Variational Minimizing Solutions for Planar Circular Restricted 3-Body Problems*

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Abstract We use some inequalities to study planar Newtonian circular restricted 3-body problems with two equal primaries, we prove that the minimizer of the Lagrangian action on "8" type symmetric loop spaces of the rotational coordinate systems is just at the center of masses, which implies that we must add topological conditions in order to get the true "8"–type solutions.

Key Words: Restricted 3–body problems, "8" type loops, variational methods.
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1. Introduction

We study planar restricted 3-body problems ([1]). Suppose point masses m_1 and m_2 move around their center of mass in circular orbits. Choose units of length, time and mass so that the angular velocity of rotation, the sum of masses of m_1 and m_2, and the gravitational constant are all equal to one. Then for this choice the distance between m_1 and m_2 is also equal to 1.

Consider the motion of an asteroid m_3 in the plane of the orbits of m_1 and m_2. We assume that the mass of m_3 is considerably smaller than the masses of m_1 and m_2 so we neglect the influence of m_3 on the motion of the two larger bodies.

It is convenient to pass to a moving reference frame which rotates with unit angular velocity around the center of mass of m_1 and m_2, in this frame m_1 and m_2 are at rest. We choose coordinates x, y in the moving frame so that the points m_1 and m_2 lie invariably on the x–axis and their center of mass is the origin of the coordinate system. Then the equations governing the motion of the asteroid can be written in the following form:

$$\ddot{x} = 2\dot{y} + \frac{\partial V}{\partial x} \tag{1.1}$$

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$$\ddot{y} = -2\dot{x} + \frac{\partial V}{\partial y} \tag{1.2}$$

$$V = \frac{x^2 + y^2}{2} + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \tag{1.3}$$

Where $\mu = m_2, 1 - \mu = m_1$ and ρ_1, ρ_2 are the distances from $q_3 = (x, y)$ to q_1 and q_2, and m_1 and m_2 have coordinates $q_1 = (-\mu, 0), q_2 = (1-\mu, 0)$

$$\rho_1 = \sqrt{(x+\mu)^2 + y^2}, \rho_2 = \sqrt{(x-1+\mu)^2 + y^2} \tag{1.4}$$

Given $T > 0, \mu = 1/2, m_1 = m_2 = 1/2$, we look for "8" type noncollision $T-$ periodic solutions of (1.1)-(1.3).

We define the Lagrangian action on a "8"−type symmetry space:

$$f(q_3) = f(x,y) = \int_0^T \left[\frac{1}{2}(|\dot{x}|^2 + |\dot{y}|^2) + \frac{1}{2}(x^2+y^2) + (x\dot{y} - y\dot{x})\right] dt$$

$$+\frac{1}{2}\int_0^T \left(\frac{1}{\sqrt{(x+\frac{1}{2})^2 + y^2}} + \frac{1}{\sqrt{(x-\frac{1}{2})^2 + y^2}}\right) dt \tag{1.5}$$

$$q_3 = (x,y) \in \Lambda = \left\{ \begin{array}{l} q_3 = (x,y) | x, y \in W^{1,2}(R/TZ, R), \\ q_3(t + T/2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} q_3(t), \\ q_3(-t) = -q_3(t), \\ q_3(t) \neq q_1 = (-1/2, 0), q_2 = \left(\frac{1}{2}, 0\right). \end{array} \right\} \tag{1.6}$$

Where

$$W^{1,2}(R/TZ, R) = \left\{ x(t) \left| \begin{array}{c} x(t), \dot{x}(t) \in L^2(R, R) \\ x(t+T) = x(t) \end{array} \right. \right\} \tag{1.7}$$

Theorem 1.1 The minimizer of $f(q_3)$ on the closure $\bar{\Lambda}$ of Λ is just the center of masses.

2. The Proof of Theorem 1.1

We define the inner product and norm of $W^{1,2}$:

$$\langle x, y \rangle = \int_0^T (xy + \dot{x}\dot{y}) dt \tag{2.1}$$

$$||x|| = (\int_0^T x^2 dt + \int_0^T |\dot{x}|^2 dt)^{1/2} \tag{2.2}$$

Lemma 2.1(Palais's symmetry principle ([8])) Let σ be an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for $\forall \sigma \in G, f(\sigma \circ x) = f(x)$, where $f : H \to R$.

Let $Fix = \{x \in H | \sigma \circ x = x,\ \forall \sigma \in G\}$. Then the critical point of f in Fix is also a critical point of f in H.

By Palais' symmetry Principle, we know that the critical point of $f(x,y)$ in Λ is a "8"–type noncollision periodic solution of (1.1)–(1.3).

