等约束条件下多元函数条件极值的充分条件[1]

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摘要： 在等约束条件下用 *Lagrange* 乘数法、多元隐函数求导法以及有条件极值化无条件极值的方法推导证明了多元函数极值的充分条件，并给出易于计算且切实可行的方法和定理，弥补了大学数学教材里多元函数条件极值无充分条件的空白。

关键词： 等约束；多元函数；极值；充分条件
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1.引 言

在各类大学数学经典教材中，关于等约束条件极值几乎只是给出了其必要条件，而要求出极值通常是通过 *Lagrange* 乘数法求出可能的极值点，至于是极大值还是极小值往往是结合实际问题讨论，要从理论上准确判断出其是极大、极小值往往没有办法[1]，没有充分条件的理论支撑。许多文章给出的方法计算量大，可行性不强。本文给出了相对简单便于计算的多元函数在多个等式约束条件下极值的充分条件。

2.二元函数在单等式约束条件下的极值

函数 $u=f(x,y)$ 在条件 $\varphi(x,y)=0$ 下取到极值的充分条件

定理 1: 若 $u=f(x,y)$，$\varphi(x,y)$ 在平面区域 D 上具有二阶连续偏导数，$M_0(x_0,y_0)$ 是 D 的内点， 且 *lagrange* 函数 $L(x,y)=f(x,y)+\lambda\varphi(x,y)$，$A=\begin{pmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{pmatrix}_{M_0}$；

则 A 正定，$M_0(x_0,y_0)$ 是 $u=f(x,y)$ 在条件 $\varphi(x,y)$=0 下的极小值；

A 负定，$M_0(x_0,y_0)$ 是 $u=f(x,y)$ 在条件 $\varphi(x,y)$=0 下的极大值；

A 不定，该法无法断定。

证明：不妨设 $\varphi_y\neq 0$，由 $\varphi(x,y)=0\Rightarrow\varphi(x,\psi(x))=0, y=\psi(x)$，则 $y'=-\dfrac{\varphi_1}{\varphi_2}$，　（1）

其中，$\varphi_1=\varphi_x,\varphi_2=\varphi_y$；由 $y'=-\dfrac{\varphi_1}{\varphi_2}\Rightarrow y''=-\dfrac{(\varphi_{11}+\varphi_{12}y')\varphi_2-\varphi_1(\varphi_{21}+\varphi_{22}y')}{\varphi_2{}^2}$　（2）

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(2)代入(1)，有 $y''=-\dfrac{\varphi_{11}\varphi_2^{\ 2}-\varphi_1\varphi_2\varphi_{12}-\varphi_1\varphi_2\varphi_{21}+\varphi_1^{\ 2}\varphi_{22}}{\varphi_2^{\ 3}}=-\dfrac{\varphi_{11}\varphi_2^{\ 2}-2\varphi_1\varphi_2\varphi_{12}+\varphi_1^{\ 2}\varphi_{22}}{\varphi_2^{\ 3}}$ （3）

又由 $u(x)=f(x,\psi(x))\Rightarrow u'(x)=f_1+f_2y'=f_1-f_2\dfrac{\varphi_1}{\varphi_2}$，令 $\lambda=-\dfrac{f_2}{\varphi_2}$，则

$\begin{cases}L_x=f_1(x,y)+\lambda\varphi_1(x,y)=0\\L_y=f_2(x,y)+\lambda\varphi_2(x,y)=0\\\varphi(x,y)=0\end{cases}$，设 (x_0,y_0,λ_0) 是该方程组的解，由

$u''(x_0)=f_{11}^{\ 0}+f_{12}^{\ 0}y'+(f_{21}^{\ 0}+f_{22}^{\ 0}y')y'+f_2y''$，将（1)，(3）代入此式化简有：

$$(\varphi_2^0)^3u''(x_0)=(f_{11}^{\ 0}+\lambda_0\varphi_{11}^0)(\varphi_2^0)^2-2(f_{12}^{\ 0}+\lambda_0\varphi_{12}^0)\varphi_1^0\varphi_2^0+(f_{22}^{\ 0}+\lambda_0\varphi_{22}^0)\varphi_1^0\Rightarrow$$

$$(\varphi_2^0)^3u''(x_0)=(\varphi_1^0,\varphi_2^0)\begin{pmatrix}f_{22}^{\ 0}+\lambda_0\varphi_{22}^0 & -f_{12}^{\ 0}-\lambda_0\varphi_{12}^0\\-f_{12}^{\ 0}-\lambda_0\varphi_{12}^0 & f_{11}^{\ 0}+\lambda_0\varphi_{11}^0\end{pmatrix}\begin{pmatrix}\varphi_1^0\\\varphi_2^0\end{pmatrix},$$

