Monopole Floer homology and Legendrian knots

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Abstract

We use monopole Floer homology for sutured manifolds to construct invariants of Legendrian knots in a contact 3-manifold. These invariants assign to a knot $K \subset Y$ elements of the monopole knot homology $KHM(-Y, K)$, and they strongly resemble the knot Floer homology invariants of Lisca, Ozsváth, Stipsicz, and Szabó. We prove several vanishing results, investigate their behavior under contact surgeries, and use this to construct many examples of non-loose knots in overtwisted 3-manifolds. We also show that these invariants are functorial with respect to Lagrangian concordance.

1 Introduction

A knot $\mathcal{K}$ in a contact 3-manifold (Y, ξ) is said to be *Legendrian* if the tangent vectors to $\mathcal{K}$ lie in the contact planes ξ. In recent years, a variety of invariants have been constructed to distinguish Legendrian knots which are topologically identical. Notable examples include contact homology [11], especially the combinatorial version due to Chekanov [5], and invariant elements of knot Floer homology constructed using either grid diagrams [36] or open book decompositions [30]; the last of these is due to Lisca, Ozsváth, Stipsicz, and Szabó and is thus often called the "LOSS invariant."

In order to construct a knot invariant from monopole Floer homology, Kronheimer and Mrowka defined a monopole version of Juhász's sutured Floer homology [22] and declared the monopole knot homology $KHM(Y, K)$ to be the sutured invariant of the complement of K. It is natural to ask whether the LOSS invariant can be defined in this setting, where the construction makes no use of Heegaard diagrams or open books but instead proceeds by embedding the knot complement in a closed 3-manifold $\bar{Y}$ and computing $\widecheck{HM}(\bar{Y})$ in certain Spin^c-structures.

The goal of this paper is to present such a invariant. Namely, to any Legendrian knot $\mathcal{K} \subset (Y, \xi)$ of topological type K, we associate elements

$$\ell_g(\mathcal{K}) \in KHM(-Y, K)$$

in monopole knot homology with local coefficients, which are invariant up to automorphisms of KHM, for all integers $g \geq 2$. (Conjecturally these do not depend on g, so for convenience we shall omit it throughout this introduction and the reader may fix any choice of g.) These elements are obtained by choosing a particular contact structure $\bar{\xi}$ on the closed manifold $\bar{Y}$, so that $(Y-\mathcal{K}, \xi|_{Y-\mathcal{K}})$ is a contact submanifold of $(\bar{Y}, \bar{\xi})$, and letting $\ell_g(\mathcal{K})$ be the monopole contact invariant of $\bar{\xi}$.

The construction of $\ell(\mathcal{K})$ presents some advantages and disadvantages over that of the LOSS invariant. It is hard to compute in general, and it does not come with a natural bigrading the way elements of knot Floer homology do. However, some vanishing and nonvanishing results have very simple proofs, as do several theorems involving contact surgery. For example:

Proposition 4.1. *If the complement of $\mathcal{K}$ is overtwisted, then $\ell(\mathcal{K}) = 0$.*

Proposition 4.3. *Let $S_+(\mathcal{K})$ and $S_-(\mathcal{K})$ denote the positive and negative stabilizations of a Legendrian knot $\mathcal{K}$. Then $\ell(S_+S_-(\mathcal{K})) = 0$ for all $\mathcal{K}$.*

We expect something stronger to hold, namely that $\ell(S_-(\mathcal{K})) = \ell(\mathcal{K})$ and $\ell(S_+(\mathcal{K})) = 0$, since the analogous statements are true for the LOSS invariant. (See Conjecture 4.2.)

Theorem 5.1. *Let $\mathcal{K}, \mathcal{S} \subset (Y, \xi)$ be disjoint, and let $\mathcal{K}_{\mathcal{S}} \subset Y_{\mathcal{S}}$ denote the image of $\mathcal{K}$ in the contact manifold $Y_{\mathcal{S}}$ obtained by performing a contact $(+1)$-surgery on $\mathcal{S}$. Then there is a map*

$$KHM(-Y, K) \to KHM(-Y_S, K_S)$$

sending $\ell(\mathcal{K})$ to $\ell(\mathcal{K}_{\mathcal{S}})$.

These results are all known to be true for the LOSS invariant, as are several consequences we will pursue in this paper. However, using work of Mrowka and Rollin [33, 32] on the monopole contact invariant which is not known to be true in Heegaard Floer homology, we can investigate one entirely new property of $\ell(\mathcal{K})$: its behavior under Lagrangian concordance [4].

Theorem 6.3. *Let $\mathcal{K}_0, \mathcal{K}_1 \subset (Y, \xi)$ be Legendrian knots, with Y a homology 3-sphere, and suppose that $\mathcal{K}_0$ is Lagrangian concordant to $\mathcal{K}_1$. Then there is a map*

$$KHM(-Y, K_1) \to KHM(-Y, K_0)$$

such that $\ell(\mathcal{K}_1) \mapsto \ell(\mathcal{K}_0)$.

The organization of this paper is as follows. In section 2 we review the necessary background on sutured monopole homology and the monopole contact invariant. We construct $\ell_g(\mathcal{K})$, prove its invariance, and compute it for Legendrian unknots in section 3, and prove the vanishing theorems mentioned above in section 4. In section 5 we investigate the effect of contact $(+1)$-surgery on $\ell(\mathcal{K})$, and apply this to prove some nonvanishing results and to construct many examples of non-loose knots in overtwisted contact manifolds. Finally, in section 6 we discuss the behavior of $\ell(\mathcal{K})$ with respect to Lagrangian concordance.

Throughout this paper we will adopt the convention that letters in the standard math font, such as K, refer to topological knots, whereas the same letters in a script font, such as $\mathcal{K}$, refer to Legendrian representatives of those knot types. We also remark that Lekili [29] has shown that one can replace $\widecheck{HM}$ with HF^+ in the Kronheimer-Mrowka construction of sutured monopole homology in order to recover sutured Floer homology. Thus the reader can apply the constructions in this paper to obtain a similar Legendrian invariant in knot Floer homology, and everything in this paper will still hold except the Lagrangian concordance results of section 6. In this sense we conjecture that $\ell(\mathcal{K})$ is identical to the LOSS invariant.

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2 Sutured manifolds and contact invariants in monopole Floer homology

2.1 The definition of SHM

For background on monopole Floer homology we refer to [27].

Let (M, γ) be a balanced sutured manifold. Kronheimer and Mrowka [28] defined the monopole Floer homology of (M, γ) as follows:

1. Choose an oriented, connected surface T such that the components of ∂T are in one-to-one correspondence with the components of γ. Form the product sutured manifold $(T \times I, \delta)$, where $I = [-1, 1]$, with annuli $A(\delta) = \partial T \times I$ and $R_\pm(\delta) = T \times \{\pm 1\}$.

2. Glue the annuli $A(\delta)$ to $A(\gamma)$ by some orientation-reversing map $A(\delta) \to A(\gamma)$ sending $\partial R_+(\delta)$ to $\partial R_+(\gamma)$. The resulting 3-manifold should have boundary $\bar{R}_+ \cup \bar{R}_-$ for some connected, closed, orientable surfaces $\bar{R}_\pm = R_\pm(\gamma) \cup R_\pm(\delta)$.

3. Form a closed manifold $\bar{Y}$ by gluing the boundary along some diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$, and let $\bar{R} \subset \bar{Y}$ be the image of $\bar{R}_\pm$.

We require that $\bar{R}$ has genus at least 2, and that T contains a simple closed curve c such that $c \times \{\pm 1\}$ is a non-separating curve in $\bar{R}_\pm$.

Definition 2.1. The sutured monopole homology of (M, γ) is defined as

$$SHM(M, \gamma) = \widecheck{HM}_\bullet(\bar{Y} | \bar{R}),$$

where $\widecheck{HM}_\bullet(\bar{Y}|\bar{R})$ is the direct sum of $\widecheck{HM}_\bullet(\bar{Y},\mathfrak{s})$ over all Spin^c structures $\mathfrak{s}$ satisfying $\langle c_1(\mathfrak{s}), \bar{R}\rangle = 2g(\bar{R}) - 2$.

Note that since $g(\bar{R}) \geq 2$, the class $c_1(\mathfrak{s})$ cannot be torsion if $\widecheck{HM}_\bullet(\bar{Y},\mathfrak{s})$ contributes to $\widecheck{HM}_\bullet(\bar{Y}|\bar{R})$; but then $\overline{HM}(Y,\mathfrak{s}) = 0$, so $\widecheck{HM}_\bullet(\bar{Y},\mathfrak{s})$ and $\widehat{HM}_\bullet(\bar{Y},\mathfrak{s})$ are canonically isomorphic. In [28] the authors therefore simply write $HM(\bar{Y}|\bar{R})$, but we will prefer to leave the "to" decoration in place as a reminder that we will be using the contact invariant associated to $\widecheck{HM}$.

We can also define SHM using local coefficients. Let $\mathcal{R}$ be a ring with exponential map $\exp : \mathbb{R} \to \mathcal{R}^\times$ and write $t^n = \exp(n)$ for convenience. To any smooth 1-cycle η in $\bar{Y}$ we can associate a local system Γ_η on the Seiberg-Witten configuration space $\mathcal{B}(\bar{Y},\mathfrak{s})$ whose fiber at any point is $\mathcal{R}$ and which assigns to any path $z : [0,1] \to \mathcal{B}(\bar{Y},\mathfrak{s})$ the multiplication map by $t^{r(z)}$, where

$$r(z) = \frac{i}{2\pi}\int_{[0,1]\times\eta} \mathrm{tr}(F_{A_z})$$

for A_z the connection on $[0,1] \times \bar{Y}$ arising from the path z.

Suppose that the diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ restricts to an orientation-preserving homeomorphism $c \times \{1\} \to c \times \{-1\}$, resulting in a curve $\bar{c} \subset \bar{Y}$. If η is taken to be a curve dual to $\bar{c}$, in the sense that $\bar{c}\cdot\eta = 1$, then we can define $SHM(M,\gamma;\Gamma_\eta) = \widecheck{HM}_\bullet(\bar{Y}|\bar{R};\Gamma_\eta)$. As in the case without local coefficients, if $t - t^{-1}$ is invertible in $\mathcal{R}$ then the authors simply write $HM(\bar{Y}|\bar{R};\Gamma_\eta)$ without any ambiguity but we will continue to use $\widecheck{HM}$.

Proposition 2.2 ([28])**.** *If $t - t^{-1}$ is invertible in $\mathcal{R}$, then $SHM(M,\gamma;\Gamma_\eta)$ depends only on (M,γ) and $\mathcal{R}$. In this case we can allow $\bar{R}$ to have genus 1, but if $g(\bar{R}) \geq 2$ and $\mathcal{R}$ has no $\mathbb{Z}$-torsion then we also have*

$$SHM(M,\gamma;\Gamma_\eta) \cong SHM(M,\gamma) \otimes \mathcal{R}.$$

2.2 SHM with coefficients in $\mathbb{Z}/2\mathbb{Z}$

Throughout [28] the authors work with coefficients (both local and otherwise) in $\mathbb{Z}$; however, we assert that SHM is still an invariant if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ is used instead. When using systems of local coefficients Γ_η over $\mathbb{F}$, we drop the condition that the ring $\mathcal{R}$ have no $\mathbb{Z}$-torsion and thus only require that $t - t^{-1} \in \mathcal{R}$ be invertible. This will allow us to pursue several applications involving surgery exact triangles, which are known to work with local coefficients over $\mathbb{F}$ (see [25] or [27, Chapter 42]) but which have not yet been proved with local coefficients over $\mathbb{Z}$.