Lemma 2.2 For any $(x,y) \in \Lambda$, we have the integral of the angular momentum for the third body:

$$\int_0^T (x(t)\dot{y}(t) - y(t)\dot{x}(t))dt = 0 \tag{2.3}$$

Proof For $(x,y) \in \Lambda$ we have

$$x(t + \frac{T}{2}) = -x(t) \tag{2.4}$$

$$y(t + T/2) = y(t) \tag{2.5}$$

Hence we have

$$\int_0^T x(t)\dot{y}(t)dt = -\int_0^T x(t+\frac{T}{2})\dot{y}(t+\frac{T}{2})dt = -\int_0^T x(\tau)\dot{y}(\tau)d\tau \tag{2.6}$$

$$\int_0^T y(t)\dot{x}(t)dt = y(t)x(t)|_0^T - \int_0^T x(t)\dot{y}(t)dt = -\int_0^T x(\tau)\dot{y}(\tau)d\tau \tag{2.7}$$

Lemma 2.3 For any $(x,y) \in \Lambda$, we have the integral averages:

$$\int_0^T x(t)dt = \int_0^T y(t)dt = 0 \tag{2.8}$$

Proof. $x(-t) = -x(t), y(-t) = -y(t)$ and $x(t+T) = x(t), y(t+T) = y(t)$ imply (2.8).

In order to prove Theorem 1.1, we need some inequalities:

Lemma 2.4 (Poincare–Wirtinger [6]) Let $q \in W^{1,2}(R/TZ, R^n)$ and $\int_0^T q(t)dt = 0$, then

$$(i) \quad \int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt \tag{2.9}$$

(ii) Inequality (2.9) takes the equality if and only if

$$q(t) = \alpha \cos\frac{2\pi}{T}t + \beta \sin\frac{2\pi}{t}t, \alpha, \beta \in R^n \tag{2.10}$$

Lemma 2.5(Jensen,[6]) 1°. Assume ϕ is a convex function on [r,R], $-\infty \leq r \leq R \leq +\infty$, $\hat{f}$ and p are integrable functions on $[c,d], -\infty \leq c \leq d \leq +\infty, r \leq \hat{f}(x) \leq R, \hat{p}(x) \geq 0, \forall x \in [c,d]$ and $\int_c^d \hat{p}(x)dx > 0$, then

$$\phi\left(\frac{\int_c^d \hat{p}(x)\hat{f}(x)dx}{\int_c^d \hat{p}(x)dx}\right) \leq \frac{\int_c^d \hat{p}(x)\phi(\hat{f}(x))dx}{\int_c^d \hat{p}(x)dx} \tag{2.11}$$

2° Inequality (2.11) takes the equality if and only if

$$\hat{f}(x) = const \tag{2.12}$$

Now we prove Theorem 1.1:

Lemma 2.6([7]) For $m_i > 0, \alpha > 0$, we have

$$\sum_{1\leq i\leq l, l+1\leq j\leq N} \frac{m_i m_j}{|q_i - q_j|^\alpha} \geq \left(\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j\right)^{1+\frac{\alpha}{2}} \cdot \left(\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j |q_i - q_j|^2\right)^{-\frac{\alpha}{2}}, \tag{2.13}$$

and the above inequality takes the equality if and only if

$$|q_i(t) - q_j(t)| = \lambda(t) > 0, \quad 1 \leq i \leq l,\ l+1 \leq j \leq N, \tag{2.14}$$

Proof by Cauchyy-Schwarz inequality we have

$$\begin{aligned}\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j &= \sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j \frac{|q_i - q_j|^{\alpha/2}}{|q_i - q_j|^{\alpha/2}} \\ &\leq \left(\sum_{1\leq i\leq l, l+1\leq j\leq N} \frac{m_i m_j}{|q_i - q_j|^\alpha}\right)^{1/2} \\ &\quad \cdot \left(\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j |q_i - q_j|^\alpha\right)^{1/2},\end{aligned} \tag{2.15}$$

By Holder inequality we have

$$\begin{aligned}&\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j |q_i - q_j|^\alpha \\ \leq &\left(\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j\right)^{\frac{2-\alpha}{2}} \left(\sum_{1\leq i\leq l, l+1\leq j\leq N} m_i m_j |q_i - q_j|^2\right)^{\alpha/2},\end{aligned} \tag{2.16}$$

Hence the inequality (2.13) is proved. By the necessary and sufficient conditions which make the Cauchy-Schwarz inequality and Holder inequality become the equalities, we know (2.13) takes the equality if and only if

$$|q_i(t) - q_j(t)| = \lambda(t) > 0, \quad 1 \leq i \leq l,\ l+1 \leq j \leq N.$$

Since $\int_0^T q_3(t)dt = 0$, so by Poincare–Wirtinger inequality we have

$$f(q_3) \geq [\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 + 1/2] \int_0^T |q_3|^2 dt + \frac{1}{2}\int_0^T |q_3 - q_1|^{-1} dt + \frac{1}{2}\int_0^T |q_3 - q_2|^{-1} dt \tag{2.17}$$

By (2.13), we have

$$f(q_3) \geq \left(\frac{2\pi^2}{T^2}+\frac{1}{2}\right)\int_0^T |q_3|^2 dt + 2^{1/2}\int_0^T \left[|q_3-q_1|^2+|q_3-q_2|^2\right]^{-1/2} \tag{2.18}$$

