Periodic solutions for some second order Hamiltonian systems*

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Abstract
We use the saddle point theorem of Benci–Rabinowitz to study the existence of periodic solutions with a fixed energy for second order Hamiltonian conservative systems without any symmetry; the key difficulty of the proof is proving the Palais–Smale condition and the non-constant property for the minimax critical point.

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1. Introduction

We study the Newtonian equation in which the force F is generated by a potential $V(q)$:

$$\ddot{q} = F = -\frac{\partial V(q)}{\partial q}, \tag{1.1}$$

where $q = q(t) = (q_1, \ldots, q_n) \in C^2(R, R^n)$, $V \in C^1(R^n, R)$, $\partial V(q)/\partial q$ denotes the gradient of $V(q)$ with respect to position q.

Define the Hamilton function:

$$H = H(\dot{q}, q) = \frac{1}{2}|\dot{q}|^2 + V(q),$$

then it is well known that H is an integral of the system (1.1), the corresponding Hamiltonian system is

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}, \tag{1.2}$$

where $p = \dot{q}$.

* Dedicated to the memories of Professor Shi Shuzhong and all who died in the earthquake on 12 May 2008.

It is a natural problem to ask whether (1.1) has a periodic solution with a fixed energy h. More generally, in 1948, Seifert [17] used geometric and topological methods to study Hamiltonian systems with H:

$$H\left(\frac{\mathrm{d}q}{\mathrm{d}t}, q\right) = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} a_{ij}(q) \frac{\mathrm{d}q_i}{\mathrm{d}t} \frac{\mathrm{d}q_j}{\mathrm{d}t} + V(q) = Q + V, \tag{1.3}$$

where $\{a_{ij}(q)\}$ is a positive definite matrix.

Theorem 1.1 (Seifert, 1948). *If $a_{ij}(q)$ and $V(q)$ are real analytic in $G \subset R^n$, $V = H$ and $\partial V(q)/\partial q \neq 0$ on ∂G, $V < H$ in G and $\bar{G}$ is homeomorphic to the unit ball in R^n. Then the Lagrange equations corresponding to $(Q - V)$ have a periodic solution with energy H.*

For the general first order Hamiltonian systems (1.2), it is more difficult than (1.1).

In 1978, Rabinowitz [14] used variational methods for strongly indefinite functionals to study the existence of a periodic solution of (1.2) with a fixed energy. He obtained the following famous result:

Theorem 1.2 (Rabinowitz, 1978). *Let $H \in C^1(R^{2n}, R)$. Suppose*

(H1) *for some $b \neq 0$, $H^{-1}(b)$ is radially homeomorphic to S^{2n-1}.*
(H2) $H_z(\zeta) \neq 0, \forall \zeta \in H^{-1}(b)$.

Then the Hamiltonian system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = J H_z \tag{1.4}$$

possesses a periodic solution on $H^{-1}(b)$, where $z = (p, q) \in R^{2n}$, $H = H(p, q)$, $H_z = \left(\frac{\partial H}{\partial p}, \quad \frac{\partial H}{\partial q}\right)$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}_{2n \times 2n}$.

For the Hamiltonian systems (1.1)–(1.4), after Rabinowitz, there were many works. One can refer [7, 11, 12, 16, 20] etc and references therein; here we mention only some related works to this paper.

In 1979, Rabinowitz [15] generalized Seifert's result. He studied a Hamiltonian of the form

$$H(p, q) = K(p, q) + V(q), \tag{1.5}$$

and he obtained

Theorem 1.3 (Rabinowitz 1979). *If $K(p, q) \in C^2(R^{2n}, R)$ satisfies $(\partial K/\partial p) \cdot p > 0, \forall p \neq 0$, $V(q) \in C^1(R^n, R)$ and $D = \{q \in R^n | V(q) < 1\}$ is C^2 diffeomorphic to a unit ball in R^n and $(\partial V(q)/\partial q) \neq 0$ on $\partial D, K(0, q) = 0, q \in D$ and for fixed $q \in \bar{D}$ and $p \in S^{n-1}$,*

$$\lim_{\alpha \to \infty} K(\alpha p, q) > 1 - V(q).$$

Then (1.1) has a periodic solution on the energy surface $H^{-1}(1)$.