The proofs of the invariance theorems in [28], Theorem 4.4 and Proposition 4.6, rely on several facts, most notably the excision theorems, Theorems 3.1 through 3.3, which still apply verbatim. We need to verify that a handful of important proofs still work, and in each case the only step requiring some additional care is the vanishing of a Tor group coming from an application of the Künneth theorem:

Corollary ([28, Corollary 3.4]). *Let $\Sigma \subset Y$ be a closed, oriented surface, and let η be a 1-cycle supported in Σ. If every component of Σ has genus at least 2, then*

$$\widetilde{HM}(Y|\Sigma;\Gamma_\eta) \cong \widetilde{HM}(Y|\Sigma) \otimes \mathcal{R}.$$

Proof. The only detail requiring care in the original proof is the map (14), denoted

$$HM_\bullet(Y_1|\Sigma_1;\Gamma_{\eta_0}) \otimes HM_\bullet(Y_2|\Sigma_2) \to HM_\bullet(\tilde{Y}|\tilde{\Sigma};\Gamma_{\eta_0}),$$

which comes from an application of the Künneth theorem and is expected to be an isomorphism. The cokernel of this map is

$$\mathrm{Tor}_{\mathbb{F}}(HM_\bullet(Y_1|\Sigma_1;\Gamma_{\eta_0}), HM_\bullet(Y_2|\Sigma_2)),$$

which is zero since $HM_\bullet(Y_1|\Sigma_1;\Gamma_{\eta_0}) = \mathcal{R}$ is a free $\mathbb{F}$-module, so the rest of the proof still applies. □

Lemma ([28, Lemma 4.7]). *Let Y be fibered over S^1 with closed fiber R of genus at least 2. Then $\widetilde{HM}(Y|R) \cong \mathbb{F}$.*

Proof. As before, if Y_h is the mapping torus of $h : R \to R$ then $\widetilde{HM}(Y_h|R) \cong \widetilde{HM}(Y_{h^{-1}}|R)$ and so the excision theorem applied to $Y_h \sqcup Y_{h^{-1}}$ gives an injective map $\widetilde{HM}(Y_h|R) \otimes_{\mathbb{F}} \widetilde{HM}(Y_h|R) \to \mathbb{F}$ with cokernel

$$\mathrm{Tor}_{\mathbb{F}}(\widetilde{HM}(Y_h|R), \widetilde{HM}(Y_h|R)).$$

Since $\widetilde{HM}(Y_h|R)$ is a free $\mathbb{F}$-module, this Tor term vanishes and the map is an isomorphism, hence $\widetilde{HM}(Y_h|R) \cong \mathbb{F}$. □

Corollary ([28, Corollary 4.8]). *The sutured homology group $SHM(M,\gamma)$ does not depend on the choice of gluing homeomorphism h.*

Proof. This is another application of the excision theorem, Theorem 3.1, to a disconnected manifold $Y = Y_1 \sqcup Y_2$ with Y_2 a mapping torus, hence $\widetilde{HM}(Y_2|\Sigma_2) \cong \mathbb{F}$ and again the proof is the same once we observe that

$$\mathrm{Tor}_{\mathbb{F}}(\widetilde{HM}(Y_1|\Sigma_1), \widetilde{HM}(Y_2|\Sigma_2)) \cong 0.$$

□

Proposition ([28, Proposition 4.10]). *If $t - t^{-1}$ is invertible in $\mathcal{R}$, then $SHM(M,\gamma;\Gamma_\eta)$ is independent of the genus g.*

Proof. Here we wish to show that

$$\widetilde{HM}(Y_1|\bar{R}_1;\Gamma_{\eta_1}) \cong \widetilde{HM}((Y_1 \sqcup Y_2)|(\bar{R}_1 \sqcup \bar{R}_2);\Gamma_\eta)$$

where $\eta = \eta_1 + \eta_2$ for some cycles $\eta_i \subset \bar{R}_i \subset Y_i$, and we know that $\widetilde{HM}(Y_2|\bar{R}_2;\Gamma_{\eta_2}) = \mathcal{R}$. The Künneth theorem thus gives a map

$$\widetilde{HM}(Y_1|\bar{R}_1;\Gamma_{\eta_1}) \otimes_{\mathcal{R}} \mathcal{R} \to \widetilde{HM}((Y_1 \sqcup Y_2)|(\bar{R}_1 \sqcup \bar{R}_2);\Gamma_\eta)$$

with cokernel

$$\mathrm{Tor}_{\mathcal{R}}(\widetilde{HM}(Y_1|\bar{R}_1;\Gamma_{\eta_1}),\mathcal{R}),$$

and since $\mathcal{R}$ is free as an $\mathcal{R}$-module, the Tor term vanishes and this is indeed an isomorphism. □

We conclude that both the standard and local versions of sutured monopole homology are invariants if we work over $\mathbb{F}$ rather than $\mathbb{Z}$.

2.3 Monopole knot homology

Given a knot K in a closed, oriented 3-manifold Y, we can form a sutured manifold $Y(K) = (M,\gamma)$ as in [22] by taking M to be the knot complement $Y\backslash N(K)$ and $\gamma \subset \partial M$ a pair of oppositely oriented meridians. Then monopole knot homology is defined by

$$KHM(Y,K) = SHM(M,\gamma),$$

and if we work with local coefficients we get $KHM(Y,K)\otimes\mathcal{R} \cong SHM(M,\gamma;\Gamma_\eta)$.

From now on we will fix $\mathcal{R}$ to be the Novikov ring

$$\left\{ \sum_\alpha c_\alpha t^\alpha \,\middle|\, \alpha\in\mathbb{R},\ c_\alpha\in\mathbb{F},\ \#\{\beta<n \mid c_\beta\neq 0\}<\infty \text{ for all } n \right\},$$

with $\exp(\alpha) = t^\alpha$ and $(t-t^{-1})^{-1} = -t-t^3-t^5-\dots$. Although we may drop the local system Γ_η from our notation, we are always working with local coefficients over $\mathcal{R}$.

2.4 Contact structures in monopole Floer homology

Let (Y,ξ) be a closed contact 3-manifold. Kronheimer and Mrowka [26] associate a contact invariant

$$\psi(\xi) : \Lambda(\xi) \to \widetilde{HM}_\bullet(-Y,\mathfrak{s}_\xi,c_{\mathrm{bal}},\Gamma_\eta)$$

where $c_{\mathrm{bal}} = 2\pi c_1(\mathfrak{s}_\xi)$ is a balanced perturbation of the Seiberg-Witten equations and $\Lambda(\xi)$ is the set of orientations of an appropriate moduli space. In general we will ignore the orientations $\Lambda(\xi)$, since we are working in characteristic 2, and so we will write $\psi(\xi)\in\widetilde{HM}_\bullet(-Y,\mathfrak{s}_\xi,c_{\mathrm{bal}},\Gamma_\eta)$.

Mrowka and Rollin [33, 32] investigated the behavior of the contact invariant under symplectic cobordisms.

Definition 2.3. A symplectic cobordism (W,ω) from (Y_-,ξ_-) to (Y_+,ξ_+) is said to be *left-exact* if ω is exact near Y_-, or equivalently if it is given in a collar neighborhood of Y_- by a symplectization $\frac{1}{2}d(t^2\eta_-)$ where $\xi_- = \ker\eta_-$. It is *right-exact* if the same holds near Y_+, and *boundary-exact* if it is both left- and right-exact.

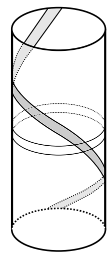


Figure 1: The convex torus $\partial(Y\backslash N(\mathcal{K}))$, cut along a meridian. The horizontal circles are sutures, while the pair of parallel arcs (or circles, once the top and bottom are identified) are dividing curves and have slope $\frac{1}{tb(\mathcal{K})}$.

Theorem 2.4 ([32, Theorem 3.5.4])**.** *Let W be a boundary-exact cobordism (W,ω) as above such that the map*

$$H^1(W;\mathbb{Z}) \to H^1(Y_+;\mathbb{Z})$$

is surjective, and let $W^\dagger$ denote W viewed as a cobordism from $-Y_+$ to $-Y_-$. Then $\psi(\xi_-) = \widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)(\psi(\xi_+))$.

Corollary 2.5. *If W is a symplectic 2-handle cobordism corresponding to Legendrian surgery, then*

$$\psi(\xi_-) = \widetilde{HM}(W^\dagger)\psi(\xi_+)$$

and in particular we can replace the map $\widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)$ of Theorem 2.4 by the total map $\widetilde{HM}(W^\dagger)$.

3 The Legendrian knot invariant

Let $\mathcal{K} \subset (Y,\xi)$ be a Legendrian knot of topological knot type K. Our goal is to construct an appropriate contact structure $\bar{\xi}$ on a closure $(\bar{Y},\bar{R})$ of the sutured knot complement $Y(K)$ so that the contact invariant $\psi(\bar{\xi})$ does not depend on any of the choices we must make. This will give us an invariant $\ell(\mathcal{K})$ of the Legendrian knot $\mathcal{K}$ which is an element of $KHM(-Y,K)$ up to automorphism.

Take a standard neighborhood $N(\mathcal{K})$ whose boundary is a convex torus. If we assign coordinates to $\partial N(\mathcal{K}) \cong \mathbb{R}^2/\mathbb{Z}^2$ so that $(\pm 1, 0)$ is a meridian and $(0,\pm 1)$ a longitude, then its dividing set Γ consists of two parallel curves of slope $\frac{1}{tb(\mathcal{K})}$, where $tb(\mathcal{K})$ is the Thurston-Bennequin invariant of $\mathcal{K}$. (See for example [16, Section 2.4].) In particular, if we view the sutured knot complement $Y(\mathcal{K})$ as the contact manifold $(Y\backslash N(\mathcal{K}), \xi|_{Y\backslash N(\mathcal{K})})$ with convex boundary, then each of the meridional sutures will intersect each dividing curve transversely in a single point as in Figure 1. Here, and in all other figures, we will color regions white and gray to represent the positive and negative regions, respectively, of a convex surface.

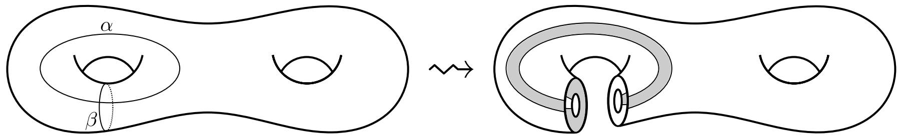


Figure 2: The construction of the auxiliary surface $(T \times I, \Xi)$.

