TRACES IN SYMMETRIC MONOIDAL CATEGORIES

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CONTENTS

INTRODUCTION

The original notion of *trace* is, of course, the trace of a square matrix with entries in a field. An important and far-reaching categorical generalization of this notion applies to any endomorphism of a *dualizable object* in a symmetric monoidal category; see [11, 23].

This generalization has a number of applications, which are often closely connected with the study of fixed points. One application of particular importance is the Lefschetz fixed point theorem and its variants and generalizations, many of which can be deduced directly from the naturality and functoriality of the canonical symmetric monoidal trace.

The purpose of this expository note is to describe this notion of trace in a symmetric monoidal category, along with its important properties (including naturality and functoriality), and to give as many examples as possible. Among other things, this note is intended as background for [35] and [34], in which the symmetric monoidal trace is generalized to the context of bicategories and indexed monoidal categories.

In §1 we describe one way to understand the connection between traces and fixed points. This provides motivation for the formal definitions in §2. In §3 we give many examples of the trace. These include topological examples connected to the Lefschetz fixed point theorem and its generalizations as well as examples arising in other contexts. In §4 we define a generalization of the trace from §2. This trace arises in many applications and is a generalization of the classical transfer.

Then in §5 we describe "coherence" properties of the trace, while in §6 we describe its functoriality and naturality, including the Lefschetz fixed point theorem as an application. Finally, in §7 we remark on some generalizations.

1. TRACES AND FIXED POINTS

A common feature of all the examples we will consider is that *traces give information about fixed points*. Thus, before embarking on formalities, in this section we will attempt to give some intuition for why this should be so.

Suppose we are working in a monoidal category, and consider a morphism whose source and target are tensor products, such as $f\colon A \otimes B \otimes C \to D \otimes B$. We think of such an f as a "process" which takes three inputs, of types A, B, and C, and produces two outputs, of types D and B. In keeping with this intuition, we draw f as follows:
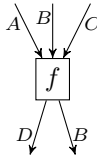
This is an example of *string diagram notation* for monoidal categories, which is "Poincaré dual" to the usual sort of diagrams: instead of drawing objects as *vertices* and morphisms as *arrows* connecting these vertices, we draw objects as *arrows* and morphisms as *vertices*, often with boxes around them. See [19, 21, 32, 37] for more about string diagram calculus. In particular, we note that Joyal and Street [19] proved that the "value" of a string diagram is invariant under deformations of diagrams, so that we can prove theorems by topological reasoning; see Proposition 2.4 for an example.

If the source and target of the morphism f above are the same, then a *fixed point* of f is a morphism $f^\dagger\colon * \to X$ (where $*$ denotes the unit for the monoidal structure) such that

We will be interested in traces of more general morphisms. For these we will need to be able to duplicate inputs and outputs, which we draw as follows:
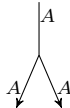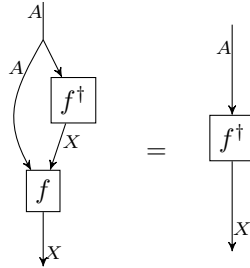

This is only possible if our monoidal category is *cartesian* monoidal, in which case the above duplication process is the diagonal $\Delta\colon A \to A \times A$.

Now suppose only *one* of the inputs of f matches its output:
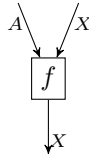
An *(A-parametrized) fixed point* of f is a morphism $f^\dagger \colon A \to X$ such that

i.e. "$f(a, f^\dagger(a)) = f^\dagger(a)$ for any $a \in A$" or "$f^\dagger(a) \in X$ is a fixed point of $f(a, -)$ for any $a \in A$."

A common way to look for fixed points in concrete situations is by iteration: we start with some $x_0 \in X$ and compute $x_1 = f(a, x_0)$, $x_2 = f(a, x_1)$, and so on. If ever $x_{n+1} = x_n$, we've found a fixed point. But even if not, we can hope that the sequence $(x_0, x_1, x_2, \dots)$ will "converge" to a fixed point. Two contexts where this works are the contraction mapping theorem in topology and the least-fixed-point combinator in domain semantics.

In order to mimic this in abstract language, we need a notion of *feedback*, i.e. a way to plug the output of a given morphism into its input. In diagrammatic terms, given a morphism one of whose inputs matches *one* of its outputs:
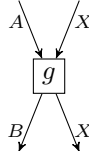
we want to construct a new morphism in which the X input and output have been "fed back into each other" somehow:
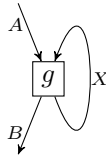
This is called a *trace* of the morphism f. In fact, Hyland [4] and Hasegawa [16] have independently observed the following.

Theorem 1.1. *In a cartesian monoidal category, to give a notion of trace is precisely the same as to give a fixed-point operator, which assigns to every morphism $A \times X \to X$ a fixed point $A \to X$ in a coherent way.*

This relationship is especially important in computer science. However, in topology, we are interested in maps which may have zero, one, or many fixed points. Thus, we can't expect to have a fixed-point operator acting on the whole category, since there is no way to specify a fixed point for a map which has no fixed points. Instead, we would like to know, given a map, does it have any fixed points, and if so, how many and what are they? Thus, we need an operation which produces, instead of a fixed point, some sort of "invariant" carrying information about whatever fixed points a map may have. A fruitful approach to this is to map our cartesian monoidal category $\mathbf{C}$ into a larger category $\mathbf{D}$ in which morphisms can be "superimposed" or "added." Then we may hope for a trace or a fixed-point operator in $\mathbf{D}$ which computes the "sum" of all the fixed points that a map may have (or "zero" if it has none).

Often our functor $Z\colon \mathbf{C} \to \mathbf{D}$ will be like the "free abelian group" functor, and the "fixed point" of $Z(f)$ will be something like $\sum_{f(a)=a} a$. And just as the "free abelian group" functor maps cartesian products not to cartesian products, but to tensor products, if we want Z to be a monoidal functor, we usually cannot expect $\mathbf{D}$ to be cartesian monoidal, only symmetric monoidal. Therefore, a trace in $\mathbf{D}$ no longer implies a fixed point operator. However, since $\mathbf{C}$ is cartesian, we still have diagonal morphisms for objects in the image of Z, and this is all we really need.

In fact, if we can find a category $\mathbf{D}$ which is suitably "additive," then it often comes with a canonical notion of trace for free. The idea is to split the "feedback" diagram into a composition of three pieces:

The morphism

(from the unit object to $M \otimes M$) is called a "coevaluation" or "unit." If M is of the form $Z(X)$ for some $X \in \mathbf{C}$, then the coevaluation is supposed to pick out a formal sum such as $\sum_{x \in X} x \otimes x$. (To be precise, the second string labeled M is actually its "dual" $M^{\star}$, and so the sum is actually $\sum_{x \in X} x \otimes x^{\star}$.) Similarly, the morphism

is called an "evaluation" or "counit." For $M = Z(X)$, the evaluation is supposed to be supported on pairs of the form $x \otimes x$ (or, more precisely, $x^\star \otimes x$), and to give zero when applied to a pair $x \otimes x'$ for $x \neq x'$. If this is the case, then the composite
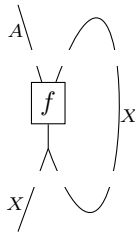
will act as follows:

$$a \mapsto \sum_{x \in X} a \otimes x \otimes x \mapsto \sum_{x \in X} f(a,x) \otimes x \mapsto \sum_{x \in X} f(a,x) \otimes f(a,x) \otimes x \mapsto \sum_{\substack{x \in X \\ f(a,x)=x}} f(a,x)$$

Thus, as desired, it picks out the sum of all the fixed points of f.

Traces constructed in this way from evaluation and coevaluation maps we call *canonical* traces. In [21], Joyal and Street showed that any traced monoidal category **D** can be embedded in a larger one Int(**D**), in such a way that the given traces in **D** are identified with canonical traces in Int(**D**). Therefore, for the purposes of finding fixed-point invariants, there is no loss in restricting our attention to canonical traces.

However, the choice of a particular **D** does restrict the maps for which we can calculate our fixed-point invariant, since the resulting objects in **D** must admit evaluations and coevaluations. This property is called being *dualizable*. For instance, in the free abelian group on X, the sum $\sum_{x \in X} x \otimes x$ is only defined when X is finite. A given functor **C** $\to$ **D** thus induces a notion of "finiteness" on objects of **C**. We will see in §3 that different choices of **D** can drastically affect the notion of finiteness, as well as the utility and computability of traces. However, in most applications the choice of **D** is straightforward, and the resulting finiteness restriction not onerous.