In the 1980s, Benci [5], Gluck and Ziller [8] and Hayashi [9] used totally different methods to prove.

Theorem 1.4 (Benci–Gluck-Ziller–Hayashi). *Suppose $V \in C^2(R^n, R)$ and*

$$\Omega = \{q \in R^n | V(q) < h\} \tag{1.6}$$

is bounded and non-empty, then the Hamiltonian system (1.1) has at least one periodic solution of energy h.

The proof of Gluck–Ziller and Hayashi used much of algebraic topology or differential geometry but rather functional analysis, Benci used the singular potential well and approximation scheme and the least action principle of Maupertuis–Jacobi which leads to a problem of differential geometry.

Let Ω be an open set in R^n with C^2-boundary, he considered the metric

$$\mathrm{d}\rho = \sqrt{h - V(x)}\,\mathrm{d}s,\ x \in \bar{\Omega}, \tag{1.7}$$

where $\mathrm{d}s$ is the Euclidean metric. Then he assumed

$$V(q) \neq 0, \qquad \forall q \in \partial\Omega \tag{1.8}$$

and proved that every closed geodesic, by a suitable re-parametrization of the independent variable (time), corresponds to a periodic solution of (1.1) of energy h. Here the closed geodesics are the critical points of the 'length' functional:

$$J(\gamma) = \int a(\gamma)|\dot{\gamma}|^2\,\mathrm{d}t, \qquad \gamma \in C^2(S^1, \bar{\Omega}). \tag{1.9}$$

Since $a(x)$ is degenerate when $x \to \partial\Omega$, so it is very difficult to study directly the functional (1.9). Benci [4, 5] used an approximation scheme which seems complex.

In the 1990s, when Ambrosetti and Coti Zelati [2, 3] studied the periodic solutions of singular Hamiltonian systems with a fixed energy, they presented a new variational functional different from $J(\gamma)$ in (1.9):

$$f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2\,\mathrm{d}t \int_0^1 (h - V(u))\,\mathrm{d}t, \qquad u \in W^{1,2}(R/Z, R^n). \tag{1.10}$$

They used the Ljusternik–Schnirelmann theory and the famous mountain-pass lemma of Ambrosetti–Rabinowitz to study the existence of the weak solutions for N–body problems ($N \geqslant 2$).

In this paper, we use the functional defined by Ambrosetti–Coti Zelati and the generalized mountain–pass lemma of Benci–Rabinowitz [6] to prove directly the existence of non-constant C^2-periodic solutions for some second order Hamiltonian systems. We notice that until now, no one has applied the famous generalized mountain–pass lemma of Benci–Rabinowitz to the nice concrete functional of Ambrosetti–Coti Zelati. The key point of our proof is to prove the Palais–Smale condition with positive level values and the non-constant property for the critical point. We discovered some intrinsic estimates for the second order Hamiltonian systems, and we notice that these estimates do not hold for the general first order Hamiltonian systems; our estimates may have other applications.

We have the following theorem:

Theorem 1.5. *Suppose $V \in C^1(R^n, R)$ satisfies:*

(V1) *There are constants $\mu_1 > 0$ and $\mu_2 > 0$ such that*

$$\langle V'(q), q\rangle \geqslant \mu_1 V(q) - \mu_2, \qquad \forall q \in R^n,$$

(V2) $V(q) \geqslant h$, *as* $|q| \to +\infty$

(V3) $V'(q) \to 0$ *as* $|q| \to +\infty$.

(V4) $V(q) \geqslant a|q|^{\mu_1} + b,\ a > 0,\ b \in R$.

(V5) $\limsup_{|q|\to 0} V(q) < h$.

Then for $\forall h > \mu_2/\mu_1$, the system (1.1) with energy h has at least a non-constant C^2-periodic solution which can be obtained by the saddle point theorem of Benci–Rabinowitz .

Remark. (V3) can be deleted.

