Finite-size corrections for dense polymers

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We study the (analytic) finite-size corrections in the dense polymer model on the strip by perturbing the critical Hamiltonian by irrelevant operators belonging to the tower of the identity. We generalize the perturbation expansion to include Jordan cells, and we examine whether the finite-size corrections are sensitive to the properties of indecomposable representations appearing in the conformal spectrum, in particular their indecomposability parameters. We find that the corrections do not depend on these parameters nor even on the presence of Jordan cells. Though the corrections themselves are not universal, the ratios are universal and correctly reproduced by the conformal perturbative approach, to first order.

I. INTRODUCTION

Finite-size scaling and corrections for critical systems have attracted much attention in recent decades. Although many theoretical results are now known about the critical exponents and universal relations among the leading critical amplitudes, not much information is available on ratios among the amplitudes in finite-size correction terms [1]. New universal amplitude ratios have been recently presented for the Ising model [2]. In this paper, we compute finite-size corrections in the critical dense polymer model as formulated in [3], and compare them with the results obtained in perturbation theory. The presence of indecomposable representations in the conformal spectrum of this model [3–5] first requires to adapt the perturbation theory to include such representations. In particular it raises the question as to whether the finite-size corrections depend on their indecomposability parameters [6–8, 10].

Conformal invariance implies that on an infinitely long strip of finite width N, the eigenstates of the critical transfer matrix, associated to conformal states, have energies which scale with N like [11]

$$E_n(N) = Nf_b + f_s + \frac{\pi\zeta(\Delta_n - c/24)}{N} + \mathcal{O}(N^{-2}), \quad (1)$$

where the bulk free energy density f_b, the surface free energy f_s, and the anisotropy factor ζ are non-universal constants; in contrast, the central charge c and the weight Δ_n of the conformal eigenstate, are universal (although boundary dependent for Δ_n).

Higher order correction terms in (1) are nonuniversal; however it has been suggested recently [2], and partially checked for the Ising universality class ($c = 1/2$), that the asymptotic expansion of the critical eigenvalues has the form ($n = 0$ will refer to the groundstate)

$$E_n(N) = Nf_b + f_s + \sum_{p=1}^{\infty} \frac{a_p^{(n)}}{N^{2p-1}}, \quad (2)$$

and that the amplitude ratios $a_p^{(n)}/a_p^{(0)}$ are universal and depend only on the boundary conditions [2, 12, 13]. The case $p = 1$ readily follows from (1).

For critical dense polymers, it is not difficult to see from the results of [3] that the finite-size energy levels have the same asymptotic expansion (2). The general calculation of all coefficients is a tedious exercise, which can however be completed for an infinite number of specific levels (the lowest-lying levels, as well as the lowest-lying logarithmic pairs in each representation). Moreover we check that the first coefficients $a_2^{(n)}$ match the perturbative values, and that the ratios $a_2^{(n)}/a_2^{(0)}$ are universal.

II. CRITICAL DENSE POLYMERS

The critical dense polymer model on a strip, with width N and height $2M$, was formulated in [3] in terms of a double-row transfer matrix $\mathbf{D}(u)$. The partition function is given by

$$Z_{N,M} = \mathrm{Tr}\,\mathbf{D}(u)^M = \sum_n e^{-2ME_n(N;u)}, \quad (3)$$

where the sum is over all eigenvalues of $\mathbf{D}(u)$, written as $e^{-2E_n(N;u)}$; u is a spectral parameter related to the anisotropy factor by $\zeta = \sin 2u$.

The configuration space on which the transfer matrix acts can be divided into sectors $\mathcal{L}_{N,\ell}$, labelled by an integer $\ell \geq 0$, of the same parity as N. The integer ℓ is a number of defect lines, and can be seen as fixing a specific boundary condition. The range of ℓ is finite when N is finite, but will be considered as unbounded in the limit $N \to \infty$. It was found in [3] that the transfer matrix is diagonalizable when restricted to each sector separately, although analytic and numerical analyses indicated that it is not in the full space when N is even. This was confirmed very recently in [4, 14], where it was shown that the sectors ℓ and $\ell + 2$, for $\ell = 0 \bmod 4$, are linked up by rank 2 Jordan blocks, while there are no Jordan cells when N is odd.