其中，$f_{ij}^0=f_{ij}(x_0,y_0),\varphi_{ij}^0=\varphi_{ij}(x_0,y_0)$；$(i,j=1,2)$

若该二次型矩阵 $A^*=\begin{pmatrix}f_{22}^{\ 0}+\lambda_0\varphi_{22}^0 & -f_{12}^{\ 0}-\lambda_0\varphi_{12}^0\\-f_{12}^{\ 0}-\lambda_0\varphi_{12}^0 & f_{11}^{\ 0}+\lambda_0\varphi_{11}^0\end{pmatrix}$ 正定，则 $u''(x_0)>0,$ 一元函数在 x_0 处取到极小值；若 A^* 负定,，则 $u''(x_0)<0,$ 一元函数在 x_0 处取到极大值；又 A^* 是

$A=\begin{pmatrix}f_{11}^{\ 0}+\lambda_0\varphi_{11}^0 & f_{12}^{\ 0}+\lambda_0\varphi_{12}^0\\f_{21}^{\ 0}+\lambda_0\varphi_{21}^0 & f_{22}^{\ 0}+\lambda_0\varphi_{22}^0\end{pmatrix}=\begin{pmatrix}f_{11}^{\ 0}+\lambda_0\varphi_{11}^0 & f_{12}^{\ 0}+\lambda_0\varphi_{12}^0\\f_{12}^{\ 0}+\lambda_0\varphi_{12}^0 & f_{22}^{\ 0}+\lambda_0\varphi_{22}^0\end{pmatrix}$ 的伴随矩阵，两矩阵有相同的正定和负定性[2]，结论得证。

3.三元函数的条件极值

3.1 单等式约束条件下极值

函数 $u=f(x,y,z)$ 在 $\varphi(x,y,z)=0$ 下取到极值的充分条件。

定理 2：设函数 $u=f(x,y,z)$，$\varphi(x,y,z)$ 存在二阶连续偏导数，且 *Lagrange* 函数 $L(x,y,z)=f(x,y,z)+\lambda\varphi(x,y,z)$，$M_0(x_0,y_0,z_0)$ 是空间区域 Ω 的内点，则

$A=(L_{xx}+2L_{xz}\dfrac{\partial z}{\partial x}+L_{zz}(\dfrac{\partial z}{\partial x})^2)\Big|_{M_0}$；$B=(L_{xy}+L_{xz}\dfrac{\partial z}{\partial x}+L_{yz}\dfrac{\partial z}{\partial y}+L_{zz}\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y})\Big|_{M_0}$；

$C=(L_{yy}+2L_{yz}\dfrac{\partial z}{\partial y}+L_{zz}(\dfrac{\partial z}{\partial y})^2)\Big|_{M_0}$ 且当 $AC-B^2>0$ 时，若 $A<0$，则 $f(x_0,y_0,z_0)$ 是极大值，若 $A>0$，则 $f(x_0,y_0,z_0)$ 是极小值；当 $AC-B^2<0$ 时，$f(x_0,y_0,z_0)$ 不是极值；

当 $AC-B^2=0$ 时,不能断定 $f(x_0,y_0,z_0)$ 是否是极值。

证明：由隐函数存在定理可知，$\varphi(x,y,z)=0$ 确定具有二阶连续偏导数的隐函数

$z=\psi(x,y)$ 且 $z_0=\psi(x_0,y_0)$，将 $z=\psi(x,y)$ 代入 $u=f(x,y,z)$ 得：

$$u=f(x,y,\psi(x,y))\Rightarrow\frac{\partial u}{\partial x}=f_x+f_z\frac{\partial z}{\partial x},$$

$$\frac{\partial u}{\partial y}=f_y+f_z\frac{\partial z}{\partial y}\Rightarrow\frac{\partial^2 u}{\partial x^2}=f_{xx}+f_{xz}\frac{\partial z}{\partial x}+(f_{zx}+f_{zz}\frac{\partial z}{\partial x})\frac{\partial z}{\partial x}+f_z\frac{\partial^2 z}{\partial x^2};$$

