{\ell+1} \textstyle\prod_{1 \le i \le s} \prod_{1 \le j \le t} (-1)^{r_{i_j}} \binom{r_{i_j} + a_{i_j}}{r_{i_j}} \prod_{t+1 \le j \le s} \binom{r_{i_j} + a_{i_j}}{r_{i_j}} \\ &= \textstyle\binom{r_i + a_i}{r_i}. \end{aligned}$$

□

Now assume that $R := k[X_{i,j} \mid 1 \le i \le n, 0 \le j \le r_i]$ is G-graded for some abelian group G, with $\deg(X_{i,j}) = \gamma_i$ for all j, and let $\mathcal{C}$ be the semi-group generated by the γ_i's.

From the standard multigraded setting, we will now derive results on the case where the γ_i's are linearly independent. In that situation

$$\dim_k R_\mu = \begin{cases} 0 & \text{if } \mu \notin \mathcal{C} \\ \prod_{i=1}^n \binom{r_i + c_i}{r_i} & \text{if } \mu = c_1\gamma_1 + \cdots + c_n\gamma_n \in \mathcal{C}. \end{cases}$$

Write $\mathbf{Z}\mathcal{C}$ for the subgroup of G generated by $\mathcal{C}$ and $\pi : G \to G/\mathbf{Z}\mathcal{C}$, then

$$M = \bigoplus_{\theta \in G/\mathbf{Z}\mathcal{C}} M_{\pi^{-1}(\theta)},$$

and only a finite number of the summands are not zero if M is finitely generated. Choosing representatives $\tilde{\theta}_i$ $(1 \leq i \leq \ell)$ of the classes of the θ's giving non zero summands, we write $M = \bigoplus_{i=1}^{\ell} M_{\tilde{\theta}_i + \mathbf{Z}\mathcal{C}}$. The modules $M_{\tilde{\theta}_i}[-\tilde{\theta}_i]$ are $\mathbf{Z}\mathcal{C}$-graded R-modules of finite type. Identifying $\mathbf{Z}\mathcal{C}$ with $\mathbf{Z}^n$, it follows from Lemma above that there exists a numerical polynomial $P_i \in \mathbf{Q}[t_1, \dots, t_n]$ and $C \in \mathbf{N}$ such that

$$\dim_k(M_\mu) = \begin{cases} 0 & \text{if } \mu \notin \{\tilde{\theta}_1, \dots, \tilde{\theta}_\ell\} + \mathbf{Z}\mathcal{C} \\ P_{\tilde{\theta}_i}(c_1, \dots, c_n) & \text{if } \mu = c_1\gamma_1 + \cdots + c_n\gamma_n + \tilde{\theta}_i \text{ and } c_i \geq C, \forall i \end{cases}$$

Now assume that $G = \mathbf{Z}^n \oplus H$ and the decomposition $\gamma_i = (\epsilon_i, h_i)$ is such that the ϵ_i's are linearly independent. Such a decomposition always exists. Set $\mathcal{E}$ for the semi-group generated by the ϵ_i's and $\mathbf{Z}\mathcal{E}$ for the lattice in $\mathbf{Z}^n$ generated by $\mathcal{E}$. Let $\tilde{\theta}_i = (\epsilon'_i, h'_i)$. By the above remarks, $M_{(\mu,g)} = 0$ unless $\mu = \epsilon.c + \epsilon'_i$ and $g = h.c + h'_i$ for some $c \in \mathbf{Z}^n$ and some $i \in \{1, \dots, \ell\}$. Choosing the representatives $\tilde{\theta}_i$ so that the ϵ'_i belong to the finite set $E := \{\lambda.\epsilon,\ \lambda \in [0,1[^n\} \cap \mathbf{Z}^n$, it follows that if $\mu \in e + \mathbf{Z}\mathcal{C}$ with $e \in E$, then

$$\dim_k(M_{(\mu,*)}) = Q_e(c_1, \dots, c_n) \text{ if } \mu = c.\epsilon + e \text{ and } c_i \geq C, \forall i,$$

with Q_e equal to the sum of the $P_{\tilde{\theta}_i}$'s over the i's such that $\epsilon'_i = e$. Finally, as the c_i's are linear forms in the coordinates of μ, one can write

