Imaginary-Scaling versus Indefinite-Metric Quantization of the Pais-Uhlenbeck Oscillator

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Using the Pais-Uhlenbeck Oscillator as a toy model, we outline a consistent alternative to the indefinite-metric quantization scheme that does not violate unitarity. We describe the basic mathematical structure of this method by giving an explicit construction of the Hilbert space of state vectors and the corresponding creation and annihilation operators. The latter satisfy the usual bosonic commutation relation and differ from those of the indefinite-metric theories by a sign in the definition of the creation operator. This change of sign achieves a definitization of the indefinite-metric that gives life to the ghost states without changing their contribution to the energy spectrum.

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1. Introduction: The recent renewal of interest in higher derivative theories of gravity [1] has provided incentive for reconsidering the old problem of quantizing the classical Pais-Uhlenbeck (PU) Oscillator [2]. Recently the authors of [3] have proposed a quantization scheme involving non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians [4–6] that yields a stable and unitary quantum system for the non-degenerate forth-order PU oscillator. A critical assessment and an alternative quantization of the PU oscillator are given in [7, 8]. In the present paper we explore the quantum analog of the imaginary scaling trick of [3] and use its mathematical properties to develop a consistent ghost-free alternative to the indefinite-metric quantization of the PU oscillator. Our method enjoys general applicability and does not involve non-Hermitian Hamiltonian operators.

The forth-order classical PU oscillator is the dynamical system defined by the equation of motion:

$$z^{(4)} + \alpha\, z^{(2)} + \beta\, z = 0, \qquad (1)$$

where $z^{(k)}$ denotes the k-th derivative of the real dynamical variable z, $\alpha := \omega_1^2 + \omega_2^2$, $\beta := \omega_1^2\omega_2^2$, and ω_1 and ω_2 are positive real numbers. The solution of (1) is a linear combination of $e^{\pm i\omega_1 t}$ and $te^{\pm i\omega_1 t}$ for the degenerate case where $\omega_1 = \omega_2$, and $e^{\pm i\omega_1 t}$ and $e^{\pm i\omega_2 t}$ for the non-degenerate case where $\omega_1 \neq \omega_2$.

Eq. (1) may be obtained from either the higher-derivative Lagrangian: $L = \frac{1}{2}(\ddot{z}^2 - \alpha\,\dot{z}^2 + \beta\, z)$ or the Lagrangian: $L = \frac{1}{2}(\dot{x}^2 - \alpha\, x^2 + \beta\, z^2) + \lambda(\dot{z} - x)$ that involves a pair of real dynamical variables x and z and a Lagrange multiplier λ. The latter enforces the constraint $x = \dot{z}$ that gives rise to (1). Applying Dirac's Hamiltonian formulation of the constrained systems to the latter Lagrangian, one finds the quadratic Hamiltonian [9, 10]:

$$H = \frac{1}{2}\left(p_x^2 + 2xp_z + \alpha x^2 - \beta z^2\right), \qquad (2)$$

that can be decoupled via a linear canonical transformations. A particular example is the transformation $(x, p_x, z, p_z) \to (x_1, p_1, x_2, p_2)$ defined by

$$x_1 := \frac{p_z + \omega_1^2 x}{\omega_1\sqrt{\omega_1^2 - \omega_2^2}}, \quad x_2 := \frac{p_x + \omega_1^2 z}{\sqrt{\omega_1^2 - \omega_2^2}}, \qquad (3)$$

$$p_1 := \frac{\omega_1(p_x + \omega_2^2 z)}{\sqrt{\omega_1^2 - \omega_2^2}}, \quad p_2 := \frac{p_z + \omega_2^2 x}{\sqrt{\omega_1^2 - \omega_2^2}}, \qquad (4)$$

where we have taken $\omega_1 > \omega_2$ without loss of generality [10]. This canonical transformation maps (2) to

$$H = \frac{1}{2}\,(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}\,(p_2^2 + \omega_2^2 q_2^2). \qquad (5)$$