$$\frac{\partial^2 u}{\partial x\partial y}=f_{xy}+f_{xz}\frac{\partial z}{\partial y}+(f_{zy}+f_{zz}\frac{\partial z}{\partial y})\frac{\partial z}{\partial x}+f_z\frac{\partial^2 z}{\partial x\partial y};$$

$$\frac{\partial^2 u}{\partial y^2}=f_{yy}+f_{yz}\frac{\partial z}{\partial y}+(f_{zy}+f_{zz}\frac{\partial z}{\partial y})\frac{\partial z}{\partial y}+f_z\frac{\partial^2 z}{\partial y^2}$$ 又将下几式

$$\frac{\partial z}{\partial x}=-\frac{\varphi_x}{\varphi_z},\frac{\partial z}{\partial y}=-\frac{\varphi_y}{\varphi_z}\Rightarrow\frac{\partial^2 z}{\partial x^2}=-\frac{\varphi_{xx}\varphi_z^2-2\varphi_{xz}\varphi_x\varphi_z+\varphi_{zz}\varphi_x^2}{\varphi_z^3};$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{\varphi_{xy}\varphi_z^2-(\varphi_{xz}\varphi_y+\varphi_{yz}\varphi_x)\varphi_z+\varphi_{zz}\varphi_x\varphi_y}{\varphi_z^3};\quad\frac{\partial^2 z}{\partial y^2}=-\frac{\varphi_{yy}\varphi_z^2-2\varphi_{yz}\varphi_y\varphi_z+\varphi_{zz}\varphi_x^2}{\varphi_z^3}$$ 代入

(4)、(5)、(6) 式且令 $\lambda=-\frac{f_z}{\varphi z}\Rightarrow\frac{\partial^2 u}{\partial x^2}=(L_{xx}\varphi_z^2-2L_{xz}\varphi_x\varphi_z+L_{zz}\varphi_x^2)\frac{1}{\varphi_z^2}$；

$$\frac{\partial^2 u}{\partial y^2}=(L_{yy}\varphi_z^2-2L_{yz}\varphi_y\varphi_z+L_{zz}\varphi_y^2)\frac{1}{\varphi_z^2};\frac{\partial^2 u}{\partial x\partial y}=(L_{xy}\varphi_z^2-L_{xz}\varphi_x\varphi_z-L_{yz}\varphi_y\varphi_z+L_{zz}\varphi_x\varphi_z)\frac{1}{\varphi_z^2}$$

$\Rightarrow\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,y_0)}=\mu A,\left.\frac{\partial^2 u}{\partial x\partial y}\right|_{(x_0,y_0)}=\mu B,\left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,y_0)}=\mu C,$ 其中 $\mu=\frac{1}{\varphi_z^2(x_0,y_0,z_0)}>0$；利用二元函数无条件极值的充分条件即可导出本定理结论。

3.2 两个等式约束条件下的极值

函数 $u=f(x,y,z)$ 在 $\varphi(x,y,z)=0$，$\psi(x,y,z)=0$ 下取到极值的充分条件。

定理 3：设函数 $u=f(x,y,z)$，$\varphi(x,y,z)$、$\psi(x,y,z)$ 存在二阶连续偏导数，且

Lagrange 函数 $L(x,y,z)=f(x,y,z)+\lambda\varphi(x,y,z)+\mu\psi(x,y,z)$，$P_0(x_0,y_0,z_0)$ 是空间区域 Ω 的内点且 M_0 为 $L(x,y,z)$ 的稳定点，$\left.\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right|_{P_0}\neq 0$ 则当

$\Delta=[L_{xx}(\frac{dx}{dz})^2+2L_{xy}\frac{dx}{dz}\frac{dy}{dz}+L_{yy}(\frac{dy}{dz})^2+2L_{xz}\frac{dx}{dz}+2L_{yz}\frac{dy}{dz}+L_{zz}]\Big|_{M_0}\neq 0$时，$f(x,y,z)$在$P_0(x_0,y_0,z_0)$取到极值，且$\Delta>0$，$f(x,y,z)$取到条件极小值；$\Delta<0$，$f(x,y,z)$取到条件极大值。其中$\frac{dx}{dz}=-\frac{\frac{\partial(\varphi,\psi)}{\partial(z,y)}}{\frac{\partial(\varphi,\psi)}{\partial(x,y)}},\frac{dy}{dz}=-\frac{\frac{\partial(\varphi,\psi)}{\partial(x,z)}}{\frac{\partial(\varphi,\psi)}{\partial(x,y)}}$。（定理的证明是定理 5 的特殊情况）

