EPJ manuscript No.
(will be inserted by the editor)

Birkhoff's Theorem in the $f(T)$ Gravity

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Received: date / Revised version: date

Abstract. Generalized from the so-called teleparallel gravity which is exactly equivalent to general relativity, the $f(T)$ gravity has been proposed as an alternative gravity model to account for the dark energy phenomena. In this letter we prove that the external vacuum gravitational field for a spherically symmetric distribution of source matter in the $f(T)$ gravity framework must be static and the conclusion is independent of the radial distribution and spherically symmetric motion of the source matter that is, whether it is in motion or static. As a consequence, the Birkhoff's theorem is valid in the general $f(T)$ theory. We also discuss its application in the de Sitter space-time evolution phase as preferred to by the nowadays dark energy observations.

PACS.98.80.Cq Modified theories of gravity

arXiv:1107.0629v1 [astro-ph.CO] 4 Jul 2011

1 Introduction

The discovery of the accelerating expansion of the Universe has stimulated great efforts to investigate the fundamental theories of gravity. As a modified gravitational theory, the $f(T)$ gravity has been proposed [1] to explain the acceleration of the cosmic expansion and attracts much attention recently. The framework is a generalization of the so-called *Teleparallel Equivalent of General Relativity* (TEGR) which was first propounded by Einstein in 1928 [2] and maturated in the 1960s (For some reviews, see [3, 4]). We know that the theory of general relativity is based on Riemann geometry which involves only curvature composed of the metric and its derivatives. On the contrary, the TEGR is based on the so named Weitzenböck geometry with the non-vanishing torsion. Owing to the definition of Weitzenböck connection rather than the Levi-Civita connection, the Riemann curvature is automatically vanishing in the TEGR framework. Therefore the parallelism of distant vectors or tensors would be independent of curves along which they are transported. That is why the theory is also called *Teleparallel Gravity*. It has been well studied that the TEGR, for a specific choice of parameters, behaves completely equivalent to Einstein's theory of general relativity. Furthermore, by using the torsion scalar T as the Lagrangian density, the TEGR can give a field equation of the second order only, which is simpler than Einstein's field equation and avoids the instability problems caused from higher order derivatives as from the metric framework f(R) gravity demonstrated.

The modified version of teleparallel gravity uses a general function $f(T)$ as the model Lagrangian density. Similar to the generalization of Einstein's theory of general relativity to the $f(R)$ theory (For some references, see [5, 6,7,8]), the $f(T)$ theory can be directly reduced to the TEGR if we choose a simplest case $f(T) = T$. A variety of $f(T)$ models have been proposed in succession to explain the late-time acceleration of the cosmic expansion without the mysteriously so-called dark energy, and have been fitted the cosmological data-sets very well (e.g. [1,9, 10,11,12,13,14]). In the theoretical aspect, the Lorentz invariance and conformal invariance of the $f(T)$ theory are also investigated interestingly[15,16], and present many interesting results. In this paper, we focus on the validity of Birkhoff's theorem in the $f(T)$ gravity.

The Birkhoff's theorem, presented with an explicit proof by George D. Birkhoff in 1923 [17], states that the spherically symmetric gravitational field in vacuum must be static, with a metric uniquely given by the Schwarzschild solution form of Einstein equations [18]. It is well known that the Schwarzschild metric is found in 1918 as the external (vacuum) solution of a static and spherical star. The Birkhoff's theorem claims that any spherically symmetric object possesses the same static gravitational field, as if the mass of the object were concentrated at the center. It is the same feature as holding in the classical Newtonian gravity. It means that the external gravitational field is static even if the central spherical object is moving radically, like the collapsing processes (such as a collapsing star or a violently exploding supernova), as long as the motion is spherically symmetric. As a result, there is no monopole gravitational radiation anyway, just as the case of electromagnetic radiation physics.