The standard canonical quantization of (5), $(x_1, p_1, x_2, p_2) \to (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$, with

$$\begin{aligned}(\hat{x}_j\psi)(x_1, x_2) &:= x_j\psi(x_1, x_2),\\ (\hat{p}_j\psi)(x_1, x_2) &:= -i\tfrac{\partial}{\partial x_j}\psi(x_1, x_2),\end{aligned} \qquad (6)$$

yields the Hermitian Hamiltonian operator,

$$\hat{H} = \frac{1}{2}\,(\hat{p_1}^2 + \omega_1^2\hat{q_1}^2) - \frac{1}{2}\,(\hat{p_2}^2 + \omega_2^2\hat{q_2}^2) \qquad (7)$$

with eigenvalues: $\omega_1(n_1 + \frac{1}{2}) - \omega_2(n_2 + \frac{1}{2})$. Here ψ is an square-integrable function, $n_1, n_2 = 0, 1, 2, \cdots$, and we have used units in which $\hbar = 1$. Because the spectrum of $\hat{H}$ is unbounded both from below and above, the corresponding quantum system is unstable. One can avoid this problem by performing an indefinite-metric quantization of the system, [11–13]. This yields a stable quantum theory which is, however, plagued with the presence of ghost states and the associated lack of unitarity. The same problem arises in the indefinite-metric quantization of higher-derivative theories of gravity [14].

In Ref. [3], the authors propose an alternative quantization of the classical Hamiltonian (2) that solves both the instability and non-unitarity problems associated with the standard definite- and indefinite-metric quantizations of the PU-oscillator. In the following we first review the quantization scheme developed in [3] and show that it really amounts to the imaginary scaling, $x_2 \to -ix_2$ and $p_2 \to ip_2$, that clearly maps (5) into the standard classical Hamiltonian for a pair of decoupled simple harmonic oscillators. The latter can be easily quantized to yield a stable and unitary quantum system via the standard canonical quantization scheme. We then explore the quantum analog of this imaginary scaling transformation and use its properties to develop a general quantization scheme. This is a definitization of the indefinite-metric quantization that is related to the latter via a change of sign in the expression for the creation operator of the theory.

2. Imaginary-Scaling Quantization: The main point of departure of the approach of Ref. [3] is the imaginary scaling of dynamical variable z of the Hamiltonian (2), namely $z \to y := -iz$ and $p_z \to p_y := ip_z$. This is a complex canonical transformation that maps (2) into a complex classical Hamiltonian whose standard canonical quantization, namely $(x, p_x, y, p_y) \to (\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ with

$$\begin{aligned}(\hat{x}\psi)(x,y) = x\psi(x,y), \;\; (\hat{p}_x\psi)(x,y) = -i\tfrac{\partial}{\partial x}\psi(x,y),\\ (\hat{y}\psi)(x,y) = y\psi(x,y), \;\; (\hat{p}_y\psi)(x,y) = -i\tfrac{\partial}{\partial y}\psi(x,y),\end{aligned} \tag{8}$$

yields the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian [3]:

$$\hat{H}_{\mathcal{PT}} := \frac{1}{2}\left(\hat{p}_x^2 - 2i\hat{x}\hat{p}_y + \alpha\hat{x}^2 + \beta\hat{y}^2\right). \tag{9}$$

It turns out that, for $\omega_1 \neq \omega_2$, there is a similarity transformation [5, 15] that maps this operator to the sum of two simple harmonic oscillator Hamiltonians, $\hat{h}$. This shows that the spectrum of $\hat{H}_{\mathcal{PT}}$ is real and positive. Moreover, it implies the existence of a nonstandard inner product and the associated Hilbert space $\mathscr{H}$ that restore the Hermiticity of $\hat{H}_{\mathcal{PT}}$, [5, 6]. The quantum system [6] defined by either of the unitary-equivalent Hilbert space-Hamiltonian pairs $(\mathscr{H}, \hat{H}_{\mathcal{PT}})$ and $(L^2(\mathbb{R}^2), \hat{h})$ is, therefore, both unitary and stable.