All eigenvalues have been determined in [3], for any finite value of N. They are given by

$$E_n = -\frac{1}{2}\sum_{\substack{j=1\\ j=N\ (2)}}^{N-2} \log\Big[(1+\zeta\,\varepsilon_j \sin\frac{\pi j}{2N})(1+\zeta\,\mu_j\sin\frac{\pi j}{2N})\Big] \tag{4}$$

where the summation includes the integers j of the same parity as N. The eigenvalues also depend on parameters ε_j, μ_j equal to ± 1, although not all choices of $\varepsilon_j, \mu_j = \pm 1$ are allowed. Exactly which sequences of $+1, -1$ correspond to actual eigenvalues, and for which sector, are given by the selection rules detailed in [3]. What we need is summarized below.

A main result from [3] is that the set of eigenvalues of the transfer matrix in the sector $\mathcal{L}_{N,\ell}$ is such that it leads, through (1), to a set of conformal weights Δ_n whose values and degeneracies exactly match those of a highest weight representation $\mathcal{V}_{1,s}$ of weight $h_{1,s} = \frac{(s-2)^2-1}{8}$, with $s = \ell + 1$. As a representation of the Virasoro algebra, $\mathcal{V}_{1,s}$ is believed to be the quotient of the highest weight Verma module $V_{1,s}$ by the singular vector at level s [14].

The overall groundstate E_0, found in $\mathcal{L}_{N,0}$ or $\mathcal{L}_{N,1}$, corresponds to all $\varepsilon_j = \mu_j = 1$ in (4). From the Euler-MacLaurin summation formula, the asymptotic expansion of E_0 takes the form (2) with the coefficients

$$a_p^{(0)} = \frac{\pi^{2p-1}\mathrm{B}_{2p}(\alpha)}{(2p)!}\,\lambda_{2p-1}(\zeta), \quad \alpha = \frac{N}{2} \bmod 1 \tag{5}$$

where $\mathrm{B}_n(z)$ are the Bernoulli polynomials, and where $\lambda_{2p-1}(\zeta)$ are polynomials of degree $2p-1$, starting with $\lambda_1(\zeta) = \zeta$, $\lambda_3(\zeta) = 2\zeta^3 - \zeta$, $\lambda_5(\zeta) = 24\zeta^5 - 20\zeta^3 + \zeta$.

For $p = 1$ in particular, one finds $\Delta_0 - c/24 = 1/12$ for N even and $-1/24$ for N odd. Assuming $c = -2$, this gives $\Delta_0 = h_{1,1} = 0$ for N even, and $\Delta_0 = h_{1,2} = -1/8$ for N odd.

For $p = 2$ one finds from the previous formula

$$a_2^{(0)} = \frac{\pi^3}{6}\big(\Delta_0^2 - \tfrac{1}{120}\big)\,\lambda_3(\zeta). \tag{6}$$

The excited levels are obtained by switching the ε_j, μ_j from $+1$ to -1 in a way allowed by the selection rules [3]. The specific levels we will consider here are the two lowest-lying levels in each sector $\mathcal{L}_{N,\ell}$, denoted $E_0^{(\ell)}$ and $E_1^{(\ell)}$, for all $\ell \geq 1$ except $\ell = 2$ (which is somewhat special and requires a separate treatment; the following checks however work for them too). They are non-degenerate (within their sector) and are obtained by setting to -1 the parameters μ_j with the following indices,

$$1 \leq j \leq [\tfrac{\ell-1}{2}] \qquad \text{for } E_0^{(\ell)}, \tag{7}$$

$$1 \leq j \neq [\tfrac{\ell-1}{2}] \leq [\tfrac{\ell+1}{2}] \qquad \text{for } E_1^{(\ell)}. \tag{8}$$