2. Traces

We now move on to the abstract study of canonical traces in symmetric monoidal categories; in the next section we will specialize to a number of examples and see how we obtain information about fixed points. We begin with the formal definition of dualizability.
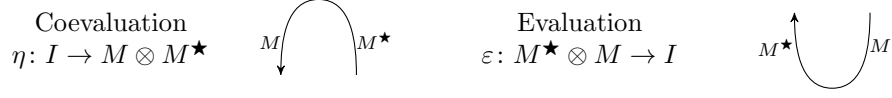
Coevaluation
$\eta\colon I \to M \otimes M^{\star}$

Evaluation
$\varepsilon\colon M^{\star} \otimes M \to I$

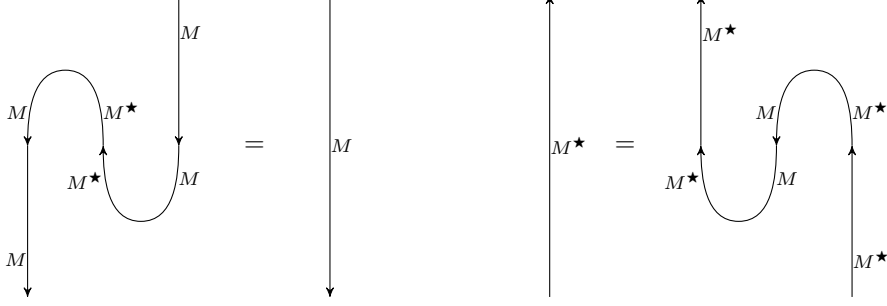FIGURE 1. Coevaluation and evaluation

FIGURE 2. The triangle identities

Let $\mathbf{C}$ be a symmetric monoidal category with product $\otimes$ and unit object I. We will omit the associativity and unit isomorphisms of $\mathbf{C}$ from the notation (effectively pretending that $\mathbf{C}$ is strict, as is allowable by the coherence theorem), and we write $\mathfrak{s}$ for any instance or composite of instances of the symmetry isomorphism of $\mathbf{C}$.

Definition 2.1. An object M of $\mathbf{C}$ is **dualizable** if there exists an object $M^{\star}$, called its **dual**, and maps

$$I \xrightarrow{\eta} M \otimes M^{\star} \qquad\qquad M^{\star} \otimes M \xrightarrow{\varepsilon} I$$

satisfying the triangle identities

$$(\mathrm{id}_M \otimes \varepsilon)(\eta \otimes \mathrm{id}_M) = \mathrm{id}_M \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_{M^{\star}})(\mathrm{id}_{M^{\star}} \otimes \eta) = \mathrm{id}_{M^{\star}}.$$

We call ε the **evaluation** and η the **coevaluation**; some authors call them the *counit* and the *unit*. We say that $\mathbf{C}$ is **compact closed** if every object is dualizable.

As suggested in §1, dual pairs are represented graphically by turning around the direction of arrows; see Figure 1. Note that the unit object I is represented by the lack of any strings, in the input to η and in the output of ε. Strictly speaking, there should be boxes at the ends of these "caps" and "cups" labeled by η and ε respectively, but these labels are almost universally omitted in string diagram notation (this is also justified by a theorem of Joyal and Street). The triangle identities for a dual pair translate graphically as "bent strings can be straightened;" see Figure 2.

Any two duals of an object M are isomorphic; an isomorphism can be constructed from η and ε. If $M^{\star}$ is a dual of M, then M is a dual of $M^{\star}$. And if M and N are dualizable, any map $f\colon Q \otimes M \to N \otimes P$ has a **dual** or **mate**

$$f^{\star}\colon N^{\star} \otimes Q \to P \otimes M^{\star},$$

given by the composite

$$N^{\star} \otimes Q \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \eta} N^{\star} \otimes Q \otimes M \otimes M^{\star} \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} N^{\star} \otimes N \otimes P \otimes M^{\star} \xrightarrow{\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}} P \otimes M^{\star}.$$
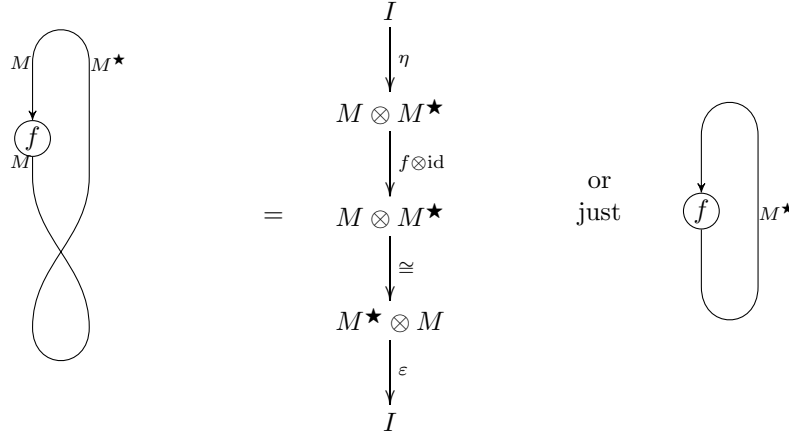
FIGURE 3. The trace

In particular, if M is dualizable, any endomorphism $f\colon M \to M$ has a dual $f^{\star}\colon M^{\star} \to M^{\star}$.

There are various equivalent characterizations of dualizable objects. For example, when $\mathbf{C}$ is closed, M is dualizable if and only if the canonical map

$$M \otimes \operatorname{Hom}(M, I) \to \operatorname{Hom}(M, M)$$

is an isomorphism. However, for us the above definition is most appropriate.

We now move on to the simplest form of trace.

Definition 2.2. Let $\mathbf{C}$ be a symmetric monoidal category, M a dualizable object of $\mathbf{C}$ and $f\colon M \to M$ an endomorphism of M. The **trace** of f, denoted $\operatorname{tr}(f)$, is the following composite:

$$(2.3) \qquad I \xrightarrow{\eta} M \otimes M^{\star} \xrightarrow{f \otimes \mathrm{id}} M \otimes M^{\star} \xrightarrow{\mathfrak{s}}_{\sim} M^{\star} \otimes M \xrightarrow{\varepsilon} I.$$

The **Euler characteristic** of a dualizable M is the trace of its identity map.

The trace of a morphism translates as graphically as "feeding its output into its input;" see Figure 3.

References for this notion of trace include [11, 14, 20, 21, 23]. It is an endomorphism of the unit object I, and does not depend on the choice of dual for M or on the choice of the maps η and ε. It also has the following fundamental property.

Proposition 2.4 (Cyclicity). *For any $f\colon M \to N$ and $g\colon N \to M$ with M, N both dualizable, we have $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.*

Proof. The following proof is really only rendered comprehensible by string diagram notation (see Figure 4).

$\operatorname{tr}(fg) = \varepsilon \mathfrak{s}(fg \otimes \mathrm{id})\eta = \varepsilon \mathfrak{s}(f \otimes \mathrm{id})(g \otimes \mathrm{id})\eta = \varepsilon \mathfrak{s}(f \otimes \mathrm{id})(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\eta \otimes \mathrm{id} \otimes \mathrm{id})(g \otimes \mathrm{id})\eta =$

$\varepsilon \mathfrak{s}(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(f \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes g \otimes \mathrm{id})(\eta \otimes \eta) = (\varepsilon \otimes \varepsilon)\mathfrak{s}(\mathrm{id} \otimes \mathrm{id} \otimes g \otimes \mathrm{id})(f \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(\eta \otimes \eta) =$

$(\varepsilon \otimes \varepsilon)\mathfrak{s}(\mathrm{id} \otimes \mathrm{id} \otimes g \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)(f \otimes \mathrm{id})\eta = \varepsilon \mathfrak{s}(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(g \otimes \mathrm{id} \otimes \mathrm{id})(\eta \otimes \mathrm{id} \otimes \mathrm{id})(f \otimes \mathrm{id})\eta =$

$\varepsilon \mathfrak{s}(g \otimes \mathrm{id})(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\eta \otimes \mathrm{id} \otimes \mathrm{id})(f \otimes \mathrm{id})\eta = \varepsilon \mathfrak{s}(g \otimes \mathrm{id})(f \otimes \mathrm{id})\eta = \varepsilon \mathfrak{s}(gf \otimes \mathrm{id})\eta = \operatorname{tr}(gf) \qquad \square$

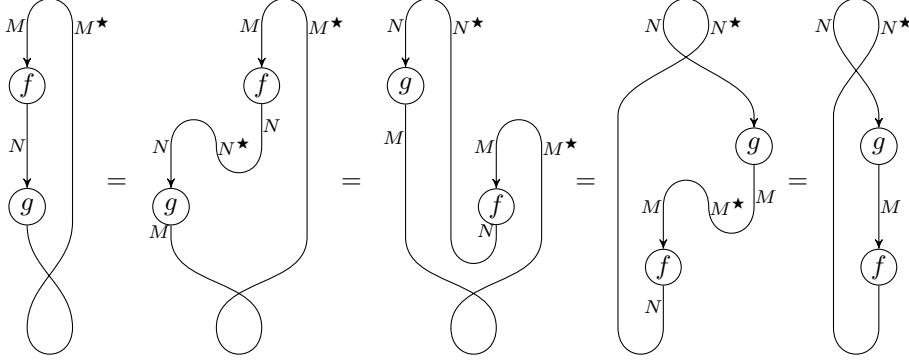We will consider additional properties of the trace in §5.

FIGURE 4. Cyclicity of the trace

3. EXAMPLES OF TRACES

Example 3.1. Let $\mathbf{C} = \mathbf{Vect}_k$ be the category of vector spaces over a field k. A vector space is dualizable if and only if it is finite-dimensional, and its dual is the usual dual vector space. We have $I = k$ and $\mathbf{C}(I, I) \cong k$ by multiplication. Using this identification, Definition 2.2 recovers the usual trace of a matrix. The Euler characteristic of a vector space is its dimension.

