INVERSION FORMULAS FOR THE SPHERICAL MEANS IN CONSTANT CURVATURE SPACES

YURI A. ANTIPOV, RICARDO ESTRADA, AND BORIS RUBIN

Abstract. The work develops further the theory of the following inversion problem, which plays the central role in the rapidly developing area of thermoacoustic tomography and has intimate connections with PDEs and integral geometry: *Reconstruct a function f supported in an n-dimensional ball B, if the spherical means of f are known over all geodesic spheres centered on the boundary of B.* We propose a new unified approach based on the idea of analytic continuation. This approach gives explicit inversion formulas not only for the Euclidean space $\mathbb{R}^n$ (as in the original set-up) but also for arbitrary constant curvature space X, including the n-dimensional sphere and the hyperbolic space. The results are applied to inverse problems for a large class of Euler-Poisson-Darboux equations in constant curvature spaces of arbitrary dimension.

1. Introduction

The paper deals with the spherical mean operator, which is also known as the spherical mean Radon transform. Importance of this transformation in analysis and geometry and many of its properties (which are still surprising!) were indicated by many authors; see, e.g., [11, p. 699], [31, 57]. In recent years an interest to this object has grown tremendously in view of a series of challenging problems. One of them is characterization of sets of injectivity (and non-injectivity) of this transform; see [15, 4, 5, 56] and references therein. Another source of mathematical problems related to the spherical means is the rapidly developing thermoacoustic tomography (TAT), the revolutionary role of which in medical imaging was pointed out in many publications; see

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[1]-[3], [18, 19], [20]-[23], [30, 35], [34]-[36], [38], [46]-[50], [59, 61]. The present investigation belongs to this area.

Setting of the problem and motivation. Let f be an infinitely differentiable function with compact support in the open ball $B = \{x \in \mathbb{R}^n : |x| < R\}$; ∂B is the boundary of B. We consider the spherical mean Radon transform Mf which integrates f over spheres centered on ∂B:

$$(Mf)(\xi, t) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(\xi - t\theta)\, d\theta, \tag{1.1}$$

where

$$\xi \in \partial B, \qquad t \in \mathbb{R}_+ = (0, \infty),$$

S^{n-1} is the unit sphere in $\mathbb{R}^n$ with the area σ_{n-1}, and $d\sigma$ stands for the usual Lebesgue measure on S^{n-1}. For the classical Radon transforms, their modifications, and applications see, e.g., [12, 14, 26, 13, 29, 42].

The general problem of reconstructing f from known data $(Mf)(\xi, t)$ on the cylinder $\partial B \times \mathbb{R}_+$ is an immediate consequence of the following commonly accepted mathematical model of TAT in $\mathbb{R}^3$ (see, e.g., [36, 59, 30] and references therein):

Given a function $c(x)$, the speed of the ultrasound propagation in the tissue, and a function $g(\xi, t)$, the measured value of the pressure at the time t at the transducers location $\xi \in S^2$, find a function $f(x)$, the initial pressure distribution $p(x, 0)$ (the TAT image), if

$$\begin{cases} p_{tt} = c^2(x)\, \Delta p & \text{for all } t \geq 0,\ x \in \mathbb{R}^3, \\ p(x,0) = f(x),\ p_t(x,0) = 0 & \text{for all } x \in \mathbb{R}^3, \\ p(\xi, t) = g(\xi, t) & \text{for all } \xi \in S^2\, (\subset \mathbb{R}^3),\ t \geq 0. \end{cases} \tag{1.2}$$

Here, p_t and p_{tt} are the first and second time derivatives, and Δ is the Laplace operator with respect to the spatial variable x.

This problem admits immediate generalization to arbitrary dimensions, more general Riemannian spaces, and a broad class of differential equations of the Euler-Poisson-Darboux (EPD) type. In the Euclidean case, a thorough discussion of main inversion methods in terms of their assumptions and computational features can be found in [36, 59]. In the important particular case of constant speed $c(x)$, solution to this problem is equivalent to reconstruction of f from its spherical mean (1.1).

Explicit inversion formulas for Mf are of particular interest. For n odd, such formulas were obtained by Finch, Patch, and Rakesh in [19]. Another derivation was suggested by Palamodov [47, Section 7.5]; see also [21, 22, 61]. The corresponding formulas for n even were obtained by Finch, Haltmeier, and Rakesh in [18]. An explicit inversion formula which relies on completely different ideas and covers both odd and even cases, was suggested by Kunyansky [38].

In spite of the elegance and ingenuity, the derivation of the existing inversion formulas for Mf is pretty involved, and basic ideas behind it remain mysterious. In view of practical importance, it would be desirable to find an independent simple proof of known formulas and thus check their correctness. Moreover, the prospective new method should be applicable to more general geometric and analytic settings and thus lead to further progress.

A simple proof for n odd was suggested by the third author [54], who suggested to treat M as a member of a certain analytic family of operators and applied the results to the inverse problem of type (1.2) for the more general EPD equation in the case $c(x) \equiv \text{const}$.

In the present article we suggest a new approach, which is conceptually simple and leads to inversion formulas for Mf *for all* $n \geq 2$. As in [54], the key idea is analytic continuation, however, the reasoning is different. We extend our method to the similar problem for spherical means on the n-dimensional sphere and the hyperbolic space, where the theory of EPD equations is also well-developed. This extension seems to be new and paves the way to diverse settings, when the relevant geodesic balls and spherical means are considered in more general Riemannian spaces.

Regarding generalizations to general Riemannian spaces, some comments are in order. The corresponding wave equations and their EPD generalizations were studied in [39, 32, 33], [43]-[45]. For example, the wave equation on the n-dimensional sphere S^n has the form [39]

$$\delta_x u = u_{\omega\omega} + \left(\frac{n-1}{2}\right)^2 u, \qquad (x,\omega) \in S^n \times (0,\pi), \tag{1.3}$$

where δ_x denotes the Beltrami-Laplace operator. The Cauchy problem for the relevant EPD equation

$$\tilde{\Box}_\alpha u = 0, \qquad u(x,0) = f(x), \quad u_\omega(x,0) = 0, \tag{1.4}$$

where

$$\tilde{\Box}_\alpha u = \delta_x u - u_{\omega\omega} - (n-1+2\alpha)\cot\omega\, u_\omega + \alpha(n-1+\alpha)u, \tag{1.5}$$

and various modifications were discussed in [9, 10, 24, 32, 33, 44]. The problem (1.4) for the case $\alpha = 0$, corresponding to the usual Darboux equation, was studied by Olevskii [43] and also by Kipriyanov and Ivanov [32]. Our definition of the EPD-equation on S^n differs from that in [32] and agrees with [44]. The particular case $\alpha = (1-n)/2$, corresponding to the wave equation (1.3), can be regarded as the spherical analogue of the TAT model (1.2) with constant speed.

Plan of the paper and main results. Section 2 contains preliminaries. Here the main statement is Lemma 2.2. For convenience of the reader and better treatment of the subject, we supply this lemma with alternative proofs, which are based on different ideas and use different tools, while leading to the same result. All these are presented in Appendix. Section 3 contains derivation of inversion formulas for Mf in the Euclidean case. The main inversion results are presented in Theorems 3.4 and 3.7; see also modified inversion formulas (3.21), (3.22). The results of Section 3 are applied in Section 4 to the Cauchy problem for the Euler-Poisson-Darboux equation

$$(1.6) \quad \square_\alpha u \equiv \Delta u - u_{tt} - \frac{n+2\alpha-1}{t} u_t = 0, \quad u(x,0)=f(x), \; u_t(x,0)=0,$$

where f is a smooth function with compact support in the ball B. Using the results of Section 3 combined with known properties of Erdélyi-Kober fractional integrals, we give explicit solution (Theorem 4.1) to the following inverse problem:

Given the trace $u(\xi,t)$ of the solution of (1.6) for all (ξ,t) on the cylindrical surface $\partial B \times \mathbb{R}_+$, reconstruct $f(x)$.

The particular case $\alpha = (1-n)/2$ gives explicit solution to the TAT problem (1.2) with constant speed $c(x) \equiv 1$.

The spherical mean Radon transform on the n-dimensional unit sphere S^n in $\mathbb{R}^{n+1}$ is studied in Section 5. Inversion formulas for this transform are given in Theorems 5.3 and 5.5. The relevant inverse problem for the EPD equation on S^n is solved in Section 6. Section 7 contains derivation of inversion formulas for the spherical mean Radon transform in the n-dimensional hyperbolic space. Here the main results are given by Theorems 7.3 and 7.5 .

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2. Auxiliary statements

Notation. We use abbreviation *a.c.* to denote analytic continuation; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$. We write $d\theta$ $(d\xi)$ for the usual Lebesgue measure on S^{n-1} (on ∂B, resp.); $[a]$ denotes the integer part of a real number a; $(\cdot)^\lambda_+$ means $(\cdot)^\lambda$ if the expression in parentheses is positive and zero, otherwise.

We will need the following lemmas.

Lemma 2.1. *Let $\varphi \in C_c^\infty(\mathbb{R})$.*
(i) *If $m = 0, 1, 2, \ldots,$ then*

$$\underset{\alpha=-2m}{a.c.} \int\limits_{\mathbb{R}} \frac{|t|^{\alpha-1}}{\Gamma(\alpha/2)}\, \varphi(t)\, dt = c_{m,1}\, \varphi^{(2m)}(0), \qquad c_{m,1} = \frac{(-1)^m\, m!}{(2m)!}. \tag{2.1}$$

(ii) *If $m = 1, 2, \ldots,$ then*

$$\underset{\alpha=1-2m}{a.c.} \int\limits_{\mathbb{R}} \frac{|t|^{\alpha-1}}{\Gamma(\alpha/2)}\, \varphi(t)\, dt \;=\; c_{m,2} \int\limits_{\mathbb{R}} \frac{\varphi^{(2m-1)}(t)}{t}\, dt \tag{2.2}$$

$$= \; -c_{m,2} \int\limits_{\mathbb{R}} \varphi^{(2m)}(t)\, \log|t|\, dt, \tag{2.3}$$

where $c_{m,2} = (\Gamma(1/2-m)(2m-1)!)^{-1}$ and the integral on the right-hand side of (2.2) is understood in the principal value sense.