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As a relatively newly proposed modified gravitational theory, we do not know clearly for the $f(T)$ gravity whether the vacuum solution of spherically symmetric field is still static before. In this present paper, first we briefly review the $f(T)$ theories in the following section, and in section three we prove the validity of Birkhoff's theorem in the context of the $f(T)$ gravity. For the sake of clarity, we firstly demonstrate the proof for a concrete form of a $f(T)$ model, and then complete it for the general case. The conclusions and discussions are devoted in the last section.

2 Elements of $f(T)$ Gravity

Instead of the metric tensor, the vierbein field $\mathbf{e}_i(x^\mu)$ is the dynamical variable in the teleparallel gravity. It is defined as the orthonormal basis of the tangent space at each point x^μ in the manifold, namely, $\mathbf{e}_i \cdot \mathbf{e}_j = \eta_{ij}$, where $\eta_{ij} = diag(1, -1, -1, -1)$ is the Minkowski metric. The vierbein vector can be expanded in spacetime coordinate basis: $\mathbf{e}_i = e_i^\mu \partial_\mu$, $\mathbf{e}^i = e^i_\mu \mathrm{d}x^\mu$. According to the convention, Latin indices and Greek indices, both running from 0 to 3, label the tangent space coordinates and the spacetime coordinates respectively. The components of vierbein are related by $e^i_\mu e_j^\mu = \delta_j{}^i, \quad e^i_\mu e_i^\nu = \delta_\mu{}^\nu$.

The metric tensor is determined uniquely by the vierbein as

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^i_\nu, \tag{1}$$

which can be equivalently expressed as: $\eta_{ij} = g_{\mu\nu} e^i_\mu e^j_\nu$. The definition of torsion tensor is given by then

$$T^\rho_{\ \mu\nu} = \Gamma^\rho_{\ \nu\mu} - \Gamma^\rho_{\ \mu\nu}, \tag{2}$$

where $\Gamma^\rho_{\ \mu\nu}$ is the connection. Evidently, $T^\rho_{\ \mu\nu}$ vanishes in the Riemann geometry since the Levi-Civita connection is symmetric with respect to the two covariant indices. Differing from that in Einstein's theory of general relativity, the teleparallel gravity uses Weitzenböck connection defined directly from the vierbein:

$$\Gamma^\rho_{\ \mu\nu} = e_i^\rho \partial_\nu e^i_\mu. \tag{3}$$

Accordingly the antisymmetric non-vanishing torsion is

$$T^\rho_{\ \mu\nu} = e_i^\rho (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu). \tag{4}$$

It can be confirmed that the Riemann curvature in this framework is precisely vanishing:

$$R^\rho_{\ \theta\mu\nu} = \partial_\mu \Gamma^\rho_{\ \theta\nu} - \partial_\nu \Gamma^\rho_{\ \theta\mu} + \Gamma^\rho_{\ \sigma\mu}\Gamma^\sigma_{\ \theta\nu} - \Gamma^\rho_{\ \sigma\nu}\Gamma^\sigma_{\ \theta\mu} = 0. \tag{5}$$

In order to get the action of the teleparallel gravity, it is convenient to define other two tensors:

$$K^{\mu\nu}_{\ \ \ \rho} = -\frac{1}{2}(T^{\mu\nu}_{\ \ \ \rho} - T^{\nu\mu}_{\ \ \ \rho} - T_\rho^{\ \mu\nu}), \tag{6}$$

and

$$S_\rho^{\ \mu\nu} = \frac{1}{2}(K^{\mu\nu}_{\ \ \ \rho} + \delta_\rho^{\ \mu} T^{\theta\nu}_{\ \ \ \theta} - \delta_\rho^{\ \nu} T^{\theta\mu}_{\ \ \ \theta}). \tag{7}$$

Then the torsion scalar as the teleparallel Lagrangian is defined by

$$T = T^\rho_{\ \mu\nu} S_\rho^{\ \mu\nu}. \tag{8}$$

The action of teleparallel gravity is expressed as

$$I = \frac{1}{16\pi G} \int \mathrm{d}^4x \ e\, T, \tag{9}$$

where $e = \det(e^i_\mu) = \sqrt{-g}$. Performing variation of the action with respect to the vierbein, one can get the equations of motion which are equivalent to the results of Einstein's theory of general relativity.