Working out the asymptotic expansion of $E_r^{(\ell)}$ for $r = 0, 1$, yields the first coefficients

$$a_1^{(n)} = \pi\,\big(h_{1,s} + r + \tfrac{1}{12}\big)\,\lambda_1(\zeta), \tag{9}$$

and

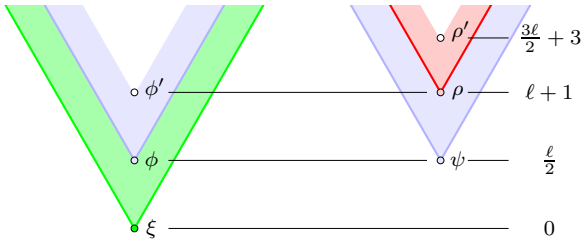
$$a_2^{(n)} = \frac{\pi^3}{6}\big(h_{1,s}^2 - \tfrac{1}{120}\big)\,\lambda_3(\zeta), \quad r = 0, \tag{10}$$

$$a_2^{(n)} = \frac{\pi^3}{6}\big(h_{1,s}^2 + 6\,h_{1,s} + \tfrac{119}{120}\big)\,\lambda_3(\zeta), \quad r = 1. \tag{11}$$

Before comparing them with the perturbative approach, it is useful to know where the Jordan blocks appear when one considers the full configuration space, namely the union of all sectors.

As mentioned above, Jordan cells appear only for N even, in which case the full configuration space is $\cup_{\ell\geq 0,\text{even}}\,\mathcal{L}_{N,\ell}$. It turns out that within each sector, the eigenstates are generically non-degenerate, implying that the transfer matrix is diagonalizable in each sector separately; however every sector $\mathcal{L}_{N,\ell}$ has a subset of eigenstates which are pairwise degenerate with eigenstates of $\mathcal{L}_{N,\ell-2}$, while the complementary subset is pairwise degenerate with eigenstates of $\mathcal{L}_{N,\ell+2}$; these double degeneracies occur *for all values of* N *and* ζ.

In the conformal limit (the order N^{-1} in the expansion of the eigenvalues), the degeneracy of the eigenstates increases. For N odd, each sector ℓ odd carries an irreducible highest weight representation $\mathcal{V}_{1,\ell+1}$. For N even, it is believed [3, 4] that the pairs of sectors ℓ and $\ell+2$, for $\ell = 0 \bmod 4$, and their associated representations $\mathcal{V}_{1,\ell+1}$ and $\mathcal{V}_{1,\ell+3}$, are tied together by rank 2 Jordan cells to form staggered modules [6–8].



The two cone-like figures above represent the representations $\mathcal{V}_{1,\ell+1}$ and $\mathcal{V}_{1,\ell+3}$. The left one has highest weight state ξ, a first singular vector ϕ, and a second singular vector ϕ', set to zero, and similarly on the right for ψ, ρ and ρ', also set to zero. The states in the two blue regions are conformal states coming from eigenstates of the transfer matrix which are pairwise degenerate for all (large enough) values of N; they form logarithmic pairs and are related by the non-diagonalizable action of L_0, like f.i. $(L_0 - \Delta)\phi = 0$, $(L_0 - \Delta)\psi = \phi$.

The full projective representation, denoted by $\mathcal{R}_{1+\ell/2}$, spans the two sectors $\mathcal{L}_{N,\ell}$ and $\mathcal{L}_{N,\ell+2}$, and contains two subsets : S_1 contains eigenstates of L_0 which are not parts of a logarithmic pair (green and red regions), and S_2 contains logarithmic pairs (in blue). Each such representation $\mathcal{R}_{1+\ell/2}$ comes with an intrinsic parameter $\beta_{1+\ell/2}$, which fixes its equivalence class [6–9].

III. CONFORMAL PERTURBATION THEORY

The finite-size corrections in (2) can in principle be computed in perturbation theory [15]. In general, the critical lattice Hamiltonian will contain correction terms to the fixed-point Hamiltonian $H_c = \frac{\pi}{N}(L_0 - \frac{c}{24})$,

$$H = \zeta H_c + \sum_k g_k \int_0^N \varphi_k(v)\,\mathrm{d}v, \tag{12}$$

where the φ_k are irrelevant quasi-primary fields, and g_k are non-universal constants. On general grounds, a perturbation φ_k with scaling dimension x_k brings a first-order correction to the energy proportional to N^{1-x_k}.