4.多元函数条件极值的充分条件

对于目标函数$f(x_1,x_2,\cdots,x_n)$在等式约束条件$\varphi_k(x_1,x_2,\cdots x_n)=0$，$k=1,2,\cdots,m;(m<n)$下的条件极值的充分条件[3]。

设$f(x_1,x_2,\cdots,x_n)$，$\varphi_k(x_1,x_2,\cdots x_n)$，$k=1,2,\cdots,m;(m<n)$在 n 维空间区域$\Omega$上具有连续的一阶偏导数，$P_0(x_1^0,x_2^0,\cdots,x_n^0)\in\Omega$，且$M_0(x_1^0,x_2^0,\cdots,x_n^0,\lambda_1^0,\lambda_2^0,\cdots,\lambda_k^0)$为$Lagrange$函数$L(x_1,x_2,\cdots,x_n,\lambda_1,\lambda_2,\cdots,\lambda_k)=f(x_1,x_2,\cdots,x_n)+\sum_{k=1}^{m}\lambda_k\varphi_k(x_1,x_2,\cdots,x_n)$的稳定点。则：

4.1 当$n=m+2$时，有如下定理：

定理 4： 设$f(x_1,x_2,\cdots,x_n)$，$\varphi_k(x_1,x_2,\cdots x_n)$，$k=1,2,\cdots,m;(m<n)$在 n 维空间区域$\Omega$上具有连续的二阶偏导数，$P_0(x_1^0,x_2^0,\cdots,x_n^0)\in\Omega$，$\frac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}\Big|_{P_0}\neq 0$，$M_0(x_1^0,x_2^0,\cdots,x_n^0,\lambda_1^0,\lambda_2^0,\cdots,\lambda_k^0)$为$L(x_1,x_2,\cdots,x_n,\lambda_1,\lambda_2,\cdots,\lambda_k)$的稳定点。令

$A=\sum_{i=1}^{n}\sum_{j=1}^{n}(L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+1}})\Big|_{M_0}$；$B=\sum_{i=1}^{n}\sum_{j=1}^{n}(L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+2}})\Big|_{M_0}$；

$C=\sum_{i=1}^{n}\sum_{j=1}^{n}(L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+2}}\frac{\partial x_j}{\partial x_{m+2}})\Big|_{M_0}$；$D=AC-B^2$；其中

$$\frac{\partial x_i}{\partial x_{m+1}}=-\frac{\frac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_{i-1},\varphi_i,\varphi_{i+1},\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_{i-1},x_{m+1},x_{i+1},\cdots,x_m)}}{\frac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}},\quad i=1,2,\cdots,m;$$

$$\frac{\partial x_i}{\partial x_{m+2}}=-\frac{\dfrac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_{i-1},\varphi_i,\varphi_{i+1},\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_{i-1},x_{m+2},x_{i+1},\cdots,x_m)}}{\dfrac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}},\quad i=1,2,\cdots,m;\quad \frac{\partial x_{m+1}}{\partial x_{m+2}}=0,\quad \frac{\partial x_{m+2}}{\partial x_{m+1}}=0。$$

则当 $D=AC-B^2>0$，若 $A<0$，则 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 取到条件极大值，若 $A>0$，则 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 取到条件极小值；当 $D=AC-B^2<0$ 时，$f(x_1,x_2,\cdots,x_n)$ 在 P_0 不取条件极值；当 $D=AC-B^2=0$ 时,不能断定 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 是否取到条件极值。