Proof. Both statements summarize known facts from [27, Chapter 1, Sec. 3]. For instance, (ii) can be proved as follows. Using the equality

$$|t|^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+2m-1)}\, (|t|^{\alpha+2m-2}\, \mathrm{sgn} t)^{(2m-1)},$$

we write the left-hand side of (2.2) in the form

$$-\underset{\alpha=1-2m}{a.c.} \frac{\Gamma(\alpha)}{\Gamma(\alpha+2m-1)\,\Gamma(\alpha/2)}\, (|t|^{\alpha+2m-2}\, \mathrm{sgn} t, \varphi^{(2m-1)}(t)).$$

The latter yields the principal value integral

$$\frac{1}{\Gamma(1/2-m)(2m-1)!} \int\limits_{\mathbb{R}} \frac{\varphi^{(2m-1)}(t)}{t}\, dt,$$

which coincides with (2.3). □

Lemma 2.2. *Let $n > 2$, $|h| < 1$.*
(i) *The integral*

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{-1}^{1} |t-h|^{\alpha-1}\, (1-t^2)^{(n-3)/2}\, dt, \qquad Re\, \alpha > 0, \tag{2.4}$$

extends as an entire function of α and this extension represents a C^∞ function of h uniformly in $\alpha \in K$ for any compact subset K of the complex plane.
(ii) *Moreover,*

$$\underset{\alpha=3-n}{a.c.}\, g_\alpha(h) = \Gamma((n-1)/2). \tag{2.5}$$

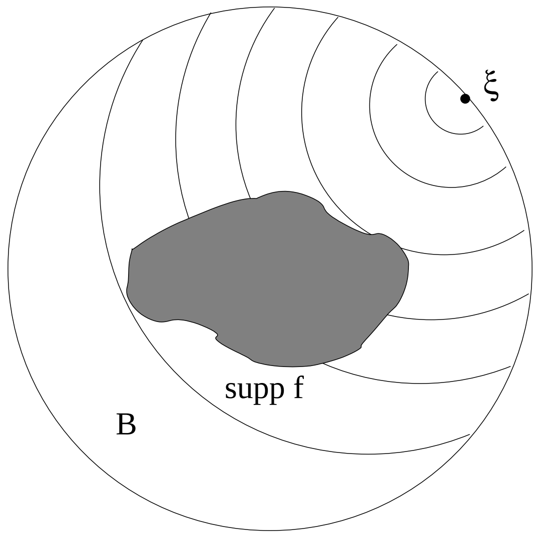

FIGURE 1. The Euclidean case.

The proof of this lemma is given in Appendix.

3. THE EUCLIDEAN CASE. DERIVATION OF THE INVERSION FORMULA

We recall that our aim is to reconstruct a C^∞ function f supported in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$ provided that the spherical means

$$(Mf)(\xi, t) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(\xi - t\sigma)\, d\sigma, \qquad (\xi, t) \in \partial B \times \mathbb{R}_+,$$

are known for all spheres centered on the boundary ∂B of B (Fig. 1).

We introduce the "back-projection" operator P that sends a function $F(\xi, t)$ on $\partial B \times \mathbb{R}_+$ to a function $(PF)(x)$ on B by the formula

$$(PF)(x) = \frac{1}{|\partial B|} \int\limits_{\partial B} F(\xi, |x - \xi|)\, d\xi, \qquad x \in B, \tag{3.1}$$

where $d\xi$ stands for the surface element of ∂B and $|\partial B|$ denotes the area of ∂B.

3.1. **The case** $n > 2$**.** Consider the following analytic family of operators

$$(N^\alpha f)(\xi, t) = \int\limits_B \frac{|t^2 - |y - \xi|^2|^{\alpha - 1}}{\Gamma(\alpha/2)} f(y)\, dy, \tag{3.2}$$

$$(\xi, t) \in \partial B \times \mathbb{R}_+, \qquad Re\, \alpha > 0.$$

Lemma 3.1. *Let f be an infinitely differentiable function supported in $B = \{x \in \mathbb{R}^n : |x| < R\}$. Then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \lambda_n \int\limits_B \frac{f(y)}{|x-y|^{n-2}}\, dy, \tag{3.3}$$

$$\lambda_n = (2R)^{2-n}\pi^{-1/2}\,\Gamma(n/2). \tag{3.4}$$

Proof. For $Re\,\alpha > 0$, changing the order of integration, we obtain

$$(PN^\alpha f)(x) = \int\limits_B f(y)\, k_\alpha(x,y)\, dy,$$

where

$$\begin{aligned} k_\alpha(x,y) &= \frac{1}{|\partial B|\,\Gamma(\alpha/2)} \int\limits_{\partial B} ||x-\xi|^2 - |y-\xi|^2|^{\alpha-1}\, d\xi \\ &= \frac{1}{\sigma_{n-1}\,\Gamma(\alpha/2)} \int\limits_{S^{n-1}} ||x|^2 - |y|^2 - 2R\theta\cdot(x-y)|^{\alpha-1}\, d\theta \\ &= \frac{(2R\,|x-y|)^{\alpha-1}}{\sigma_{n-1}} \int\limits_{S^{n-1}} \frac{|\theta\cdot\sigma - h|^{\alpha-1}}{\Gamma(\alpha/2)}\, d\theta, \end{aligned} \tag{3.5}$$

$$\sigma = \frac{x-y}{|x-y|}, \qquad h = \frac{|x|^2 - |y|^2}{2R\,|x-y|}.$$

By the rotation invariance of the inner product, the integral in (3.5) is independent of σ and can be written as

$$\frac{\sigma_{n-2}}{\Gamma(\alpha/2)} \int\limits_{-1}^{1} |t-h|^{\alpha-1}\,(1-t^2)^{(n-3)/2}\, dt = \sigma_{n-2}\, g_\alpha(h);$$

cf. (2.4). Note that $|h| < 1-\delta$ for some $\delta > 0$ because x and y belong to the support of f and the latter is separated from the boundary of B. Hence, Lemma 2.2 yields

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{(2R)^{2-n}\,\sigma_{n-2}}{\sigma_{n-1}} \int\limits_B \frac{f(y)}{|x-y|^{n-2}}\, \underset{\alpha=3-n}{a.c.}\, g_\alpha(h)\, dy$$

$$= \lambda_n \int\limits_B \frac{f(y)}{|x-y|^{n-2}}\, dy, \qquad \lambda_n = (2R)^{2-n}\pi^{-1/2}\,\Gamma(n/2)$$

(to justify interchange of integration and analytic continuation, the reader may consult, e.g., [52, Lemma 1.17]). □

Let us obtain another representation of the analytic continuation of $PN^\alpha f$, now, in terms of the spherical means of f.

Lemma 3.2. *Let f be an infinitely differentiable function supported in $B = \{x \in \mathbb{R}^n : |x| < R\}$,*

$$D = \frac{1}{2t}\frac{d}{dt}, \qquad \delta_n = \frac{(-1)^{[n/2-1]}\,\Gamma((n-1)/2)}{(n-3)!}. \tag{3.6}$$

(i) *If $n = 3, 5, \ldots,$ then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{2R^{n-1}} \int\limits_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|} d\xi. \tag{3.7}$$

(ii) *If $n = 4, 6, \ldots,$ then*

(3.8)

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi\, R^{n-1}} \int\limits_{\partial B} d\xi \int\limits_0^{2R} t\, D^{n-2}[t^{n-2}(Mf)(\xi,t)]\, \log|t^2 - |x-\xi|^2|\, dt.$$

Proof. Passing to polar coordinates, we have

$$(N^\alpha f)(\xi,t) = \sigma_{n-1} \int\limits_0^{2R} \frac{|t^2-r^2|^{\alpha-1}}{\Gamma(\alpha/2)}\,(Mf)(\xi,r)\, r^{n-1}\, dr$$

$$= \int\limits_0^{4R^2} \frac{|t^2-\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\,\varphi_\xi(\tau)\, d\tau, \quad \varphi_\xi(\tau) = \frac{\sigma_{n-1}}{2}\,\tau^{n/2-1}(Mf)(\xi,\tau^{1/2}).$$

Since the support of f is separated from the boundary of B, there is an $\varepsilon > 0$ such that $\varphi_\xi(\tau) \equiv 0$ when $\tau \notin (\varepsilon, 4R^2-\varepsilon)$. Hence, $\varphi_\xi(\tau)$ can be regarded as a function in $C_c^\infty(\mathbb{R})$ and we can write

$$(N^\alpha f)(\xi,t) = \int\limits_{\mathbb{R}} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\,\varphi_\xi(\tau+t^2)\, d\tau.$$

Now, Lemma 2.1 yields the following equalities.
For $n = 3, 5, \ldots$:

$$\underset{\alpha=3-n}{a.c.}\,(N^\alpha f)(\xi,t) = \delta_{n,1}\,\varphi_\xi^{(n-3)}(t^2), \quad \delta_{n,1} = \frac{(-1)^{(n-3)/2}\,((n-3)/2)!}{(n-3)!}.$$

For $n = 4, 6, \ldots$:

$$\underset{\alpha=3-n}{a.c.}\,(N^\alpha f)(\xi,t) = \delta_{n,2} \int\limits_{\mathbb{R}} \varphi_\xi^{(n-2)}(\tau)\, \log|\tau - t^2|\, d\tau,$$

$$\delta_{n,2} = -\frac{1}{\Gamma((3-n)/2)\,(n-3)!}.$$

Combining these formulas with the backprojection P and noting that operations *a.c.* and P commute, we obtain

For $n = 3, 5, \ldots$:

$$\underset{\alpha=3-n}{a.c.}(PN^{\alpha}f)(x) = \frac{\delta_{n,1}}{|\partial B|}\int\limits_{\partial B}\varphi_{\xi}^{(n-3)}(|x-\xi|^2)\,d\xi.$$

For $n = 4, 6, \ldots$:

$$\underset{\alpha=3-n}{a.c.}(PN^{\alpha}f)(x) = \frac{\delta_{n,2}}{|\partial B|}\int\limits_{\partial B}d\xi\int\limits_{0}^{4R^2}\varphi_{\xi}^{(n-2)}(\tau)\,\log|\tau-|x-\xi|^2|\,d\tau.$$

These formulas give the desired result. □

Comparing different forms of the analytic continuation in Lemmas 3.1 and 3.2, we obtain the following statement.

Lemma 3.3. *Let f be an infinitely differentiable function supported in the unit ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, $D = \frac{1}{2t}\frac{d}{dt}$. Then*

$$(3.9)\quad \lambda_n\int\limits_{B}\frac{f(y)}{|x-y|^{n-2}}\,dy$$

$$= \begin{cases} \dfrac{\delta_n}{2R^{n-1}}\displaystyle\int\limits_{\partial B}D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|}\,d\xi, \\ \text{if } n=3,5,\ldots, \\ \\ -\dfrac{\delta_n}{\pi\,R^{n-1}}\displaystyle\int\limits_{\partial B}d\xi\int\limits_{0}^{2R}t\,D^{n-2}[t^{n-2}(Mf)(\xi,t)]\,\log|t^2-|x-\xi|^2|\,dt, \\ \text{if } n=4,6,\ldots, \end{cases}$$

where λ_n and δ_n are defined by (3.4) and (3.6), respectively.

The left-hand side of (3.9) is a constant multiple of the Riesz potential of order 2 defined by

$$(3.10)\qquad (I^2f)(x) = \frac{\Gamma(n/2-1)}{4\pi^{n/2}}\int_B\frac{f(y)\,dy}{|x-y|^{n-2}}$$

and satisfying

$$(3.11)\qquad -\Delta I^2 f = f, \qquad \Delta = \sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}.$$

Thus, we arrive at the following result.