Example 3.2. Let $\mathbf{C} = \mathbf{Mod}_R$ be the category of modules over a commutative ring R. The dualizable objects are the finitely generated projectives. As in $\mathbf{Vect}_k$, we have $I = R$ and $\mathbf{C}(R, R) \cong R$, so every endomorphism of a finitely generated projective module has a trace which is an element of R. The Euler characteristic of such a module is its rank, regarded as an element of R (so that, for instance, the Euler characteristic of a rank-p free $(\mathbb{Z}/p)$-module is zero).

Example 3.3. Again, let R be a commutative ring and consider the category $\mathbf{Ch}_R$ of chain complexes of R-modules, with its symmetric monoidal tensor product. The "correct" symmetry isomorphism $M \otimes N \cong N \otimes M$ introduces a sign: $a \otimes b \mapsto (-1)^{|a||b|}(b \otimes a)$. The dualizable objects are again the finitely generated projectives, the unit is again R itself (in degree 0), and endomorphisms of the unit can again be identified with elements of R. The trace of an endomorphism of a finitely generated projective chain complex, called its *Lefschetz number*, is the alternating sum of its degreewise traces. Likewise, the Euler characteristic of such a chain complex is the alternating sum of its ranks. This generalizes straightforwardly to modules over a DGA.

Example 3.4. There is also a symmetric monoidal category $\mathrm{Ho}(\mathbf{Ch}_R)$, also called the *derived category* of R, obtained from $\mathbf{Ch}_R$ by formally inverting the quasi-isomorphisms (morphisms which induce isomorphisms on all homology groups). The dualizable objects in $\mathrm{Ho}(\mathbf{Ch}_R)$ are those that are quasi-isomorphic to an object that is dualizable in $\mathbf{Ch}_R$, and the two kinds of traces also agree.

Example 3.5. Let $n\mathbf{Cob}$ be the category whose objects are closed $(n-1)$-dimensional manifolds, and whose morphisms are diffeomorphism classes of cobordisms. Composition is by gluing, cylinders $M \times [0, 1]$ give identities, and disjoint union supplies a symmetric monoidal structure. The unit object is the empty set $\emptyset$, and an endomorphism of $\emptyset$ is just a closed n-manifold.

Every object of $n\mathbf{Cob}$ is dualizable: the evaluation and coevaluation are both $M \times [0, 1]$, regarded either as a cobordism from $\emptyset$ to $M \sqcup M$ or from $M \sqcup M$ to $\emptyset$. The trace of a cobordism from M to M is the closed n-manifold obtained by gluing the two components of its boundary together. In particular, the Euler characteristic of a closed $(n-1)$-manifold M is $M \times S^1$, regarded as a cobordism from $\emptyset$ to itself.

Example 3.6. In a *cartesian* monoidal category, the only dualizable object is the terminal object. Thus, in this case there are no interesting traces. However, as suggested in §1, often we can obtain useful dualities and traces by applying a functor from such a category to a non-cartesian monoidal category. Such a functor F induces a notion of "finiteness" on its domain in the evident way: X is "finite" if $F(X)$ is dualizable. For an endomorphism f of such an X, we can then compute the trace of $F(f)$.

Probably the easiest such functor is the "free abelian group" functor $\mathbb{Z}[-] \colon \mathbf{Set} \to \mathbf{Ab}$. Of course, $\mathbb{Z}[X]$ is dualizable in $\mathbf{Ab}$ if and only if X is a finite set. If $f \colon X \to X$ is an endomorphism of a finite set, then the trace of $\mathbb{Z}[f] \colon \mathbb{Z}[X] \to \mathbb{Z}[X]$ is easily seen to be simply the number of fixed points of f. This justifies the hope expressed in §1 that by mapping a cartesian monoidal category into an "additive" one, we could extract information about fixed points which may or may not be present. The next few examples can also be viewed in this light.

Example 3.7. Suppose that instead of a set we start with a topological space. The category $\mathbf{Top}$ is, of course, cartesian monoidal, so in order to obtain interesting dualities we need to apply a functor landing in some non-cartesian monoidal category. One obvious guess, by analogy with Example 3.6, would be the category of abelian topological groups and the free abelian topological group functor. It usually turns out to be better, however, to use a more refined notion: the category $\mathbf{Sp}$ of *spectra*.

For the reader unfamiliar with spectra some intuition can be gained as follows. A connective spectrum can be thought of as analogous to an abelian topological group, except that its group structure is only associative, unital, and commutative up to homotopy and all higher homotopies. The passage from connective spectra to arbitrary spectra is then analogous to the passage from bounded-below chain complexes to arbitrary ones. There is a symmetric monoidal category $\mathbf{Sp}$ of spectra and a "free" functor $\Sigma_+^\infty \colon \mathbf{Top} \to \mathbf{Sp}$, usually called the *suspension spectrum* functor. (Actually, there are many such categories $\mathbf{Sp}$, all equivalent up to homotopy, but each having different technical advantages and disadvantages; see for instance [13, 28, 29]. We will generally gloss over such distinctions.) The monoidal structure of $\mathbf{Sp}$ is called the "smash product" $\wedge$, and its unit object is the *sphere spectrum* S (which can be identified with Σ_+^∞ of a point).

Since $\mathbf{Sp}$ is not cartesian monoidal, we can hope for it to have an interesting duality theory. However, it turns out that in $\mathbf{Sp}$ it is only reasonable to ask for duality up to homotopy. Thus, instead of $\mathbf{Sp}$ we usually work with the category $\mathrm{Ho}(\mathbf{Sp})$ obtained from it by inverting the "stable equivalences;" this is called the *stable homotopy category*. We still have a functor $\Sigma_+^\infty \colon \mathbf{Top} \to \mathrm{Ho}(\mathbf{Sp})$ which factors through $\mathrm{Ho}(\mathbf{Top})$ (in which we invert the weak homotopy equivalences). The reason for the use of "stable" is that for compact spaces M and N, the homset $\mathrm{Ho}(\mathbf{Sp})(\Sigma_+^\infty(M), \Sigma_+^\infty(N))$ can be identified with the set of stable homotopy classes of maps from M to N, i.e. the colimit over n of the sets of homotopy classes of maps $\Sigma^n(M_+) \to \Sigma^n(N_+)$.

One can now show that if M is a closed smooth manifold, or more generally a compact ENR (Euclidean Neighborhood Retract), then $\Sigma_+^\infty(M)$ is dualizable in $\mathrm{Ho}(\mathbf{Sp})$; its dual is the Thom spectrum $T\nu$ of its stable normal bundle. This is proven in [1, 11, 25]. The set of endomorphisms of the sphere spectrum S is $\mathrm{colim}_n \pi_n(S^n)$, which is isomorphic to $\mathbb{Z}$; thus traces in $\mathrm{Ho}(\mathbf{Sp})$ can be identified with integers.

Using this identification, the trace of an endomorphism can be identified with its *fixed point index*. The fixed point index is an integer which is defined classically, for a map with isolated fixed points, as the sum over all fixed points x of the degree of the self-map induced by the "difference" of the identity map and the endomorphism on a sufficiently small sphere surrounding x. This turns out to be homotopy invariant, and so for an arbitrary map it can be defined by homotoping to a map with isolated fixed points. In particular, this is necessary for the identity map, whose fixed point index is the Euler characteristic of the manifold (this is, of course, the origin of the term "Euler characteristic" for traces of identity maps in general). See [5,7] for the classical approach to the index and [8,9,11] for the identification of this trace with the classical fixed point index and Euler characteristic.

For compact spaces the induced notions of duality and trace can also be formulated without using the stable homotopy category, by replacing S with the n-sphere S^n for some large enough finite n. In this guise it is called *n-duality*; references include [11, 25].

$\mathbf{Sp}$ and $\mathbf{Ch}_R$ are two instances of a general phenomenon: a symmetric monoidal category that has an associated symmetric monoidal homotopy category. A general theory of when and how symmetric monoidal structures descend to homotopy categories is given by the study of *monoidal model categories*, as in [17, Ch. 4].

Example 3.8. For a compact Lie group G, there is an *equivariant stable homotopy category* $\mathrm{Ho}(G\text{-}\mathbf{Sp})$, which is related to the category $G\text{-}\mathbf{Top}$ of G-spaces in the same way that $\mathrm{Ho}(\mathbf{Sp})$ is related to $\mathbf{Top}$; see for instance [28]. Now the suspension and stabilization take place not relative to ordinary spheres S^n, but relative to representations of G. The category $\mathrm{Ho}(G\text{-}\mathbf{Sp})$ is also symmetric monoidal and admits a suspension G-spectrum functor from G-spaces.

The dualizable objects in $\mathrm{Ho}(G\text{-}\mathbf{Sp})$ include the equivariant suspension spectra of closed smooth G-manifolds and compact G-ENR's. Such dual pairs can be also described using V-duality for a representation V. A reference for equivariant duality is [25]. Traces in $\mathrm{Ho}(G\text{-}\mathbf{Sp})$ are again called fixed point indices; see [38].