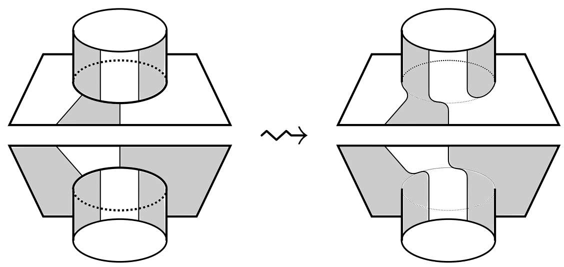


Figure 3: Gluing $T \times I$ to $Y \backslash N(\mathcal{K})$ and rounding edges in a cylindrical neighborhood of one of the sutures on $\partial N(\mathcal{K})$, as viewed from $T \times \{1\}$ on top and $T \times \{-1\}$ on the bottom.

3.1 Closure of the sutured knot complement

Our construction follows the definition of the sutured invariant as in Section 2. We must first pick an auxiliary surface T whose boundary components are in one-to-one correspondence with the sutures of $Y(\mathcal{K})$ and glue the annuli $\partial T \times I \subset T \times I$ to neighborhoods $A(\gamma)$ of the sutures. In order to form a contact structure on this glued manifold, we must assign a contact structure to $T \times I$ whose restriction to a neighborhood of $\partial T \times I$ agrees with ξ in a neighborhood of $A(\gamma)$. By Giroux's flexibility theorem [20] it suffices to ensure that the dividing curves match, sending the positive region of $A(\gamma)$ to the negative region of $\partial T \times I$ and vice versa.

Let T_0 be a closed surface of genus at least 2, and pick a pair of dual curves $\alpha, \beta \subset T_0$ such that $\alpha \cdot \beta = 1$. Give $T_0 \times I$ the I-invariant contact structure Ξ_α whose dividing curves consist of two parallel disjoint copies of α on each surface $T_0 \times \{\pm 1\}$ and for which the negative region of $T_0 \times \{+1\}$ is an annulus. We define $(T \times I, \Xi)$ as the contact manifold obtained by cutting $T_0 \times I$ along a convex perturbation of the annulus $\beta \times I$, as in Figure 2; it may also be viewed as a product sutured manifold with sutures $\delta = \partial T \times \{0\}$.

We now form a contact manifold $(Y', \xi') = (Y \backslash N(\mathcal{K})) \cup (T \times I)$ by gluing along some orientation-reversing diffeomorphism $A(\delta) \to A(\gamma)$ as described above. This manifold has edges, corresponding to the corners $\partial T \times \partial I$, which we smooth using edge-rounding [21], under which dividing curves turn to the left (as viewed from outside Y') as they approach an edge. See Figure 3.

Lemma 3.1. *The contact manifold $Y' = (Y \backslash N(\mathcal{K})) \cup (T \times I)$ depends only on $\mathcal{K}$, (Y, ξ), and the genus of T_0.*

Proof. The construction of $T \times I$ depends only on $g(T_0)$ and on the curves $\alpha, \beta \subset T_0$. Given any other pair of curves α' and β' which intersect once, there is a diffeomorphism $\varphi : T_0 \to T_0$ with $\varphi(\alpha) = \alpha'$ and $\varphi(\beta) = \varphi(\beta')$, and this extends to a contactomorphism $\varphi \times Id : (T_0 \backslash N(\beta)) \times I \to (T_0 \backslash N(\beta')) \times I$. $\square$

Finally, we close up Y' to get a contact manifold $(\bar{Y}, \bar{\xi})$ with distinguished convex surface $\bar{R}$ of genus $g \geq 2$. The boundary of Y' consists of two convex surfaces $\bar{R}_+$ and $\bar{R}_-$ determined by $T \times \{\pm 1\} \subset \bar{R}_\pm$. These are split by pairs of parallel dividing curves $\Gamma_\pm \subset \bar{R}_\pm$ into positive and negative regions $(\bar{R}_+)_\pm \subset \bar{R}_+$ and $(\bar{R}_-)_\pm \subset \bar{R}_-$, and each of $(\bar{R}_+)_-$ and $(\bar{R}_-)_+$ is an annulus. Fix any diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ which sends $(\bar{R}_+)_\pm$ to $(\bar{R}_-)_\mp$, and hence also Γ_+ to Γ_-, and such that $h(x \times \{1\})$ and $x \times \{-1\}$ lie in the same component of Γ_- for any $x \times \{1\}$ in $\Gamma_+ \cap (\text{int}(T) \times \{1\})$. In other words, a dividing curve $c \subset \Gamma_+$ corresponds to one of the two copies of $\alpha \subset T_0$, and we require $h(c)$ to be the dividing curve of Γ_- corresponding to the same copy of α.

We glue $\bar{R}_+$ to $\bar{R}_-$ via h. The resulting contact manifold is the desired $(\bar{Y}, \bar{\xi})$.

Definition 3.2. The Legendrian invariant of $\mathcal{K}$ is defined as $\ell(\mathcal{K}) = \psi(\bar{Y}, \bar{\xi}) \in \widetilde{HM}(-\bar{Y}, \mathfrak{s}_{\bar{\xi}}; \Gamma_\eta)$.

We can compute that

$$\langle c_1(\bar{\xi}), \bar{R} \rangle_{\bar{Y}} = \chi((\bar{R}_+)_+) - \chi((\bar{R}_+)_-) = 2 - 2g(\bar{R})$$

and so $\langle c_1(\mathfrak{s}_{\bar{\xi}}), \bar{R} \rangle_{-\bar{Y}} = 2g(\bar{R}) - 2$. This means that $\ell(\mathcal{K})$ is in fact an element of $\widetilde{HM}(-\bar{Y}|\bar{R}; \Gamma_\eta) = SHM(-Y(K); \Gamma_\eta)$, which is by definition the knot homology with local coefficients. Therefore

$$\ell(\mathcal{K}) \in KHM(-Y, K) \otimes \mathcal{R}.$$

Remark 3.3. The desire to arrange that $\langle c_1(\bar{\xi}), \bar{R} \rangle = 2 - 2g$ motivated our choice of contact structure on $T \times I$. In particular, $\chi((\bar{R}_+)_+)$ and $\chi((\bar{R}_+)_-)$ sum to $\chi(\bar{R}_+) = 2 - 2g$, and if we fix their difference as above then we must have $\chi((\bar{R}_+)_-) = 0$. But now $(\bar{R}_+)_-$ does not have any sphere or torus components, and if it had disk components then $\bar{R}_+$ would not have a tight neighborhood [20], so $(\bar{R}_+)_-$ is forced to be a union of annuli.

In addition to the Legendrian knot $\mathcal{K} \subset (Y, \xi)$, the construction of $\ell(\mathcal{K})$ from a closure $(\bar{Y}, \bar{\xi})$ with distinguished convex surface $\bar{R}$ potentially depends on both the genus $g = g(\bar{R})$ and the choice of diffeomorphism $\bar{R}_+ \to \bar{R}_-$. Our goal in the next section is to prove that in fact it is independent of the diffeomorphism.

3.2 Invariance under diffeomorphism

In this section we establish that $\ell(\mathcal{K})$ is independent of the choice of diffeomorphism $\bar{R}_+ \to \bar{R}_-$.

Proposition 3.4. *Let $(\bar{Y}', \bar{\xi}')$ be the contact manifold obtained from $\bar{Y}$ by cutting along the convex surface $\bar{R}$ and regluing along some orientation-preserving diffeomorphism h such that $h(\gamma) = \gamma$ for each dividing curve γ of $\bar{R}$. Then there is an isomorphism $\widetilde{HM}(-\bar{Y}'|\bar{R};\Gamma_\eta) \to \widetilde{HM}(-\bar{Y}|\bar{R};\Gamma_\eta)$ which sends $\psi(\bar{Y}', \bar{\xi}')$ to $\psi(\bar{Y}, \bar{\xi})$.*

Lemma 3.5. *Proposition 3.4 holds when h is a Dehn twist along some nonseparating curve c which does not intersect the dividing curves Γ of $\bar{R}$.*

Proof. We observe that c is nonisolating, i.e. that every component of $\bar{R}\backslash(\Gamma \cup c)$ has a boundary component which intersects Γ, and thus by the Legendrian Realization Principle [24, 21] we can take c to be Legendrian. Indeed, the complement $\bar{R}\backslash\Gamma$ has two connected components; if c is nonseparating within its component then it is clearly nonisolating. Otherwise, c divides its component of $\bar{R}\backslash\Gamma$ into two components, say A and B. Since c is nonseparating in $\bar{R}$ there is a path in $\bar{R}\backslash c$ which connects A to B, and this path must pass through the other component of $\bar{R}\backslash\Gamma$. In particular, the path crosses Γ, so both ∂A and ∂B intersect Γ and thus c is nonisolating.

Suppose now that h is a positive Dehn twist along c, and that c has been realized as a Legendrian curve. Then h can be realized by (-1)-surgery on c with respect to the framing induced by $\bar{R}$, and since $tw(c, \bar{R}) = -\frac{1}{2}|c \cap \Gamma| = 0$ this is a Legendrian surgery. If W is the corresponding symplectic cobordism, and $W^\dagger$ is the oppositely oriented cobordism from $-\bar{Y}'$ to $-\bar{Y}$, then

$$\widetilde{HM}(W^\dagger)(\psi(\bar{Y}', \bar{\xi}')) = \psi(\bar{Y}, \bar{\xi})$$

by Corollary 2.5. The fact that $\widetilde{HM}(W^\dagger)$ gives an isomorphism $\widetilde{HM}(-\bar{Y}'|\bar{R};\Gamma_\eta) \to \widetilde{HM}(-\bar{Y}|\bar{R};\Gamma_\eta)$ is an easy consequence of the surgery exact triangle for $\widetilde{HM}$ and the fact that $\bar{R}$ becomes compressible in the manifold $-\bar{Y}_0$ obtained by 0-surgery along c, hence $\widetilde{HM}(-\bar{Y}_0|\bar{R}) = 0$ by the adjunction inequality [27].

If instead h is a negative Dehn twist, we note that $\bar{Y}$ can be obtained from $\bar{Y}$' by a positive Dehn twist along c, so we construct a cobordism W from $\bar{Y}'$ to $\bar{Y}$ as above and then $\widetilde{HM}(W^\dagger)^{-1}$ is the desired isomorphism. □

Proof of Proposition 3.4. In general, we can arrange by an isotopy that the diffeomorphism h is actually the identity on each dividing curve. Then h restricts to a boundary-fixing diffeomorphism on the closure of each component of $\bar{R}\backslash\Gamma$. One component is an annulus A, so up to isotopy $h|_A$ is a composition of Dehn twists about the core of A. The other component is a surface Σ of genus $g(\bar{R}) - 1 \geq 1$ with two boundary components, and so $h|_\Sigma$ can also be expressed as a product of Dehn twists about nonseparating curves which do not intersect $\Gamma = \partial\Sigma$. Since $h = h|_A \circ h|_\Sigma$, repeated application of Lemma 3.5 completes the proof of Proposition 3.4. □

We have now shown that the construction of $\ell(\mathcal{K}) \in KHM(-Y, K) \otimes \mathcal{R}$ is independent of all choices except possibly the genus $g = g(\bar{R})$. Thus we have constructed a sequence of Legendrian knot invariants $\ell_g(\mathcal{K})$ for $g \geq 2$.