Theorem 3.4. *Let f be an infinitely differentiable function supported in the unit ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, $D = \frac{1}{2t}\frac{d}{dt}$.*
(i) *If $n = 3, 5, \dots,$ then*

$$f(x) = d_{n,1}\,\Delta \int\limits_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|}\,d\xi, \tag{3.12}$$

$$d_{n,1} = \frac{(-1)^{(n-1)/2}\,\pi^{1-n/2}}{4R\,\Gamma(n/2)}.$$

(ii) *If $n = 4, 6, \dots,$ then*

$$f(x) = d_{n,2}\,\Delta \int\limits_{\partial B} d\xi \int\limits_0^{2R} t\,D^{n-2}[t^{n-2}(Mf)(\xi,t)]\,\log|t^2 - |x-\xi|^2|\,dt, \tag{3.13}$$

$$d_{n,2} = \frac{(-1)^{n/2-1}\,\pi^{-n/2}}{2R\,(n/2-1)!}.$$

3.2. **The case $n = 2$.** Let $\mathcal{D}$ be the open disk in $\mathbb{R}^2$ of radius R centered at the origin. In this section, for the sake of completeness, we reproduce (with minor changes) the argument from [18], keeping in mind that the Riesz potential of order 2 in the previous section is substituted by the logarithmic potential

$$(I_* f)(x) = \frac{1}{2\pi}\int\limits_{\mathcal{D}} f(y)\log|x-y|\,dy\,, \tag{3.14}$$

satisfying $\Delta I_* f = f$.

The following statement is a substitute for Lemma 2.2.

Lemma 3.5. *Let $-1 < h < 1$, $\sigma \in S^1$. Then*

$$g_* \equiv \int\limits_{S^1} \log|\theta\cdot\sigma - h|\,d\theta = -2\pi\log 2. \tag{3.15}$$

Proof. Owing to rotational invariance, we can write

$$g_* = 2\int_{-1}^{1} \frac{\log|t-h|}{\sqrt{1-t^2}}\,dt\,. \tag{3.16}$$

This integral is known; see, e.g., [17, p. 296], [18].[1] □

[1] A more general integral was evaluated in [7, Lemma 6.1].

Lemma 3.6. *Let f be a C^∞ function supported in $\mathcal{D}$. Then*

$$(3.17)\qquad (I_* f)(x) = \frac{1}{2\pi R} \int\limits_{\partial\mathcal{D}} \int\limits_0^{2R} (Mf)(\xi,t) \log\left|t^2 - |x-\xi|^2\right| \, t\,dt\,d\xi + c_f,$$

$$c_f = -\frac{\log R}{2\pi} \int\limits_{\mathcal{D}} f(y)\,dy.$$

Proof. Let

$$(N_* f)(\xi,t) = \int\limits_{\mathcal{D}} f(y) \log\left|t^2 - |y-\xi|^2\right| \, dy.$$

Changing the order of integration and making use of (3.15) with

$$\sigma = \frac{x-y}{|x-y|}, \qquad h = \frac{|x|^2 - |y|^2}{2R\,|x-y|},$$

we obtain

$$(PN_* f)(x) = \int\limits_{\mathcal{D}} f(y)\, k_*(x,y)\,dy,$$

where

$$\begin{aligned} k_*(x,y) &= \frac{1}{2\pi R} \int\limits_{\partial\mathcal{D}} \log\left||x-\xi|^2 - |y-\xi|^2\right| \, d\xi \\ &= \frac{1}{2\pi R} \int\limits_{\partial\mathcal{D}} \log\left||x|^2 - |y|^2 - 2\xi\cdot(x-y)\right| \, d\xi \\ &= \frac{1}{2\pi} \int\limits_{S^1} \left(\log(2R\,|x-y|) + \log|h - \theta\cdot\sigma|\right) \, d\theta \\ &= \log(2R\,|x-y|) + \frac{g_*}{2\pi} = \log R + \log|x-y|. \end{aligned}$$

This gives

$$(3.18)\qquad (PN_* f)(x) = \int\limits_{\mathcal{D}} f(y)\, \log|x-y|\,dy + \log R \int\limits_{\mathcal{D}} f(y)\,dy.$$

On the other hand, $(PN_* f)(x)$ can be expressed in terms of the spherical means. Indeed, passing to polar coordinates, we have

$$(N_* f)(\xi,t) = 2\pi \int\limits_0^{2R} (Mf)(\xi,r) \log|r^2 - t^2|\, r dr$$

and therefore,

$$(PN_*f)(x) = \frac{1}{R}\int_{\partial\mathcal{D}}\int_0^{2R} (Mf)\,(\xi, r)\log\left|r^2 - |x-\xi|^2\right|\, r\,dr\,d\xi. \tag{3.19}$$

Comparing (3.18) and (3.19), we arrive at (3.17). □

Lemma 3.6 allows us complete Theorem 3.4 in the following way.

Theorem 3.7. *Let f be an infinitely differentiable function supported in the disk $\mathcal{D} = \{x \in \mathbb{R}^2 : |x| < R\}$. Then*

$$f\,(x) = \Delta\left(\frac{1}{2\pi R}\int_{\partial\mathcal{D}}\int_0^{2R} (Mf)\,(\xi, t)\log\left|t^2 - |x-\xi|^2\right|\, t\,dt\,d\xi\right). \tag{3.20}$$

Formula (3.20) can be formally obtained from (3.13) by setting $n = 2$. It coincides with formula (1.4) in [18].

3.3. **Modified inversion formulas.** We can replace f by Δf in (3.9) Since f is smooth and supp$f$ is separated from the boundary of B, then $I^2\Delta f = -f$. Furthermore, since $u(x,t) \equiv (Mf)(x,t)$ satisfies the Darboux equation

$$\Box u \equiv \Delta u - u_{tt} - \frac{n-1}{t}\,u_t = 0$$

and Δ commutes with rotations and translations, then

$$(M\Delta f)(x,t) = (\Delta Mf)(x,t) = L[(Mf)(x,\cdot)](t), \qquad \forall x \in \mathbb{R}^n, \quad t > 0,$$

where

$$L = \frac{d^2}{dt^2} + \frac{n-1}{t}\,\frac{d}{dt};$$

see, e.g., [29, p. 17]. This reasoning and its obvious analogue for $n = 2$ give the following modifications of inversion formulas (3.12), (3.13), and (3.20) with the same constant factors:

(i) If $n = 3, 5, \ldots$, then

$$f(x) = d_{n,1}\int_{\partial B} D^{n-3}[t^{n-2}(LMf)(\xi,t)]\Big|_{t=|x-\xi|}\,d\xi. \tag{3.21}$$

(ii) If $n = 2, 4, 6, \ldots$, then

$$f(x) = d_{n,2}\int_{\partial B} d\xi \int_0^{2R} t\,D^{n-2}[t^{n-2}(LMf)(\xi,t)]\,\log|t^2 - |x-\xi|^2|\,dt. \tag{3.22}$$

Formula (3.21) agrees with [54, formula (3.8)].

4. Spherical means and EPD equations

Consider the Cauchy problem for the Euler-Poisson-Darboux equation:

$$\Box_\alpha u \equiv \Delta u - u_{tt} - \frac{n+2\alpha-1}{t}\, u_t = 0\,, \tag{4.1}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{4.2}$$

As in the previous section, we assume that f is a smooth function with compact support in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$. If $\alpha \geq (1-n)/2$, then (4.1)-(4.2) has a unique solution

$$u(x,t) = (M^\alpha f)(x,t), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{4.3}$$

where $M^\alpha f$ is defined as analytic continuation of the integral

$$(M^\alpha f)(x,t) = \frac{\Gamma\,(\alpha+n/2)}{\pi^{n/2}\Gamma\,(\alpha)} \int\limits_{|y|<1} (1-|y|^2)^{\alpha-1} f(x-ty)\, dy, \qquad Re\,\alpha > 0;$$

see [8] for details. If $\alpha = 0$, then $M^0 f \equiv \underset{\alpha=0}{a.c.} M^\alpha f$ represents the spherical mean (1.1).

Consider the following problem:

Given the trace $u\,(\xi,t)$ of the solution of (4.1) - (4.2) for all $(\xi,t) \in \partial B \times \mathbb{R}_+$, reconstruct $f(x)$.

To solve this problem we need some facts from fractional calculus; see, e.g., [55, Sec. 18.1] or [17, Sec. 9.6]. For $Re\,\alpha > 0$ and $\eta \geq -1/2$, the Erdélyi-Kober fractional integral of a function φ on $\mathbb{R}_+$ is defined by

$$(I^\alpha_\eta \varphi)(t) = \frac{2t^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t (t^2 - r^2)^{\alpha-1} r^{2\eta+1} \varphi(r)\, dr, \qquad t > 0. \tag{4.4}$$

In our case it suffices to assume that φ is infinitely smooth and supported away from the origin. Then $I^\alpha_\eta \varphi$ extends as an entire function of α and η, so that $I^0_\eta \varphi = \varphi$, $\left(I^\alpha_\eta\right)^{-1} \varphi = I^{-\alpha}_{\eta+\alpha} \varphi$,

$$(I^{-m}_\eta \varphi)(t) = t^{-2(\eta-m)} D^m\, t^{2\eta} \varphi(t), \qquad D = \frac{1}{2t}\,\frac{d}{dt}.$$

Assuming $x = \xi \in \partial B$ and passing to polar coordinates, we obtain

$$u_\xi(t) = (M^\alpha f)(\xi,t) = \frac{\Gamma\,(\alpha+n/2)}{\Gamma\,(n/2)} \left(I^\alpha_\eta \varphi_\xi\right)(t)\,, \tag{4.5}$$

where $u_\xi\,(t) = u(\xi, t)$, $\varphi_\xi\,(t) = (Mf)(\xi, t)$, $\eta = n/2 - 1$. This gives

$$\varphi_\xi = \frac{\Gamma(n/2)}{\Gamma(\alpha + n/2)} \left(I^\alpha_\eta\right)^{-1} u_\xi = \frac{\Gamma(n/2)}{\Gamma(\alpha + n/2)} I^{-\alpha}_{\eta+\alpha} u_\xi. \tag{4.6}$$

Now, since $\varphi_\xi(t) = (Mf)(\xi, t)$ is known, we can use Theorem 3.4 to reconstruct f by the following formulas.

Theorem 4.1. *Let f be an infinitely differentiable function supported in the unit ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, $D = \frac{1}{2t}\frac{d}{dt}$.*
(i) *If $n = 3, 5, \ldots,$ then*

$$f(x) = \widetilde{d}_{n,1}\, \Delta \int\limits_{\partial B} D^{n-3}[t^{n-2}(I^{-\alpha}_{\eta+\alpha} u_\xi)\,(t)]\Big|_{t=|x-\xi|}\, d\xi,$$

$$\widetilde{d}_{n,1} = \frac{(-1)^{(n-1)/2}\, \pi^{1-n/2}}{4R\,\Gamma(\alpha + n/2)}.$$

(ii) *If $n = 2, 4, 6, \ldots,$ then*

$$f(x) = \widetilde{d}_{n,2}\, \Delta \int\limits_{\partial B} d\xi \int\limits_0^{2R} t\, D^{n-2}[t^{n-2}(I^{-\alpha}_{\eta+\alpha} u_\xi)\,(t)]\, \log|t^2 - |x - \xi|^2|\, dt,$$

$$\widetilde{d}_{n,2} = \frac{(-1)^{n/2-1}\, \pi^{-n/2}}{2R\,\Gamma(\alpha + n/2)}.$$

The case $\alpha = (1 - n)/2$ in this theorem gives explicit solution to the TAT problem (see Introduction) with constant speed $c(x) \equiv 1$. Moreover, after f has been found, we can reconstruct $u(x, t)$ in the whole space by setting $u(x, t) = (M^\alpha f)(x, t)$. The latter gives an explicit solution to the Cauchy problem for the generalized EPD equation (4.1) with initial data on the cylinder $\partial B \times \mathbb{R}_+$.