Next, we recall that the similarity transformations mapping $\hat{H}_{\mathcal{PT}}$ to $\hat{h}$ correspond to (complex) linear canonical transformations. Because these commute with the canonical quantization of the underlying classical Hamiltonian, one may perform the necessary linear canonical transformations in the classical level and then quantize the system. This is particularly desirable, because it avoids dealing with the non-Hermitian Hamiltonian operator (9) and gives the same Hermitian Hamiltonian operator $\hat{h}$. Following this approach we first apply the real linear canonical transformation (3)-(4) on the classical Hamiltonian (2) to obtain (5). We then perform the imaginary scaling transformation:

$$(x_2, p_2) \to (-ix_2, ip_2) =: (\tilde{x}_2, \tilde{p}_2), \tag{10}$$

that flips the unwanted sign in (5). Finally we quantize the resulting classical Hamiltonian to obtain:

$$\hat{h} := \frac{1}{2}\left(\hat{p}_1^2 + \omega_1^2\hat{x}_1^2 + \hat{\tilde{p}}_2^2 + \omega_2^2\hat{\tilde{x}}_2^2\right). \tag{11}$$

Clearly, this defines a unitary and stable quantum system. The procedure outlined in [3] can therefore be reduced to a simple imaginary scaling transformation.

Another equivalent prescription is to perform the imaginary scaling transformation on the quantum Hamiltonian operator (7). This is affected by a linear operator with rather peculiar properties.

Let $\mathscr{C}^\infty$ denote the vector space of smooth complex-valued functions defined on the real line and $\hat{x}, \hat{p}$ be the standard position and momentum operators acting in $\mathscr{C}^\infty$; for all $f \in \mathscr{C}^\infty$, $(\hat{x}f)(x) := xf(x)$ and $(\hat{p}f)(x) := -if'(x)$. Consider the operator $A : \mathscr{C}^\infty \to \mathscr{C}^\infty$ that is defined by

$$A := \exp\left(\frac{\pi}{4}\{\hat{x}, \hat{p}\}\right). \tag{12}$$

It is not difficult to show that [16]

$$(Af)(x) = f(-ix). \tag{13}$$

We can view A as a linear operator acting in the Hilbert space of square-integrable functions $L^2(\mathbb{R})$. This is a densely-defined unbounded one-to-one linear operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with a dense range [21] that realizes the quantum imaginary scaling transformation:

$$\begin{aligned}\hat{x} \xrightarrow{A} \hat{\tilde{x}} := A\,\hat{x}\,A^{-1} = -i\hat{x},\\ \hat{p} \xrightarrow{A} \hat{\tilde{p}} := A\,\hat{p}\,A^{-1} = i\hat{p}.\end{aligned} \tag{14}$$

Expression (12) suggests that A is a Hermitian operator acting in $L^2(\mathbb{R})$. This is however not true. One way of seeing this is to notice that if A was a Hermitian operator, its square A^2, that is also densely defined, would have been a positive operator [17]. This contradicts the fact that $g(x) := xe^{-x^4}$ is an eigenfunction of A^2 with eigenvalue -1, [22]. Therefore, A is not Hermitian.

Another unusual but obvious property of A is that the harmonic oscillator eigenfunctions do not belong to its domain, unless we consider it as mapping $L^2(\mathbb{R})$ into another function space. Let $\psi_n(x) := N_n H_n(x) e^{-x^2/2}$ be the normalized eigenfunctions of the simple harmonic oscillator Hamiltonian $\hat{\mathcal{H}} := \frac{1}{2}(\hat{p}^2 + \hat{x}^2)$, with N_n and H_n being the normalization constants and Hermit polynomials. Clearly, $\psi_n \in \mathscr{C}^\infty$ and

$$\tilde{\psi}_n(x) := (A\psi_n)(x) = N_n H_n(-ix) e^{x^2/2} \notin L^2(\mathbb{R}). \tag{15}$$

This observation suggests the definition of an appropriate Hilbert space $\tilde{\mathscr{H}}$ containing $\tilde{\psi}_n$ and viewing A as a linear operator mapping $L^2(\mathbb{R})$ to $\tilde{\mathscr{H}}$. In order to construct $\tilde{\mathscr{H}}$, we endow the linear span $\tilde{\mathscr{S}}$ of $\tilde{\psi}_n$ with the inner product:

$$\langle\!\langle \tilde{\phi}, \tilde{\psi} \rangle\!\rangle := \langle A^{-1}\tilde{\phi} | A^{-1}\tilde{\psi}\rangle = \int_{-\infty}^{\infty} \tilde{\phi}(ix)^* \tilde{\psi}(ix) dx, \tag{16}$$

and identify $\tilde{\mathscr{H}}$ with the Cauchy completion of the resulting inner-product space [18, 19].