Conjecture 3.6. *The elements $\ell_g(\mathcal{K})$, $g \geq 2$, are all equal as elements of $KHM(-Y, K) \otimes \mathcal{R}$ up to automorphism.*

Since we will show in section 3.3 that $\ell_g(\mathcal{U}) = 1 \in \mathcal{R}$ where $\mathcal{U} \subset (S^3, \xi_{\mathrm{std}})$ is the Legendrian unknot with $tb = -1$, this conjecture would follow from a connected sum formula of the form

$$\ell_g(\mathcal{K}) \otimes \ell_{g'}(\mathcal{K}') = \ell_{g+g'-1}(\mathcal{K}\#\mathcal{K}')$$

which we expect to be true by comparison with the LOSS invariant [30, Theorem 7.1].

From now on we will drop the g subscript where convenient and simply write $\ell(\mathcal{K})$ to mean $\ell_g(\mathcal{K})$ for some fixed $g \geq 2$.

3.3 The Legendrian unknot

The Legendrian representatives of the topological unknot $U \subset S^3$ were classified by Eliashberg and Fraser [12]: they are completely determined by their classical invariants tb and r, and there is a representative $\mathcal{U}$ with $(tb, r) = (-1, 0)$ so that all others are stabilizations of $\mathcal{U}$. In this subsection we will prove that the Legendrian invariant of $\mathcal{U}$ is a unit of $KHM(U) \otimes \mathcal{R} \cong \mathcal{R}$.

Our strategy is to explicitly determine the contact structure on a particular closure $\bar{Y}$ of $S^3(\mathcal{U})$.

Lemma 3.7. *Let ξ be the I-invariant contact structure on $(S^1 \times I) \times I$ whose dividing curves on the annulus $S^1 \times I$ are a pair of parallel arcs $\{t_1\} \times I$ and $\{t_2\} \times I$. Then after edge-rounding, ξ is contactomorphic to the complement of $\mathcal{U}$.*

Proof. By the classification of tight contact structures on solid tori [21, Theorem 2.3], there is a unique tight contact structure Ξ on $S^1 \times D^2$ for which the dividing curves on the boundary have slope -1; since $tb(\mathcal{U}) = -1$, the complement of $\mathcal{U}$ must be $(S^1 \times D^2, \Xi)$. But if we round the edges on $((S^1 \times I) \times I, \xi)$, we get a tight contact structure on $S^1 \times D^2$ for which the dividing curves on the boundary $S^1 \times S^1$ have slope -1, and so this contact structure must be Ξ as well. ☐

Proposition 3.8. *The invariant $\ell(\mathcal{U})$ is a unit in $KHM(U) \otimes \mathcal{R} \cong \mathcal{R}$.*

Proof. We glue a surface $T \times I$ to $(S^1 \times I) \times I$ as in Section 3.1, identifying the annuli $\partial T \times I$ with $(S^1 \times I) \times \partial I$, to get an I-invariant contact manifold $Y' = \Sigma_g \times I$ which is universally tight by Giroux's criterion [20] and has convex boundary. Gluing $\Sigma_g \times \{1\}$ to $\Sigma_g \times \{-1\}$ via the identity map, we get the closure $\bar{Y} = \Sigma_g \times S^1$ with S^1-invariant, universally tight contact structure $\bar{\xi}$ and distinguished surface $\bar{R} = \Sigma_g \times \{*\}$. Since no component of $\Gamma \subset \Sigma_g$ is separating, Theorem 5 of [34] asserts that $\bar{\xi}$ is weakly fillable.

The claim that $KHM(U) \otimes \mathcal{R} = \widecheck{HM}(\bar{Y}|\bar{R}; \Gamma_\eta) \cong \mathcal{R}$ now follows from Corollary 2.3 of [28]. Furthermore, since $\bar{\xi}$ is weakly fillable we know that

$\psi(\bar{\xi})$ is a unit of $\widetilde{HM}(\bar{Y};\Gamma_\eta)$ [26, 32], and since $\psi(\bar{\xi}) \in \widetilde{HM}(\bar{Y}|\bar{R};\Gamma_\eta) \cong \mathcal{R}$ the proposition follows. □

Remark 3.9. Wendl [40, Corollary 2] has shown that $(\bar{Y},\bar{\xi})$ has vanishing untwisted ECH contact invariant. By work of Taubes [39] it follows that the untwisted contact invariant $\psi(\bar{\xi}) \in \widetilde{HM}(\bar{Y}|\bar{R})$ is also zero, so we must work with local coefficients for $\ell(\mathcal{U})$ to be nonzero.

We can also compute $\ell(\mathcal{U}_Y)$ if $\mathcal{U}_Y$ is a Legendrian unknot in a Darboux ball of some contact manifold (Y,ξ). Observe that both $S^3(U)$ and $S^3(1)$ have $(\Sigma_g \times S^1, \Sigma_g \times \{*\})$ as a closure, where $M(1)$ denotes the complement of a ball in M with a single suture, and since $Y(U_Y) \cong Y\#S^3(U)$ we conclude that $KHM(Y,U_Y) \cong SHM(Y(1))$.

Let $\widetilde{HM}(Y) = SHM(Y(1)) \cong KHM(Y,U_Y)$. Then clearly $\widetilde{HM}$ is analogous to the hat version of Heegaard Floer homology, since $\widehat{HF}(Y) \cong SFH(Y(1))$ virtually by definition [22]. In fact, it is equivalent to define $\widetilde{HM}(Y)$ as the homology of the mapping cone of $\check{C}(Y) \xrightarrow{U_\dagger} \check{C}(Y)$ ([1]), just as $\widehat{HF}(Y)$ comes from the Heegaard Floer complex $CF^+(Y)$.

We will define a contact invariant $\tilde{\psi}_g(\xi) \in \widetilde{HM}(-Y)\otimes\mathcal{R}$ up to automorphism as $\ell_g(\mathcal{U}_Y)$. (Having noted that $\tilde{\psi}_g$ potentially depends on g just as ℓ_g does, we will similarly drop the subscript and write $\tilde{\psi}(\xi)$ from now on.) This seems to be a reasonable choice by analogy with the LOSS invariant $\hat{\mathcal{L}}(\mathcal{U}_Y) \in \widehat{HFK}(-Y,U_Y)$, which is identified with the Heegaard Floer contact invariant $\hat{c}(\xi) \in \widehat{HF}(-Y)$ as argued in the proof of [30, Corollary 7.3].

Proposition 3.10. *There is a map*

$$\widetilde{HM}(-Y) \otimes_{\mathbb{F}} \mathcal{R} \to \widetilde{HM}(-Y) \otimes_{\mathbb{F}} \mathcal{R}$$

which sends $\tilde{\psi}(\xi) = \ell(\mathcal{U}_Y)$ *to* $\psi(\xi) \otimes 1$.

Proof. Recall that $\mathcal{U} \subset (S^3,\xi_{\mathrm{std}})$ has closure $(\bar{Y},\bar{R}) = (\Sigma_g \times S^1, \Sigma_g \times \{*\})$ with S^1-invariant contact structure $\bar{\xi}$ and its homology is twisted by a 1-cycle $\eta \subset \bar{Y}$. Thus the Legendrian unknot $\mathcal{U}_Y \subset Y$ has closure $(Y\#\bar{Y},\bar{R})$.

Build a symplectic cobordism (W,ω) from $(Y,\xi) \sqcup (\bar{Y},\bar{\xi})$ to $(Y\#\bar{Y},\xi\#\bar{\xi})$ by attaching a symplectic 1-handle to the symplectization $(Y \sqcup \bar{Y}) \times I$. The induced map

$$\widetilde{HM}(W_\dagger,\mathfrak{s}_\omega) : \widetilde{HM}(-(Y\#\bar{Y}),\mathfrak{s}_\xi\#\mathfrak{s}_{\bar{\xi}};\Gamma_\eta) \to \widetilde{HM}(-Y \sqcup -\bar{Y},\mathfrak{s}_\xi \sqcup \mathfrak{s}_{\bar{\xi}};\Gamma_\eta)$$

sends $\ell(\mathcal{U}_Y) = \psi(\xi\#\bar{\xi}) \in \widetilde{HM}(-(Y\#\bar{Y});\Gamma_\eta)$ to $\psi(\xi \sqcup \bar{\xi})$ by Theorem 2.4. But the map

$$\widetilde{HM}(-Y,\mathfrak{s}_\xi) \otimes_{\mathbb{F}} \widetilde{HM}(-\bar{Y},\mathfrak{s}_{\bar{\xi}};\Gamma_\eta) \to \widetilde{HM}(-Y \sqcup -\bar{Y},\mathfrak{s}_\xi \sqcup \mathfrak{s}_{\bar{\xi}};\Gamma_\eta)$$

coming from the Künneth theorem is an isomorphism since $\widetilde{HM}(-\bar{Y},\mathfrak{s}_{\bar{\xi}};\Gamma_\eta) \cong \mathcal{R}$ is free, so in fact $\widetilde{HM}(W_\dagger,\mathfrak{s}_\omega)$ can be expressed as a map

$$\widetilde{HM}(-(Y\#\bar{Y}),\mathfrak{s}_\xi\#\mathfrak{s}_{\bar{\xi}};\Gamma_\eta) \to \widetilde{HM}(-Y,\mathfrak{s}_\xi) \otimes_{\mathbb{F}} \mathcal{R}$$

sending $\psi(\xi\#\bar{\xi})$ to $\psi(\xi)\otimes\psi(\bar{\xi}) = \psi(\xi)\otimes 1$. The source and target of this map are summands of $\widetilde{HM}(-(Y\#\bar{Y})|\bar{R};\Gamma_\eta) = \widetilde{HM}(-Y)\otimes\mathcal{R}$ and $\widetilde{HM}(-Y)\otimes\mathcal{R}$, respectively, and $\psi(\xi\#\bar{\xi})$ is $\ell(\mathcal{U}_Y)$, so we are done. □

Corollary 3.11. *If $\psi(\xi)\in\widetilde{HM}(-Y)$ is nonzero, then so is $\tilde{\psi}(\xi)\in\widetilde{HM}(-Y)\otimes\mathcal{R}$.*

For example, if ξ is strongly symplectically fillable then $\psi(\xi)$ is nonzero and primitive [26, 33], so Proposition 3.10 implies that $\tilde{\psi}(\xi)$ is a primitive element of $\widetilde{HM}(-Y)\otimes\mathcal{R}$.

4 Vanishing results

4.1 Loose knots

Recall that a Legendrian knot $\mathcal{K}\subset(Y,\xi)$ is said to be *loose* if the complement of $\mathcal{K}$ is overtwisted.