5. Spherical Means on S^n

The suggested method of analytic continuation enables us to study the spherical mean Radon transform Mf on arbitrary constant curvature space X. In this setting, supp$f \subset B$, where B is a geodesic ball centered at the origin, and the spherical means of f are evaluated over geodesic spheres, the centers of which are located on the boundary of B. In this section we consider the case, when $X = S^n$ is the n-dimensional sphere in $\mathbb{R}^{n+1}$.

Given $x \in S^n$ and $t \in (-1, 1)$, let

$$(Mf)(x, t) = \frac{(1 - t^2)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{x \cdot y = t} f(y)\, d\sigma(y) \tag{5.1}$$

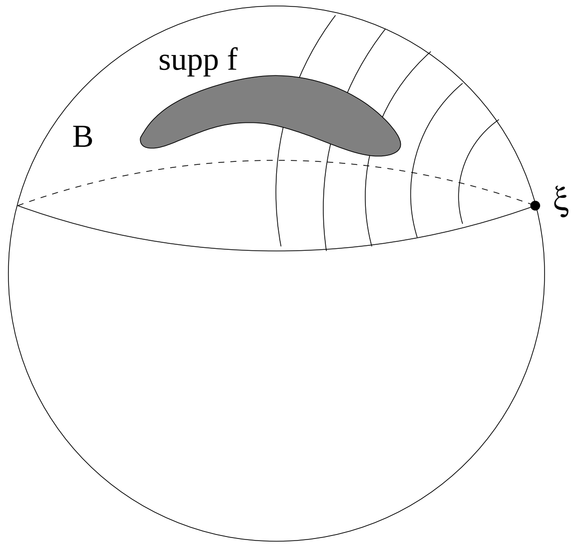

FIGURE 2. The spherical case.

be the mean value of a function $f \in C^\infty(S^n)$ over the planar section $\{y \in S^n : x \cdot y = t\}$ (Fig.2).

Our aim is to reconstruct f under the following assumptions:

(a) The support of f lies on the spherical cap

$$B_\theta = \{x \in S^n : x \cdot e_{n+1} > \cos\theta\}, \tag{5.2}$$

(the geodesic ball of radius θ), where $e_{n+1} = (0, \ldots, 0, 1)$ is the north pole of S^n and $\theta \in (0, \pi/2]$ is fixed.

(b) The mean values (5.1) are known for all $x = \xi \in \partial B_\theta$ and all $t \in (-1, 1)$, where ∂B_θ is the boundary of B_θ.

This problem can be solved using the method of the previous section. Let us fix our notation. In the following $S^n_+ = \{x \in S^n : x_{n+1} \geq 0\}$ is the upper hemisphere, B_0 denotes the unit ball in the hyperplane $x_{n+1} = 0$; S^{n-1} stands for the boundary of B_0, which is also the boundary of S^n_+. For $x \in S^n_+$ we write

$$x = (x', \sqrt{1 - |x'|^2}), \qquad x' = (x_1, \ldots, x_n, 0) \in B_0,$$

so that

$$\begin{aligned}\int\limits_{S^n_+} f(x)\, dx &= \int\limits_{B_0} f(x', \sqrt{1-|x'|^2}) \sqrt{1 + \left(\frac{\partial x_{n+1}}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial x_{n+1}}{\partial x_n}\right)^2}\, dx' \\ &= \int\limits_{B_0} f(x', \sqrt{1-|x'|^2})\, \frac{dx'}{\sqrt{1-|x'|^2}}.\end{aligned}$$

We introduce the backprojection operator P, that sends functions on $\partial B_\theta \times (-1,1)$ to functions on B_θ by the formula

$$(PF)(x) = \frac{1}{|\partial B_\theta|} \int\limits_{\partial B_\theta} F(\xi, \xi \cdot x)\, d\xi, \qquad x \in B_\theta. \tag{5.3}$$

Here $d\xi$ and $|\partial B_\theta|$ denote the surface element and the area of ∂B_θ, respectively. We denote by $\tilde{B}_\theta$ the orthogonal projection of B_θ onto the hyperplane $x_{n+1} = 0$.

5.1. **The case $n > 2$.** Assuming $(\xi, t) \in \partial B_\theta \times (-1,1)$ and $Re\,\alpha > 0$, consider the following analytic family of operators

$$(N^\alpha f)(\xi, t) = \int\limits_{B_\theta} \frac{|\xi \cdot y - t|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)\, dy. \tag{5.4}$$

Lemma 5.1. *Let $f \in C^\infty(S^n)$, $\text{supp} f \subset B_\theta$. Then*

$$\underset{\alpha=3-n}{a.c.} (PN^\alpha f)(x) = \frac{\Gamma(n/2)\,(\sin\theta)^{2-n}}{\pi^{1/2}} \int\limits_{\tilde{B}_\theta} \frac{\tilde{f}(y')\, dy'}{|x'-y'|^{n-2}}, \tag{5.5}$$

$$\tilde{f}(y') = (1-|y'|^2)^{-1/2} f(y', (1-|y'|^2)^{1/2}). \tag{5.6}$$

Proof. For $Re\,\alpha > 0$, changing the order of integration, we obtain[2]

$$(PN^\alpha f)(x) = \int\limits_{B_\theta} f(y)\, k_\alpha(x,y),\, dy, \quad k_\alpha(x,y) = \frac{1}{|\partial B_\theta|} \int\limits_{\partial B_\theta} \frac{|\xi \cdot (x-y)|^{\alpha-1}}{\Gamma(\alpha/2)} d\xi.$$

Since ξ has the form $\xi = e_{n+1} \cos\theta + \omega \sin\theta$, $\omega \in S^{n-1}$, then

$$\begin{aligned} |\xi \cdot (x-y)| &= |(x_{n+1} - y_{n+1}) \cos\theta + (x'-y') \cdot \omega \ \sin\theta| \\ &= |h - \omega \cdot \sigma|\, |x'-y'| \sin\theta, \end{aligned} \tag{5.7}$$

$$h = \frac{x_{n+1} - y_{n+1}}{|x'-y'|} \cot\theta, \qquad \sigma = \frac{x'-y'}{|x'-y'|}. \tag{5.8}$$

Hence,

$$k_\alpha(x,y) = \frac{(|x'-y'| \sin\theta)^{\alpha-1}}{\sigma_{n-1}} \int\limits_{S^{n-1}} \frac{|h-\omega\cdot\sigma|^{\alpha-1}}{\Gamma(\alpha/2)} d\omega. \tag{5.9}$$

[2]For the sake of convenience, we use some notations which mimic analogous expressions in the Euclidean case.

The integral in (5.9) is independent of ξ and can be written as

$$\frac{\sigma_{n-2}}{\Gamma(\alpha/2)}\int_{-1}^{1}|t-h|^{\alpha-1}\,(1-t^2)^{(n-3)/2}\,dt=\sigma_{n-2}\,g_\alpha(h);$$

cf. (2.4). This gives

$$k_\alpha(x,y)=\frac{\sigma_{n-2}\,(|x'-y'|\,\sin\theta)^{\alpha-1}}{\sigma_{n-1}}\,g_\alpha(h). \tag{5.10}$$

Let us show that $|h|<1$. We write

$$x=e_{n+1}\,\cos\,\gamma+u\sin\,\gamma,\qquad y=e_{n+1}\,\cos\,\delta+v\sin\,\delta,$$

$$\gamma,\delta\in(0,\theta);\qquad u,v\in S^{n-1};\qquad \tilde h=\frac{x_{n+1}-y_{n+1}}{|x'-y'|}.$$

Then

$$\begin{aligned}|\tilde h|^2 &= \frac{(\cos\,\gamma-\cos\,\delta)^2}{|u\sin\gamma-v\sin\delta|^2}\\ &= \frac{(\cos\,\gamma-\cos\,\delta)^2}{\sin^2\gamma-2(u\cdot v)\sin\gamma\,\sin\delta+\sin^2\delta}\\ &\le \frac{(\cos\,\gamma-\cos\,\delta)^2}{\sin^2\gamma-2\sin\gamma\,\sin\delta+\sin^2\delta}=\frac{(\cos\,\gamma-\cos\,\delta)^2}{(\sin\gamma-\sin\delta)^2}.\end{aligned}$$

Without loss of generality, suppose that $\gamma\le\delta$. Then

$$|\tilde h|\le\frac{\cos\,\gamma-\cos\,\delta}{\sin\delta-\sin\gamma}=\tan\frac{\gamma+\delta}{2}<\tan\theta,$$

and therefore, $|h|=|\tilde h|\,\cot\theta<1$.

Since $|h|<1$, Lemma 2.2 yields

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x)=c_n\int_{B_\theta}\frac{f(y)\,dy}{|x'-y'|^{n-2}}=c_n\int_{\tilde B_\theta}\frac{\tilde f(y')\,dy'}{|x'-y'|^{n-2}},$$

$$\tilde f(y')=(1-|y'|^2)^{-1/2}\,f(y',(1-|y'|^2)^{1/2}),\quad c_n=\frac{\Gamma(n/2)\,(\sin\theta)^{2-n}}{\pi^{1/2}}.$$

□

As before, we need one more representation of $\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x)$, now in terms of the spherical means $(Mf)(\xi,t)$.

Lemma 5.2. *Let $f\in C^\infty(S^n)$, supp$f\subset B_\theta$. Then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x)=\frac{\delta_n}{(\sin\theta)^{n-1}}\int_{\partial B_\theta}(d/dt)^{n-3}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\Big|_{t=\xi\cdot x}\,d\xi,$$

if $n = 3, 5, \ldots,$ *and*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi\,(\sin\theta)^{n-1}} \int\limits_{\partial B_\theta} d\xi$$

$$\times \int\limits_{\cos\,2\theta}^{1} (d/dt)^{n-2}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\,\log|t-\xi\cdot x|\,dt,$$

if $n = 4, 6, \ldots,$ *where* δ_n *is defined by (3.6).*

Proof. For $Re\,\alpha > 0$, by making use of the formula

$$\int\limits_{S^n} f(y)\,a(\xi\cdot y)\,dy = \sigma_{n-1}\int\limits_{-1}^{1} a(\tau)(Mf)(\xi,\tau)\,(1-\tau^2)^{n/2-1}\,d\tau, \tag{5.11}$$

we have

$$(N^\alpha f)(\xi,t) = \frac{\sigma_{n-1}}{\Gamma(\alpha/2)}\int\limits_{-1}^{1}(Mf)(\xi,\tau)\,|\tau-t|^{\alpha-1}(1-\tau^2)^{n/2-1}\,d\tau$$

$$= \int\limits_{\mathbb{R}} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\,\varphi_\xi(\tau+t)\,d\tau, \qquad \varphi_\xi(\tau) = \sigma_{n-1}\,(Mf)(\xi,\tau)\,(1-\tau^2)_+^{n/2-1}.$$

Since f is smooth and its support is separated from the boundary ∂B_θ, then $(Mf)(\xi,\tau)$ is smooth in the τ-variable uniformly in ξ and vanishes identically in the respective neighborhoods of $\tau = \pm 1$. Thus, we can invoke Lemma 2.1 which yields the following equalities.
For $n = 3, 5, \ldots;$ $\cos\,2\theta < t < 1$:

$$\underset{\alpha=3-n}{a.c.}\,(N^\alpha f)(\xi,t) = \delta_n\,\varphi_\xi^{(n-3)}(t),$$

For $n = 4, 6, \ldots$:

$$\underset{\alpha=3-n}{a.c.}\,(N^\alpha f)(\xi,t) = -\frac{\delta_n}{\pi}\int\limits_{\cos\,2\theta}^{1}\varphi_\xi^{(n-2)}(\tau)\,\log|\tau-t|\,d\tau,$$

δ_n being defined by (3.6). The above formulas mimic those in the proof of Lemma 3.2 and the result follows. □

Lemmas 5.1 and 5.2 imply the following inversion result for the spherical means on S^n. In the statement below, $\Delta_{x'} = \partial_1^2 + \ldots + \partial_n^2$ is the usual Laplace operator in the x'-variable.