By construction, for all $\phi, \psi \in L^2(\mathbb{R})$ satisfying $A\phi, A\psi \in \tilde{\mathscr{H}}$, we have $\langle\phi|\psi\rangle = \langle\!\langle A\phi, A\psi\rangle\!\rangle$. Therefore,

$A: L^2(\mathbb{R}) \to \tilde{\mathscr{H}}$ is a densely-defined isometry [18]. In particular, it is a bounded operator with a bounded inverse whose domain includes ψ_n. As we will see momentarily this is a necessary step in relating the above imaginary-scaling quantization of the PU oscillator to its indefinite-metric quantization.

Now, consider $\hat{\tilde{\mathcal{H}}} := -\hat{\mathcal{H}} = -\frac{1}{2}(\hat{p}^2 + \hat{x}^2)$. If we view $\hat{\tilde{\mathcal{H}}}$ as an operator acting in $L^2(\mathbb{R})$, it is a Hermitian operator with a negative spectrum, $\{-(n+\frac{1}{2})|n = 0, 1, 2, \cdots\}$. The situation is different if we identify $\hat{\tilde{\mathcal{H}}}$ with an operator acting in $\tilde{\mathscr{H}}$. In this case, performing the imaginary scaling transformation on $\hat{\tilde{\mathcal{H}}}$, we find the operator

$$\hat{\mathfrak{h}} := A^{-1}\hat{\tilde{\mathcal{H}}}A = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) = \hat{\mathcal{H}}, \tag{17}$$

that acts in $L^2(\mathbb{R})$ and has a positive spectrum, $\{n + \frac{1}{2}|n = 0, 1, 2, \cdots\}$. Because $A : L^2(\mathbb{R}) \to \tilde{\mathscr{H}}$ is a bounded operator with a bounded inverse and domain of A contains the eigenfunctions ψ_n of $\hat{\mathfrak{h}}$, the operators $\hat{\tilde{\mathcal{H}}} : \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$ and $\hat{\mathfrak{h}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are isospectral. In particular, the spectrum of the former operator is also real and positive. It is easy to see that $\tilde{\psi}_n$ are the eigenfunctions of $\hat{\tilde{\mathcal{H}}}$ with eigenvalue $n + \frac{1}{2}$.

3. Connection to Indefinite-Metric Quantization: Consider the standard annihilation and creation operators, $\hat{a} := (\hat{x}+i\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger := (\hat{x}-i\hat{p})/\sqrt{2}$, that satisfy $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{a}\psi_0 = 0$, and $\psi_n = (n!)^{-1/2}\hat{a}^{\dagger n}\psi_0$. We can view ψ_n as elements of $\mathscr{C}^\infty$ and try to reconstruct the Hilbert space of the state vectors of the simple Harmonic oscillator $\hat{\mathfrak{h}}$, namely $L^2(\mathbb{R})$, using ψ_n. This is done by endowing the linear span $\mathscr{S}$ of ψ_n with an inner product $\langle\cdot,\cdot\rangle$ that renders the number operator $\hat{N} := \hat{a}^\dagger\hat{a}$ Hermitian. This holds if $\hat{a}^\dagger$ is the adjoint of $\hat{a}$, i.e., for all $\phi, \psi \in \mathscr{S}$, $\langle\phi, a\psi\rangle = \langle a^\dagger\phi, \psi\rangle$. Once we select an inner product fulfilling this condition, we can identify the Hilbert space with the Cauchy completion of $\mathscr{S}$. If we use the L^2-inner product, $\langle\psi|\phi\rangle = \int_{-\infty}^{\infty}\phi(x)^*\psi(x)dx$, that has the above property, we obtain $L^2(\mathbb{R})$ as the Hilbert space of the system.