Just as in the $f(R)$ theory, the generalized version of teleparallel gravity could be obtained by extending the Lagrangian density directly to a general function of the scalar torsion T :

$$I = \frac{1}{16\pi G} \int \mathrm{d}^4x \ e\, f(T). \tag{10}$$

This modification is expected to provide a natural way to understand the cosmological observations, especially for the dark energy phenomena, as a motivation. The variation of the action with respect to vierbein leads to the following equations:

$$\begin{aligned} \left[e^{-1} e^i_\mu \partial_\sigma (e S_i^{\ \sigma\nu}) - T^\rho_{\ \sigma\mu} S_\rho^{\ \nu\sigma}\right] f_T + S_\mu^{\ \rho\nu} \partial_\rho T f_{TT} \\ - \frac{1}{4}\delta_\mu^{\ \nu} f = 4\pi G T_\mu^{\ \nu}, \end{aligned} \tag{11}$$

where f_T and f_{TT} represent the first and second order derivative with respect to T respectively, and $S_i^{\ \sigma\nu} = e_i^\rho S_\rho^{\ \sigma\nu}$. $T_\mu^{\ \nu}$ is the energy-momentum tensor of the particular mater, with assuming that matter couples to the metric in the standard form.

3 The Validity of Birkhoff's Theorem

We consider the external vacuum gravitational field solution of a spherically symmetric object. The spherically symmetric metric can always be written in the following form:

$$\mathrm{d}s^2 = A^2(t,r)\ \mathrm{d}t^2 - B^2(t,r)\ \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta\ \mathrm{d}\phi^2, \tag{12}$$

where $A(t,r)$, $B(t,r)$ are arbitrary functions of the coordinates t and r. The corresponding vierbein field directly reads

$$e^i_\mu = \mathrm{diag}\big(A(t,r),\ B(t,r),\ r,\ r\sin\theta\big), \tag{13}$$

and the determinant of vierbein is $e = A(t,r)B(t,r)r^2\sin\theta$. Then the tensors defined in Eqs. (4,6,7) are determined, and the torsion scalar is given by

$$T = \frac{2\big(A(t,r) + 2A'(t,r)\, r\big)}{A(t,r)B^2(t,r)\, r^2}, \tag{14}$$

where a prime denotes the derivative with respect to r while the derivative with respect to t will be denoted by a dot overhead, according to which we will follow these convention throughout this work.

For convenience, we introduce the tensor $E_\mu{}^\nu$ to stand for the left hand side of Eq. (11), and the field equation can be re-expressed then as

$$E_\mu{}^\nu = 4\pi G T_\mu{}^\nu. \tag{15}$$

Firstly, we consider a typically concrete $f(T)$ model in the following form, which has been studied frequently in literature(e.g. [1,9,10]):

$$f(T) = T + \alpha(-T)^n, \tag{16}$$

where α and n are real constants with arbitrary sign and the negative sign for convenience in the second term can be absorbed in the coefficient α if one likes. It will be shown later that other more complicated forms of $f(T)$ models possess the same conclusion of this form. Then we work out all the components of $E_\mu{}^\nu$, and find half of them are not vanishing, including some quite complicated ones. The two components we used, fortunately not very complex, are given by respectively

$$E_1{}^0 = \frac{\left[\alpha n(-T)^{n-1} - 1\right]\dot{B}}{A^2 B\, r}, \tag{17}$$

$$E_1{}^1 = \frac{1}{2B^2r^2}\Big[\big((nB^2 + 1 - 2n)A + 2(1-2n)rA'\big) \cdot\alpha\big(-T\big)^{n-1} + 2rA' - \big(B^2 - 1\big)A\Big]. \tag{18}$$