Proposition 4.1. *If $\mathcal{K}\subset Y$ is loose, then $\ell(\mathcal{K})=0$.*

Proof. By assumption $Y\backslash\mathcal{K}$ has an overtwisted disk, so any closure $(\bar{Y},\bar{\xi})$ does as well. Then $\psi(\bar{Y},\bar{\xi})$ vanishes (see [33, Corollary B]), hence $\ell(\mathcal{K})$ does as well. □

4.2 Stabilization

Let $S_+(\mathcal{K})$ and $S_-(\mathcal{K})$ denote the positive and negative stabilizations of a Legendrian knot $\mathcal{K}$, which may also be thought of as the connected sums $\mathcal{K}\#\mathcal{U}_\pm$ where $\mathcal{U}_\pm\subset S^3$ is the topologically trivial knot with $tb=-2$ and $r=\pm1$. We expect the following conjecture to be true:

Conjecture 4.2. *For any Legendrian knot $\mathcal{K}\subset Y$ we have $\ell(S_-(\mathcal{K}))=\ell(\mathcal{K})$ and $\ell(S_+(\mathcal{K}))=0$.*

A theorem of Epstein, Fuchs, and Meyer [13] characterizes transverse knots as pushoffs of Legendrian knots up to negative stabilization, and so ℓ would then give a transverse knot invariant as well.

We can prove a slightly weaker result than the desired $\ell(S_+(\mathcal{K}))=0$ of Conjecture 4.2.

Proposition 4.3. *If $\mathcal{K}$ is any Legendrian knot, then $\ell(S_+S_-(\mathcal{K}))=0$.*

Proof. We will construct a closure $\bar{Y}$ of $\mathcal{K}'=S_+S_-(\mathcal{K})$ with an overtwisted disk, so that the vanishing follows from Corollary B of [33]. Stabilization of a Legendrian knot $\mathcal{K}$ corresponds to attaching a bypass to its complement: if we stabilize to get $S_\pm(\mathcal{K})$ inside a standard neighborhood $N(\mathcal{K})\subset Y$ and fix a standard neighborhood $N(\mathcal{K}^\pm)\subset N(\mathcal{K})$, then $Y\backslash N(\mathcal{K}^\pm)$ is obtained from $Y\backslash N(\mathcal{K})$ by a bypass attachment. See [15] or the proof of Theorem 1.5 in [38] for discussion.

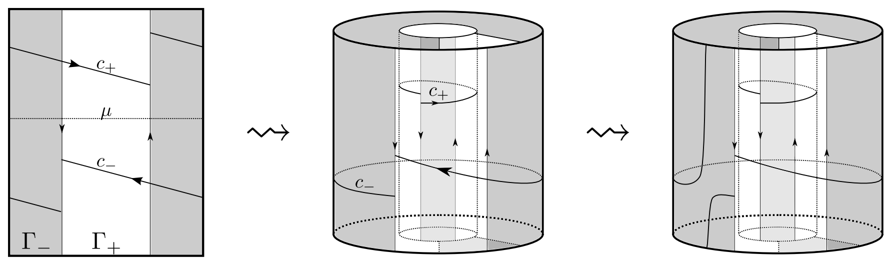


Figure 4: Attaching curves for bypasses in the complement of $S_+S_-(\mathcal{K})$ and its closure.

In the leftmost part of Figure 4 we have indicated the attaching arcs c_+ and c_- of bypasses corresponding to positive and negative stabilizations on $\partial(Y\backslash N(\mathcal{K}'))$, as in Figure 10 of [38], with a single meridional suture μ between them. The dividing curves are shown with orientation for convenience, so that they have the same orientation as the boundary $\partial\Gamma_+$ of the positive region. If we start to form the closure of $Y\backslash N(\mathcal{K}')$ by attaching a surface $T \times I$ to neighborhoods of the sutures and then rounding edges, we may then cut out $T\times I$ to get a contact manifold with corners as in the middle figure; this indicates the positions of the arcs $c_\pm$ on the boundary components $\bar{R}_\pm \subset Y'$.

We wish to glue $\bar{R}_+$ to $\bar{R}_-$ so that the arcs c_+ and c_- are glued together, but as shown in the middle of Figure 4 we cannot do this by identifying the inside and outside regions in the obvious way. Indeed, we must identify the white component $(\bar{R}_+)_+$ on the outside with the gray component $(\bar{R}_-)_-$ on the inside, identifying the left dividing curve on the outside with the left dividing curve on the inside and likewise for the right dividing curves, but then c_+ and c_- cannot be made parallel so that they end up identified. The problem is that as we follow them leftward and around the back of the cylinder from the leftmost dividing curves, the arc c_- ends "above" its starting point whereas c_+ ends "below" its starting point. However, we can glue c_+ to c_- by applying a Dehn twist to the outer gray annulus $(\bar{R}_+)_-$ along its core as shown on the right side of Figure 4. We can then "untwist" c_- by reparametrizing $(\bar{R}_+)_-$, sliding the lower endpoint of c_- downward along its dividing curve until it has nearly traversed the entire curve and lies just above the other endpoint; this allows us to identify it with $(\bar{R}_-)_+$ so that c_- is sent to c_+. Now we can glue $\bar{R}_+$ to $\bar{R}_-$ so that c_- and c_+ are identified, and the union of their respective bypasses is an overtwisted disk in the closure $(\bar{Y},\bar{\xi})$. We conclude that $\psi(\bar{Y},\bar{\xi})$, and hence $\ell(\mathcal{K}')$, is zero. □

5 Contact surgery

5.1 Behavior under contact $(+1)$-surgery

The following is a direct analog of Theorem 1.1 of [35], which concerns the LOSS invariant $\hat{\mathcal{L}}(\mathcal{K}) \in \widehat{HFK}(-Y, K)$ (or more generally $\mathcal{L}(\mathcal{K}) \in HFK^-(-Y, K)$) but is much harder to prove.

Theorem 5.1. *Let $\mathcal{K}$ and $\mathcal{S}$ be disjoint Legendrian knots in (Y, ξ), and let $(Y_{\mathcal{S}}, \xi_{\mathcal{S}})$ denote the contact manifold obtained by performing contact $(+1)$-surgery along $\mathcal{S}$. Let $\mathcal{K}_{\mathcal{S}}$ be the image of the Legendrian knot $\mathcal{K}$ in $Y_{\mathcal{S}}$. Then there is a map $KHM(-Y, K) \otimes \mathcal{R} \to KHM(-Y_{\mathcal{S}}, K_{\mathcal{S}}) \otimes \mathcal{R}$ such that $\ell(\mathcal{K}) \mapsto \ell(\mathcal{K}_{\mathcal{S}})$.*

Proof. We may obtain (Y, ξ) by performing contact (-1)-surgery on $\mathcal{S} \subset Y_{\mathcal{S}}$ (see [8, Proposition 8]). Since $\mathcal{S}$ and $\mathcal{K}$ are disjoint it is easy to see that we can fix a closure $\bar{Y}_{\mathcal{S}}$ of the complement $Y_{\mathcal{S}}(\mathcal{K}_{\mathcal{S}})$ so that contact (-1)-surgery on $\mathcal{S} \subset \bar{Y}_{\mathcal{S}}$ gives a closure $\bar{Y}$ of $Y(\mathcal{K})$, and the surface $\bar{R}$ and cycle $\eta \subset \bar{R}$ are the same in both closures. The Weinstein cobordism (W, ω) from $\bar{Y}_{\mathcal{S}}$ to $\bar{Y}$ coming from this handle attachment gives a map

$$\widetilde{HM}(W^\dagger) : \widetilde{HM}(-\bar{Y}; \Gamma_\eta) \to \widetilde{HM}(-\bar{Y}_{\mathcal{S}}; \Gamma_\eta)$$

carrying $\ell(\mathcal{K})$ to $\ell(\mathcal{K}_{\mathcal{S}})$ by Corollary 2.5, and $\widetilde{HM}(W^\dagger, \mathfrak{s})(\ell(\mathcal{K}))$ is zero for all Spin^c structures $\mathfrak{s} \neq \mathfrak{s}_\omega$. If we restrict $\widetilde{HM}(W^\dagger)$ to the Spin^c structures on W which are extremal with respect to $\bar{R}$ on each component of the boundary, then we have a map

$$F_{W^\dagger} : KHM(-Y, K) \otimes \mathcal{R} \to KHM(-Y_{\mathcal{S}}, K_{\mathcal{S}}) \otimes \mathcal{R}$$

such that $F_{W^\dagger, \mathfrak{s}}(\ell(\mathcal{K}))$ is $\ell(\mathcal{K}_{\mathcal{S}})$ for a unique Spin^c structure (again, $\mathfrak{s}_\omega$) and zero for all others. ☐

We can use Theorem 5.1 to recover an analog of a theorem of Sahamie [37, Theorem 6.1].

Theorem 5.2. *Let $\mathcal{K} \subset (Y, \xi)$ be Legendrian, and let $(Y_\pm, \xi_\pm)$ be the result of a contact (± 1)-surgery along $\mathcal{K}$. These surgeries induce maps*

$$\begin{aligned} KHM(-Y, K) \otimes \mathcal{R} &\to \widetilde{HM}(-Y_+) \otimes \mathcal{R} \\ \widetilde{HM}(-Y_-) \otimes \mathcal{R} &\to KHM(-Y, K) \otimes \mathcal{R} \end{aligned}$$

sending $\ell(\mathcal{K}) \mapsto \tilde{\psi}(\xi_+)$ and $\tilde{\psi}(\xi_-) \mapsto 0$, respectively.

Proof. Let $\mathcal{K}^\pm$ be Legendrian push-offs of $\mathcal{K}$ with an extra positive or negative twist around $\mathcal{K}$ as in Figure 5, so that $\mathcal{K}^+$ (resp. $\mathcal{K}^-$) is Legendrian (resp. topologically) isotopic to $\mathcal{K}$. As explained in the proof of Proposition 1 of [9], performing a contact (± 1)-surgery on $\mathcal{K}$ turns $\mathcal{K}^\pm$ into a meridian of the surgery torus with $tb = -1$, so that $\mathcal{K}^\pm$ becomes a Legendrian unknot in $(Y_\pm, \xi_\pm)$. In particular, we have $KHM(-Y_\pm, K^\pm) \cong \widetilde{HM}(-Y_\pm)$ and $\ell(\mathcal{K}^\pm \subset Y_\pm) = \tilde{\psi}(\xi_\pm)$.

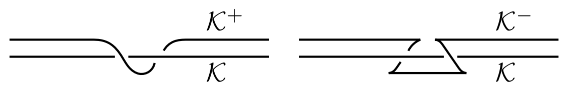


Figure 5: The knots $\mathcal{K}^+$ and $\mathcal{K}^-$ are constructed by adding a positive twist and a negative twist, respectively, to a Legendrian push-off of $\mathcal{K}$.