Next, following the standard argument used in the indefinite-metric quantization scheme, we propose to construct a Hilbert space using a different set of functions, $\tilde{\phi}_n$, that we define as follows: $\tilde{\phi}_0$ is (up to the choice of a normalization constant) a solution of the differential equation $\hat{a}^\dagger\tilde{\phi}_0 = 0$, and $\tilde{\phi}_n := (n!)^{-1/2}\hat{a}^n\tilde{\phi}_0$ for all $n = 1, 2, \cdots$. In other words, we now interpret $\hat{\tilde{a}} := \hat{a}^\dagger$ and $\hat{\tilde{a}}^\dagger := \hat{a}$ as the annihilation and creation operators, respectively. This is because

$$\hat{\tilde{a}}\,\tilde{\phi}_0 = 0, \qquad \tilde{\phi}_n = (n!)^{-1/2}\hat{\tilde{a}}^{\dagger n}\tilde{\phi}_0. \tag{18}$$

The new annihilation and creation operators, $\hat{\tilde{a}}$ and $\hat{\tilde{a}}$ satisfy the "abnormal" bosonic commutation relation [12]:

$$[\hat{\tilde{a}}, \hat{\tilde{a}}^\dagger] = -1. \tag{19}$$

We will denote the linear span of $\tilde{\phi}_n$ with $\tilde{\mathscr{T}}$ and try to construct an appropriate inner product $\prec\cdot,\cdot\succ$ on $\tilde{\mathscr{T}}$ that makes $\hat{\tilde{a}}^\dagger$ adjoint of $\hat{\tilde{a}}$, i.e., for all $\tilde{\xi}, \tilde{\zeta} \in \tilde{\mathscr{T}}$,

$$\prec\tilde{\xi}, \hat{\tilde{a}}\,\tilde{\zeta}\succ = \prec\hat{\tilde{a}}^\dagger\tilde{\xi}, \tilde{\zeta}\succ. \tag{20}$$

It turns out that the number operator for $\tilde{\psi}_n$ is the operator $\hat{\tilde{N}} : \tilde{\mathscr{T}} \to \tilde{\mathscr{T}}$ that is defined by

$$\hat{\tilde{N}} := -\hat{\tilde{a}}^\dagger\hat{\tilde{a}} = -\hat{a}\hat{a}^\dagger. \tag{21}$$

We can use (18) and (19) to show that $\hat{\tilde{N}}\tilde{\phi}_n = n\tilde{\phi}_n$. Furthermore, in light of (21), we have $\hat{\tilde{\mathcal{H}}} = \hat{\tilde{N}} + \frac{1}{2}$. This shows that if we can find an inner product respecting (20), we can complete $\tilde{\mathscr{T}}$ into a Hilbert space $\mathscr{H}'$ and view $\hat{\tilde{\mathcal{H}}}$ as an operator acting in $\mathscr{H}'$. The spectrum of this operator will then consist of the eigenvalues $n + \frac{1}{2}$ with $n = 0, 1, 2, \cdots$. In particular, it will be positive.

The main difficulty with the above construction is that the condition (20) conflicts with the positive-definiteness of the inner product $\prec\cdot,\cdot\succ$; one can use (18) and (19) to show that for all $m, n = 0, 1, 2, \cdots$,

$$\prec\tilde{\phi}_m, \tilde{\phi}_n\succ = (-1)^n\delta_{mn}\prec\tilde{\phi}_0, \tilde{\phi}_0\succ. \tag{22}$$

Therefore, $\prec\cdot,\cdot\succ$ is an indefinite inner product. It gives $\tilde{\mathscr{S}}$ the structure of a Krein space. It is customary to choose $\prec\tilde{\phi}_0, \tilde{\phi}_0\succ = 1$ and view $\tilde{\phi}_{2n+1}$, that have an imaginary norm, as defining "ghost states."

A key observation that links the imaginary-scaling and indefinite-metric quantization schemes is to notice that up to a constant coefficient $\tilde{\phi}_n$ coincides with $\tilde{\psi}_n$; as elements of $\mathscr{C}^\infty$, they are related according to

$$\tilde{\phi}_n = i^n\tilde{\psi}_n. \tag{23}$$

This in turn implies that $\tilde{\phi}_n$ belong to the Hilbert space $\tilde{\mathscr{H}}$ of the preceding section, and $\tilde{\mathscr{T}} = \tilde{\mathscr{S}}$.