Fields in the tower of the identity are available in any conformal theory. On the strip, the lowest contributing field is $\varphi = L_{-2}^2(v)$, of dimension 4 [16]; it corrects the conformal energies by a N^{-3} term, and should reproduce the first correction coefficient $a_2^{(n)}$ in (2).

Perturbation techniques have been studied for a long time, and successfully applied in concrete models [17]. Write $H = H_0 + gV$. A basic result states that the first-order correction to the energy $E_{c,n}$ of a non-degenerate eigenstate $|n\rangle$ of H_0 is given by $\langle n|gV|n\rangle$. Here it is sufficient that the state $|n\rangle$ be non-degenerate within its own representation since φ preserves each representation.

This scheme can be readily applied to the polymer model when N is odd, or to the two lowest-lying states in the indecomposable representations $\mathcal{R}_k$ (ξ and $L_{-1}\xi$) when N is even, but should be adapted for states in Jordan cells. In addition the representations $\mathcal{R}_k$ carry an intrinsic parameter β_k, which may affect the corrections.

It has been appropriately emphasized in [5, 10] that the conformal states should be equipped with a scalar product for which the conformal Hamiltonian is hermitian. When the Hamiltonian is not diagonalizable, this bilinear form cannot be positive definite and differs from the canonical scalar product usually used in the lattice configuration space. For a typical rank 2 Jordan cell in the subset $S_2 \subset \mathcal{R}_k$, one can take the scalar product to be such that $\langle\phi|\phi\rangle = 0$, $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle = \beta$, $\langle\psi|\psi\rangle = \gamma$ where $\beta = \beta_k$ if (ϕ,ψ) is the lowest logarithmic pair of $\mathcal{R}_k$, otherwise β is a calculable multiple of β_k for a descendant logarithmic pair [5]. For states in S_1, the usual scalar product is used.

Assume that ϕ, ψ are the conformal states of a non-degenerate logarithmic pair (only pair at their level),

$$(H_0 - \lambda_0)\phi = 0, \qquad (H_0 - \lambda_0)\psi = \phi, \tag{13}$$

with $\lambda_0 = \Delta$ being their conformal weight (we take $H_0 = L_0$ for simplicity). Since ϕ, ψ are the conformal shadows of states which are degenerate for all (sufficiently large) values of N, we may assume that the Jordan cells are preserved by the perturbation, to any order, that is,

$$(H - \lambda)\Phi = 0, \qquad (H - \lambda)\Psi = \Phi. \tag{14}$$

Expanding in power series in g, $\lambda = \Delta + g\lambda_1 + \ldots$, $\Phi = \phi + g\phi_1 + \ldots$ and $\Psi = \psi + g\psi_1 + \ldots$, we obtain at first order the consistency condition $\langle\phi|V|\phi\rangle = 0$ and

$$\beta\lambda_1 = \langle\phi|V|\psi\rangle = \langle\psi|V|\phi\rangle. \tag{15}$$

Since ψ is the only state to have a non-zero scalar product with ϕ, the r.h.s. is proportional to β, and so λ_1 does not depend on β.

The required matrix element can be computed by the procedure used in [16] for highest weight representations. To avoid confusions, we denote hereafter by (ϕ_k, ψ_k) the lowest logarithmic pair in $\mathcal{R}_k$. The main idea is to compare the chiral 3-point function on a strip of width N,

$$F_{\psi\varphi\psi}(w_i) = \langle\psi_k(w_1)\,(L_{-2}^2\mathbb{I})(w_2)\,\psi_k(w_3)\rangle_{\text{strip}} \tag{16}$$

with its expression in the operator formalism. The previous correlator is first computed on the upper-half plane and then conformally transformed to the strip. It can be shown that the logarithmic part of $F_{\psi\varphi\psi}$ is

$$F_{\psi\varphi\psi}(w_i)\Big|_{\log} = \beta_k\Big\{\log\frac{z_3}{z_1} - 2\log\Big(1 - \frac{z_3}{z_1}\Big)\Big\}F_{\pi\varphi\pi}(w_i), \tag{17}$$

where $F_{\pi\varphi\pi}$ is the correlator (16) with ψ_k replaced by a primary field π of the same weight as ψ_k, and where $z_i = \exp(\frac{\pi w_i}{N})$ relates the planar variables to the strip coordinates. By conformal invariance, $F_{\pi\varphi\pi} = F_{\pi\varphi\pi}(x_1, x_2)$ is a function of $x_1 = z_2/z_1$ and $x_2 = z_3/z_2$.