Example 3.9. Another variation is to consider *parametrized* duality, which instead of spaces or G-spaces starts from spaces *over* a base space B. In [31], May and Sigurdsson construct a category $\mathbf{Sp}_B$ of *parametrized spectra* over B, which is symmetric monoidal, has a symmetric monoidal homotopy category $\mathrm{Ho}(\mathbf{Sp}_B)$, and admits a suspension functor $\Sigma_{B,+}^\infty$ from $\mathbf{Top}/B$ that is similar to Σ_+^∞.

If M is a fibration over B, then $\Sigma_{B,+}^\infty(M)$ is dualizable in $\mathrm{Ho}(\mathbf{Sp}_B)$ if and only if each of its fibers is dualizable in the usual stable homotopy category. In particular, a fibration of closed smooth manifolds gives rise to a dualizable parametrized spectrum. The trace of a fiberwise endomorphism is once again called its fixed point index; see [9].

Remark 3.10. For parametrized spaces and spectra it is often more illuminating to consider a different type of duality called *Costenoble-Waner duality*, and its

associated notion of trace. These notions of duality and trace do not take place in a symmetric monoidal category, but rather in a bicategory arising from an indexed symmetric monoidal category; see [31, Ch. 18] and [34, 35].

Here are some more "toy" examples.

Example 3.11. Let **Rel** be the category whose objects are sets and whose morphisms from M to N are relations $R \subset M \times N$; we write $R \colon M \nrightarrow N$ to avoid confusion with functions $M \to N$. If $S \colon N \nrightarrow P$ is another relation, their composite is

$$SR = \Big\{ (m, p) \,\Big|\, \exists n \text{ with } (m, n) \in R \text{ and } (n, p) \in S \Big\}.$$

A symmetric monoidal structure on **Rel** is induced by the cartesian product of sets (which is *not* the cartesian product in **Rel**).

There is a functor **Set** $\to$ **Rel** which is the identity on objects and takes a function $f \colon X \to Y$ to its graph $\Gamma_f = \{ (x, f(x)) \mid x \in X \}$. Moreover, *every* set is dualizable in **Rel**, and moreover is its own dual; the relations η and ε are both the identity relation on X, considered as a relation $* \nrightarrow X \times X$ or $X \times X \nrightarrow *$, respectively. Thus every set is "finite" relative to this functor. However, the tradeoff is that traces contain correspondingly less information, since the only endomorphisms of the unit $*$ in **Rel** are the empty relation and the full one. If we regard these as the *truth values* "false" and "true," respectively, then the trace of a relation $R \colon M \nrightarrow M$ is the truth value of the statement "$\exists m : (m, m) \in R$." In particular, for a function $f \colon M \to M$, $\operatorname{tr}(\Gamma_f)$ is true if and only if f has a fixed point.

This example can be generalized to internal relations in any suitably well-behaved category.

Example 3.12. Let **Sup** denote the category of *suplattices*: that is, its objects are posets with all suprema and its morphisms are supremum-preserving maps. (Of course, a suplattice also has all infima, but suplattice maps need not preserve infima.) There is a tensor product of suplattices, concisely described by saying that suplattice maps $M \otimes N \to P$ represent functions $M \times N \to P$ which preserve suprema in each variable separately. The unit object is the suplattice $I = (0 \le 1)$.

We can see an analogy between **Sup** and **Ab** by regarding suprema in a suplattice as similar to sums in an abelian group. For instance, there is a "free suplattice" functor **Set** $\to$ **Sup** which simply takes a set A to its power set $\mathcal{P}(A)$; the "free generators" are the singleton sets, and a subset $B \subseteq A$ is the "formal sum" $\sum_{x \in B} \{x\}$. Every such power set is dualizable, so every set is "finite" relative to the functor $\mathcal{P}$. Explicitly, we have $\mathcal{P}(A)^{\star} \cong \mathcal{P}(A)$ with coevaluation

$$\eta(1) = \bigvee_{a \in A} \{a\} \otimes \{a\}$$

and evaluation

$$\varepsilon(X \otimes Y) = \begin{cases} 1 & \text{if } X \cap Y \ne \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that a suplattice map $f \colon \mathcal{P}(A) \to \mathcal{P}(A)$ is equivalent to a function $A \to \mathcal{P}(A)$, and thereby to a relation $R_f \subset A \times A$. The trace of such a map in **Sup** is easily verified to be id_I if there is an $a \in A$ with $(a, a) \in R_f$ and 0 otherwise; thus we essentially recapture Example 3.11.

However, not all dualizable suplattices are power sets. For instance, if A is an Alexandrov topological space (one where arbitrary intersections of open sets are

open), then its open-set lattice $\mathcal{O}(A)$ is a dualizable suplattice. In this case, for a continuous map $f\colon A \to A$, the trace of $f^{-1}\colon \mathcal{O}(A) \to \mathcal{O}(A)$ is id_I if there is an $a \in A$ such that $a \le f(a)$ in the specialization order, and 0 otherwise. (Recall that the *specialization order* of a topological space is defined so that $x \le y$ if and only if every open set containing y also contains x.)

Example 3.13. Continuing the analogy between **Sup** and **Ab**, it is natural to consider commutative monoid objects in **Sup** as analogous to commutative rings. A commutative monoid object in a symmetric monoidal category **C** is an object R with morphisms $R \otimes R \to R$ and $I \to R$ satisfying evident axioms. In particular, for any topological space B, the open-set lattice $\mathcal{O}(B)$ is a commutative monoid object in **Sup** whose "multiplication" map is intersection $\cap\colon \mathcal{O}(B) \otimes \mathcal{O}(B) \to \mathcal{O}(B)$. Likewise, for any continuous $f\colon A \to B$, we have a monoid homomorphism $f^{-1}\colon \mathcal{O}(B) \to \mathcal{O}(A)$. In this way a category of suitably nice topological spaces can be identified with the opposite of a subcategory of commutative monoids in **Sup**; see [22] and [18, Ch. C1]. This is analogous to how the category of affine schemes can be identified with the opposite of the category of commutative rings.

If **C** has coequalizers preserved on both sides by $\otimes$, then for any commutative monoid object R the category of R-modules in **C** is itself symmetric monoidal under the tensor product given by the usual coequalizer

$$M \otimes R \otimes N \rightrightarrows M \otimes N \to M \otimes_R N.$$

In particular, this applies to $\mathcal{O}(B)$-modules in **Sup** for any space B. And given any continuous map $p\colon X \to B$, $\mathcal{O}(X)$ becomes a $\mathcal{O}(B)$-algebra (that is, there is a monoid homomorphism $p^{-1}\colon \mathcal{O}(B) \to \mathcal{O}(X)$) and thus a $\mathcal{O}(B)$-module. If p is a local homeomorphism (that is, X is the "espace etale" of a sheaf over B), then $\mathcal{O}(X)$ is a dualizable $\mathcal{O}(B)$-module that is its own dual: the evaluation is

$$\mathcal{O}(X) \otimes_{\mathcal{O}(B)} \mathcal{O}(X) \xrightarrow{\varepsilon} \mathcal{O}(B)$$
$$(W, W') \mapsto p(W \cap W')$$

and the coevaluation is

$$\mathcal{O}(B) \xrightarrow{\eta} \mathcal{O}(X) \otimes_{\mathcal{O}(B)} \mathcal{O}(X)$$
$$U \mapsto \bigvee_{\substack{p(W) \subseteq U \\ p|_W \text{ is a homeomorphism}}} W \otimes W.$$

The unit $\mathcal{O}(B)$-module is $\mathcal{O}(B)$, and an $\mathcal{O}(B)$-module map $\mathcal{O}(B) \to \mathcal{O}(B)$ is determined uniquely by where it sends $B \in \mathcal{O}(B)$ (the unit for $\cap$). Any map $f\colon X \to X$ over B gives a map $f^{-1}\colon \mathcal{O}(X) \to \mathcal{O}(X)$ of $\mathcal{O}(B)$-modules, whose trace is characterized by

$$B \mapsto \big\{ b \in B \mid \exists x \in p^{-1}(b) : f(x) = x \big\};$$

that is, the set of $b \in B$ such that $f|_{p^{-1}(b)}$ has a fixed point. In particular, the Euler characteristic of a sheaf X is its *support* $p(X) \subseteq B$. When B is the one-point space, X must be discrete, and we recapture power sets in **Sup**. For more on this point of view, see [36].

We have so far considered only *symmetric* monoidal categories, but the definitions we have given make sense with only a braiding, and there are interesting examples which are not symmetric.

Example 3.14. Let **Tang** be the category of *tangles*: its objects are natural numbers $0, 1, 2, \ldots$, and its morphisms from n to m are tangles from n points to m points. A tangle is like a braid, except that strings can be turned around, so that n need not equal m; see [14]. **Tang** is braided but not symmetric monoidal; its product is disjoint union and its unit object is 0. Every object is its own dual, the endomorphisms of the unit are links, and the trace of an endo-tangle is its "tangle closure" into a link. There are oriented and framed variants.