Theorem 5.3. *Let* $f \in C^\infty(S^n)$, $\mathrm{supp} f \subset B_\theta$. *Then*

$$(5.12)\quad f(x) = \frac{d_n\, x_{n+1}}{\sin\theta}\, \Delta_{x'} f_0(x', \sqrt{1-|x'|^2}), \quad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $f_0(x) \equiv f_0(x', \sqrt{1-|x'|^2})$ *has the following form:*

$$f_0(x) = -\int\limits_{\partial B_\theta} (d/dt)^{n-3}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\Big|_{t=\xi\cdot x} d\xi$$

if $n = 3, 5, \ldots,$ *and*

$$f_0(x) = \frac{1}{\pi}\int\limits_{\partial B_\theta} d\xi \int\limits_{\cos 2\theta}^{1} (d/dt)^{n-2}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\,\log|t-\xi\cdot x|\,dt$$

if $n = 4, 6, \ldots.$

5.2. **The case** $n = 2$. We keep the notation of section 5.1. Let

$$(5.13)\quad (I_* f)(x) \equiv (I_* f)(x', \sqrt{1-|x'|^2}) = \frac{1}{2\pi}\int\limits_{B_\theta} f(y)\log|x'-y'|\,dy,$$

so that

$$(5.14)\qquad \Delta_{x'}(I_* f)(x) = (1-|x'|^2)^{-1/2} f(x) = f(x)/x_3.$$

Lemma 5.4. *Let* f *be a* C^∞ *function supported in* B_θ. *Then*

$$(5.15)\qquad (I_* f)(x) = \frac{1}{|\partial B_\theta|}\int\limits_{\partial B_\theta}\int\limits_{-1}^{1} (Mf)(\xi,\tau)\log|\tau-\xi\cdot x|\,d\tau\,d\xi + c_f,$$

$$c_f = -\frac{1}{2\pi}\left(\log\frac{\sin\theta}{2}\right)\int\limits_{B_\theta} f(y)\,dy.$$

Proof. Let

$$(N_* f)(\xi,t) = \int\limits_{B_\theta} f(y)\log|\xi\cdot y - t|\,dy, \qquad (\xi,t)\in\partial B_\theta\times(-1,1).$$

Changing the order of integration, owing to (5.7), we obtain

$$(PN_* f)(x) = \int\limits_{B_\theta} f(y)\,k_*(x,y)\,dy,$$

where

$$\begin{aligned} k_*(x,y) &= \frac{1}{|\partial B_\theta|} \int\limits_{\partial B_\theta} \log|\xi\cdot(x-y)|\, d\xi \\ &= \frac{1}{2\pi}\int\limits_{S^1} [\log|x'-y'| + \log\sin\theta + \log|h-\omega\cdot\sigma|]\, d\omega \\ &= \log|x'-y'| + \log\sin\theta + \frac{g_*}{2\pi}, \end{aligned}$$

$$g_* \equiv \int\limits_{S^1} \log|h-\omega\cdot\sigma|\, d\omega = 2\int_{-1}^{1} \frac{\log|t-h|}{\sqrt{1-t^2}}\, dt = -2\pi\log 2;$$

cf. Lemma 3.5. This gives

$$(PN_* f)(x) = 2\pi\, (I_* f)(x) + \left(\log\frac{\sin\theta}{2}\right) \int\limits_{B_\theta} f(y)\, dy. \tag{5.16}$$

On the other hand, by (5.11),

$$(PN_* f)(x) = \frac{1}{\sin\theta} \int\limits_{\partial B_\theta} \int\limits_{-1}^{1} (Mf)(\xi,\tau)\log|\tau-\xi\cdot x|\, d\tau d\xi. \tag{5.17}$$

Comparing (5.17) with (5.16), we obtain the result. □

Lemma 5.4 allows us complete Theorem 5.3 in the following way.

Theorem 5.5. *Let f be an infinitely differentiable function supported in the spherical cap $B_\theta = \{x \in S^2 : x\cdot e_3 > \cos\theta\}$, $\theta \in (0,\pi/2]$. Then*

$$f(x) = \frac{x_3}{2\pi\sin\theta}\, \Delta_{x'} \int\limits_{\partial B_\theta} \int\limits_{-1}^{1} (Mf)(\xi,\tau)\log|\tau-\xi\cdot x|\, d\tau\, d\xi. \tag{5.18}$$

Formula (5.18) can be formally obtained from (5.12) by setting $n = 2$.

6. The inverse problem for the EPD equation on S^n

The Euler-Poisson-Darboux equation on S^n has the form

$$\tilde{\Box}_\alpha u \equiv \delta_x u - u_{\omega\omega} - (n-1+2\alpha)\cot\omega\, u_\omega + \alpha(n-1+\alpha)u = 0. \tag{6.1}$$

Here $x \in S^n$ is the space variable, $\omega \in (0,\pi)$ is the time variable, δ_x is the relevant Beltrami-Laplace operator. For the sake of simplicity, we restrict ourselves to the case $Re\,\alpha > -n/2$. In this case the corresponding Cauchy problem

$$\tilde{\Box}_\alpha u = 0, \qquad u(x,0) = f(x), \quad u_\omega(x,0) = 0, \tag{6.2}$$

with $f \in C^\infty(S^n)$ has a solution $u(x,\omega) = (M^\alpha f)(x, \cos\omega)$, where $(M^\alpha f)(x,t)$ is defined as analytic continuation of the integral

$$(M^\alpha f)(x,t) = \frac{c_{n,\alpha}}{(1-t^2)^{\alpha-1+n/2}} \int\limits_{x\cdot y>t} (x\cdot y - t)^{\alpha-1} f(y)dy, \tag{6.3}$$

$$c_{n,\alpha} = 2^{\alpha-1}\pi^{-n/2}\Gamma(\alpha+n/2)/\Gamma(\alpha), \quad Re\,\alpha > 0, \quad t\in(-1,1);$$

see [44], [53, p. 179], and references therein.

Let $B_\theta = \{x \in S^n : x\cdot e_{n+1} > \cos\theta\}$ be the spherical cap of a fixed radius $\theta \in (0, \pi/2]$ and let ∂B_θ be the boundary of B_θ. Our aim is to solve the following

Inverse problem. *Suppose that the values $g(\xi,\omega)$ of the solution of (6.2) are known for all $(\xi,\omega) \in \partial B_\theta \times (0,\pi)$. Reconstruct the initial function $f \in C^\infty(S^n)$, provided that the support of f lies in B_θ.*

This problem can be solved using the results of the previous section. Assuming $Re\,\alpha > 0$, we pass to spherical polar coordinates and write (6.3) as

$$(M^\alpha f)(\xi,t) = \frac{c_{n,\alpha}\,\sigma_{n-1}}{(1-t^2)^{\alpha-1+n/2}} \int\limits_t^1 (\tau-t)^{\alpha-1}(Mf)(\xi,\tau)\,(1-\tau^2)^{n/2-1}\,d\tau.$$

Then we set

$$F_\xi(t) = (Mf)(\xi,t)\,(1-t^2)^{n/2-1},$$

$$G_{\alpha,\xi}(t) = \frac{2^{1-\alpha}\pi^{n/2}}{\Gamma(\alpha+n/2)\,\sigma_{n-1}}\,(1-t^2)^{\alpha-1+n/2} g(\xi, \cos^{-1} t),$$

and invoke Riemann-Liouville fractional integrals [55]

$$(I^\alpha_- u)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^1 (\tau-t)^{\alpha-1} u(\tau)\,d\tau, \qquad Re\,\alpha > 0. \tag{6.4}$$

Thus, if $Re\,\alpha > 0$, then

$$(I^\alpha_- F_\xi)(t) = G_{\alpha,\xi}(t). \tag{6.5}$$

Since f is infinitely differentiable and the support of f is separated from the boundary ∂B_θ, then F_ξ is infinitely differentiable on $(-1,1)$ uniformly in ξ and $\operatorname{supp} F_\xi$ does not meet the endpoints ± 1. It follows that (6.5) extends by analyticity to all complex α, and we have

$$(Mf)(\xi,t) = (1-t^2)^{1-n/2}(I^{-\alpha}_- G_{\alpha,\xi})(t)$$

where $I^{-\alpha}_-$ is understood in the sense of analytic continuation. Now Theorem 5.3 yields the following explicit solution of our inverse problem.

Theorem 6.1. *Let B_θ be the spherical cap on S^n of radius $\theta \in (0, \pi/2]$,*

$$G_{\alpha,\xi}(t)=\frac{2^{1-\alpha}\pi^{n/2}}{\Gamma(\alpha+n/2)\,\sigma_{n-1}}\,(1-t^2)^{\alpha-1+n/2}g(\xi,\cos^{-1}t), \qquad Re\,\alpha>-n/2,$$

where g is a given function on $\partial B_\theta \times (0,\pi)$. If the initial function f in the Cauchy problem (6.2) is infinitely differentiable and the support of f lies in the interior of B_θ, then f can be reconstructed by the formula

$$(6.6) \quad f(x)=\frac{d_n\, x_{n+1}}{\sin\theta}\,\Delta_{x'} f_0(x', \sqrt{1-|x'|^2}), \quad d_n=\frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $f_0(x) \equiv f_0(x', \sqrt{1-|x'|^2})$ has the following form:

$$f_0(x) = -\int\limits_{\partial B_\theta} (d/dt)^{n-3}[(I_-^{-\alpha}G_{\alpha,\xi})(t)]\Big|_{t=\xi\cdot x}\, d\xi$$

if $n = 3, 5, \ldots,$ and

$$f_0(x) = \frac{1}{\pi}\int\limits_{\partial B_\theta} d\xi \int\limits_{\cos 2\theta}^{1} (d/dt)^{n-2}[(I_-^{-\alpha}G_{\alpha,\xi})(t)]\,\log|t-\xi\cdot x|\,dt$$

if $n = 2, 4, 6, \ldots$.