Writing $\mathcal{S} = \mathcal{K}$ and applying Theorem 5.1 to (Y, ξ), we now have a map

$$KHM(-Y, K^+) \otimes \mathcal{R} \to KHM(-Y_+, K^+) \otimes \mathcal{R}$$

sending $\ell(\mathcal{K}^+) = \ell(\mathcal{K})$ to $\ell(\mathcal{K}^+ \subset Y_+) = \tilde{\psi}(\xi_+)$. Similarly, if we let $\mathcal{S} \subset (Y_-, \xi_-)$ be the core of the contact (-1)-surgery torus, then a contact $(+1)$-surgery on $\mathcal{S}$ cancels the original (-1)-surgery, leaving the original contact manifold (Y, ξ). Theorem 5.1 then produces a map

$$KHM(-Y_-, K^-) \otimes \mathcal{R} \to KHM(-Y, K^-) \otimes \mathcal{R}$$

which sends $\ell(\mathcal{K}^- \subset Y_-) = \tilde{\psi}(\xi_-)$ to $\ell(\mathcal{K}^- \subset Y)$. But $\mathcal{K}^-$ is Legendrian isotopic in Y to the double stabilization $S_+S_-(\mathcal{K})$, hence $\ell(\mathcal{K}^- \subset Y) = 0$ by Proposition 4.3 and we are done. ☐

Corollary 5.3. *If the result of contact $(+1)$-surgery on $\mathcal{K} \subset (Y, \xi)$ has nonzero contact invariant $\psi(\xi_+)$, then $\ell(\mathcal{K}) \neq 0$.*

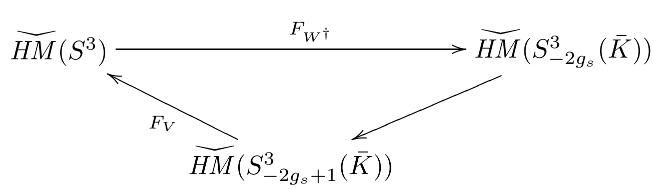
Proof. Proposition 3.10 provides a map $\widetilde{HM}(-Y) \otimes \mathcal{R} \to \widetilde{HM}(-Y) \otimes \mathcal{R}$ sending $\tilde{\psi}(\xi_+)$ to $\psi(\xi_+) \otimes 1$, so if $\psi(\xi_+) \neq 0$ then $\tilde{\psi}(\xi_+) \neq 0$ and hence $\ell(\mathcal{K}) \neq 0$ as well. ☐

For example, let $K \subset S^3$ be a knot with smooth slice genus $g_s \geq 1$, and suppose we have a Legendrian representative $\mathcal{K} \subset (S^3, \xi_{\mathrm{std}})$ of K with $tb(\mathcal{K}) = 2g_s - 1$. Let (Y_+, ξ_+) denote the result of contact $(+1)$-surgery on $\mathcal{K}$. The following argument of Lisca and Stipsicz [31], translated directly from Heegaard Floer to monopole Floer homology, shows that $\psi(\xi_+) \neq 0$.

Letting W denote the Weinstein cobordism from (Y_+, ξ_+) to (S^3, ξ_{std}) which reverses the contact $(+1)$-surgery along $\mathcal{K}$, we have a map

$$\widetilde{HM}(-S^3) \xrightarrow{F_{W^\dagger}} \widetilde{HM}(-Y_+)$$

sending $\psi(\xi_{\mathrm{std}}) \neq 0$ to $\psi(\xi_+)$ by Corollary 2.5, so we wish to show that $F_{W^\dagger}$ is injective. Now Y_+ is the result of a topological $2g_s$-surgery on K, so $-Y_+$ is the result of a $-2g_s$-surgery on the mirror image $\bar{K}$ and thus $F_{W^\dagger}$ fits into a surgery exact triangle



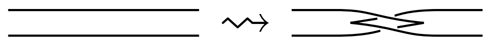


Figure 6: Constructing a Legendrian Whitehead double from $\mathcal{K}$ and its push-off.

where V is a 2-handle cobordism. Lisca and Stipsicz show that V contains a closed surface Σ of genus $g_s > 0$ and self-intersection $2g_s - 1 \geq 0$. If $\mathfrak{s}$ is a Spin^c-structure for which $F_{V,\mathfrak{s}} \neq 0$, then the adjunction inequality for cobordisms says that

$$|\langle c_1(\mathfrak{s}), \Sigma\rangle| + \Sigma \cdot \Sigma \leq 2g(\Sigma) - 2,$$

hence $|\langle c_1(\mathfrak{s}), \Sigma\rangle| \leq -1$, a contradiction. Therefore F_V is zero and $F_{W^\dagger}$ is injective by exactness.

Corollary 5.4. *If $\mathcal{K} \subset (S^3, \xi_{\mathrm{std}})$ is a Legendrian representative of a knot K with slice genus $g_s > 0$ and $tb(\mathcal{K}) = 2g_s - 1$, then $\ell(\mathcal{K}) \neq 0$. Examples include any topologically nontrivial $\mathcal{K}$ for which $tb(\mathcal{K}) = 2g(K) - 1$, where $g(K)$ is the Seifert genus of K.*

For example, in [31] the authors remark that $\overline{tb}(K) = 2g(K) - 1$ for any algebraic knot, where $\overline{tb}$ denotes maximal Thurston-Bennequin number. More generally, if $\mathcal{K}$ is the Legendrian closure of a positive braid [23] with n strands and c crossings then it is easy to compute that $tb(\mathcal{K}) = c - n = 2g(K) - 1$, hence closures of positive braids have the same property.

For any Legendrian knot $\mathcal{K}$, the Legendrian Whitehead double $W(\mathcal{K})$ (due to Eliashberg, and denoted Γ_{dbl} by Fuchs in [18]) is constructed by taking $\mathcal{K}$ and a slight push-off $\mathcal{K}'$ in the z-direction, and then replacing a pair of parallel segments with a clasp as in Figure 6; it has genus 1 and $tb = 1$.

Thus $\ell(\mathcal{K}) \neq 0$ if $\mathcal{K}$ is a tb-maximizing representative of the closure of a positive braid or a Legendrian Whitehead double. Similarly, there are many examples of knots with $tb(\mathcal{K}) = 2g_s(K) - 1$ where $1 \leq g_s(K) < g(K)$, and these all have $\ell(\mathcal{K}) \neq 0$; according to KnotInfo [3], the smallest examples have topological types $m(8_{21})$ and $m(9_{45})$.

5.2 Non-loose knots

By Proposition 4.1, a Legendrian knot $\mathcal{K}$ in an overtwisted manifold is non-loose if $\ell(\mathcal{K}) \neq 0$. Our goal in this section is to apply Theorem 5.1 to construct examples where this is the case. In order to do so, we will first need the following lemma on monopole knot homology and surgery.

Lemma 5.5. *Let $K, S \subset Y$ be disjoint knots, and for any integral framing f, let K_f denote the image of K in the manifold Y_f obtained by f-surgery along S. For each f there is a map $s_f : KHM(Y, K) \to KHM(Y_f, K_f)$ corresponding to a 2-handle attachment along S in a closure $\bar{Y}$ of $Y \backslash K$, and these maps satisfy the following:*

1. *If s_{f+1} is either injective or surjective, then s_f is injective.*

2. *If s_f is either surjective or zero, then s_{f+1} is zero.*

Proof. We have a surgery exact triangle

$$\begin{array}{ccc} \widecheck{HM}(\bar{Y}|\bar{R}) & \xrightarrow{F_1} & \widecheck{HM}(\bar{Y}_f|\bar{R}) \\ & {}_{F_3}\nwarrow \quad \swarrow_{F_2} & \\ & \widecheck{HM}(\bar{Y}_{f+1}|\bar{R}) & \end{array}$$

where $\bar{Y}$, $\bar{Y}_f$, and $\bar{Y}_{f+1}$ are closures of the complements of K in Y, Y_f, and Y_{f+1}; note that $\widecheck{HM}(\bar{Y}_f|\bar{R}) = KHM(Y_f, K_f)$ by definition. Similarly, we have a second triangle of the form

$$\begin{array}{ccc} \widecheck{HM}(\bar{Y}|\bar{R}) & \xrightarrow{G_1} & \widecheck{HM}(\bar{Y}_{f+1}|\bar{R}) \\ & {}_{G_3}\nwarrow \quad \swarrow_{G_2} & \\ & \widecheck{HM}(\bar{Y}_{f+2}|\bar{R}) & \end{array}$$

and by [25, Proposition 7.2] we have $G_1 \circ F_3 = F_3 \circ G_1 = 0$, since each composition comes from a cobordism created by a pair of 2-handles which contains a homologically nontrivial sphere of self-intersection zero. This implies that G_1 (resp. F_3) is zero if F_3 (resp. G_1) is either injective or surjective.

Suppose G_1 is either injective or surjective; then $F_3 = 0$, hence by exactness F_1 is injective. Similarly, if F_1 is surjective then F_2 is zero, hence F_3 is injective, and if F_1 is zero then F_3 is surjective; either of these imply $G_1 = 0$. Since $F_1 = s_f$ and $G_1 = s_{f+1}$, we are done. $\square$

Proposition 5.6. *Let $\mathcal{K}, \mathcal{S} \subset (Y, \xi)$ be nullhomologous Legendrian knots such that S is homotopic to a meridian of K in $Y \backslash K$. If $\mathcal{K}_{\mathcal{S}} \subset (Y_{\mathcal{S}}, \xi_{\mathcal{S}})$ is the image of $\mathcal{K}$ in the manifold obtained by contact $(+1)$-surgery on $\mathcal{S}$, then $\ell(\mathcal{K}_{\mathcal{S}}) \neq 0$ if and only if $\ell(\mathcal{K}) \neq 0$ and $tb(\mathcal{S}) \geq 0$.*

Proof. Observe that $Y_{\mathcal{S}}$ is obtained from Y by a topological $(tb(\mathcal{S})+1)$-surgery along S, so $-Y_{\mathcal{S}}$ is related to $-Y$ by a k-surgery along S, where $k = -tb(\mathcal{S}) - 1$. In the notation of Lemma 5.5, the map

$$s_k : KHM(-Y, K) \otimes \mathcal{R} \to KHM(-Y_{\mathcal{S}}, K_{\mathcal{S}}) \otimes \mathcal{R}$$

carries $\ell(\mathcal{K})$ to $\ell(\mathcal{K}_{\mathcal{S}})$ as in Theorem 5.1, so we will show that s_k is injective if $k \leq -1$ (i.e. if $tb(\mathcal{S}) \geq 0$) and zero if $k \geq 0$. By Lemma 5.5 it will be enough to show that $KHM((-Y)_0, K_0) = \widecheck{HM}((-\bar{Y})_0|\bar{R}) = 0$, where $(\bar{Y}, \bar{R})$ is a closure of $Y \backslash \mathcal{K}$ and $(-\bar{Y})_0$ is obtained by 0-surgery on $S \subset -\bar{Y}$. Indeed, this implies that s_k is zero for $k = 0$ and hence for all $k \geq 0$, and since s_0 is also surjective it follows that s_k is injective for all $k \leq -1$.