However, the trace as we have defined it is not quite correct in the non-symmetric case. For instance, with our definitions, the trace of the identity id_2 in **Tang** is two *linked* circles, while the trace of the braiding $\mathfrak{s}_2$ is two *unlinked* circles; clearly it would make more sense for this to be the other way round. This can be remedied with the notion of a *balanced* monoidal category, which is a braided monoidal category in which each object is equipped with a "double-twist" automorphism; see [20, 21]. Symmetric monoidal categories can be identified with balanced ones in which every double-twist is the identity. In a balanced monoidal category, we define the trace of an endomorphism f by including a double-twist with f in between η and ε; this remedies the problem noted above with **Tang**. For simplicity, however, in this paper we will consider only the symmetric case.

4. Twisted traces and transfers

The examples in §3 show that the canonical trace defined in §2 does give useful information about fixed points, but usually it only indicates their presence or absence, or at best counts the number of them (with multiplicity). However, in §1 we saw that in the presence of "diagonals", we could hope to extract not merely the number of fixed points, but the fixed points themselves. The use of diagonals in this way turns out to be a special case of the following more general notion of "twisted trace."

Definition 4.1. Let **C** be a symmetric monoidal category, M a dualizable object of **C**, and $f: Q \otimes M \to M \otimes P$ a morphism in **C**. The **trace** $\mathrm{tr}(f)$ of f is the following composite:

$$(4.2) \qquad Q \xrightarrow{\eta} Q \otimes M \otimes M^{\star} \xrightarrow{f} M \otimes P \otimes M^{\star} \xrightarrow{\mathfrak{s}} M^{\star} \otimes M \otimes P \xrightarrow{\varepsilon} P$$

This trace is displayed graphically in Figure 5.

Remark 4.3. Since **C** is symmetric, it may seem odd to write the domain of f as $Q \otimes M$ but its codomain as $M \otimes P$. Indeed, in the literature on symmetric monoidal traces it is more common to align the M's on one side, as in the right-hand version of Figure 5. Our notation is chosen instead to match that of the bicategorical generalization presented in [35], in which case the order we have chosen here is the only possibility.

Of course, when $Q = P = I$ is the unit object, this reduces to the previous notion of trace. It is also cyclic, in a suitable sense.

Lemma 4.4. *If M and N are dualizable and $f: Q \otimes M \to N \otimes P$ and $g: K \otimes N \to M \otimes L$ are morphisms, then*

$$\mathrm{tr}\left((g \otimes \mathrm{id}_P)(\mathrm{id}_K \otimes f)\right) = \mathrm{tr}\left(\mathfrak{s}(f \otimes \mathrm{id}_L)(\mathrm{id}_Q \otimes g)\mathfrak{s}\right).$$
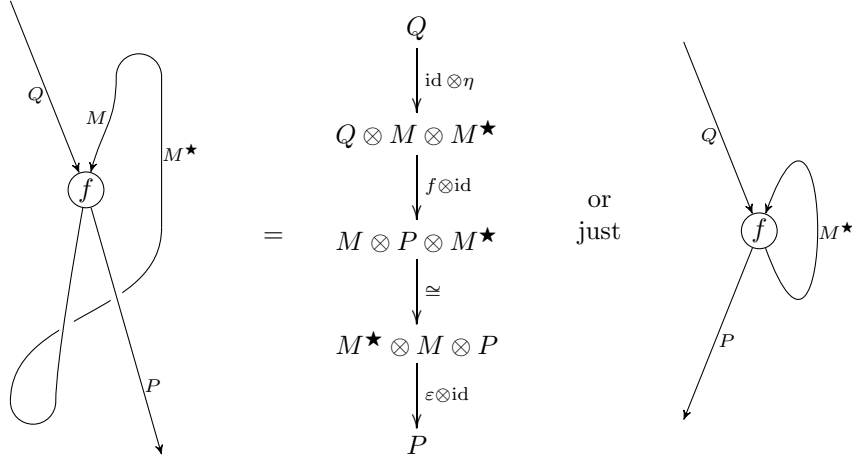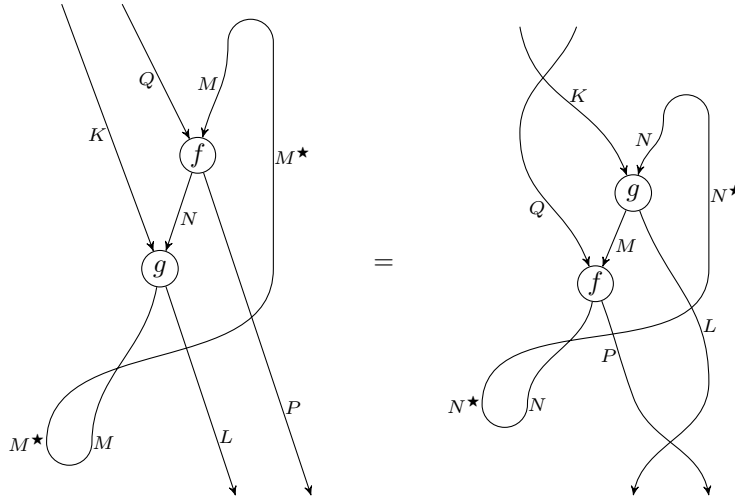
FIGURE 5. The "twisted" trace

FIGURE 6. Cyclicity of the "twisted" trace

The string diagram for this lemma is Figure 6, which should be compared to Figure 4.

As promised, the trace using a "diagonal morphism" is a special case of the general notion of twisting.

Definition 4.5. Let $M \in \mathbf{C}$ be a dualizable object equipped with a "diagonal" morphism $\Delta \colon M \to M \otimes M$, and let $f \colon M \to M$ be an endomorphism of M. Then the **trace of f with respect to Δ** is the trace of $\Delta \circ f \colon M \to M \otimes M$. The trace of id_M with respect to Δ is called the **transfer** of M.

The trace of f with respect to Δ is a morphism $I \to M$; by cyclicity it is also equal to the trace of $(f \otimes \mathrm{id}) \circ \Delta$.

Where do such diagonals come from? Of course, if $\mathbf{C}$ is *cartesian* monoidal, then any object M has such a diagonal, but we have seen (Example 3.6) that in this case there are few dualizable objects. However, we have also seen that often the dualizable objects for which we are interested in traces are in the image of a symmetric monoidal functor whose domain *is* cartesian monoidal, and such a functor preserves the existence of diagonals. That is, if X is an object of a cartesian monoidal category, and Z a symmetric monoidal functor, then the diagonal $X \to X \times X$ gives rise to a diagonal $Z(X) \to Z(X) \otimes Z(X)$. This is usually the source of "diagonals" in examples.

Example 4.6. Let $\mathbf{S} = \mathbf{Set}$, $\mathbf{C} = \mathbf{Ab}$, and Σ be the free abelian group functor $\mathbb{Z}[-]$. In this case the induced diagonal $\mathbb{Z}[X] \to \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ sends a generator x to $x \otimes x$. If X is finite, so that $\mathbb{Z}[X]$ is dualizable, the trace of $\mathbb{Z}[f]$ with respect to this diagonal is $\sum_{f(x)=x} x \in \mathbb{Z}[X]$. In particular, the transfer of $\mathbb{Z}[X]$ is $\sum_{x \in X} x \in \mathbb{Z}[X]$.

Note that while the ordinary trace of $\mathbb{Z}[f]$ records only the *number* of fixed points of f, its trace with respect to Δ records *what* those fixed points are (as elements of $\mathbb{Z}[X]$).

Example 4.7. As a more sophisticated version of the previous example, let $\mathbf{S} = \mathbf{Top}$ and let $\mathbf{C} = \mathrm{Ho}(\mathbf{Sp})$ be the stable homotopy category. Since $\mathbf{Top}$ is cartesian monoidal and the suspension spectrum functor Σ^∞_+ is strong monoidal, the diagonal $M \to M \times M$ of any space induces a diagonal

$$\Delta \colon \Sigma^\infty_+(M) \to \Sigma^\infty_+(M) \wedge \Sigma^\infty_+(M).$$

Thus, when M is n-dualizable, we can define traces and transfers with respect to Δ. This example is the original use of the term *transfer*. In this case, the transfer of an n-dualizable space M is a map $S \to \Sigma^\infty_+(M)$, which is by definition an element of $\pi^s_0(M_+)$, the 0th stable homotopy group of M. If M is connected, there is an isomorphism

$$\pi^s_0(M_+) \cong H_0(M_+)$$

under which the image of the transfer is $\chi(M)$ times the generator of $H_0(M_+)$; see [25, III.8.4]. More generally, if f is an endomorphism of M, then the trace of f with respect to Δ is the fixed point transfer defined by Dold in [10]. We have the same intuition as for the previous example; while the fixed point index only *counts* the fixed points, the fixed point transfer *records* them.

There are also equivariant and parametrized transfers. For example, the Becker-Gotleib transfer [3] is the parametrized transfer of a fibration with compact manifold fibers.