Since the non-diagonal elements of energy-momentum tensor are equal to zero, $E_1{}^0$ always vanishes, restricting $B(t,r)$ to be only the function of r, that is,

$$B(t,r) = B(r). \tag{19}$$

There is no density or pressure of matter in the external vacuum space, implying that $E_1{}^1$ is also equal to zero. After some manipulation, Eq. (18) leads to

$$\frac{2^n\alpha\, n}{(B^2 r)^{n-1}}\bigg(-\frac{2A'}{A} - \frac{1}{r}\bigg)^{n-1}\bigg[\frac{B^2 - 2 + \frac{1}{n}}{r} - \Big(4 - \frac{2}{n}\Big)\frac{A'}{A}\bigg] + \frac{4A'}{A} + \frac{2(1-B^2)}{r} = 0. \tag{20}$$

It can be regarded as an algebraic equation of degree n for (A'/A), with no analytical solutions generally if $n > 2$. Nevertheless, for B is independent of t, Eq. (20) determines (A'/A) as an implicit function of r. So far as the solution exists, it could be expressed as

$$\frac{A'(t,r)}{A(t,r)} = G\big(r, B(r), \alpha, n\big) \equiv g(r). \tag{21}$$

The integration of the above equation with respect to the variable r gives that

$$\ln A(t,r) = \int g(r)\mathrm{d}r + C(t), \tag{22}$$

where $C(t)$ as the integral constant, is an arbitrary function of t. Therefore the function $A(t,r)$ can be written as

$$A(t,r) = \mathrm{e}^{\int g(r)\mathrm{d}r}\mathrm{e}^{C(t)}. \tag{23}$$

The factor $\mathrm{e}^{C(t)}$ can always be absorbed in the metric through a coordinate transformation $t \to t'$, where t' is the new time coordinate defined as:

$$\mathrm{d}t' = \mathrm{e}^{C(t)}\mathrm{d}t. \tag{24}$$

Defining $\widetilde{A}(r) \equiv \mathrm{e}^{\int g(r)\mathrm{d}r}$, the metric presented in Eq. (12) becomes

$$\mathrm{d}s^2 = \widetilde{A}^2(r)\,\mathrm{d}t'^2 - B^2(r)\,\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{25}$$

This is exactly a static metric which is required by the Birkhoff's theorem validity. In the following we will discuss a general case for the f(T) modified gravity model.

The $f(T)$ models investigated by several authors before take variant forms, most of which are more complex than that we just considered in (16) form. It is impracticable to express the field equations of all these cases so specifically as in the (20). Nonetheless, without being concerned with the particular forms of $f(T)$ models, we give the two field equations in vacuum $E_1{}^0 = 0$, $E_1{}^1 = 0$ directly as

$$\frac{\dot{B}f_T}{A^2Br} = 0, \tag{26}$$

$$\frac{B^2r^2f + \big(2B^2 - 4 - 8r(\frac{A'}{A})\big)f_T}{4B^2r^2} = 0. \tag{27}$$

Eq.(26) also gives eq.(19). Noting that f and f_T are both functions of torsion T which according to eq.(14) can be re-expressed as

$$T = \frac{4}{B^2r}\Big(\frac{1}{2r} + \frac{A'}{A}\Big), \tag{28}$$

it is clear that the A and A' in Eq. (27) only present in the form of (A'/A), just as in the case of Eq. (20). Consequently, the relation (21) is preserved, and the static metric so (25) is obtained again.

Note that the integral in (22) is performed over the vacuum region, therefore the distribution and motion of the internal source matter can not influence $\widetilde{A}(r)$ any way. The only property of the source matter may appear in $\widetilde{A}(r)$ is the total mass, or, more generally speaking, the total charge. We then come to the conclusion that the spherically symmetric vacuum solution of the $f(T)$ gravity must be static, and is independent of the radial distribution and motion of the source matter, implying that the Birkhoff's theorem still holds generally.