We recall that the case $\alpha = (1-n)/2$ in this theorem gives a solution of the relevant inverse problem for the wave equation on S^n in the framework of the formal spherical TAT model.

7. Spherical Means in the Hyperbolic Space

Most of the facts listed below can be found in [60]. Let $\mathbb{E}^{n,1}$, $n \ge 2$, be the real pseudo-Euclidean space of points $x = (x_1, \ldots, x_{n+1})$ with the inner product

$$(7.1) \qquad [x,y] = -x_1y_1 - \cdots - x_ny_n + x_{n+1}y_{n+1}.$$

The hyperbolic space $\mathbb{H}^n$ is interpreted as the "upper" sheet of the two-sheeted hyperboloid

$$(7.2) \qquad \mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : [x,x] = 1, x_{n+1} > 0\}.$$

The hyperbolic coordinates of a point $x = (x_1, \ldots, x_{n+1}) \in \mathbb{H}^n$ are defined by

$$(7.3) \quad \begin{cases} x_1 = \sinh r \sin\omega_{n-1} \ldots \sin\omega_2 \sin\omega_1, \\ x_2 = \sinh r \sin\omega_{n-1} \ldots \sin\omega_2 \cos\omega_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = \sinh r\, \cos\omega_{n-1}, \\ x_{n+1} = \cosh r, \end{cases}$$

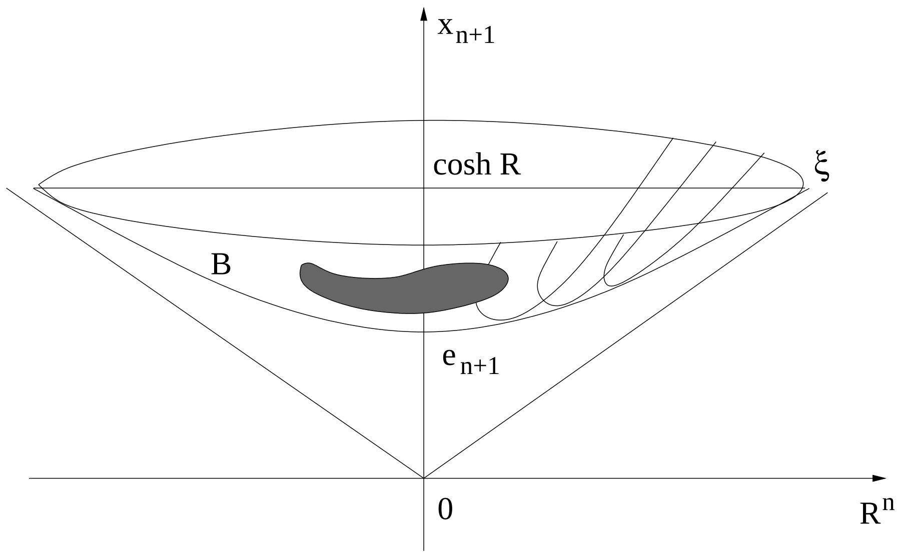

FIGURE 3. The hyperbolic case.

where

$$0 \le \omega_1 < 2\pi; \quad 0 \le \omega_j < \pi, \quad 1 < j \le n-1; \quad 0 \le r < \infty.$$

By (7.3), each $x \in \mathbb{H}^n$ can be represented as

$$x = \omega \sinh r + e_{n+1} \cosh r \ = (\omega \sinh r, \cosh r) \tag{7.4}$$

where ω is a point of the unit sphere S^{n-1} in $\mathbb{R}^n$ with Euler angles $\omega_1, \ldots, \omega_{n-1}$. We regard $\mathbb{R}^n$ as the hyperplane $x_{n+1} = 0$ in $E^{n,1}$. The invariant (with respect to hyperbolic motions) measure dx in $\mathbb{H}^n$ is given by $dx = \sinh^{n-1} r \, d\omega dr$, where $d\omega$ is the surface element of S^{n-1}. The geodesic distance between points x and y in $\mathbb{H}^n$ is defined by

$$\text{dist}(x, y) = \cosh^{-1}[x, y] \qquad (\text{i.e., } \cosh \text{dist}(x, y) = [x, y]).$$

Given $x \in \mathbb{H}^n$ and $t > 1$, let

$$(Mf)(x, t) = \frac{(t^2-1)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{[x,y]=t} f(y) \, d\sigma(y) \tag{7.5}$$

be the mean value of f over the planar section $\{y \in \mathbb{H}^n : [x, y] = t\}$.

As before, our aim is to reconstruct a function $f \in C^\infty(\mathbb{H}^n)$ under the following assumptions:

(a) The support of f lies in the geodesic ball (Fig.3)

$$B = \{x \in \mathbb{H}^n : \text{dist}(x, e_{n+1}) < R\} = \{x \in \mathbb{H}^n : x_{n+1} < \cosh R\},$$

where $e_{n+1} = (0, \ldots, 0, 1)$ is the origin of $\mathbb{H}^n$ and $R > 0$ is fixed.

(b) The mean values (7.5) are known for all $x = \xi \in \partial B$ and all $t > 1$, where ∂B is the boundary of B.

Owing to (7.4), we can write $x \in \mathbb{H}^n$ as

$$x = (x', \sqrt{1 + |x'|^2}), \qquad x' = (x_1, \dots, x_n, 0) \in \mathbb{R}^n,$$

so that

$$\int\limits_{\mathbb{H}^n} f(x)\, dx = \int\limits_{\mathbb{R}^n} f(x', \sqrt{1+|x'|^2})\, \rho(x')\, dx', \quad \rho(x') = \sqrt{\frac{1+2|x'|^2}{1+|x'|^2}}. \tag{7.6}$$

We introduce the "back-projection" operator P that sends functions on $\partial B \times (1, \infty)$ to functions on B by the formula

$$(PF)(x) = \frac{1}{|\partial B|} \int\limits_{\partial B} F(\xi, [\xi, x])\, d\sigma(\xi), \qquad x \in B. \tag{7.7}$$

Let

$$\tilde{B} = \{x' \in \mathbb{R}^n : |x'| < \sinh R\}$$

be the orthogonal projection of B onto the hyperplane $x_{n+1} = 0$. If $\xi = e_{n+1} \cosh R + \omega \sinh R$, $\omega \in S^{n-1}$, and $x = (x', x_{n+1}) \in B$, then

$$[\xi, x] = \sqrt{1 + |x'|^2}\, \cosh R - (x' \cdot \omega)\, \sinh R,$$

and $(PF)(x)$ is actually a function of $x' \in \tilde{B}$. We denote this function by $(\tilde{P}F)(x')$.

7.1. **The case $n > 2$.** Let, as above, $\xi \in \partial B$, $t > 1$. Consider the analytic family of operators

$$(N^\alpha f)(\xi, t) = \int\limits_B \frac{|[\xi, y] - t|^{\alpha - 1}}{\Gamma(\alpha/2)}\, f(y)\, dy, \qquad Re\, \alpha > 0, \tag{7.8}$$

Lemma 7.1. *If $f \in C^\infty(\mathbb{H}^n)$, $\text{supp} f \subset B$, then*

$$\underset{\alpha = 3-n}{a.c.} (\tilde{P} N^\alpha f)(x') = \frac{\Gamma(n/2)\, (\sinh R)^{2-n}}{\pi^{1/2}} \int\limits_{\tilde{B}} \frac{\tilde{f}(y')}{|x' - y'|^{n-2}}\, dy', \tag{7.9}$$

$$\tilde{f}(y') = f(y', \sqrt{1 + |y'|^2}) \sqrt{\frac{1 + 2|y'|^2}{1 + |y'|^2}}. \tag{7.10}$$

Proof. For $Re\, \alpha > 0$, changing the order of integration, we obtain

$$(PN^\alpha f)(x) = \int\limits_B f(y)\, k_\alpha(x, y)\, dy, \quad k_\alpha(x, y) = \frac{1}{|\partial B|} \int\limits_{\partial B} \frac{|[\xi, (x - y)]|^{\alpha - 1}}{\Gamma(\alpha/2)} d\sigma(\xi).$$

Since ξ has the form $\xi = e_{n+1}\cosh R + \omega \sinh R$, $\omega \in S^{n-1}$, then

$$\begin{aligned} |[\xi,(x-y)]| &= |(x_{n+1}-y_{n+1})\cosh\, R - (x'-y')\cdot\omega\ \sinh R| \\ &= |h-\omega\cdot\sigma|\ |x'-y'|\sinh R, \end{aligned}$$

$$h = \frac{x_{n+1}-y_{n+1}}{|x'-y'|}\coth R, \qquad \sigma = \frac{x'-y'}{|x'-y'|}. \tag{7.11}$$

Hence,

$$\begin{aligned} k_\alpha(x,y) &= \frac{(|x'-y'|\sinh R)^{\alpha-1}}{\sigma_{n-1}} \int\limits_{S^{n-1}} \frac{|h-\omega\cdot\sigma|^{\alpha-1}}{\Gamma(\alpha/2)}\,d\omega \\ &= \frac{\sigma_{n-2}\,(|x'-y'|\sinh R)^{\alpha-1}}{\sigma_{n-1}}\,g_\alpha(h), \quad \text{cf. (5.10)}. \end{aligned}$$

If $|h| < 1$, we can apply Lemma 2.2 and write the integral over B as that over $\tilde{B} \subset \mathbb{R}^n$. This will give the result.

It remains to show that $|h| < 1$. By symmetry we may suppose that $|y'| \le |x'|$, which we shall do from now on. Let

$$a = |y'|, \qquad b = |x'|, \qquad b_0 = \sinh R.$$

Since $|x'-y'| \ge |x'| - |y'|$, then

$$h \le f_a(b)\coth R, \qquad f_a(b) = \frac{\sqrt{1+b^2}-\sqrt{1+a^2}}{b-a}.$$

For a fixed, the function $f_a(b)$ is increasing in (a,∞), because

$$f_a'(b) = \frac{D(a,b)}{(b-a)^2\sqrt{1+b^2}}, \quad D(a,b) = \sqrt{(1+b^2)(1+a^2)} - 1 - ab > 0.$$

Hence, $h \le f_a(\sinh R)\coth R$. The right-hand side of this inequality is less than 1. Indeed, setting $b_0 = \sinh R$, we have

$$\begin{aligned} & f_a(\sinh R)\coth R = \frac{\sqrt{1+b_0^2}-\sqrt{1+a^2}}{b_0-a}\,\frac{\sqrt{1+b_0^2}}{b_0} < 1, \\ & \Leftrightarrow \frac{1+b_0^2-\sqrt{(1+a^2)(1+b_0^2)}}{(b_0-a)\,b_0} < 1, \\ & \Leftrightarrow 1+b_0^2-\sqrt{(1+a^2)(1+b_0^2)} < (b_0-a)\,b_0, \\ & \Leftrightarrow 0 < D(a,b_0). \end{aligned}$$