Example 4.8. Recall from Example 3.11 that we also have a functor $\mathbf{Set} \to \mathbf{Rel}$ which is the identity on objects and sends a function f to its graph Γ_f. In this case, for any endofunction $f \colon M \to M$, the trace of Γ_f with respect to Γ_Δ is the set of all fixed points of f, regarded as a relation from $\star$ to M. In particular, the transfer of M is M itself so regarded.

Example 4.9. Likewise, if Σ is the free suplattice functor $\mathcal{P} \colon \mathbf{Set} \to \mathbf{Sup}$ from Example 3.12, then for any endofunction $f \colon M \to M$ the trace of $\mathcal{P}(f)$ with respect to $\mathcal{P}(\Delta)$ is also the set of fixed points of f, now regarded as an element of $\mathcal{P}(M)$.

Twisted traces not arising from diagonals are less common, but they do occur.

Example 4.10. Let $f\colon Q \times M \to M$ be a function between sets, where M is finite. Then the trace of $\mathbb{Z}[f]\colon \mathbb{Z}[Q]\otimes\mathbb{Z}[M] \to \mathbb{Z}[M]$ in **Ab** is the homomorphism $\mathbb{Z}[Q] \to \mathbb{Z}$ which sends each generator $q \in Q$ to the number of fixed points of $f(q,-)$.

We can also combine this with a transfer, by considering $(f, \mathrm{pr}_2)\colon Q \times M \to M \times M$. This induces a map $\mathbb{Z}[Q]\otimes\mathbb{Z}[M] \to \mathbb{Z}[M]\otimes\mathbb{Z}[M]$, whose trace $\mathbb{Z}[Q] \to \mathbb{Z}[M]$ sends each generator $q \in Q$ to the sum of the fixed points of $f(q,-)$.

Similarly, for any set M and any function $f\colon Q \times M \to M$, the trace of Γ_f in **Rel** is the relation from Q to $*$ defined by

$$\mathrm{tr}(\Gamma_f) = \{\, q \in Q \mid f(q,-) \text{ has a fixed point} \,\},$$

and the trace of the induced relation $Q \times M \to M \times M$ is

$$\{\, (q,m) \in Q \times M \mid m \text{ is a fixed point of } f(q,-) \,\}.$$

Example 4.11. If $f\colon Q \times M \to M$ is a continuous map of topological spaces and $\Sigma^\infty_+ M$ is dualizable, the trace of f is the stable homotopy class of a map

$$Q \to S^0.$$

Using explicit descriptions of the coevaluation and evaluation for $\Sigma^\infty_+ M$, it is not difficult to verify this stable map is homotopically trivial if the set

$$\{(q,m) \in Q \times M | f(q,m) = m\}$$

is empty.

Let $\pi\colon Q \times M \to Q$ be the first coordinate projection. Note that f determines a fiberwise map $F\colon Q \times M \to Q \times M$ by $F(q,m) = (q, f(q,m))$. The trace of F as described in Example 3.9 coincides with the trace of f under a corresponding comparison of fiberwise stable endomorphism of the unit object over Q with the stable maps from Q to S^0.

This trace is related to the *higher Euler characteristics* in [15].

Example 4.12. For any topological space A we have an "intersection" morphism $\cap\colon \mathcal{O}(A) \otimes \mathcal{O}(A) \to \mathcal{O}(A)$ in **Sup**. If A is moreover Alexandrov, so that $\mathcal{O}(A)$ is dualizable in **Sup**, then for any $f\colon A \to A$, the trace of $f^{-1} \circ \cap$ is the function $\mathcal{O}(A) \to I$ that takes $U \subset A$ to 1 if U contains a point x with $x \le f(x)$ and to 0 otherwise. If we identify a suplattice map $g\colon \mathcal{O}(A) \to I$ with the closed subset

$$\bigcap_{\substack{K \text{ closed} \\ g(A\setminus K)=0}} K$$

(which determines it), then the trace of $f^{-1} \circ \cap$ is identified with the closure of

$$\{\, a \in A \mid a \le f(a) \,\}.$$

On the other hand, if B is another Alexandrov space with a map $m\colon A \times B \to A$, then we have an induced suplattice map $m^{-1}\colon \mathcal{O}(A) \to \mathcal{O}(A \times B) \cong \mathcal{O}(A) \otimes \mathcal{O}(B)$. Its trace is the suplattice map $I \to \mathcal{O}(B)$ which takes 1 to the open set

$$\{\, b \in B \mid (\exists a \in A)(a \le m(a,b)) \,\}.$$

Example 4.13. If $p\colon X \to B$ is any local homeomorphism, then regarding $\mathcal{O}(X)$ as a $\mathcal{O}(B)$-module we again have an "intersection" morphism $\cap\colon \mathcal{O}(X)\otimes_{\mathcal{O}(B)}\mathcal{O}(X) \to \mathcal{O}(X)$ (corresponding to the diagonal $X \to X \times_B X$). For any $f\colon X \to X$ over B,

the trace of $f^{-1} \circ \cap$ is the $\mathcal{O}(B)$-module map $\mathcal{O}(X) \to \mathcal{O}(B)$ that sends $V \in \mathcal{O}(X)$ to

$$\left\{\, b \in B \mid \exists x \in p^{-1}(b) \cap V : f(x) = x \,\right\}.$$

In particular, the trace of $\cap$ itself sends each $V \in \mathcal{O}(X)$ to its support.

On the other hand, if $q \colon Y \to B$ is another local homeomorphism and $f \colon X \times_B Y \to X$ is a map over B, then the trace of $f^{-1} \colon \mathcal{O}(X) \to \mathcal{O}(X) \otimes_{\mathcal{O}(B)} \mathcal{O}(Y)$ is the map $\mathcal{O}(B) \to \mathcal{O}(Y)$ sending the unit B to the open set

$$\left\{\, y \in Y \mid \exists x \in p^{-1}(q(y)) : f(x,y) = x \,\right\}.$$

5. Properties of traces

In addition to cyclicity, the (twisted) symmetric monoidal trace satisfies many useful naturality properties which we summarize here. We omit most proofs, which are straightforward diagram chases (and are especially easy in string diagram notation). References include [11, 21, 25, 30].

We begin with invariance under dualization. Recall that any $f \colon Q \otimes M \to M \otimes P$ has a mate $f^{\star} \colon Q \otimes M^{\star} \to M^{\star} \otimes P$, and since $M^{\star}$ is also dualizable (with dual M), after composing with symmetry isomorphisms on either side we can take the trace of $f^{\star}$ as well.

Proposition 5.1. *If M is dualizable and $f \colon Q \otimes M \to M \otimes P$ is any morphism, then $\mathrm{tr}(f) = \mathrm{tr}(\mathfrak{s} f^{\star} \mathfrak{s})$.*

Next we have a naturality property.

Proposition 5.2. *Let M be dualizable, let $f \colon Q \otimes M \to M \otimes P$ be a map, and let $g \colon Q' \to Q$ and $h \colon P \to P'$ be two maps. Then*

$$h \circ \mathrm{tr}(f) \circ g = \mathrm{tr}\left((\mathrm{id}_M \otimes h) \circ f \circ (g \otimes \mathrm{id}_M)\right).$$

In other words, the function

$$\mathrm{tr} \colon \mathbf{C}(Q \otimes M, M \otimes P) \longrightarrow \mathbf{C}(Q, P)$$

is natural in Q and P.

This implies that quite generally, traces "calculate fixed points", as described informally in §1.

Corollary 5.3 (Fixed point property). *If M is dualizable, $\Delta \colon M \to M \otimes P$ is a map and $f \colon M \to M$ is an endomorphism, and $h \colon P \to P$ is such that $(f \otimes h)\Delta = \Delta f$, then*

$$h \circ \mathrm{tr}(\Delta f) = \mathrm{tr}(\Delta f).$$

In particular, we may take $h = f$, in which case $(f \otimes f)\Delta = \Delta f$ will hold automatically if f and Δ both come from a cartesian monoidal category. In this case the conclusion is exactly that $\mathrm{tr}(\Delta f)$ is a fixed point of f, as promised.

An additional naturality property follows directly from cyclicity.

Proposition 5.4. *Let M and N be dualizable and let $f \colon Q \otimes M \to N \otimes P$ and $h \colon N \to M$ be maps. Then*

$$\mathrm{tr}((h \otimes \mathrm{id}_P)f) = \mathrm{tr}(f(\mathrm{id}_Q \otimes h)).$$

In fancier language, this says that the function

$$\operatorname{tr} \colon \mathbf{C}(Q \otimes M, M \otimes P) \longrightarrow \mathbf{C}(Q, P)$$

is "extraordinary-natural" (see [12]) in the dualizable object M.

We now consider compatibility of traces with the monoidal structure. Note that the unit I is always dualizable with $I^\star = I$.

Proposition 5.5. *If $f \colon Q \otimes I \to I \otimes P$ is a morphism in $\mathbf{C}$, then $\operatorname{tr}(f) = f$ (modulo unit isomorphisms).*

If M and N are dualizable, then so is $M \otimes N$, with dual $M^\star \otimes N^\star$. In this case, if we have a map

$$f \colon Q \otimes N \otimes M \longrightarrow M \otimes N \otimes P,$$

we can either take the trace of $f\mathfrak{s}$ with respect to $M \otimes N$, or we can first take the trace of f with respect to M and then with respect to N; either way results in a map $Q \to P$.