2. Some lemmas

In order to prove theorem 1.1, it is well known [2] that we can define functional

$$f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2\,\mathrm{d}t \cdot \int_0^1 (h - V(u))\,\mathrm{d}t, \qquad \forall u \in H^1, \tag{2.1}$$

where

$$H^1 = W^{1,2}(R/Z, R^n). \tag{2.2}$$

Lemma 2.1 ([2,3]). *Let $\tilde{u} \in H^1$ be such that $f'(\tilde{u}) = 0$ and $f(\tilde{u}) > 0$.*
Set

$$\frac{1}{T^2} = \frac{\displaystyle\int_0^1 (h - V(\tilde{u}))\,\mathrm{d}t}{\displaystyle\frac{1}{2}\int_0^1 |\dot{\tilde{u}}|^2\,\mathrm{d}t}. \tag{2.3}$$

Then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant T-periodic solution for (1.1) and (1.2)in section 1.

By lemma 2.1, we have

Lemma 2.2. *If $\bar{u} \in H^1$ is a critical point of $f(u)$ and $f(\bar{u}) > 0$, then $\bar{q}(t) = \bar{u}(t/T)$ is a non-constant T-periodic solution of (1.1) and (1.2) in section 1.*

Lemma 2.3 (Sobolev–Rellich–Kondrachov, compact imbedding theorem [1,12,20,22]).

$$W^{1,2}(R/TZ, R^n) \subset C(R/TZ, R^n)$$

and the imbedding is compact.

Lemma 2.4 (Eberlein–Shmulyan [21] [10]). *A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.*

Lemma 2.5 ([12,22]). *Let $q \in W^{1,2}(R/TZ, R^n)$,*

(i) if $\int_0^T q(t)\,\mathrm{d}t = 0$, then we have the Poincare–Wirtinger's inequality

$$\int_0^T |\dot{q}(t)|^2\,\mathrm{d}t \geqslant \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2\,\mathrm{d}t;$$

(ii) if $q(0) = q(T) = 0$, then we have the Friedrics–Poincaré inequality:

$$\int_0^T |\dot{q}(t)|^2\,\mathrm{d}t \geqslant \left(\frac{\pi}{T}\right)^2 \int_0^T |q(t)|^2\,\mathrm{d}t;$$

(iii) if $\int_0^T q(t)\,\mathrm{d}t = 0$, we have the Sobolev's inequality:

$$\max_{0\leqslant t\leqslant T} |q(t)| = \|q\|_\infty \leqslant \sqrt{\frac{T}{12}}\left(\int_0^T |\dot{q}(t)|^2\,\mathrm{d}t\right)^{1/2}.$$

We define the equivalent norms in $H^1 = W^{1,2}(R/TZ, R^n)$:

$$\|u\|_{H^1} = \left(\int_0^1 |\dot{u}|^2\,\mathrm{d}t\right)^{1/2} + |u(0)|$$

or

$$\|u\|_{H^1} = \left(\int_0^1 |\dot{u}|^2\,\mathrm{d}t\right)^{1/2} + \left|\int_0^1 u(t)\,\mathrm{d}t\right|.$$

Lemma 2.6 (Benci–Rabinowitz [6], generalized mountain-pass lemma). *Let X be a Banach space, $f \in C(X, R)$ satisfies $(\mathrm{PS})^+$ condition. Let $X = X_1 \bigoplus X_2$, $\dim X_1 < +\infty$,*

$$B_a = \{x \in X \mid \|x\| \leqslant a\},$$

$$S = \partial B_\rho \cap X_2, \rho > 0,$$

$$\partial Q = (B_R \cap X_1) \cup (\partial B_R \cap (X_1 \bigoplus R^+ e)), R > \rho,$$

where $e \in X_2$, $\|e\| = 1$,

$$\partial B_R \cap (X_1 \bigoplus R^+ e) = \{x_1 + se \mid (x_1, s) \in X_1 \times R^+, \|x_1\|^2 + s^2 = R^2\},$$

$$Q = \{x_1 + se \mid (x_1, s) \in X_1 \times R^1, s \geqslant 0, \|x_1\|^2 + s^2 \leqslant R^2\}.$$

If

$$f|_S \geqslant \alpha > 0,$$

and

$$f|_{\partial Q} \leqslant 0,$$

then $C = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)) \geqslant \alpha$ and is a critical value for f, where,

$$\Gamma = \{\phi \in C(Q, X), \phi|_{\partial Q} = \mathrm{id}\}.$$

References [7, 11, 13, 18, 19] gave simpler proofs, or applications, of Benci–Rabinowitz's theorem.