This completes the proof. □

Lemma 7.2. *Let* $f \in C^\infty(\mathbb{H}^n)$, $\text{supp} f \subset B$. *Then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{(\sinh R)^{n-1}} \int\limits_{\partial B} (d/dt)^{n-3}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\Big|_{t=[\xi,x]} d\xi,$$

if $n = 3, 5, \dots,$ *and*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi\,(\sinh R)^{n-1}} \int\limits_{\partial B} d\xi$$

$$\times \int\limits_1^{\cosh 2R} (d/dt)^{n-2}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\,\log|t-[\xi,x]|\,dt,$$

if $n = 4, 6, \dots,$ *where* δ_n *is defined by (3.6).*

Proof. For $Re\,\alpha > 0$, by making use of the formula

$$\text{(7.12)} \quad \int\limits_{\mathbb{H}^n} f(y)\,a([\xi,y])\,dy = \sigma_{n-1}\int\limits_1^\infty a(\tau)(Mf)(\xi,\tau)\,(\tau^2-1)^{n/2-1}\,d\tau,$$

we have

$$(N^\alpha f)(\xi,t) = \frac{\sigma_{n-1}}{\Gamma(\alpha/2)}\int\limits_1^\infty (Mf)(\xi,\tau)\,|\tau-t|^{\alpha-1}(\tau^2-1)^{n/2-1}\,d\tau$$

$$= \int\limits_{\mathbb{R}} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\,\varphi_\xi(\tau+t)\,d\tau, \quad \varphi_\xi(\tau) = \sigma_{n-1}\,(Mf)(\xi,\tau)\,(\tau^2-1)_+^{n/2-1}.$$

Since f is smooth and the support of f is separated from the boundary ∂B, then $(Mf)(\xi,\tau)$ is smooth in the τ-variable uniformly in ξ and vanishes identically in the respective neighborhood of $\tau = 1$. Thus, Lemma 2.1 yields the following equalities:
For $n = 3, 5, \dots$:

$$\underset{\alpha=3-n}{a.c.}(N^\alpha f)(\xi,t) = \delta_n\,\varphi_\xi^{(n-3)}(t),$$

For $n = 4, 6, \dots$:

$$\underset{\alpha=3-n}{a.c.}(N^\alpha f)(\xi,t) = -\frac{\delta_n}{\pi}\int\limits_1^{\cosh 2R} \varphi_\xi^{(n-2)}(\tau)\,\log|\tau-t|\,d\tau,$$

δ_n being the constant from (3.6). Now the result follows; cf. Lemmas 3.2 and 5.2. $\square$

Lemmas 7.1 and 7.2 imply the following inversion result for the spherical means on $\mathbb{H}^n$. We recall that $\Delta_{x'}$ denotes the usual Laplace operator in the x'-variable.

Theorem 7.3. *Let $n > 2$. An infinitely differentiable function f supported in the geodesic ball $B = \{x \in \mathbb{H}^n : \operatorname{dist}(x, e_{n+1}) < R\}$, can be reconstructed from its spherical means $(Mf)(\xi, \tau)$, $(\xi, t) \in \partial B \times (1, \infty)$, by the formula*

$$f(x) = \frac{d_n\, x_{n+1}}{|x|\, \sinh R}\, \Delta_{x'} f_0(x', \sqrt{1 - |x'|^2}), \qquad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $|x| = \sqrt{|x'|^2 + x_{n+1}^2}$ and $f_0(x) \equiv f_0(x', \sqrt{1 + |x'|^2})$ has the following form:

$$f_0(x) = -\int_{\partial B} (d/dt)^{n-3}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\Big|_{t=[\xi,x]}\, d\xi,$$

if $n = 3, 5, \dots,$ and

$$f_0(x) = \frac{1}{\pi} \int_{\partial B} d\xi \int_1^{\cosh 2R} (d/dt)^{n-2}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\, \log|t - [\xi, x]|\, dt$$

if $n = 4, 6, \dots.$

7.2. **The case $n = 2$.** The argument follows Section 5.2 almost verbatim. Let

$$(I_* f)(x) = \frac{1}{2\pi} \int_B f(y) \log|x' - y'|\, dy, \tag{7.13}$$

so that

$$\Delta_{x'}(I_* f)(x) = \tilde{f}(x') = |x|\, f(x)/x_3. \tag{7.14}$$

Lemma 7.4. *If f be a C^∞ function supported in B, then*

$$(I_* f)(x) = \frac{1}{|\partial B|} \int_{\partial B} \int_1^\infty (Mf)(\xi, \tau) \log|\tau - [\xi, x]|\, d\tau\, d\xi + c_f, \tag{7.15}$$

$$c_f = -\frac{1}{2\pi}\left(\log \frac{\sinh R}{2}\right) \int_B f(y)\, dy.$$

Proof. Let

$$(N_* f)(\xi, t) = \int_B f(y) \log|[\xi, y] - t|\, dy, \qquad (\xi, t) \in \partial B \times (1, \infty).$$

Changing the order of integration, we obtain

$$(PN_* f)(x) = \int\limits_B f(y)\, k_*(x,y)\, dy,$$

$k_*(x, y = \log|x' - y'| + \log\sinh R - \log 2$ (see the proof of Lemma 5.4) This gives

$$(PN_* f)(x) = 2\pi\, (I_* f)(x) + \Big(\log \frac{\sinh R}{2}\Big) \int\limits_B f(y)\, dy. \tag{7.16}$$

On the other hand, by (7.12),

$$(PN_* f)(x) = \frac{1}{\sinh R} \int\limits_{\partial B} \int\limits_1^\infty (Mf)(\xi, \tau) \log|\tau - [\xi, x]|\, d\tau d\xi. \tag{7.17}$$

Comparing (7.17) with (7.16), we obtain (7.15). □

Owing to (7.14), Lemma 7.4 allows us complete Theorem 7.3 as follows.

Theorem 7.5. *An infinitely differentiable function f supported in the geodesic ball $B = \{x \in \mathbb{H}^2 : \mathrm{dist}(x, e_3) < R\}$ can be reconstructed from its spherical means $(Mf)(\xi, \tau)$, $(\xi, t) \in \partial B \times (1, \infty)$, by the formula*

$$f(x) = \frac{x_3}{2\pi\, |x|\, \sinh R}\, \Delta_{x'} \int\limits_{\partial B} \int\limits_1^\infty (Mf)(\xi, \tau) \log|\tau - [\xi, x]|\, d\tau\, d\xi. \tag{7.18}$$

Remark 7.6. As in Section 6, Theorems 7.3 and 7.5 can be applied to solution of inverse problems for the EPD equation in the hyperbolic space. The reasoning follows the same lines as before. We leave it to the interested reader.

8. Appendix: Proof of Lemma 2.2

It is convenient to split the proof in two parts.
(i) We recall the notation

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{-1}^{1} |t - h|^{\alpha - 1} (1 - t^2)^{(n-3)/2}\, dt, \qquad Re\, \alpha > 0, \tag{8.1}$$

where $n > 2$ and $|h| \le 1-\delta,\ \delta > 0$. Changing variables $t = 2\tau - 1$, $h = 2\xi - 1$, we write $g_\alpha(h) \equiv G_\alpha((1+h)/2)$, where

$$\begin{aligned} G_\alpha(\xi) &= \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} \int_0^1 |\tau-\xi|^{\alpha-1}\,(1-\tau)^{(n-3)/2}\,\tau^{(n-3)/2}\,dt \\ &= U_\alpha(\xi) + U_\alpha(1-\xi), \qquad \delta/2 \le \xi \le 1-\delta/2, \end{aligned} \tag{8.2}$$

$$U_\alpha(\xi) = \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} \int_0^\xi (\xi-\tau)^{\alpha-1}\tau^{(n-3)/2}(1-\tau)^{(n-3)/2}\,d\tau.$$

The last integral expresses through the Gauss hypergeometric function so that $U_\alpha(\xi) = a_\xi(\alpha)\,b(\alpha)\,\zeta_\alpha(\xi)$, where

$$a_\xi(\alpha) = 2^{\alpha+n-3}\,\xi^{(n-3)/2+\alpha}\,\Gamma((n-1)/2), \qquad b(\alpha) = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)},$$

$$\zeta_\alpha(\xi) = \frac{1}{\Gamma(\alpha+(n-1)/2)}\,F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n-1}{2}+\alpha;\,\xi\right);$$

see, e.g., [51, 2.2.6(1)]. Owing to [16, 2.1.6], $\zeta_\alpha(\xi)$ extends as an entire function of α, which is represented by an absolutely convergent power series. Since $\delta/2 \le \xi \le 1-\delta/2$, this series converges uniformly in $\alpha \in K$ for any compact subset K of the complex plane. Furthermore, $a_\xi(\alpha)$ is also an entire function and $b(\alpha)$ is meromorphic with the only poles $-1, -3, -5, \ldots$. Since g_α is an even function, i.e., $g_\alpha(h) = g_\alpha(-h)$, these poles are eventually removable. Hence, $G_\alpha(\xi)$ extends to all complex α as an entire function of α and this extension represents a C^∞ function of ξ uniformly in $\alpha \in K$. This gives the desired result for $g_\alpha(h)$.

(ii) To compute analytic continuation of g_α at $\alpha = 3-n$, first, we assume $1/2 < Re\,\alpha < 1$ and $|Im\,\alpha| < 1$ and represent $G_\alpha(\xi)$ as a Mellin convolution

$$G_\alpha(\xi) = \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)}\,f(\xi), \quad f(\xi) = \int_0^\infty f_1(\tau) f_2\left(\frac{\xi}{\tau}\right)\frac{d\tau}{\tau}, \tag{8.3}$$

where

$$f_1(\tau) = \begin{cases} \tau^{\alpha+(n-3)/2}(1-\tau)^{(n-3)/2} & \text{if } 0<\tau<1, \\ 0, & \text{if } 1<\tau<\infty, \end{cases} \qquad f_2(\tau) = |1-\tau|^{\alpha-1}.$$

The Mellin transforms $\tilde f_j(s) = \int_0^\infty f_j(\tau)\tau^{s-1}d\tau$ $(j=1,2)$ are evaluated as

$$\tilde f_1(s) = \frac{\Gamma(s+\alpha+(n-3)/2)\Gamma((n-1)/2)}{\Gamma(s+\alpha+n-2)}, \qquad Re\,s > \frac{3-n}{2} - Re\,\alpha\,,$$

$$\tilde{f}_2(s) = \frac{\Gamma(s)\Gamma(\alpha)}{\Gamma(s+\alpha)} + \frac{\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)}, \qquad 0 < Re\, s < 1 - Re\, \alpha.$$

By applying the convolution theorem and the relevant Mellin inversion formula [58], we obtain

$$f(\xi) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \tilde{f}(s)\, \xi^{-s}\, ds, \qquad 0 < \kappa < 1 - Re\, \alpha,$$

where $\tilde{f}(s) = \tilde{f}_1(s)\tilde{f}_2(s)$. The function $\tilde{f}(s)$ has poles in the half-plane $Re\, s < \kappa$ at the points $s = -j$ and $s = -j - \alpha - (n-3)/2$, $j = 0, 1, 2, \ldots$. Since $1/2 < Re\, \alpha < 1$, all these poles are simple, and the Cauchy residue theorem yields

$$f(\xi) = \Gamma\left(\frac{n-1}{2}\right)\Gamma(\alpha) \sum_{j=0}^{\infty} \frac{(-\xi)^j}{j!} \left[\frac{\Gamma(\alpha - j + (n-3)/2)}{\Gamma(\alpha - j)\Gamma(\alpha - j + n - 2)} \right.$$
$$\left. + \frac{\xi^{\alpha + (n-3)/2}}{\Gamma((n-1)/2 - j)} \left(\frac{\Gamma(-\alpha + (3-n)/2 - j)}{\Gamma((3-n)/2 - j)} + \frac{\Gamma((n-1)/2 + j)}{\Gamma(\alpha + (n-1)/2 + j)} \right) \right].$$