The relationship between the indefinite inner product $\prec\cdot,\cdot\succ$ and the definite inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is typical of the Hilbert spaces endowed with a Krein-space structure [20]. To describe this relationship we use the parity operator $\mathcal{P}$ to introduce

$$\Pi_\pm := \frac{1}{2}(\mathcal{P}\pm 1), \quad \tilde{\mathscr{S}}_\pm := \Pi_\pm\tilde{\mathscr{S}}, \quad \mathcal{C} := \Pi_+ - \Pi_-. \tag{24}$$

It is easy to see that $\tilde{\psi}_{2n} \in \tilde{\mathscr{S}}_+$, $\tilde{\psi}_{2n+1} \in \tilde{\mathscr{S}}_-$,

$$\tilde{\mathscr{S}} = \tilde{\mathscr{S}}_+ \oplus \tilde{\mathscr{S}}_-, \tag{25}$$

$\tilde{\mathscr{S}}_\pm$ are the eigenspaces of the restriction of $\mathcal{P}$ onto $\tilde{\mathscr{S}}$ with eigenvalues ± 1, and $\mathcal{C}$ is the grading operator associated with the decomposition (25). Note also that in view of (16), (22), and (23), this is an orthogonal decomposition with orthogonality condition defined by either of $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and $\prec\cdot,\cdot\succ$. Furthermore, we can use (16), (22), and (23) to show that for all $\tilde{\xi}, \tilde{\zeta} \in \tilde{\mathscr{S}}$,

$$\langle\!\langle\tilde{\xi}, \tilde{\zeta}\rangle\!\rangle = \prec\Pi_+\tilde{\xi}, \Pi_+\tilde{\zeta}\succ - \prec\Pi_-\tilde{\xi}, \Pi_-\tilde{\zeta}\succ, \tag{26}$$

$$\prec\tilde{\xi}, \tilde{\zeta}\succ = \langle\!\langle\tilde{\xi}, \mathcal{P}\tilde{\zeta}\rangle\!\rangle = \int_{-\infty}^{\infty}\tilde{\xi}(ix)^*\tilde{\zeta}(-ix)dx. \tag{27}$$

4. Creation and Annihilation Operators: The above analysis suggests that the number operator $\tilde{N}$ has a real spectrum and an orthonormal set of eigenvectors belonging to the Hilbert space $\tilde{\mathscr{H}}$, namely $\tilde{\psi}_n$. Hence, $\tilde{N}$ is a Hermitian operator acting in $\tilde{\mathscr{H}}$. However, we showed by explicit calculation that every inner product that identifies $\tilde{\hat{a}}^\dagger$ with the adjoint of $\tilde{\hat{a}}$ is necessarily indefinite. Therefore, as operators acting in $\tilde{\mathscr{H}}$, $\tilde{\hat{a}}^\dagger$ is not the adjoint of $\tilde{\hat{a}}$, yet $\tilde{N} = -\tilde{\hat{a}}^\dagger\tilde{\hat{a}}$ is a Hermitian (self-adjoint) operator. In order to see that this strange phenomenon does not lead to any inconsistency, we make the following observations. First, we note that according to (16) and (26),

$$\prec \tilde{\xi}, \mathcal{P}\tilde{\zeta} \succ = \prec \mathcal{P}\tilde{\xi}, \tilde{\zeta} \succ . \tag{28}$$

Combining this relation with (26) and using the fact that $\mathcal{P}$ and $\tilde{\hat{a}}^\dagger$ anticommmute, we have

$$\langle\!\langle \tilde{\xi}, \tilde{\hat{a}}\tilde{\zeta} \rangle\!\rangle = \langle\!\langle \mathcal{P}\tilde{\hat{a}}^\dagger\mathcal{P}\tilde{\xi}, \tilde{\zeta} \rangle\!\rangle = \langle\!\langle -\tilde{\hat{a}}^\dagger\tilde{\xi}, \tilde{\zeta} \rangle\!\rangle. \tag{29}$$