3. The proof of theorems 1.1

Lemma 3.1. *If* (V1)–(V3) *and $h > \mu_2/\mu_1$ hold, $f(u)$ satisfies the $(\mathrm{PS})^+$ condition on H^1.*

Proof. Let $\{u_n\} \subset H^1$ satisfy

$$0 < d \leqslant f(u_n) \leqslant C, \qquad f'(u_n) \to 0. \tag{3.1}$$

Firstly, we claim $\{u_n\}$ is bounded. By $f(u_n) \leqslant C$, we have

$$-\frac{1}{2}\|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 V(u_n)\,\mathrm{d}t \leqslant C - \frac{h}{2}\|\dot{u}_n\|_{L^2}^2. \tag{3.2}$$

By (V1) we have

$$\begin{aligned}\langle f'(u_n), u_n\rangle &= \|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 \left(h - V(u_n) - \frac{1}{2}\langle V'(u_n), u_n\rangle\right)\mathrm{d}t \\ &\leqslant \|\dot{u}_n\|_{L^2}^2 \int_0^1 \left[h + \frac{\mu_2}{2} - \left(1 + \frac{\mu_1}{2}\right)V(u_n)\right]\mathrm{d}t.\end{aligned} \tag{3.3}$$

By (3.2) and (3.3) we have

$$\begin{aligned}\langle f'(u_n), u_n\rangle &\leqslant \left(h + \frac{\mu_2}{2}\right)\|\dot{u}_n\|_{L^2}^2 + \left(1 + \frac{\mu_1}{2}\right)\left(2C - h\|\dot{u}_n\|_{L^2}^2\right) \\ &= \left(-\frac{\mu_1}{2}h + \frac{\mu_2}{2}\right)\|\dot{u}_n\|_{L^2}^2 + C_1 = a\|\dot{u}\|_{L^2}^2 + C_1,\end{aligned} \tag{3.4}$$

where $C_1 = 2(1 + (\mu_1/2))C$, $a = -(\mu_1/2)h + (\mu_2/2)$.

By $f'(u_n) \to 0$, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$|\langle f'(u_n), u_n\rangle| \leqslant C_2 + C_3\|u_n\| = C_2 + C_3(\|\dot{u}_n\|_{L^2} + |u_n(0)|). \tag{3.5}$$

By (3.4) and (3.5) we have

$$-(C_2 + C_3\|u_n\|) \leqslant \left(-\frac{\mu_1}{2}h + \frac{\mu_2}{2}\right)\|\dot{u}_n\|^2_{L^2} + C_1. \tag{3.6}$$

If $\|\dot{u}_n\|_{L^2}$ is unbounded, then since $h > \mu_2/\mu_1$, $|u_n(0)|$ must be unbounded and there exists a subsequence, still denoted by $\{u_n\}$ s.t.

$$|u_n(0)| \geqslant b\|\dot{u}_n\|^2_{L^2}, \qquad b > 0. \tag{3.7}$$

By the Newton–Leibniz formula and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\min_{0\leqslant t\leqslant 1}|u_n(t)| &\geqslant |u_n(0)| - \|\dot{u}_n\|_2 \\ &\geqslant b\|\dot{u}_n\|^2_2 - \|\dot{u}_n\|_2 \to +\infty, \qquad \text{as n} \to +\infty.\end{aligned} \tag{3.8}$$

So by (V2) we have

$$\int_0^1 V(u_n)\,dt \geqslant h, \qquad \text{as n} \to +\infty. \tag{3.9}$$

$$\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty}\frac{1}{2}\int_0^1 |\dot{u}_n|^2\,dt\int_0^1 (h - V(u_n))\,dt \leqslant 0. \tag{3.10}$$

This contradicts $f(u_n) \geqslant C > 0$. So $\|\dot{u}_n\|_{L^2} \leqslant M_1$.

We notice that

$$\begin{aligned}f'(u_n)\cdot(u_n - u_n(0)) =& \int_0^1 |\dot{u}_n|^2\,dt\int_0^1 (h - V(u_n))\,dt \\ & -\frac{1}{2}\int_0^1 |\dot{u}_n|^2\,dt\int_0^1 \langle V'(u_n), u_n - u_n(0)\rangle\,dt \\ & = 2f(u_n) - \frac{1}{2}\int_0^1 |\dot{u}_n|^2\int_0^1 \langle V'(u_n), u_n - u_n(0)\rangle\,dt.\end{aligned} \tag{3.11}$$

Then we claim $|u_n(0)|$ is bounded.