Since S and a meridian of K are homotopic in $Y \backslash K$, they are homotopic in $-\bar{Y}$ as well, and in particular S is homotopic to a nonseparating curve $c \subset \bar{R}$.

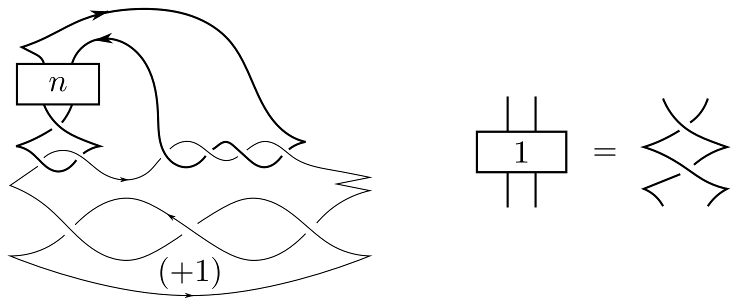


Figure 7: A Legendrian knot $\mathcal{K}_n$ in the overtwisted contact structure on $-P$ obtained from $(S^3, \xi_{\mathrm{std}})$ by contact (+1)-surgery on a stabilized right-handed trefoil $\mathcal{T}$.

When we perform 0-surgery along S to obtain $-\bar{Y}_0$, then, the curve c becomes nullhomotopic and so $[\bar{R}] \in H_2(-\bar{Y}_0)$ has a representative of genus $g(\bar{R}) - 1$. Since $g(\bar{R}) \geq 2$, the adjunction inequality tells us that $\widetilde{HM}(-\bar{Y}_0|\bar{R}) = 0$ as desired. □

Corollary 5.7. *Let $\mathcal{K} \cup \mathcal{S}$ be a two-component Legendrian link in $(S^3, \xi_{\mathrm{std}})$ satisfying the following:*

1. *$\mathcal{K}$ is a Legendrian unknot with $tb(\mathcal{K}) = -1$.*
2. *$tb(\mathcal{S}) \geq 0$.*
3. *The linking number $lk(K, S)$ is ± 1.*

Then $\mathcal{K}_{\mathcal{S}}$ is a non-loose Legendrian knot in the contact manifold $(S^3_{\mathcal{S}}, \xi_{\mathcal{S}})$, where the subscript denotes contact $(+1)$-surgery along $\mathcal{S}$.

Proof. We know that $\ell(\mathcal{K}) = 1 \in \mathcal{R}$ by Proposition 3.8, so Proposition 5.6 tells us that $\ell(\mathcal{K}_{\mathcal{S}}) \neq 0$. □

In particular, given a knot $\mathcal{S}$ with $tb(\mathcal{S}) > 0$ which satisfies all the other conditions of Corollary 5.7, we can stabilize $\mathcal{S}$ to get $\mathcal{S}'$ with $tb(\mathcal{S}') \geq 0$ and then apply the corollary to $\mathcal{K} \cup \mathcal{S}'$. Since $\mathcal{S}'$ is stabilized, the contact (+1)-surgery results in an overtwisted contact structure (see for example [10]), and $\mathcal{K}_{\mathcal{S}'}$ is non-loose.

For example, the right handed trefoil has a unique Legendrian representative with $(tb, r) = (1, 0)$. Let $\mathcal{S}$ be a stabilization of this Legendrian knot, and let $\mathcal{K}$ be a Legendrian unknot with $lk(\mathcal{K}, \mathcal{S}) = \pm 1$. Then (+1)-surgery on $\mathcal{S}$ gives an overtwisted contact structure on the Poincaré homology sphere $-P = -\Sigma(2, 3, 5)$, which in fact does not admit tight positive contact structures [16], and the image $\mathcal{K}_{\mathcal{S}}$ of $\mathcal{K}$ in $-P$ is a non-loose knot. We exhibit a family $\mathcal{K}_n$ of such knots in Figure 7.

Proposition 5.8. *The knots K_n $(n \geq 0)$ are all distinct, and none of them are fibered.*

Proof. Let $L = U_n \cup T \subset S^3$ and $\hat{L} = K_n \cup \hat{T} \subset -P$, where $\hat{T}$ is the core of the surgery torus glued to $S^3 \backslash T$ to obtain $-P$. We will compute the Conway polynomial ∇_L and use it to determine $\nabla_{\hat{L}}$ and ∇_{K_n}, and hence the Alexander polynomial Δ_{K_n}, referring to the results of [2]; note that ∇ and Δ are related by

$$\nabla_L(s_1, \ldots, s_m) = \begin{cases} (s_1 - s_1^{-1})^{-1}\Delta_L(s_1^2), & |L| = 1 \\ \Delta_L(s_1^2, s_2^2, \ldots, s_m^2), & |L| > 1. \end{cases}$$

Since $\hat{L}$ is obtained as the cores of surgery tori for $\frac{1}{0}$-surgery on U_n and $\frac{1}{1}$-surgery on T, and $lk(U_n, T) = -1$, the link $\hat{L}$ is determined by L and the framing matrix

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

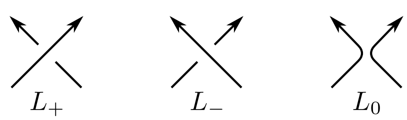
from which we can determine $lk_{-P}(K_n, \hat{T}) = -1$ and

$$\nabla_{\hat{L}}(s_1, s_2) = \nabla_L(s_1, s_1 s_2)$$

by the "variance under surgery" proposition. Then $\nabla_{\hat{L}}(s, 1) = -(s - s^{-1})\nabla_{K_n}(s)$ by "restriction," and so

$$\Delta_{K_n}(s^2) = (s - s^{-1})\nabla_{K_n}(s) = -\nabla_{\hat{L}}(s, 1) = -\nabla_L(s, s).$$

Then we have reduced the computation of $\Delta_{K_n \subset -P}$ to that of $\nabla_{L \subset S^3}(s, s)$. Note that the latter term is determined entirely by a skein relation $\nabla_{L_+} - \nabla_{L_-} = (s - s^{-1})\nabla_{L_0}$:



and by $\nabla_U(s) = (s - s^{-1})^{-1}$, where U is the unknot.

Using the skein relation at a crossing in one of the n full twists of U_n, we see that

$$\nabla_{U_n \cup T} - \nabla_{U_{n-1} \cup T} = (s - s^{-1})\nabla_{L_0}$$

and so $\nabla_L = \nabla_{U_0 \cup T} + n(s - s^{-1})\nabla_{L_0}$. Applying the skein relation to the crossing of U_n directly below the n twists, when $n = 0$, we get $\nabla_{U_0 \cup T} - \nabla_{L_1} = (s - s^{-1})\nabla_{L_0}$, hence

$$\nabla_L = \nabla_{L_1} + (n+1)(s - s^{-1})\nabla_{L_0}$$

where $L = U_n \cup T$, L_0, and L_1 are the links in Figure 8.

A straightforward computation now yields

$$\begin{aligned} \nabla_{L_0}(s, s, s) &= -2(s - s^{-1})(s^2 - 1 + s^{-2}) \\ \nabla_{L_1}(s, s) &= -(s^2 - 1 + s^{-2}) \end{aligned}$$

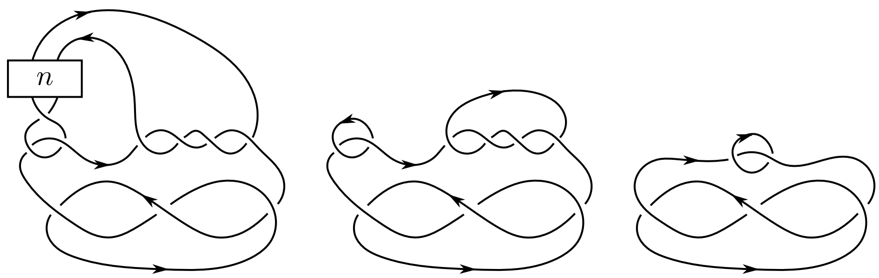


Figure 8: Links appearing in the computation of $L = U_n \cup T$ by the skein relation.

and since $\Delta_{K_n}(s^2) = -\nabla_{L_1} - (n+1)(s - s^{-1})\nabla_{L_0}$ we conclude that

$$\Delta_{K_n}(t) = (t - 1 + t^{-1})(1 + 2(n+1)(t - 2 + t^{-1})).$$

Since the Alexander polynomials $\Delta_{K_n}(t)$ are all distinct, so are the K_n; and since Δ_{K_n} is never monic, the K_n cannot be fibered. □

We remark that in general very few examples of non-loose knots in overtwisted contact manifolds have been studied. Etnyre [14] observed that if the result $(S^3_{\mathcal{K}}, \xi_{\mathcal{K}})$ of contact $(+1)$-surgery on $\mathcal{K} \subset (S^3, \xi_{\mathrm{std}})$ is overtwisted, then the core $\mathcal{K}'$ of the surgery torus in $S^3_{\mathcal{K}}$ is non-loose. (If $S^3_{\mathcal{K}}$ is the Poincaré homology sphere with either orientation, then K must be a trefoil [19] and so one can show that $\Delta_{K'} = \pm\Delta_K \neq \Delta_{K_n}$.) Furthermore, Etnyre and Vela-Vick [17] showed that given an open book decomposition which supports (Y, ξ), any Legendrian approximation of the binding is non-loose. To the best of our knowledge, these are the only known examples in manifolds other than S^3.

In particular, it seems that the non-loose knots $\mathcal{K}_n \subset -P$ were not previously known, and in fact may be the only known non-fibered examples (even in S^3) which are not the cores of surgery tori. The construction of Corollary 5.7 is of course much more general; it would be interesting to give examples of links $\mathcal{K}_i \cup \mathcal{S}$ $(i = 1, 2)$ which are topologically but not Legendrian isotopic and which give distinct non-loose knots $(\mathcal{K}_i)_{\mathcal{S}}$.

6 Lagrangian concordance

Chantraine [4] defined an interesting notion of concordance on the set of all Legendrian knots in a contact 3-manifold Y.

Definition 6.1. Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be Legendrian knots parametrized by embeddings $\gamma_i : S^1 \to Y$, and let $Y \times \mathbb{R}$ be the symplectization of Y. We say that $\mathcal{K}_0$ is *Lagrangian concordant* to $\mathcal{K}_1$, denoted $\mathcal{K}_0 \prec \mathcal{K}_1$, if there is a Lagrangian embedding $L : S^1 \times \mathbb{R} \hookrightarrow Y \times \mathbb{R}$ and a $T > 0$ such that $L(s,t) = (\gamma_0(s), t)$ for $t < -T$ and $L(s,t) = (\gamma_1(s), t)$ for $t > T$.