This identifies the adjoint of $\tilde{\hat{a}}$ with $-\tilde{\hat{a}}^\dagger$. Denoting the latter by $\tilde{\hat{a}}^{\tilde{\dagger}}$, we find that $\tilde{N} = -\tilde{\hat{a}}^\dagger\tilde{\hat{a}} = \tilde{\hat{a}}^{\tilde{\dagger}}\tilde{\hat{a}}$. Therefore $\tilde{N}$ is indeed a positive (and in particular Hermitian) operator acting in $\tilde{\mathscr{H}}$. Another outcome of this calculation is that we can reconstruct the Hilbert space of the imaginary-scaling quantization by, respectively, adopting

$$\tilde{\hat{a}} := \hat{a}^\dagger, \qquad \tilde{\hat{a}}^{\tilde{\dagger}} := -\hat{a}, \tag{30}$$

as the annihilation and creation operators. Note that these satisfy the standard bosonic commutation relation, $[\tilde{\hat{a}}, \tilde{\hat{a}}^{\tilde{\dagger}}] = 1$. Therefore, the algebra of the creation and annihilation operators of the imaginary-scaling quantization is identical with the usual canonical quantization of a bosonic system. What makes the difference is how the creation and annihilation operators related to the position and momentum operators (fields and their time-derivative in field theory).

5. Concluding Remarks: In this paper we have established the imaginary scaling transformation as the main ingredient of the quantization scheme developed in Ref. [3]. We have subsequently simplified this scheme in such a way that it avoids dealing with non-Hermitian $\mathcal{PT}$-symmetric operators. We have examined the consequences of performing the quantum imaginary scaling transformation. This is realized by a linear operator with rather unusual properties. We have exploited these properties to identify the appropriate Hilbert space of state vectors of the quantum PU oscillator. We have also shown how these constructions reappear in the indefinite-metric quantization scheme. In particular, we have given an explicit expression for the standard indefinite inner product and show that it gives the Hilbert space of the imaginary-scaling quantization the structure of a Krein space.

We have revealed the relationship between the definite (Hilbert space) inner product and the indefinite (Krein space) inner product and obtained an appropriate pair of creation and annihilation operators for the imaginary-scaling quantization scheme. These differ from the creation and annihilation operators of the indefinite-metric theories in the sign of the creation operator. This seemingly minor difference is responsible for transforming the ghosts into physical states and the subsequent restoration of unitarity. It is important to note that this is achieved at no energy cost; the contribution of the ghost states to the spectrum of the Hamiltonian is the same as in the corresponding indefinite-metric theory. In particular, if we consider the sum of two harmonic oscillator Hamiltonians with unit mass and frequency, and quantize the first of these using the standard canonical quantization and the second via the imaginary scaling transformation, the energy spectrum turns out to be the difference of the mode numbers. In particular, similarly to the case of indefinite-metric quantization of the second oscillator, the vacuum energy vanishes identically [11]. This observation seems to indicate that the imaginary-scaling quantization shares the niceties of the indefinite-metric quantization while not suffering from the lack of a consistent probabilistic interpretation. We plan to examine the prospects of this scheme in dealing with the quantum mechanical and field theoretical models that were treated in the context of indefinite-metric quantum theories.

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[19] It is tempting to view equation (16) as an indication that $\langle\!\langle\tilde\xi,\tilde\zeta\rangle\!\rangle$ can be expressed as $\langle\tilde\xi|\eta_+\tilde\zeta\rangle$ with η_+ being the metric operator $\eta_+ = A^{-1\dagger}A^{-1}$. This argument is erroneous, because in general $\tilde\xi$ and $\eta_+\tilde\zeta$ do not belong to $L^2(\mathbb{R})$, and $\langle\tilde\xi|\eta_+\tilde\zeta\rangle$ may be divergent.

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[21] Smooth functions having a compact support belong to the domain of A. Because these functions constitute a dense subset of $L^2(\mathbb{R})$, A is densely defined. It is also easy to see that $A^{-1} = \mathcal{P}A$. Therefore, as operators acting in $L^2(\mathbb{R})$, the domains of A and A^{-1} coincide.

[22] A^2 is the restriction of the parity operator $\mathcal{P}$ on the domain of A^2. The parity operator is defined by $(\mathcal{P}\psi)(x) := \psi(-x)$ for every function $\psi : \mathbb{R} \to \mathbb{C}$ defined on $\mathbb{R}$.