Otherwise, there a subsequence, still denoted by u_n s.t. $|u_n(0)| \to +\infty$. Since $\|\dot{u}_n\| \leqslant M_1$, then

$$\min_{0\leqslant t\leqslant 1}|u_n(t)| \geqslant |u_n(0)| - \|\dot{u}_n\|_2 \to +\infty, \qquad \text{as n} \to +\infty. \tag{3.12}$$

Then by (V3) we have

$$V'(u_n) \to 0, \tag{3.13}$$

By Friedrics–Poincaré's inequality ,we have

$$\int_0^1 |\dot{u}_n(t)|^2\,dt \geqslant \pi^2\int_0^1 |u_n(t) - u_n(0)|^2\,dt, \tag{3.14}$$

$$\int_0^1 V'(u_n)(u_n - u_n(0))\,dt \to 0, \tag{3.15}$$

$$f'(u_n)\cdot(u_n - u_n(0)) \to 0. \tag{3.16}$$

So $f(u_n) \to 0$, this is a contradication, hence $u_n(0)$ is bounded, and $\|u_n\| = \|\dot{u}_n\|_{L^2} + |u_n(0)|$ is bounded.

By the embedding theorem, $\{u_n\}$ has a weakly convergent subsequence which is uniformly converges to $u \in H^{1,2}$.

Hence

$$V(u_n) \to V(u), \qquad \langle V'(u_n), u_n \rangle \to \langle V'(u), u \rangle. \tag{3.17}$$

Furthermore, it is similar to the one by Ambrosetti–Coti Zelati [3]; the weakly convergent subsequence is also strongly convergent to $u \in H^{1,2}$.

Since (PS) sequence u_n is bounded in H^1, so by Sobolev's embedding inequality, we know it is also bounded in the maximum norm. By the continuity of V, $V(u_n)$ is also uniformly bounded in maximum norm, so by $f(u_n) \geqslant d > 0$, we have

$$0 < d \leqslant f(u_n) = \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 \int_0^1 (h - V(u_n))\, dt \leqslant \frac{e}{2}\|\dot{u}_n\|_{L^2}^2, \tag{3.18}$$

that is

$$\|\dot{u}_n\|_{L^2}^2 \geqslant \frac{2d}{e} > 0, \qquad \forall n \in N. \tag{3.19}$$

It is easy to know that

$$\langle f'(u_n), u_n \rangle = \|\dot{u}_n\|_{L^2}^2 \int_0^1 \left[h - V(u_n) - \frac{1}{2}\langle V'(u_n), u_n \rangle \right] dt. \tag{3.20}$$

Hence by (3.20), we have

$$\int_0^1 (h - V(u_n))\, dt = \frac{1}{2}\int_0^1 \langle V'(u_n), u_n \rangle\, dt + \frac{\langle f'(u_n), u_n \rangle}{\|u_n\|^2}. \tag{3.21}$$

From (3.17), (3.19) and (3.21) and $\langle f'(u_n), u_n \rangle \leqslant \|f'(u_n)\| \cdot \|u_n\| \to 0$, we deduce

$$\int_0^1 (h - V(u_n))\, dt \to \frac{1}{2}\int_0^1 \langle V'(u), u \rangle\, dt. \tag{3.22}$$

From $f(u_n) > 0$, we deduce

$$\int_0^1 (h - V(u_n))\, dt \geqslant 0. \tag{3.23}$$

Since $\|\dot{u}_n\|_{L^2}^2$ is bounded, so if

$$\int_0^1 (h - V(u_n))\, dt \to 0 \tag{3.24}$$

then

$$f(u_n) = \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 \int_0^1 (h - V(u_n))\, dt \to 0. \tag{3.25}$$

This is impossible by $f(u_n) \geqslant d > 0$. Hence from (3.17) and (3.22), we have

$$\int_0^1 (h - V(u))\, dt = \frac{1}{2}\int_0^1 \langle V'(u), u \rangle\, dt > 0. \tag{3.26}$$