Ultimately, we arrive at the following expression for $G_\alpha(\xi)$:

$$\begin{aligned} G_\alpha(\xi) &= \lambda_1\, F\left(1 - \alpha, 3 - \alpha - n; \frac{5-n}{2} - \alpha; \xi\right) \\ &+ \lambda_2\, F\left(\frac{3-n}{2}, \frac{n-1}{2}; \frac{n-1}{2} + \alpha; \xi\right), \end{aligned} \tag{8.4}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\Gamma((n-1)/2)}{2^{3-\alpha-n}\Gamma(\alpha/2)} \frac{(-1)^n\, \Gamma(3 - \alpha - n)\, \sin \alpha\pi}{\Gamma((5-n)/2 - \alpha)\, \cos(\alpha + n/2)\pi}, \\ \lambda_2 &= \frac{\Gamma((n-1)/2)}{2^{3-\alpha-n}\Gamma(\alpha/2)} \frac{\xi^{\alpha + (n-3)/2}\, \Gamma(\alpha)}{\Gamma(\alpha + (n-1)/2)} \left(1 + \frac{\cos n\pi/2}{\cos(\alpha + n/2)\pi}\right). \end{aligned}$$

Case 1. Let $n = 2m$, $m = 2, 3, \ldots$. Then

$$G_\alpha(\xi) = \frac{\pi\, \Gamma(m - 1/2)}{2^{3-\alpha-2m}\, \Gamma(\alpha/2)\, \cos \alpha\pi} [D_1(\xi; \alpha) + D_2(\xi; \alpha)], \tag{8.5}$$

where

$$\begin{aligned} D_1(\xi; \alpha) &= \frac{F(1 - \alpha, 3 - \alpha - 2m; 5/2 - \alpha - m; \xi)}{(-1)^m \Gamma(\alpha + 2m - 2)\, \Gamma(5/2 - \alpha - m)}, \\ D_2(\xi; \alpha) &= \frac{\cot(\alpha\pi/2)\, F(3/2 - m, m - 1/2, \alpha + m - 1/2; \xi)}{\xi^{3/2 - \alpha - m}\, \Gamma(1 - \alpha)\, \Gamma(\alpha + m - 1/2)}. \end{aligned}$$

A simple computation yields $\underset{\alpha = 3 - 2m}{a.c.}\, G_\alpha(\xi) = \Gamma(m - 1/2) = \Gamma((n-1)/2)$.

Case 2. Let $n = 2m+1$, $m = 1, 2, \ldots$. Then

$$G_\alpha(\xi) = \frac{\Gamma(m)\,\Gamma(1-\alpha/2)}{2^{2-\alpha-2m}\,\cos(\alpha\pi/2)}\,[E_1(\xi;\alpha) + E_2(\xi;\alpha)], \tag{8.6}$$

where

$$\begin{aligned} E_1(\xi;\alpha) &= \frac{F(1-\alpha, 2-\alpha-2m; 2-\alpha-m; \xi)}{(-1)^{m+1}\,\Gamma(\alpha-1+2m)\,\Gamma(2-\alpha-m)}, \\ E_2(\xi;\alpha) &= \frac{\xi^{\alpha+m-1} F(1-m, m; \alpha+m; \xi)}{\Gamma(1-\alpha)\,\Gamma(\alpha+m)}. \end{aligned}$$

Passing to the limit as $\alpha \to 2-2m$, we obtain $\underset{\alpha=2-2m}{a.c.}\, G_\alpha(\xi) = \Gamma(m) = \Gamma((n-1)/2)$. This completes the proof. □

Remark 8.1. The basic equality (8.4) can be proved in a different way if we rearrange hypergeometric functions in (8.2) using known formulas. Specifically, the second term in (8.2) can be transformed by formulas (33), (6), and (21) from [16, Section 2.9]. This gives

$$U_\alpha(1-\xi) = \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)}\, B\left(\frac{n-1}{2}, \alpha\right)\,(A(\xi) + B(\xi)),$$

$$\begin{aligned} A(\xi) &= \gamma_1(\alpha)\, F\left(1-\alpha, 3-n-\alpha; \frac{5-n}{2} - \alpha; \xi\right), \\ B(\xi) &= (\xi(1-\xi))^{\alpha+(n-3)/2}\gamma_2(\alpha)\, F\left(\alpha, \alpha+n-2; \frac{n-1}{2} + \alpha; \xi\right), \end{aligned}$$

$$\begin{aligned} \gamma_1(\alpha) &= \frac{\Gamma(\alpha+(n-1)/2)\,\Gamma(\alpha+(n-3)/2)}{\Gamma(\alpha)\,\Gamma(\alpha+n-2)}, \\ \gamma_2(\alpha) &= \frac{\Gamma(\alpha+(n-1)/2)\,\Gamma((3-n)/2-\alpha)}{\Gamma((n-1)/2)\,\Gamma((3-n)/2)} = \frac{\sin(n-1)/2)\pi}{\sin(n-1)/2+\alpha)\pi}. \end{aligned}$$

Owing to [16, 2.9(2)],

$$B(\xi) = \gamma_2(\alpha)\,\xi^{(n-3)/2+\alpha}\, F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n-1}{2} + \alpha; \xi\right).$$

This gives (8.4).

An alternative proof of (ii) The following alternative proof is instructive and leads to the same result. In fact, it suffices to prove the equality

$$\underset{\alpha=3-n}{a.c.}\, g_\alpha(h) = \Gamma((n-1)/2) \tag{8.7}$$

in the weak sense. Indeed, suppose that

$$\text{(8.8)} \quad \underset{\alpha=3-n}{a.c.}\,(g_\alpha,\psi)\equiv \underset{\alpha=3-n}{a.c.}\int\limits_{\mathbb{R}} g_\alpha(h)\,\psi(h)\,dh=\Gamma((n-1)/2)\int\limits_{\mathbb{R}}\psi(h)\,dh$$

for any C^∞ function ψ with compact support in the interval $(-1,1)$. Since, by Part (i), the analytic continuation of $g_\alpha(h)$ represents a C^∞ function of h uniformly in $\alpha\in K$ for any compact subset K of the complex plane, then (use, e.g., [52, Lemma 1.17])

$$\underset{\alpha=3-n}{a.c.}\,(g_\alpha,\psi)=(\underset{\alpha=3-n}{a.c.}\,g_\alpha,\psi)$$

and (8.8) yields $(\underset{\alpha=3-n}{a.c.}\,g_\alpha,\psi)=\Gamma((n-1)/2)\,(1,\psi)$. This implies (8.7).

Let us prove (8.8). We denote

$$\rho_\alpha(t)=\frac{|t|^{\alpha-1}}{\Gamma(\alpha/2)},\qquad \omega(t)=(1-t^2)_+^{(n-3)/2},$$

where $(\cdot)_+$ stands for zero when the expression in brackets is non-positive. We interpret these functions as $\mathcal{D}'$-distributions on $\mathbb{R}$. Then g_α is a convolution of ρ_α with the compactly supported distribution ω so that $(g_\alpha,\psi)=(\rho_\alpha(s),(\omega(t),\psi(s+t)))$; see, [27, Ch. I, Sec. 4(2)].

If n is odd, $n=2m+3$, $m=0,1,\dots$, then (2.1) yields

$$\begin{aligned}\underset{\alpha=3-n}{a.c.}\,(g_\alpha,\psi) &= \underset{\alpha=-2m}{a.c.}\,(g_\alpha,\psi)=c_{m,1}\left(\frac{d}{ds}\right)^{2m}(\omega(t),\psi(s+t))\Big|_{s=0}\\ &= c_{m,1}\,(\omega,\psi^{(2m)})=c_{m,1}\,([(1-t^2)^m]^{(2m)},\psi(t))\\ &= (-1)^m c_{m,1}\,(2m)!=m!=\Gamma((n-1)/2).\end{aligned}$$

Let now n be even. Since the convolution is commutative,

$$a.c.\,(g_\alpha,\psi)=a.c.\,(\omega(t),(\rho_\alpha(s),\psi(s+t)))=(\omega(t),a.c.\,(\rho_\alpha(s),\psi(s+t))).$$

If $n=2m+2$ $(m=1,2,\dots)$, then, applying (2.2) and changing variables, we have

$$\underset{\alpha=3-n}{a.c.}\,(g_\alpha,\psi)=\underset{\alpha=1-2m}{a.c.}\,(g_\alpha,\psi)=c_{m,2}\left(\omega(t),\text{p.v.}\int\limits_{\mathbb{R}}\frac{\psi^{(2m-1)}(s+t)}{s}\,ds\right)$$

$$=\ (-1)^{m+1}c_{m,2}\int\limits_{-1}^{1}\psi^{(2m-1)}(h)\,q(h)\,dh=(-1)^m c_{m,2}\int\limits_{-1}^{1}\psi(h)\,q^{(2m-1)}(h)\,dh,$$

$$c_{m,2}=\frac{1}{\Gamma(1/2-m)\,(2m-1)!},\qquad q(h)=\text{p.v.}\int\limits_{-1}^{1}\frac{(t^2-1)^m}{(t-h)\sqrt{1-t^2}}\,dt$$

(interchange of the order of integration can be justified if we complete $[-1,1]$ to a closed contour and use [25, Section 7.1]). Now, $q(h)$ is a polynomial with leading term πh^{2m-1}. This follows from the well known relation for Chebyshev polynomials

$$\text{p.v.} \int_{-1}^{1} \frac{T_n(t)\,dt}{(t-h)\sqrt{1-t^2}} = \pi U_{n-1}(h), \quad -1 < h < 1, \tag{8.9}$$

and the fact that the leading terms of $T_n(h)$ and $U_n(h)$ are $2^{n-1}h^n$ and $2^n h^n$, respectively; see formulas 10.11(47), 10.11(22), and 10.11(23) in [16, vol. II)]. Hence integration by parts yields

$$\underset{\alpha=3-n}{a.c.} (g_\alpha, \psi) = (-1)^m c_{m,2}\, \pi\, (2m-1)! \int_{-1}^{1} \psi(h)\,dh = \Gamma(m+1/2) \int_{-1}^{1} \psi(h)\,dh,$$

where $\Gamma(m+1/2) = \Gamma((n-1)/2)$. Thus, we are done. □

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Department of Mathematics, Louisiana State University, Baton Rouge, LA, 70803 USA
E-mail address: antipov@math.lsu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, LA, 70803 USA
E-mail address: restrada@math.lsu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, LA, 70803 USA
E-mail address: borisr@math.lsu.edu