Proposition 5.6. *In the above situation, we have $\operatorname{tr}(f\mathfrak{s}) = \operatorname{tr}(\operatorname{tr}(f))$.*

A different situation is when M and N are still both dualizable, but we have two maps $f \colon Q \otimes M \to M \otimes P$ and $g \colon K \otimes N \to N \otimes L$.

Proposition 5.7. *In the above situation, we have $\operatorname{tr}\big(\mathfrak{s}(f \otimes g)\mathfrak{s}\big) = \operatorname{tr}(f) \otimes \operatorname{tr}(g)$.*

Taking $N = I$ we obtain the following.

Corollary 5.8. *If M is dualizable and $f \colon Q \otimes M \to M \otimes P$ and $g \colon K \to L$ are maps, then $\operatorname{tr}(\mathfrak{s}(f \otimes g)) = \operatorname{tr}(f) \otimes g$.*

On the other hand, it is not hard to show that Proposition 5.7 follows from Corollary 5.8 together with Proposition 5.6.

Finally, if M and N are dualizable and we have maps $f \colon Q \otimes M \to M \otimes P$ and $g \colon P \otimes N \to N \otimes K$, then we have the composite

$$(\operatorname{id}_M \otimes g)(f \otimes \operatorname{id}_N) \colon Q \otimes M \otimes N \longrightarrow M \otimes N \otimes K.$$

The next result then follows from Proposition 5.2 and Proposition 5.6.

Corollary 5.9. *In the above situation, we have*

$$\operatorname{tr}\big((\operatorname{id}_M \otimes g)(f \otimes \operatorname{id}_N)\big) = \operatorname{tr}(g) \circ \operatorname{tr}(f).$$

In particular, we can apply these results to untwisted traces. Note that by the Eckmann-Hilton argument, the two operations $\otimes$ and $\circ$ on $\mathbf{C}(I, I)$ agree (up to unit isomorphisms) and make it a commutative monoid. We thereby obtain the following.

Corollary 5.10. *If $\mathbf{C}$ is symmetric monoidal, M and N are dualizable, and $f \colon M \to M$ and $g \colon N \to N$ are endomorphisms, then*

$$\operatorname{tr}(f \otimes g) = \operatorname{tr}(f) \otimes \operatorname{tr}(g) = \operatorname{tr}(f) \circ \operatorname{tr}(g).$$

One final property of traces that should be mentioned here is the following.

Proposition 5.11. *If M is dualizable, then the trace of $\operatorname{id}_{M \otimes M} \colon M \otimes M \to M \otimes M$ is $\operatorname{id}_M \colon M \to M$.*

In [21] the above properties were taken to constitute the following definition.

Definition 5.12. A symmetric monoidal category $\mathbf{C}$ is **traced** if it is equipped with functions

$$\mathrm{tr}\colon \mathbf{C}(Q \otimes M, M \otimes P) \to \mathbf{C}(Q, P)$$

satisfying the conclusions of Propositions 5.2, 5.4, 5.5, 5.6, and 5.11 as well as Corollary 5.8.

Actually, [21] deals with the more general case of a balanced monoidal category; we have simplified things by treating only the symmetric case. A similar set of axioms is considered in [27].

Evidently if $\mathbf{C}$ is compact closed (every object is dualizable), then it is traced in a canonical (and, in fact, unique) way. Conversely, it is shown in [21] that any traced symmetric monoidal category can be embedded in a compact closed one, in a way that identifies the given trace with the canonical trace. On the other hand, much of the interest of the canonical symmetric monoidal trace lies in its applicability to particular interesting dualizable objects in categories where not every object is dualizable.

6. Functoriality of traces

One of the main advantages of having an abstract formulation of trace is that disparate notions of trace which all fall into the general framework can be compared functorially. In this section we summarize the relevant results and their applicability in some examples, including the Lefschetz fixed point theorem.

Recall that a **lax symmetric monoidal functor** $F\colon \mathbf{C} \to \mathbf{D}$ between symmetric monoidal categories consists of a functor F and natural transformations

$$\mathfrak{c}\colon F(M) \otimes F(N) \longrightarrow F(M \otimes N)$$

$$\mathfrak{i}\colon I_{\mathbf{D}} \longrightarrow F(I_{\mathbf{C}})$$

satisfying appropriate coherence axioms. We say F is **normal** if $\mathfrak{i}$ is an isomorphism, and **strong** if $\mathfrak{c}$ and $\mathfrak{i}$ are both isomorphisms.

Proposition 6.1. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a normal lax symmetric monoidal functor, let $M \in \mathbf{C}$ be dualizable with dual $M^{\star}$, and assume that $\mathfrak{c}\colon F(M) \otimes F(M^{\star}) \to F(M \otimes M^{\star})$ is an isomorphism (as it is when F is strong). Then $F(M)$ is dualizable with dual $F(M^{\star})$.*

Proof. Suppose given M with dual $M^{\star}$ exhibited by η and ε. Then the maps

$$I_{\mathbf{D}} \xrightarrow{\mathfrak{i}} F(I_{\mathbf{C}}) \xrightarrow{F(\eta)} F(M \otimes M^{\star}) \xrightarrow{\mathfrak{c}^{-1}} F(M) \otimes F(M^{\star})$$

and

$$F(M^{\star}) \otimes F(M) \xrightarrow{\mathfrak{c}} F(M^{\star} \otimes M) \xrightarrow{F(\varepsilon)} F(I_{\mathbf{C}}) \xrightarrow{\mathfrak{i}^{-1}} I_{\mathbf{D}}$$

show that $F(M)$ is dualizable with dual $F(M^{\star})$. $\square$

In the above situation, we say that F **preserves** the dual $M^{\star}$ of M. Actually, a slightly weaker condition on F suffices for the above conclusion; see [6].

Proposition 6.2. *If F preserves the dual $M^{\star}$ of M, and moreover $\mathfrak{c}\colon F(P) \otimes F(M) \to F(P \otimes M)$ is an isomorphism (as it is whenever $P = I$ and F is normal), then for any map $f\colon Q \otimes M \to M \otimes P$, we have*

$$F(\mathrm{tr}(f)) = \mathrm{tr}\left(\mathfrak{c} \circ F(f) \circ \mathfrak{c}^{-1}\right).$$

In particular, for an endomorphism $f\colon M \to M$, we have

$$F(\mathrm{tr}(f)) = \mathfrak{i} \circ \mathrm{tr}(F(f)) \circ \mathfrak{i}^{-1}$$

Proof. Use the dual $F(M^\star)$ of $F(M)$ to evaluate the right hand side. $\square$

Example 6.3. If R and S are commutative rings and $\phi\colon R \to S$ is a ring homomorphism, then extension of scalars defines a strong symmetric monoidal functor $(-\otimes_R S)$ from R-modules to S-modules. If M is a dualizable R-module and $f\colon Q \otimes M \to M \otimes P$ is a map of R-modules, Proposition 6.2 implies $\mathrm{tr}(f \otimes_R S) = \mathrm{tr}(f) \otimes_R S$. If Q and P are both the ring R, then as usual, we can think of the traces of f and $f \otimes_R S$ as elements of R and S, respectively; in this case, we have $\mathrm{tr}(f \otimes_R S) = \mathrm{tr}(f) \otimes_R S = \phi(\mathrm{tr}(f))$.

Example 6.4. Homology is a normal lax symmetric monoidal functor from $\mathbf{Ch}_R$ or $\mathrm{Ho}(\mathbf{Ch}_R)$ to the category $\mathbf{GrMod}_R$ of graded R-modules. The Künneth theorem implies that the natural transformation

$$\mathfrak{c}\colon H(M_*) \otimes H(N_*) \to H(M_* \otimes N_*)$$

is an isomorphism if M_p and $H_p(M_*)$ are projective for each p. When these conditions are satisfied (such as when the ground ring R is a field), Proposition 6.2 implies

$$\mathrm{tr}(H_*(f)) = H_*(\mathrm{tr}(f))$$

for any map of chain complexes $f\colon M_* \to M_*$. In other words, the Lefschetz number is the same whether it is calculated at the level of chain complexes or homology.

Example 6.5. By composing the rational cellular chain complex functor with a functorial CW approximation, we obtain a functor $\mathbf{Top} \to \mathbf{Ch}_{\mathbb{Q}}$. In fact, this functor factors through $\mathbf{Sp}$ via a similar construction of CW spectra, and we have moreover an induced functor $\mathrm{Ho}(\mathbf{Sp}) \to \mathrm{Ho}(\mathbf{Ch}_{\mathbb{Q}})$ which is strong symmetric monoidal. It follows that the fixed point index of a continuous map is equal to the Lefschetz number of the induced map on chain complexes. Combining this with the previous example, and using rational coefficients so the Künneth theorem holds, we obtain the *Lefschetz fixed point theorem*: if $f\colon M \to M$ is a continuous map, where $\Sigma_+^\infty(M)$ is dualizable, and $\mathrm{tr}\, H_*(f, \mathbb{Q}) \neq 0$, then $\mathrm{tr}(f) \neq 0$, and thus f has a fixed point. This example was one of the original motivations for the abstract study of traces in [11].