In the operator formalism, and for $w = u + iv$, the correlator is

$$F_{\psi\varphi\psi} = \langle 0|\hat\psi_k(0,v_1)\,T^{u_1-u_2}\,\hat\varphi(0,v_2)\,T^{u_2-u_3}\,\hat\psi_k(0,v_3)|0\rangle, \tag{18}$$

with the transfer matrix $T = \mathrm{e}^{-\pi L_0/N}$. Inserting a complete set of states of $\mathcal{R}_k$, the contribution coming from the pair (ϕ,ψ) reads

$$F \sim \langle 0|\hat\psi_k(0,v_1)T^{u_1-u_2}P_\psi\hat\varphi(0,v_2)P_\psi T^{u_2-u_3}\hat\psi_k(0,v_3)|0\rangle, \tag{19}$$

where

$$P_\psi = \frac{|\psi\rangle\langle\phi|}{\beta} + \frac{|\phi\rangle\langle\psi|}{\beta} - \gamma\frac{|\phi\rangle\langle\phi|}{\beta^2} \tag{20}$$

is a projector on the subspace spanned by ϕ, ψ, namely $P_\psi|\psi\rangle = |\psi\rangle$, $P_\psi|\phi\rangle = |\phi\rangle$ and $P_\psi|n\rangle = 0$ for any other state $|n\rangle$. Using the action of the transfer operator, namely $\mathrm{e}^{xL_0}|\phi\rangle = \mathrm{e}^{x\Delta}|\phi\rangle$ and $\mathrm{e}^{xL_0}|\psi\rangle = \mathrm{e}^{x\Delta}(|\psi\rangle + x|\phi\rangle)$, we find that (19) has a term quadratic in the u's,

$$\frac{\pi^2}{\beta^2N^2}\,\mathrm{e}^{-\frac{\pi\Delta}{N}(u_1-u_3)}\,(u_1-u_2)(u_2-u_3) \times\langle 0|\hat\psi_k(0,v_1)|\phi\rangle\,\langle\phi|\hat\varphi(0,v_2)|\phi\rangle\,\langle\phi|\hat\psi_k(0,v_3)|0\rangle. \tag{21}$$

As $u \sim \log z$, this term would require a corresponding $\log^2$ term in the 3-point function $F_{\psi\phi\psi}$, which does not exist. We thus have $\langle\phi|\hat\varphi(0,v_2)|\phi\rangle = 0$, which fulfills the consistency condition we found in perturbation theory.

Looking now at the terms linear in the u's, we find

$$-\frac{\pi}{\beta^2N}\,\mathrm{e}^{-\frac{\pi\Delta}{N}(u_1-u_3)}\times \Big\{(u_1-u_2)\langle 0|\hat\psi_k(0,v_1)|\phi\rangle\,\langle\phi|\hat\varphi(0,v_2)|\psi\rangle\,\langle\phi|\hat\psi_k(0,v_3)|0\rangle + (u_2-u_3)\langle 0|\hat\psi_k(0,v_1)|\phi\rangle\,\langle\psi|\hat\varphi(0,v_2)|\phi\rangle\,\langle\phi|\hat\psi_k(0,v_3)|0\rangle\Big\}. \tag{22}$$

The dependence in u_2 must vanish, which implies $\langle\phi|\hat\varphi(0,v_2)|\psi\rangle = \langle\psi|\hat\varphi(0,v_2)|\phi\rangle$, and a single term remains,

$$-\frac{\pi}{\beta^2 N}\,\mathrm{e}^{-\frac{\pi\Delta}{N}(u_1-u_3)}\,(u_1-u_3)\times \\ \langle 0|\hat\psi_k(0,v_1)|\phi\rangle\,\langle\phi|\hat\varphi(0,v_2)|\psi\rangle\,\langle\phi|\hat\psi_k(0,v_3)|0\rangle. \quad (23)$$

Since $\mathrm{e}^{-\frac{\pi}{N}(u_1-u_3)} \sim z_3/z_1$, a comparison with (17) implies, provided the pair (ϕ,ψ) is not degenerate, that the product of the three matrix elements in the previous equation equals the coefficient of $(x_1x_2)^\Delta = (z_3/z_1)^\Delta$ in $\beta^2\beta_k F_{\pi\varphi\pi}(x_1,x_2)$, and does not depend on v_2.