Theorem 6.2 ([4]). *The relation $\prec$ descends to a relation on Legendrian isotopy classes of Legendrian knots. If $\mathcal{K}_0 \prec \mathcal{K}_1$ then $tb(\mathcal{K}_0) = tb(\mathcal{K}_1)$ and $r(\mathcal{K}_0) = r(\mathcal{K}_1)$.*

Our goal in this section is to investigate the behavior of $\ell(\mathcal{K})$ under Lagrangian concordance:

Theorem 6.3. *Let $\mathcal{K}_0, \mathcal{K}_1$ be Legendrian knots in a contact homology 3-sphere Y satisfying $\mathcal{K}_0 \prec \mathcal{K}_1$. Then there is a homomorphism*

$$KHM(-Y, K_1) \otimes \mathcal{R} \to KHM(-Y, K_0) \otimes \mathcal{R}$$

sending $\ell(\mathcal{K}_1)$ to $\ell(\mathcal{K}_0)$.

We compare this with the remarks in [4, Section 5.2], where it is observed that Lagrangian concordance induces a map $LCH(\mathcal{K}_1) \to LCH(\mathcal{K}_0)$ on Legendrian contact homology.

Proof. We fix a particular closure $(\bar{Y}_i, \bar{R}_i)$ of the sutured knot complements $Y(\mathcal{K}_i)$: place the meridional sutures close together so that in $\partial(Y \backslash \mathcal{K}_i)$ they bound an annulus A in which the dividing curves are parallel to a longitude. In the other annulus A' bounded by the sutures, the dividing curves twist around the meridional direction a total of $tb(\mathcal{K}_i)$ times; recall that $tb(\mathcal{K}_0) = tb(\mathcal{K}_1)$. We glue a surface $T \times I$ to each complement and round edges, resulting in a manifold with boundary $\bar{R}_+ \sqcup \bar{R}_-$ and $\mathrm{int}(A) \subset \bar{R}_+$. Finally, we glue $\bar{R}_+$ to $\bar{R}_-$ by identifying $(x, 1) \in T \times \{1\}$ to $(x, -1) \in T \times \{-1\}$ for all $x \in \mathrm{int}(T)$, and identifying A with A' by a homeomorphism composed of enough Dehn twists around the core of A to make the dividing curves match.

This construction guarantees that $Z_0 = \bar{Y}_0 \backslash \mathrm{int}(Y \backslash \mathcal{K}_0)$ and $Z_1 = \bar{Y}_1 \backslash \mathrm{int}(Y \backslash \mathcal{K}_1)$ are contactomorphic as 3-manifolds with torus boundary. In the symplectization $Y \times \mathbb{R}$, the cylinder $\mathcal{K}_0 \times \mathbb{R}$ is Lagrangian, hence it has a standard neighborhood symplectomorphic to a neighborhood N of the 0-section in $T^*(S^1 \times \mathbb{R})$. Then a neighborhood of the boundary $T^2 \times \mathbb{R}$ of the symplectization $Z_0 \times \mathbb{R}$, can be identified with the complement of the 0-section in N.

Now consider the Lagrangian cylinder $\mathcal{L} \subset Y \times \mathbb{R}$ defining the concordance from $\mathcal{K}_0$ to $\mathcal{K}_1$. Once again, $\mathcal{L}$ has a neighborhood symplectomorphic to N; if we remove a sufficiently small neighborhood of $\mathcal{L}$, then there is a collar neighborhood of $\partial((Y \times \mathbb{R}) \backslash \mathcal{L})$ which is orientation-reversing symplectomorphic to N with the 0-section removed. Thus we can glue $(Y \times \mathbb{R}) \backslash \mathcal{L}$ to $Z_0 \times \mathbb{R}$ to get a symplectic manifold W with two infinite ends. One of these ends is a piece $\bar{Y}_0 \times (-\infty, T]$ of the symplectization of $\bar{Y}_0$, and since Z_0 is contactomorphic to Z_1 the other end is $\bar{Y}_1 \times [T, \infty)$. Thus W is a boundary-exact symplectic cobordism from $\bar{Y}_0$ to $\bar{Y}_1$.

Finally, we wish to show that the map $i^* : H^1(W, \bar{Y}_1) \to H^1(\bar{Y}_0)$ is zero. By Poincaré duality it suffices to show that $H_3(W, \bar{Y}_0) \to H_2(\bar{Y}_0)$ is zero, or equivalently (by the long exact sequence of the pair $(W, \bar{Y}_0)$) that the map $H_2(\bar{Y}_0) \to H_2(W)$ is injective. But there is a natural isomorphism $H_2((Y \times \mathbb{R}) \backslash \mathcal{L}) \cong H_2(Y \backslash \mathcal{K}_0)$ by Alexander duality, hence by the Mayer-Vietoris sequence and the five lemma it follows that $H_2(\bar{Y}_0) \to H_2(W)$ is an isomorphism as well, and so i^* is indeed zero.

Since W is a boundary-exact symplectic cobordism and $H^1(W, \bar{Y}_1) \to H^1(\bar{Y}_0)$ is zero, we apply Theorem 2.4 to conclude that

$$\psi(\bar{Y}_0, \bar{\xi}_0) = \widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)\psi(\bar{Y}_1, \bar{\xi}_1).$$

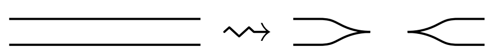
Thus $\widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)$ induces a map $f : KHM(-Y, K_1) \otimes \mathcal{R} \to KHM(-Y, K_0) \otimes \mathcal{R}$ satisfying $f(\ell(\mathcal{K}_1)) = \ell(\mathcal{K}_0)$, as desired. □

Corollary 6.4. *If $\mathcal{K}_0 \prec \mathcal{K}_1$ and $\ell(\mathcal{K}_0)$ is nonzero, then so is $\ell(\mathcal{K}_1)$.*

Corollary 6.5. *If a Legendrian knot $\mathcal{K} \subset (S^3, \xi_0)$ bounds a Lagrangian disk in the standard symplectic 4-ball B^4, then $\ell(\mathcal{K})$ is a unit of $KHM(-S^3, K)$.*

Proof. The Legendrian unknot $\mathcal{U}$ is Lagrangian concordant to $\mathcal{K}$, and $\ell(\mathcal{U})$ is a generator of $KHM(-S^3, U) \otimes \mathcal{R} \cong \mathcal{R}$ by Proposition 3.8, so by Theorem 6.3 there is a map $KHM(-S^3, K) \otimes \mathcal{R} \to \mathcal{R}$ such that the image of $\ell(\mathcal{K})$ is a unit. □

It is observed in the addendum to [4] that the following tangle replacement in the front projection (obtained from a 1-smoothing of a crossing in the Lagrangian projection) can be realized by a Lagrangian saddle cobordism:



If such a move turns a Legendrian knot $\mathcal{K}$ into a Legendrian unlink whose components are both $\mathcal{U}$, we can cap both components with Lagrangian disks and thus build a Lagrangian slice disk for $\mathcal{K}$, proving that $\ell(\mathcal{K})$ is a primitive element of $KHM(-S^3, K)$. Figure 9 shows grid diagrams for seven such knots, of topological types $m(9_{46})$, $m(10_{140})$, $m(10_{140})$, $11n_{139}$, $m(12n_{582})$, $m(12n_{768})$, and $m(12n_{838})$, which were discovered using a combination of KnotInfo [3], the Legendrian knot atlas [6], and Gridlink [7]. As usual, these may be turned into front projections of Legendrian knots by smoothing out all northeast and southwest corners and then rotating 45 degrees counterclockwise. The dotted line in each diagram indicates where to perform the tangle replacement.

Conjecture 6.6. *Given a Lagrangian cobordism $\mathcal{K}_0 \prec_\Sigma \mathcal{K}_1$ of arbitrary genus, there is a map $KHM(-Y, K_1) \otimes \mathcal{R} \to KHM(-Y, K_0) \otimes \mathcal{R}$ sending $\ell(\mathcal{K}_1)$ to $\ell(\mathcal{K}_0)$.*

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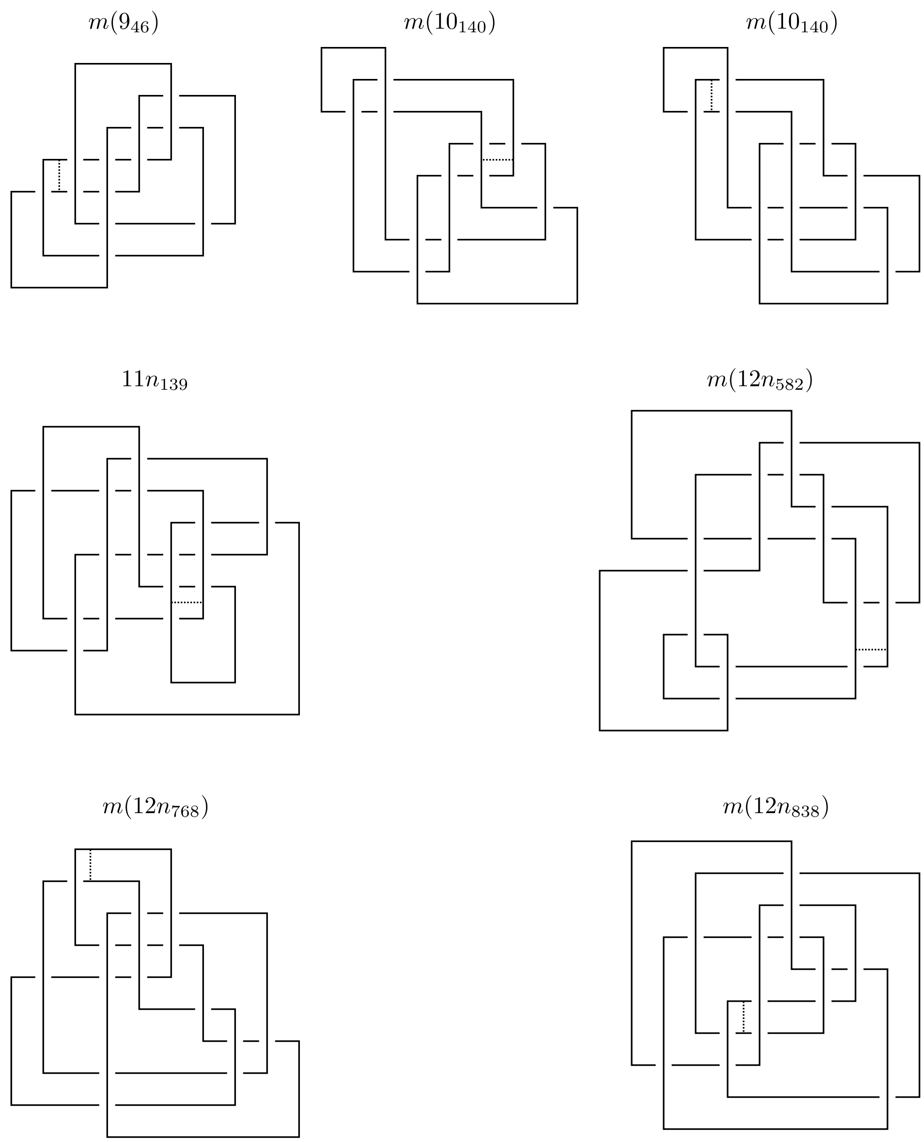

Figure 9: Seven Lagrangian knots $\mathcal{K}$ which bound Lagrangian disks in B^4 and thus satisfy $\ell(\mathcal{K}) \neq 0$.

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