Example 6.6. Generalizing the previous example, if $\Sigma_+^\infty(M)$ is dualizable in $\mathrm{Ho}(\mathbf{Sp})$ and $f\colon Q \times M \to M$ is a continuous map, then the trace of $\Sigma_+^\infty(f)$ in $\mathrm{Ho}(\mathbf{Sp})$ is a morphism $\Sigma_+^\infty(Q) \to S$ in $\mathrm{Ho}(\mathbf{Sp})$. We can then take the rational homology of this map to obtain a map $\mathrm{tr}(H_*(f_+))\colon H_*(Q_+) \to \mathbb{Z}$. On the other hand, we can also apply rational homology before taking the trace; this way we obtain a map

$$H_*(f_+)\colon H_*(Q_+) \otimes H_*(M_+) \to H_*(M_+)$$

whose trace is a map $H_*(Q_+) \to \mathbb{Z}$. Proposition 6.2 then shows

$$H_*(\mathrm{tr}\, f_+)) = \mathrm{tr}(H_*(f_+)).$$

When Q is a point, the set of morphisms $\Sigma_+^\infty(Q) \to S$ in $\mathrm{Ho}(\mathbf{Sp})$ and the set of morphisms $H_0(Q_+) \to \mathbb{Z}$ in $\mathbf{Ab}$ can both be identified with $\mathbb{Z}$, so no information about traces is lost by passage to rational homology. For general Q, information is lost, but this is not necessarily a bad thing: the set of maps $\Sigma_+^\infty(Q) \to S$ can

be difficult to calculate, while the set of maps $H_0(Q_+) \to \mathbb{Z}$ will usually be much easier to describe.

Example 6.7. An *n-dimensional topological field theory* [2] with values in a symmetric monoidal category **C** (such as $\mathbf{Vect}_k$) is a strong symmetric monoidal functor $Z: n\mathbf{Cob} \to \mathbf{C}$. Since every object M of $n\mathbf{Cob}$ is dualizable, so is each object $Z(M)$. Thus, the trace of a cobordism B from M to itself is mapped to an endomorphism of the unit of **C**, which can be regarded as an algebraic invariant of B computed by the field theory Z.

If $n = 1$, then $1\mathbf{Cob}$ is the free symmetric monoidal category on a dualizable object; thus a 1-dimensional TFT is just a dualizable object. Likewise, if $n = 2$, then $2\mathbf{Cob}$ is the free symmetric monoidal category on a Frobenius algebra; see, for instance, [24]. For a higher-dimensional generalization, see [26].

Finally, since monoidal categories form not just a category but a 2-category, it is natural to ask also how traces interact with monoidal natural transformations. Recall that if $F, G: \mathbf{C} \to \mathbf{D}$ are lax symmetric monoidal functors, a **monoidal natural transformation** is a natural transformation $\alpha: F \to G$ which is compatible with the monoidal constraints of F and G in an evident way.

Proposition 6.8. *Let $F, G: \mathbf{C} \to \mathbf{D}$ be normal lax symmetric monoidal functors, let $\alpha: F \to G$ be a monoidal natural transformation, let M be dualizable in $\mathbf{C}$, and assume that F and G preserve its dual $M^\star$. Then $\alpha_M: F(M) \to G(M)$ is an isomorphism, and for any $f: Q \otimes M \to M \otimes P$, the square*

$$
\begin{array}{ccc}
F(Q) & \xrightarrow{\mathrm{tr}\left(\mathfrak{c} \circ F(f) \circ \mathfrak{c}^{-1}\right)} & F(P) \\
\alpha_Q \downarrow & & \downarrow \alpha_P \\
G(Q) & \xrightarrow[\mathrm{tr}\left(\mathfrak{c} \circ G(f) \circ \mathfrak{c}^{-1}\right)]{} & G(P)
\end{array}
$$

commutes. In particular, for an endomorphism $f: M \to M$, we have

$$\mathrm{tr}(F(f)) = \mathrm{tr}(G(f)).$$

Proof. Since $F(M)$ and $G(M)$ have duals $F(M^\star)$ and $G(M^\star)$ respectively, the morphism $\alpha_{M^\star}: F(M^\star) \to G(M^\star)$ has a dual $(\alpha_{M^\star})^\star: G(M) \to F(M)$. A diagram chase (see [6, Prop. 6]) shows that this is an inverse to α_M. Then since $G(f) = \alpha_M \circ (F(f)) \circ (\alpha_M)^{-1}$, cyclicity implies that $\mathrm{tr}(F(f)) = \mathrm{tr}(G(f))$. $\square$

Remark 6.9. In particular, if **C** is compact closed and $F, G: \mathbf{C} \to \mathbf{D}$ are strong monoidal, then any monoidal transformation $F \to G$ is an isomorphism. Thus, that when we say $1\mathbf{Cob}$ is the free symmetric monoidal category on a dualizable object, as in Example 6.7, we really mean that the category of strong monoidal functors $1\mathbf{Cob} \to \mathbf{D}$ is equivalent to the *groupoid* of dualizable objects in **D** and isomorphisms between them. This also generalizes to higher dimensions.

Proposition 6.8 implies some useful "comparisons between comparisons" of ways to compute traces.

Example 6.10. As in Example 6.4, since $\mathbb{Q}$ is a field, the Künneth theorem implies that the functor $H_*(-; \mathbb{Q})$ from $\mathrm{Ho}(\mathbf{Sp})$ to the category $\mathbf{GrVect}_{\mathbb{Q}}$ of graded $\mathbb{Q}$-vector spaces is strong symmetric monoidal. While integral homology $H_*(-; \mathbb{Z})$ is

not strong symmetric monoidal, the Künneth theorem implies that it becomes so if we quotient by torsion; thus $H_*(-;\mathbb{Z})/\text{Torsion}$ is a strong symmetric monoidal functor from $\text{Ho}(\mathbf{Sp})$ to $\mathbf{GrMod}_\mathbb{Z}$. Hence we can compute Lefschetz numbers using integral homology as well, and it is natural to want to compare the two results.

Now as in Example 6.3, extension of scalars along the inclusion $\iota\colon \mathbb{Z} \to \mathbb{Q}$ defines a strong symmetric monoidal functor from $\mathbf{GrMod}_\mathbb{Z}$ to $\mathbf{GrVect}_\mathbb{Q}$. Thus we have two functors

$$H_*(-;\mathbb{Q}) \qquad \text{and} \qquad \big(H_*(-;\mathbb{Z})/\text{Torsion}\big) \otimes \mathbb{Q}$$

from $\text{Ho}(\mathbf{Sp})$ to $\mathbf{GrVect}_\mathbb{Q}$, and the same inclusion also defines a natural transformation

$$\alpha\colon \big(H_*(-;\mathbb{Z})/\text{Torsion}\big) \otimes \mathbb{Q} \longrightarrow H_*(-;\mathbb{Q}).$$

Therefore, we can combine Propositions 6.2 and 6.8 to compare the Lefschetz numbers computed using $H_*(-;\mathbb{Z})$ and $H_*(-;\mathbb{Q})$.

Explicitly, suppose $\Sigma_+^\infty(M)$ is dualizable and $f\colon M \to M$ is an endomorphism in $\text{Ho}(\mathbf{Sp})$. Then Proposition 6.8 implies that, first of all, α is an isomorphism

$$\big(H_*(M;\mathbb{Z})/\text{Torsion}\big) \otimes \mathbb{Q} \cong H_*(M;\mathbb{Q}),$$

and secondly, the trace of $\big(H_*(f;\mathbb{Z})/\text{Torsion}\big) \otimes \mathbb{Q}$ is the same as the trace of $H_*(f;\mathbb{Q})$. Since this trace is not twisted, the observation at the end of Example 6.3 implies

$$\text{tr}\left(\big(H_*(f;\mathbb{Z})/\text{Torsion}\big) \otimes \mathbb{Q}\right) = \iota\Big(\text{tr}\big(H_*(f;\mathbb{Z})/\text{Torsion}\big)\Big);$$

thus the Lefschetz number of f computed using $H_*(-;\mathbb{Z})/\text{Torsion}$ is the same as the Lefschetz number computed using $H_*(-;\mathbb{Q})$.

7. Vistas

The symmetric monoidal trace described in this paper can be generalized in various directions. We have already mentioned its generalizations to *balanced monoidal categories* (at the end of §3) and to *traced monoidal categories* (Definition 5.12). There are also straightforward generalizations to symmetric monoidal 2-categories and symmetric monoidal n-categories (modulo a definition of the latter).

Categorifying in a different direction, in [33] the first author introduced a general notion of trace for *bicategories*, regarded as "monoidal categories with many objects". This type of trace applies to *noncommutative* situations such as modules over a noncommutative ring, and has applications to refinements of the Lefschetz fixed point theorem which use versions of the *Reidemeister trace*. Bicategorical traces are studied further in [35], including a suitable notion of string diagram.

Finally, the forthcoming [34] deals with an abstract context that gives rise to both bicategories and symmetric monoidal categories (including parametrized spectra as a prime example), and the relationships of the traces therein.

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