Repeating the same steps for the 2-point function $F_{\psi\psi}(w_1,w_3) = \langle\psi_k(w_1)\,\psi_k(w_3)\rangle_{\rm strip}$, we obtain similarly that the product $\langle 0|\hat\psi_k(0,v_1)|\phi\rangle\,\langle\phi|\hat\psi_k(0,v_3)|0\rangle$ is equal to the coefficient of $(z_3/z_1)^\Delta$ in $\beta\beta_k F_{\pi\pi}(z_3/z_1)$.

The ratio of these two product yields

$$\langle\phi|\hat\varphi(0,v_2)|\psi\rangle = \beta\frac{F_{\pi\varphi\pi}(x_1,x_2)\Big|_{(x_1x_2)^\Delta}}{F_{\pi\pi}(x_1x_2)\Big|_{(x_1x_2)^\Delta}}. \quad (24)$$

This formula shows that the first correction λ_1 to the conformal energy of ψ can be computed as if ψ was a (non-degenerate) descendant of a primary field. Being degenerate with ψ, the first correction for ϕ is equal to that of ψ. Thus all Jordan couplings between sectors for N even can be neglected for the calculation of λ_1, and each sector $\mathcal{L}_{N,\ell}$ can be considered separately, as carrying a highest weight representation $\mathcal{V}_{1,s}$.

Let $|n\rangle = |\Delta, r\rangle$ be a non-degenerate, level r descendant state in a representation of highest weight Δ. The normalized matrix elements $C_{n\varphi n} \equiv (N/\pi)^4\langle n|\hat\varphi(0,0)|n\rangle$ for $\varphi = (L^2_{-2}\mathbb{I})$ have been computed in [16, 18],

$$C_{n\varphi n} = (\Delta+r)\Big(\Delta + \frac{r(2\Delta+r)(5\Delta+1)}{(\Delta+1)(2\Delta+1)} - \frac{2+c}{12}\Big) \\ + \Big(\frac{c}{24}\Big)^2 + \frac{11c}{1440} + \frac{r}{30}\Big[r^2(5c-8) - (5c+28)\Big]\delta_{\Delta,0}. \quad (25)$$

From this we obtain the first correction to the conformal energy of a state $|n\rangle$ as $\lambda_1 = \langle n|gV|n\rangle = \frac{g\pi^4}{N^3}C_{n\varphi n}$.

We may now apply this to the two lowest levels in each sector $\mathcal{L}_{N,\ell}$ (the only ones to be generically non-degenerate). For those states, $\Delta = h_{1,s}$ and $r = 0$ or $r = 1$, so that we find from the formula (25) with $c = -2$,

$$\lambda_1 = \frac{g\pi^4}{N^3}\big(h_{1,s}^2 - \tfrac{1}{120}\big), \qquad r = 0, \quad (26)$$

$$\lambda_1 = \frac{g\pi^4}{N^3}\big(h_{1,s}^2 + 6h_{1,s} + \tfrac{119}{120}\big), \qquad r = 1. \quad (27)$$

The corrections obtained earlier in (6), (10) and (11) from the lattice transfer matrix are clearly reproduced by setting $g = \lambda_3(\zeta)/6\pi$. We note that $c = -2$ is the only value for which the perturbation theory correctly reproduces the lattice finite-size corrections. If we do not fix c, the weight Δ is shifted to $\Delta = h_{1,s} + \frac{c+2}{24}$, and from the formula (25), the first order λ_1 for $r = 0$ and $r = 1$ gets an extra term, inside the parentheses, equal to $\frac{c+2}{1440}$ and $\frac{241(c+2)}{1440}$ respectively.

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