证明：因 $\varphi_k(x_1,x_2,\cdots x_n)$ 具有连续的二阶偏导数，且 $\left.\dfrac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}\right|_{P_0}\neq 0$，由隐函数存在定理，在 P_0 的某邻域内方程组 $\varphi_k(x_1,x_2,\cdots x_n)=0$，$k=1,2,\cdots,m;$ 可确定出唯一的一组连续且有连续偏导数的隐函数组 $\begin{cases}x_1=x_1(x_{m+1},x_{m+2});\\ x_2=x_2(x_{m+1},x_{m+2});\\ \cdots\cdots\\ x_m=x_m(x_{m+1},x_{m+2});\end{cases}$ （*），将（*）式代入 *Lagrange* 函数 $L(x_1,x_2,\cdots,x_n,\lambda_1^0,\lambda_2^0,\cdots,\lambda_k^0)$ 中，得：

$$L(x_1,x_2,\cdots,x_n,\lambda_1^0,\lambda_2^0,\cdots,\lambda_k^0)=f(x_1(x_{m+1},x_{m+2}),\cdots,x_m(x_{m+1},x_{m+2}),x_{m+1}(x_{m+1},x_{m+2}))$$
$$+\sum_{k=1}^m\lambda_k^0\varphi_k(x_1(x_{m+1},x_{m+2}),\cdots,x_m(x_{m+1},x_{m+2}),x_{m+1},x_{m+2})\triangleq H(x_{m+1},x_{m+2})\quad\Rightarrow$$

$$H_{x_{m+1}}=\sum_{i=1}^m f_{x_i}\frac{\partial x_i}{\partial x_{m+1}}+f_{x_{m+1}}+\sum_{k=1}^m\lambda_k^0\left(\sum_{i=1}^m\frac{\partial\varphi_k}{\partial x_i}\frac{\partial x_i}{\partial x_{m+1}}+\frac{\partial\varphi_k}{\partial x_{m+1}}\right);$$

$$H_{x_{m+2}}=\sum_{i=1}^m f_{x_i}\frac{\partial x_i}{\partial x_{m+2}}+f_{x_{m+2}}+\sum_{k=1}^m\lambda_k^0\left(\sum_{i=1}^m\frac{\partial\varphi_k}{\partial x_i}\frac{\partial x_i}{\partial x_{m+2}}+\frac{\partial\varphi_k}{\partial x_{m+2}}\right);$$

$$H_{x_{m+1}^2}=\sum_{i=1}^m\left(\sum_{j=1}^m f_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+1}}+f_{x_ix_{m+1}}\frac{\partial x_i}{\partial x_{m+1}}+f_{x_i}\frac{\partial^2 x_i}{\partial x_{m+1}^2}\right)+\sum_{i=1}^m f_{x_{m+1}x_i}\frac{\partial x_i}{\partial x_{m+1}}+f_{x_{m+1}^2}+$$
$$\sum_{k=1}^m\lambda_k^0\left[\sum_{i=1}^m\left(\sum_{j=1}^n\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}\frac{\partial x_j}{\partial x_{m+1}}\frac{\partial x_i}{\partial x_{m+1}}+\frac{\partial^2\varphi_k}{\partial x_i\partial x_{m+1}}\frac{\partial x_i}{\partial x_{m+2}}+\frac{\partial\varphi_k}{\partial x_i}\frac{\partial^2 x_i}{\partial x_{m+1}^2}\right)+\sum_{i=1}^m\frac{\partial^2\varphi_k}{\partial x_{m+1}\partial x_i}\frac{\partial x_i}{\partial x_{m+1}}+\frac{\partial^2\varphi_k}{\partial x_{m+1}^2}\right]$$
$$=\sum_{i=1}^m\sum_{j=1}^m\left(f_{x_ix_j}+\sum_{k=1}^m\lambda_k^0\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}\right)\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+1}}+\sum_{i=1}^m\left(f_{x_ix_j}+\sum_{k=1}^m\lambda_k^0\frac{\partial^2\varphi_k}{\partial x_i\partial x_{m+1}}\right)\frac{\partial x_i}{\partial x_{m+1}}$$
$$+\sum_{i=1}^m\left(f_{x_{m+1}x_i}+\sum_{k=1}^m\lambda_k^0\frac{\partial^2\varphi_k}{\partial x_{m+1}\partial x_i}\right)\frac{\partial x_i}{\partial x_{m+1}}+\left(f_{x_{m+1}^2}+\sum_{k=1}^m\lambda_k^0\frac{\partial^2\varphi_k}{\partial x_{m+1}^2}\right)+\sum_{i=1}^m\left(f_{x_i}+\sum_{k=1}^m\lambda_k^0\frac{\partial\varphi_k}{\partial x_i}\right)\frac{\partial^2 x_i}{\partial x_{m+1}^2}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{m}L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+1}}+\sum_{i=1}^{m}L_{x_ix_{m+1}}\frac{\partial x_i}{\partial x_{m+1}}+\sum_{i=1}^{m}L_{x_{m+1}x_i}\frac{\partial x_i}{\partial x_{m+1}}+L_{x_{m+1}^2}+\sum_{i=1}^{m}L_{x_i}\frac{\partial^2 x_i}{\partial x_{m+1}^2}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+1}}+\sum_{i=1}^{m}L_{x_i}\frac{\partial^2 x_i}{\partial x_{m+1}^2};$$