4 Discussions and Conclusions

The Birkhoff's theorem is a significant feature of the theory of general relativity, in analogy to the Gauss theorem in electromagnetism or classical Newtonian gravity. It has been confirmed to be valid in the Palatini formalism of $f(R)$ gravity, while it no longer holds in the metric formalism generally[7,19,20]. Similar to the origination of $f(R)$ theories, the $f(T)$ gravity is extended from the teleparallel gravity which is equivalent to the theory of general relativity. We prove in this brief report that the Birkhoff's theorem holds in a typical $f(T)$ model with the form of power law, and also holds in general $f(T)$ gravities. The validity of this conclusion is independent of the concrete form of the $f(T)$ models. As a consequence, the significant inference in the theory of general relativity, such as the in-existence of the monopole gravitational wave, is also possessed in the $f(T)$ gravities.

The Birkhoff's theorem in the theory of general relativity leads to a second inference which is often discussed in literature as that the vacuum space-time inside a spherically distributed matter is flat. It has few actual applications in astrophysics observations indeed, since there is usually no vacuum cavity in celestial bodies. Nevertheless, it is often considered as a help to understand the fashion of the cosmic expansion [21,22]. Because of the globally isotropy and homogeneity properties of the observed universe in large scales, the distribution of distant matter is close to spherically symmetric about us. Though a great number of galaxies at Hubble distance is moving away from us at the relativistic speeds, the local effect can be neglected. That is to say, in the background of the expanding Friedmann-Robertson-Walker(FRW) universe, the ambient vacuum of a spherically symmetric star or galaxy can be regarded as in the static Schwarzschild geometry. It explains that the celestial bodies move along Newtonian trajectories which are impervious to the cosmic expansion.

It should be mentioned that the above inference, the second one, which is not involved in the proof of this work directly, is authentically valid only in Newtonian gravity and the theory of general relativity without the cosmological constant. It generally does not hold in modified gravity [23]. It is just approximately correct even in the standard ΛCDM model of cosmology. The non-vanishing cosmological constant Λ in Einstein's field equation is equivalent to a special matter with the state parameter $\omega = -1$ everywhere, which breaks the vacuum condition of the inside space. From the local point of view, with the presence of cosmological constant Λ, the vacuum gravitational field is not described by the Schwarzschild solution but by the Schwarzschild-de Sitter solution:

$$\mathrm{d}s^2 = \Big(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\Big)\mathrm{d}t^2-\Big(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\Big)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2 \tag{29}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The spacetime is not absolutely flat in the vacuum cavity where the total mass or charge $M = 0$. For the current value given by the standard model of cosmology, $\Lambda \sim 10^{-52}\mathrm{m}^{-2}$, the term $\frac{\Lambda}{3}r^2$ in the above metric can be neglected in the scale of solar system where $r \sim 10^{13}$m when compared with the first term O(1). Other topics involving the local influences of the cosmic expansion have also been studied (e.g. [24,25]), showing that the effects are inconsiderable. In the context of the $f(T)$ gravity, the validity of the second inference is beyond the scope of this present paper, for it involves the concrete solution of particular $f(T)$ models, which still needs and is worth of further studies. We will leave it for future work.

Besides the popular studies of the f(T) gravity models to cosmology for mimicking the dark energy behaviors, there are also still lots of interesting topics in its astrophysics applications, which in some sense might be more practical and obviously is worthy of further investigations.

Acknowledgement

We thank Prof. Lewis H Ryder for lots of interesting discussions on possible roles the torsion may play in gravity and cosmology physics during the project over years. This work is partly supported by Natural Science Foundation of China under Grant Nos.11075078 and 10675062 and by the project of knowledge Innovation Program (PKIP) of Chinese Academy of Sciences (CAS) under the grant No. KJCX2.YW.W10 through the KITPC where we have initiated this present work.

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