By $f'(u_n) \to 0$, we have $\langle f'(u_n), v \rangle \to 0$, that is

$$\int_0^1 \dot{u}_n \dot{v}\, dt \int_0^1 (h - V(u_n))\, dt - \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 \int_0^1 \langle V'(u_n), v \rangle\, dt \to 0, \qquad \forall v \in H^1. \tag{3.27}$$

Take $v = u$ in (3.27) and we use (3.26) to get

$$\lim_{n\to\infty} \int_0^1 \dot{u}_n \cdot \dot{u}\, dt = \lim_{n\to\infty} \|\dot{u}_n\|_{L^2}^2. \tag{3.28}$$

By $u_n \rightharpoonup u$ weakly, we have

$$\int_0^1 \dot{u}_n \cdot \dot{u} \, dt + |u_n(0) \cdot u(0)| \to \int_0^1 |\dot{u}|^2 \, dt + |u(0)|^2. \tag{3.29}$$

By the Sobolev embedding theorem, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$ s.t. $u_n(0) \to u(0)$.

We notice

$$\begin{aligned} \|u_n - u\| &= \left(\int_0^1 |\dot{u}_n - \dot{u}|^2 \, dt \right)^{1/2} + |u_n(0) - u(0)| \\ &= \left(\int_0^1 |\dot{u}_n|^2 \, dt - 2 \int_0^1 \dot{u}_n \dot{u} \, dt + \int_0^1 |\dot{u}|^2 \, dt \right)^{1/2} + |u_n(0) - u(0)| \\ &\to (\|\dot{u}\|_{L^2}^2 - 2\|\dot{u}\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2 t)^{1/2} + 0 = 0. \end{aligned} \tag{3.30}$$

that is, $u_n \to u$ strongly in E.

Remark. If (V3) is deleted, lemma 3.1 is still true.

Now we prove theorem 1.1. In Benci–Rabinowitz's saddle point theorem, we take

$$X_1 = R^n, X_2 = \left\{ u \in W^{1,2}(R/Z, R^n), \int_0^1 u \, dt = 0 \right\},$$

$$S = \left\{ u \in X_2 | \left(\int_0^1 |\dot{u}_2|^2 \, dt \right)^{1/2} = \rho \right\},$$

$$\partial Q = \{u_1 \in R^n || u_1| = R\} \cup$$

$$\{u = u_1 + se, u_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|u\| = (|u_1(0)|^2 + s^2)^{1/2} = R > \rho\}.$$

If $u \in X_2$, by Sobolev's inequality we have

$$\|u\| \geqslant \sqrt{12} |u|_\infty.$$

Hence if $\|u\| \leqslant \delta \to 0$, then $\|u\|_\infty \leqslant \delta \to 0$. By V(5), for $\|u\|_\infty$ small, there exists $\epsilon > 0$ such that $V(q) \leqslant h - \epsilon$, so we have

$$f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 \, dt \cdot \int_0^1 (h - V(u)) \, dt \geqslant \frac{1}{2} \epsilon \|u\|^2.$$

$$f|_S \geqslant \frac{1}{2} \varepsilon \rho^2 > 0.$$

On the other hand, if $u \in X_1$, then we have

$$-\int_0^1 V(u) \, dt \to -\infty, |u| = R \to +\infty;$$

if

$$u \in \{u = u_1 + se, u_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|u\| = (|u_1(0)|^2 + s^2)^{1/2} = R > \rho\},$$

then by (V4) and Jensen's inequality, we have

$$\begin{aligned}
&-\int_0^1 V(u_1+se)\,\mathrm{d}t\\
&\leqslant -\int_0^1 (a|u_1+se|^{\mu_1}+b)\,\mathrm{d}t\\
&\leqslant -\left[a\left(\int_0^1 |u_1+se|^2\,\mathrm{d}t\right)^{\mu_1/2}+b\right]\\
&= -a\left[|u_1|^2+s^2\int_0^1 |e(t)|^2\,\mathrm{d}t\right]^{\mu_1/2}-b\\
&\to -\infty, s\to+\infty(R\to+\infty).
\end{aligned}$$

So if R is large enough, we have

$$f|_{\partial Q}\leqslant 0.$$

By lemma (3.1), f satisfies $(\mathrm{PS})^+$, so f has a critical value $C>0$ the corresponding critical point is non-constant by the definition of the functional $f(u)$.

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