$$\dim_k(M_{(\mu,*)}) = Q^\sharp_e(\mu_1, \dots, \mu_n) \text{ if } \mu \in e + C\epsilon_1 + \cdots + C\epsilon_n + \mathcal{E},$$

where $Q^\sharp_e$ is a polynomial in $\mathbf{Q}[t_1, \dots, t_n]$ that takes integral values on $e + \mathbf{Z}\mathcal{E}$. Such a function is called a quasi-polynomial with respect to the lattice $\mathbf{Z}\mathcal{E}$ in $\mathbf{Z}^n$. It should be noticed that the functions $Q^\sharp_e$ depend very much on the decomposition $G = \mathbf{Z}^n \oplus H$. In the case $G = \mathbf{Z}^n$, closely related results can be find in [Rob98], [Fie02], and [CN08].

Notice that with some more generality, if $\deg(X_{i,j}) = m_{i,j}\gamma_i$ for some positive integers $m_{i,j}$ –the γ_i's being linearly independent– then setting $l_i := \mathrm{lcm}(m_{i,j})$ and $\delta := (l_1\gamma_1, \dots, l_n\gamma_n)$, R is finite over the subring $\oplus_{\mu \in \mathbf{N}^n} R_{\mu.\delta}$, which is generated by elements of degrees $l_1\gamma_1, \dots, l_n\gamma_n$.

Hence for any finitely generated graded R-module, the construction above shows that there exists quasi-polynomials $Q^\sharp_e$ with respect to $\mathbf{Z}l_1\epsilon_1 \oplus \cdots \oplus \mathbf{Z}l_n\epsilon_n$ such that $\dim_k(M_{(\mu,*)}) = Q^\sharp_e(\mu_1, \dots, \mu_n)$ if $\mu = e + u_1l_1\epsilon_1 + \cdots u_nl_n\epsilon_n$ and $u_i \geq C$ for all i.

The following result summarizes our discussion above.

Theorem 6.5. *Let R be a polynomial ring over a field in finitely many variables, graded by an abelian group G. Assume that the subgroup generated by the degrees is $\gamma_1\mathbf{Z} \oplus \cdots \oplus \gamma_n\mathbf{Z}$ and that the degree of each variable is in $\gamma_i\mathbf{Z}_+$ for some i. Let B_i be the ideal generated by the variables whose degree is a multiple of γ_i and $B := \cap_{i=1}^n B_i$. There exists positive integers $l_1, \dots, l_n$ such that the following holds.*

For any decomposition $G = \mathbf{Z}^n \oplus H$ such that the images $\epsilon_1, \dots, \epsilon_n$ of $\gamma_1, \dots, \gamma_n$ in $\mathbf{Z}^n = G/H$ are linearly independent, set $E := \{\sum_{i=1}^n \lambda_i\epsilon_i l_i,\ 0 \leq \lambda_i < 1\} \cap \mathbf{Z}^n$, denote by $\mathcal{E}$ the semi-group generated by the $l_i\epsilon_i$'s and by $\mathbf{Z}\mathcal{E}$ for the lattice in $\mathbf{Z}^n$ generated by $\mathcal{E}$.

Then for any finitely generated graded R-module M and any $e \in E$, there exist a polynomial P_e in $\mathbf{Q}[t_1, \dots, t_n]$ such that

$$[M](\mu) = P_e(\mu), \text{ for all } \mu \in e + \mathbf{Z}\mathcal{E} \text{ such that } \mu \oplus H \subset \mathrm{reg}_B(M).$$

Notice that by Theorem 4.5, in the context of the above Theorem, there exists $\mu_M \in \mathbf{Z}^n$ such that $\mu \oplus H \subset \mathrm{reg}_B(M)$ for any $\mu \in \mu_M + \mathbf{Q}_+\mathcal{E}$.

The main example where the above result applies correspond to a product of anisotropic projective spaces over a field.

Also remark that Theorem 6.5 can be stated with R being a polynomial ring over a Noetherian local ring $(S, \mathfrak{m}, k)$, and for finitely generated modules that are annihilated by a power of $\mathfrak{m}$. Indeed, in this case M is filtered by the modules $0 :_M \mathfrak{m}^i$, whose successive quotients are finitely generated graded modules over $R \otimes_S k$, which is a polynomial ring over k. This condition on M is automatically satisfied if S is Artinian.

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