同理；$H_{x_{m+2}^2}=\sum_{i=1}^{n}\sum_{j=1}^{n}L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+2}}\frac{\partial x_j}{\partial x_{m+2}}+\sum_{i=1}^{m}L_{x_i}\frac{\partial^2 x_i}{\partial x_{m+2}^2}$；

$H_{x_{m+1}x_{m+2}}=\sum_{i=1}^{n}\sum_{j=1}^{n}L_{x_ix_j}\frac{\partial x_i}{\partial x_{m+1}}\frac{\partial x_j}{\partial x_{m+2}}+\sum_{i=1}^{m}L_{x_i}\frac{\partial^2 x_i}{\partial x_{m+1}\partial x_{m+2}}$；因为 M_0 为 L 的稳定点，故 $L_{x_i}\big|_{M_0}=0$，因此，$H_{x_{m+1}^2}\big|_{P_0}=A$；$H_{x_{m+2}^2}\big|_{P_0}=C$；$H_{x_{m+1}x_{m+2}}\big|_{P_0}=B$；由二元函数无条件极值的充分条件可知，当 $D=AC-B^2>0$，且 $A<0$，则 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 取到条件极大值，且 $A>0$，则 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 取到条件极小值；当 $D=AC-B^2<0$ 时，$f(x_1,x_2,\cdots,x_n)$ 在 P_0 不取条件极值；当 $D=AC-B^2=0$ 时,不能断定 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 是否取到条件极值。证毕。

4.2 当 $n=m+1$ 时，有如下定理：（类似的证明方法）得：

定理 5： 设 $f(x_1,x_2,\cdots,x_n)$，$\varphi_k(x_1,x_2,\cdots x_n)$，$k=1,2,\cdots,m;(m<n)$ 在 n 维空间区域 Ω 上具有连续的二阶偏导数，$P_0(x_1^0,x_2^0,\cdots,x_n^0)\in\Omega$，$\left.\frac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}\right|_{P_0}\neq 0$，$M_0(x_1^0,x_2^0,\cdots,x_n^0,\lambda_1^0,\lambda_2^0,\cdots,\lambda_k^0)$ 为 $L(x_1,x_2,\cdots,x_n,\lambda_1,\lambda_2,\cdots,\lambda_k)$ 的稳定点。则当 $\Delta=\left.\sum_{i=1}^{n}\sum_{j=1}^{n}(L_{x_ix_j}\frac{dx_i}{dx_n}\frac{dx_j}{dx_n})\right|_{M_0}\neq 0$ 时，则 $f(x_1,x_2,\cdots,x_n)$ 在 P_0 取到条件极大值，且 $\Delta>0$，$f(x_1,x_2,\cdots,x_n)$ 取到条件极小值；$\Delta<0$，$f(x_1,x_2,\cdots,x_n)$ 取到条件极大值；其中

$$\frac{dx_i}{dx_n}=-\frac{\dfrac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_{i-1,}\varphi_i,\varphi_{i+1},\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_{i-1,}x_n,x_{i+1},\cdots,x_m)}}{\dfrac{\partial(\varphi_1,\varphi_2,\cdots,\varphi_m)}{\partial(x_1,x_2,\cdots,x_m)}},\quad i=1,2,\cdots,m。$$

5.结论

本文给出了相对简单便于计算的多元函数在多个等式约束条件下极值的充分条件。弥补了大学数学教材里多元函数条件极值无充分条件的空白。

Sufficient Conditions Of Multivariate Function For Conditional Extreme Value Under The Equality Constraints

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Abstract： Under the equality constraints, by using Lagrange multiplier method, derivation of multivariate implicit function, and the method of conditional extremum reversing unconditional extremum, it has deduced the sufficient conditions of the multi-function extremum, and also presents the theorem as well as a practical method which is easy to calculate. And it fills in the blank in the college book that there is no sufficient conditions for multi-function conditions extremum.

Key words: equality constraints,multivatrate function,extreme,sufficient conditions

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