By Jensen's inequality we have

$$\begin{aligned}
f(q_3) &\geq (\frac{2\pi^2}{T^2}+1/2)\int_0^T |q_3|^2 dt + 2^{\frac{1}{2}}T^{3/2}\left[\int_0^T (|q_3-q_1|^2+|q_3-q_2|^2)dt\right]^{-1/2} \\
&= \left(\frac{2\pi^2}{T^2}+\frac{1}{2}\right)\int_0^T \left[|q_3-q_1|^2+|q_3-q_2|^2\right]dt \\
&\quad +2^{\frac{1}{2}}\cdot T^{3/2}\cdot\left\{\int_0^T \left[|q_3-q_1|^2+|q_3-q_2|^2\right]dt\right\}^{-\frac{1}{2}} \\
&\quad -\left(\frac{2\pi^2}{T^2}+\frac{1}{2}\right)\cdot\left[-2\int_0^T q_3\cdot q_1+\int_0^T |q_1|^2-2\int_0^T q_3\cdot q_2+\int_0^T |q_2|^2\right] \\
&= \varphi(s)=\left(\frac{2\pi^2}{T^2}+\frac{1}{2}\right)s^2+2^{\frac{1}{2}}T^{3/2}\cdot s^{-1} \\
&\quad -\left(\frac{2\pi^2}{T^2}+\frac{1}{2}\right)\left(\frac{T}{2}\right)\geq \inf\left\{\varphi(\mathrm{s}),\mathrm{s}>0\right\}, && (2.19) \\
& && (2.20)
\end{aligned}$$

where

$$s^2=\int_0^T \left[|q_3-q_1|^2+|q_3-q_2|^2\right]dt \tag{2.21}$$

We notice that $\varphi(s)$ is a strictly convex smooth function on $s>0$ and $\varphi(s)\to+\infty$ as $s\to 0^+$ and $s\to+\infty$, so $\varphi(s)$ attains its infimum at some $s_0>0$.

We notice that the inequality (2.20)take the equalities if and only if Poincare–Wirtinger's inequality and (2.13) and Jensen's inequality take the equalities simultaneously, hence we have

$$q_3(t)=\alpha\cos\frac{2\pi}{T}t+\beta\sin\frac{2\pi}{T}t, \alpha,\beta\in R^n, \tag{2.22}$$

$$|q_3(t)-q_1|=|q_3(t)-q_2|, \tag{2.23}$$

$$|q_3(t)-q_1|^2+|q_3(t)-q_2|^2=const, \tag{2.24}$$

By (2.23) and (2.24) we have

$$|q_3(t)-q_1|^2=|q_3(t)-q_2|^2=\mathrm{const} \tag{2.25}$$

Let $\alpha=(a_1,b_1),\beta=(a_2,b_2)$. Then

$$\begin{aligned}
|q_3(t)-q_1|^2 &= (a_1\cos\frac{2\pi}{T}t+a_2\sin\frac{2\pi}{T}t+\frac{1}{2})^2 \\
&\quad +(b_1\cos\frac{2\pi}{T}t+b_2\sin\frac{2\pi}{T}t)^2=\mathrm{const} && (2.26)
\end{aligned}$$

Let $t=0$ and $t=T/2$ we have

$$(a_1+\frac{1}{2})^2+b_1^2=(-a_1+\frac{1}{2})^2+(-b_1)^2 \tag{2.27}$$

Then

$$a_1=0 \tag{2.28}$$

Let $t=T/4, \frac{3}{4}T$, we have

$$\left(\frac{1}{2}+a_2\right)^2+b_2^2=\left(\frac{1}{2}-a_2\right)^2+(-b_2)^2, \tag{2.29}$$

Hence

$$a_2=0 \tag{2.30}$$

By $a_1=a_2=0$ and (2.26), we have

$$\left|b_1\cos\frac{2\pi}{T}t+b_2\sin\frac{2\pi}{T}t\right|^2=\text{const} \tag{2.31}$$

Let $t=0, \frac{T}{4}$, we have

$$b_1^2=b_2^2 \tag{2.32}$$

Hence by (2.31) and (2.32) we have

$$b_1^2+b_1b_2\sin\frac{4\pi}{T}t=\text{const} \tag{2.33}$$

$$b_1=b_2=0 \tag{2.34}$$

So

$$q_3(t)=0 \tag{2.35}$$

3. A Remark

From Theorem 1.1, we know that the pure symmetry conditions can't imply the existence of the true "8"–type noncollision periodic solutions for planar circular restricted 3–body problems, so in order to obtain the true "8"–type noncollision periodic solutions, we must add suitable topological conditions.

We define

$$\Lambda_8=\{q_3\in\Lambda, deg(q_3-q_1)=1, deg(q_3-q_2)=-1\} \tag{3.1}$$

Conjecture There are $a,b>0$ such that for $T\in[a,b]$, the minimizer of $f(q_3)$ on $\overline{\Lambda_8}$ is a noncollision T–periodic solution of (1.1)–(1.3).

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